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**International Baccalaureate
Diploma Programme**

Roger Dixon
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Robert Haese
Michael Haese
Sandra Haese



Haese & Harris Publications

MATHEMATICS FOR THE INTERNATIONAL STUDENT

Mathematics HL (Core) – WORKED SOLUTIONS

International Baccalaureate Diploma Programme

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FOREWORD

This book gives you fully worked solutions for every question in each chapter of the Haese & Harris Publications textbook **Mathematics HL (Core)** which is one of three textbooks in our series ‘Mathematics for the International Student’. The other two textbooks are **Mathematics SL** and **Mathematical Studies SL**, and books of fully worked solutions are available for those textbooks also.

Correct answers can sometimes be obtained by different methods. In this book, where applicable, each worked solution is modeled on the worked example in the textbook.

Be aware of the limitations of calculators and computer modelling packages. Understand that when your calculator gives an answer that is different from the answer you find in the book, you have not necessarily made a mistake, but the book may not be wrong either.

We have a list of errata for **Mathematics HL (Core)** on our website. Please contact us if you have any additions to this list.

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Background knowledge

EXERCISE A

- 1**
- a** $\sqrt{3} \times \sqrt{5}$
 $= \sqrt{3 \times 5}$
 $= \sqrt{15}$
- b** $(\sqrt{3})^2$
 $= \sqrt{3} \times \sqrt{3}$
 $= 3$
- c** $2\sqrt{2} \times \sqrt{2}$
 $= 2(\sqrt{2} \times \sqrt{2})$
 $= 2 \times 2$
 $= 4$
- d** $3\sqrt{2} \times 2\sqrt{2}$
 $= (3 \times 2)(\sqrt{2} \times \sqrt{2})$
 $= 6 \times 2$
 $= 12$
- e** $3\sqrt{7} \times 2\sqrt{7}$
 $= (3 \times 2)(\sqrt{7} \times \sqrt{7})$
 $= 6 \times 7$
 $= 42$
- f** $\frac{\sqrt{12}}{\sqrt{2}}$
 $= \sqrt{\frac{12}{2}}$
 $= \sqrt{6}$
- g** $\frac{\sqrt{12}}{\sqrt{6}}$
 $= \sqrt{\frac{12}{6}}$
 $= \sqrt{2}$
- h** $\frac{\sqrt{18}}{\sqrt{3}}$
 $= \sqrt{\frac{18}{3}}$
 $= \sqrt{6}$
- 2**
- a** $2\sqrt{2} + 3\sqrt{2}$
 $= (2 + 3)\sqrt{2}$
 $= 5\sqrt{2}$
- b** $2\sqrt{2} - 3\sqrt{2}$
 $= (2 - 3)\sqrt{2}$
 $= -\sqrt{2}$
- c** $5\sqrt{5} - 3\sqrt{5}$
 $= (5 - 3)\sqrt{5}$
 $= 2\sqrt{5}$
- d** $5\sqrt{5} + 3\sqrt{5}$
 $= (5 + 3)\sqrt{5}$
 $= 8\sqrt{5}$
- e** $3\sqrt{5} - 5\sqrt{5}$
 $= (3 - 5)\sqrt{5}$
 $= -2\sqrt{5}$
- f** $7\sqrt{3} + 2\sqrt{3}$
 $= (7 + 2)\sqrt{3}$
 $= 9\sqrt{3}$
- g** $9\sqrt{6} - 12\sqrt{6}$
 $= (9 - 12)\sqrt{6}$
 $= -3\sqrt{6}$
- h** $\sqrt{2} + \sqrt{2} + \sqrt{2}$
 $= 3 \times \sqrt{2}$
 $= 3\sqrt{2}$
- 3**
- a** $\sqrt{8}$
 $= \sqrt{4 \times 2}$
 $= \sqrt{4} \times \sqrt{2}$
 $= 2\sqrt{2}$
- b** $\sqrt{12}$
 $= \sqrt{4 \times 3}$
 $= \sqrt{4} \times \sqrt{3}$
 $= 2\sqrt{3}$
- c** $\sqrt{20}$
 $= \sqrt{4 \times 5}$
 $= \sqrt{4} \times \sqrt{5}$
 $= 2\sqrt{5}$
- d** $\sqrt{32}$
 $= \sqrt{16 \times 2}$
 $= \sqrt{16} \times \sqrt{2}$
 $= 4\sqrt{2}$
- e** $\sqrt{27}$
 $= \sqrt{9 \times 3}$
 $= \sqrt{9} \times \sqrt{3}$
 $= 3\sqrt{3}$
- f** $\sqrt{45}$
 $= \sqrt{9 \times 5}$
 $= \sqrt{9} \times \sqrt{5}$
 $= 3\sqrt{5}$
- g** $\sqrt{48}$
 $= \sqrt{16 \times 3}$
 $= \sqrt{16} \times \sqrt{3}$
 $= 4\sqrt{3}$
- h** $\sqrt{54}$
 $= \sqrt{9 \times 6}$
 $= \sqrt{9} \times \sqrt{6}$
 $= 3\sqrt{6}$
- i** $\sqrt{50}$
 $= \sqrt{25 \times 2}$
 $= \sqrt{25} \times \sqrt{2}$
 $= 5\sqrt{2}$
- j** $\sqrt{80}$
 $= \sqrt{16 \times 5}$
 $= \sqrt{16} \times \sqrt{5}$
 $= 4\sqrt{5}$
- k** $\sqrt{96}$
 $= \sqrt{16 \times 6}$
 $= \sqrt{16} \times \sqrt{6}$
 $= 4\sqrt{6}$
- l** $\sqrt{108}$
 $= \sqrt{36 \times 3}$
 $= \sqrt{36} \times \sqrt{3}$
 $= 6\sqrt{3}$
- 4**
- a** $4\sqrt{3} - \sqrt{12}$
 $= 4\sqrt{3} - \sqrt{4 \times 3}$
 $= 4\sqrt{3} - 2 \times \sqrt{3}$
 $= 4\sqrt{3} - 2\sqrt{3}$
 $= 2\sqrt{3}$
- b** $3\sqrt{2} + \sqrt{50}$
 $= 3\sqrt{2} + \sqrt{25 \times 2}$
 $= 3\sqrt{2} + 5 \times \sqrt{2}$
 $= 3\sqrt{2} + 5\sqrt{2}$
 $= 8\sqrt{2}$
- c** $3\sqrt{6} + \sqrt{24}$
 $= 3\sqrt{6} + \sqrt{4 \times 6}$
 $= 3\sqrt{6} + 2 \times \sqrt{6}$
 $= 3\sqrt{6} + 2\sqrt{6}$
 $= 5\sqrt{6}$
- d** $2\sqrt{27} + 2\sqrt{12}$
 $= 2\sqrt{9 \times 3} + 2\sqrt{4 \times 3}$
 $= 6\sqrt{3} + 4\sqrt{3}$
 $= 10\sqrt{3}$
- e** $\sqrt{75} - \sqrt{12}$
 $= \sqrt{25 \times 3} - \sqrt{4 \times 3}$
 $= 5\sqrt{3} - 2\sqrt{3}$
 $= 3\sqrt{3}$
- f** $\sqrt{2} + \sqrt{8} - \sqrt{32}$
 $= \sqrt{2} + \sqrt{4 \times 2} - \sqrt{16 \times 2}$
 $= \sqrt{2} + 2\sqrt{2} - 4\sqrt{2}$
 $= -\sqrt{2}$

5 a	$\frac{1}{\sqrt{2}}$	b	$\frac{6}{\sqrt{3}}$	c	$\frac{7}{\sqrt{2}}$	d	$\frac{10}{\sqrt{5}}$	e	$\frac{10}{\sqrt{2}}$
	$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$		$= \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$		$= \frac{7}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$		$= \frac{10}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$		$= \frac{10}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
	$= \frac{\sqrt{2}}{2}$		$= \frac{6\sqrt{3}}{3}$		$= \frac{7\sqrt{2}}{2}$		$= \frac{10\sqrt{5}}{5}$		$= \frac{10\sqrt{2}}{2}$
			$= 2\sqrt{3}$				$= 2\sqrt{5}$		$= 5\sqrt{2}$
f	$\frac{18}{\sqrt{6}}$	g	$\frac{12}{\sqrt{3}}$	h	$\frac{5}{\sqrt{7}}$	i	$\frac{14}{\sqrt{7}}$	j	$\frac{2\sqrt{3}}{\sqrt{2}}$
	$= \frac{18}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$		$= \frac{12}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$		$= \frac{5}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$		$= \frac{14}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$		$= \frac{2\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$
	$= \frac{18\sqrt{6}}{6}$		$= \frac{12\sqrt{3}}{3}$		$= \frac{5\sqrt{7}}{7}$		$= \frac{14\sqrt{7}}{7}$		$= \frac{2\sqrt{6}}{2}$
	$= 3\sqrt{6}$		$= 4\sqrt{3}$				$= 2\sqrt{7}$		$= \sqrt{6}$

EXERCISE B

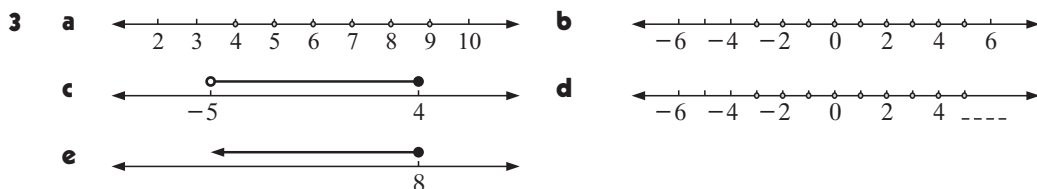
1 a	259	b	259 000	c	2.59
	$= \overbrace{2.59} \times 10^2$		$= \overbrace{2.59000} \times 10^5$		$= 2.59 \times 1$
	$= 2.59 \times 10^2$		$= 2.59 \times 10^5$		$= 2.59 \times 10^0$
d	0.259	e	0.000 259	f	40.7
	$= \overbrace{02.59} \div 10$		$= \overbrace{00002.59} \div 10^4$		$= \overbrace{4.07} \times 10$
	$= 2.59 \times 10^{-1}$		$= 2.59 \times 10^{-4}$		$= 4.07 \times 10^1$
g	4070	h	0.0407	i	407 000
	$= \overbrace{4.070} \times 10^3$		$= \overbrace{004.07} \div 10^2$		$= \overbrace{4.07000} \times 10^5$
	$= 4.07 \times 10^3$		$= 4.07 \times 10^{-2}$		$= 4.07 \times 10^5$
j	407 000 000	k	0.000 040 7		
	$= \overbrace{4.07000000} \times 10^8$		$= \overbrace{000004.07} \div 10^5$		
	$= 4.07 \times 10^8$		$= 4.07 \times 10^{-5}$		
2 a	149 500 000 000 m	b	0.0003 mm	c	0.001 mm
	$= \overbrace{1.49500000000} \times 10^{11}$		$= \overbrace{0003.} \times 10^{-4}$		$= \overbrace{001.} \times 10^{-3}$
	$= 1.495 \times 10^{11}$ m		$= 3 \times 10^{-4}$ mm		$= 1 \times 10^{-3}$ mm
d	15 million degrees	e	300 000 times		
	$= 15\,000\,000\,^{\circ}\text{C}$		$= 3 \times 100\,000$		
	$= 1.500\,000\,0 \times 10^7\,^{\circ}\text{C}$		$= 3 \times 10^5$ times		
	$= 1.5 \times 10^7\,^{\circ}\text{C}$				
3 a	4×10^3	b	5×10^2	c	2.1×10^3
	$= 4 \times 1000$		$= 5 \times 100$		$= \overbrace{2.100} \times 10^3$
	$= 4000$		$= 500$		$= 2100$
d	7.8×10^4	e	3.8×10^5	f	8.6×10^1
	$= \overbrace{7.8000} \times 10^4$		$= \overbrace{3.80000} \times 10^5$		$= \overbrace{8.6} \times 10$
	$= 78\,000$		$= 380\,000$		$= 86$
g	4.33×10^7	h	6×10^7		
	$= \overbrace{4.3300000} \times 10^7$		$= 6 \times 10\,000\,000$		
	$= 43\,300\,000$		$= 60\,000\,000$		

- 4 a** 4×10^{-3}
 $= \overline{004} \div 10^3$
 $= 0.004$
- b** 5×10^{-2}
 $= \overline{05} \div 10^2$
 $= 0.05$
- c** 2.1×10^{-3}
 $= \overline{002.1} \div 10^3$
 $= 0.0021$
- d** 7.8×10^{-4}
 $= \overline{0007.8} \div 10^4$
 $= 0.00078$
- e** 3.8×10^{-5}
 $= \overline{00003.8} \div 10^5$
 $= 0.000038$
- f** 8.6×10^{-1}
 $= \overline{8.6} \div 10^1$
 $= 0.86$
- g** 4.33×10^{-7}
 $= \overline{0000004.33} \div 10^7$
 $= 0.000000433$
- h** 6×10^{-7}
 $= \overline{0000006} \div 10^7$
 $= 0.0000006$
- 5 a** $9 \times 10^{-7} \text{ m}$
 $= \overline{0000009} \div 10^7$
 $= 0.0000009 \text{ m}$
- b** $6.130 \times 10^9 \text{ people}$
 $= \overline{6.13000000} \times 10^9$
 $= 6\,130\,000\,000 \text{ people}$
- c** $1 \times 10^5 \text{ light years}$
 $= 1 \times 100\,000$
 $= 100\,000 \text{ light years}$
- d** $1 \times 10^{-5} \text{ mm}$
 $= \overline{00001} \div 10^5$
 $= 0.00001 \text{ mm}$
- 6 a** $(3.42 \times 10^5) \times (4.8 \times 10^4)$
 $= (3.42 \times 4.8) \times (10^5 \times 10^4)$
 $= 16.416 \times 10^9$
 $= 1.6416 \times 10^{10}$
 $= 1.64 \times 10^{10} \text{ (2 d.p.)}$
- b** $(6.42 \times 10^{-2})^2$
 $= (6.42)^2 \times (10^{-2})^2$
 $= 41.2164 \times 10^{-4}$
 $= 4.12164 \times 10^{-3}$
 $= 4.12 \times 10^{-3} \text{ (2 d.p.)}$
- c** $\frac{3.16 \times 10^{-10}}{6 \times 10^7}$
 $= \frac{3.16}{6} \times \frac{10^{-10}}{10^7}$
 $= 0.52\overline{6} \times 10^{-17}$
 $= 5.2\overline{6} \times 10^{-18}$
 $= 5.27 \times 10^{-18} \text{ (2 d.p.)}$
- d** $(9.8 \times 10^{-4}) \div (7.2 \times 10^{-6})$
 $= \frac{9.8 \times 10^{-4}}{7.2 \times 10^{-6}}$
 $= \frac{9.8}{7.2} \times \frac{10^{-4}}{10^{-6}}$
 $= 1.3\overline{6} \times 10^2$
 $= 1.36 \times 10^2 \text{ (2 d.p.)}$
- e** $\frac{1}{3.8 \times 10^5}$
 $= \frac{1}{3.8} \times 10^{-5}$
 $= 2.63 \times 10^{-6} \text{ (2 d.p.)}$
- f** $(1.2 \times 10^3)^3$
 $= (1.2)^3 \times (10^3)^3$
 $= 1.728 \times 10^9$
 $= 1.73 \times 10^9 \text{ (2 d.p.)}$
- 7 a** 1 day = 24 hours
i.e., missile travels 5400×24
 $= 129\,600$
 $= 1.296 \times 10^5$
 $\div 1.30 \times 10^5 \text{ km}$
- b** 1 week = 7 days
 $= 7 \times 24 \text{ hours}$
 $= 168 \text{ hours}$
i.e., missile travels 5400×168
 $= 907\,200$
 $= 9.072 \times 10^5$
 $\div 9.07 \times 10^5 \text{ km}$
- c** 2 years = $2 \times 365.25 \text{ days}$
 $= 730.5 \text{ days}$
 $= 730.5 \times 24 \text{ hours}$
 $= 17\,532 \text{ hours}$
i.e., missile travels $5400 \times 17\,532$
 $= 94\,672\,800$
 $= 9.46728 \times 10^7$
 $\div 9.47 \times 10^7 \text{ km}$

- 8 a** distance = speed $\times$ time
time = 1 minute = 60 seconds
so, light travels $(3 \times 10^8) \times 60$
 $= 180 \times 10^8$
 $= 1.80 \times 10^{10}$ m
- b** distance = speed $\times$ time
time = 1 day = 24 hours
 $= 24 \times 60 \times 60$ seconds
 $= 86\,400$ seconds
 $= 8.64 \times 10^4$ seconds
i.e., light travels $(3 \times 10^8) \times (8.64 \times 10^4)$
 $= 3 \times 8.64 \times 10^{12}$
 $= 25.92 \times 10^{12}$
 $\div 2.59 \times 10^{13}$ m
- c** distance = speed $\times$ time
time = 1 year = 365.25 days
 $= 365.25 \times 8.64 \times 10^4$ sec {from **b**}
 $= 3155.76 \times 10^4$
 $\div 3.16 \times 10^7$ sec
i.e., light travels $(3 \times 10^8) \times (3.156 \times 10^7)$
 $= 3 \times 3.156 \times 10^{15}$
 $= 9.468 \times 10^{15}$
 $\div 9.47 \times 10^{15}$ m

EXERCISE C

- 1 a** $\{x : x > 5\}$ reads ‘the set of all x such that x is greater than 5’
b $\{x : x \leq 3\}$ reads ‘the set of all x such that x is less than or equal to 3’
c $\{y : 0 < y < 6\}$ reads ‘the set of all y such that y lies between 0 and 6’
d $\{x : 2 \leq x \leq 4\}$ reads ‘the set of all x such that x is greater than or equal to 2, but less than or equal to 4’
e $\{t : 1 < t < 5\}$ reads ‘the set of all t such that t lies between 1 and 5’
f $\{n : n < 2 \text{ or } n \geq 6\}$ reads ‘the set of all n such that n is less than 2 or greater than or equal to 6’
- 2 a** $\{x : x > 2\}$ **b** $\{x : 1 < x \leq 5\}$ **c** $\{x : x \leq -2 \text{ or } x \geq 3\}$
d $\{x : x \in \mathbb{Z}, -1 \leq x \leq 3\}$ **e** $\{x : x \in \mathbb{Z}, 0 \leq x \leq 5\}$ **f** $\{x : x < 0\}$

**EXERCISE D**

- 1 a** $3x + 7x - 10$
 $= 10x - 10$ **b** $3x + 7x - x$
 $= 9x$ **c** $2x + 3x + 5y$
 $= 5x + 5y$
- d** $8 - 6x - 2x$
 $= 8 - 8x$ **e** $7ab + 5ba$
 $= 7ab + 5ab$
 $= 12ab$ **f** $3x^2 + 7x^3$
 $= 3x^2 + 7x^3$
i.e., cannot be simplified
- 2 a** $3(2x + 5) + 4(5 + 4x)$
 $= 6x + 15 + 20 + 16x$
 $= 22x + 35$ **b** $6 - 2(3x - 5)$
 $= 6 - 6x + 10$
 $= 16 - 6x$

$$\begin{aligned} \text{c} \quad & 5(2a - 3b) - 6(a - 2b) \\ &= 10a - 15b - 6a + 12b \\ &= 4a - 3b \end{aligned}$$

$$\begin{aligned} 3 \quad \text{a} \quad & 2x(3x)^2 \\ &= 2x \times 9x^2 \\ &= 18x^3 \\ \text{b} \quad & \frac{3a^2b^3}{9ab^4} \\ &= \frac{\cancel{3} \times \cancel{a} \times a \times \cancel{b} \times \cancel{b} \times \cancel{b}}{\cancel{3} \times 3 \times \cancel{a} \times \cancel{b} \times \cancel{b} \times \cancel{b} \times b} \\ &= \frac{a}{3b} \end{aligned}$$

$$\begin{aligned} \text{d} \quad & 3x(x^2 - 7x + 3) - (1 - 2x - 5x^2) \\ &= 3x^3 - 21x^2 + 9x - 1 + 2x + 5x^2 \\ &= 3x^3 - 16x^2 + 11x - 1 \end{aligned}$$

$$\begin{aligned} \text{c} \quad & \sqrt{16x^4} \\ &= \sqrt{16} \times \sqrt{x^4} \\ &= 4 \times \sqrt{(x^2)^2} \\ &= 4x^2 \\ \text{d} \quad & (2a^2)^3 \times 3a^4 \\ &= 2^3 \times (a^2)^3 \times 3a^4 \\ &= 8 \times a^6 \times 3a^4 \\ &= 24a^{10} \end{aligned}$$

EXERCISE E

$$\begin{aligned} 1 \quad \text{a} \quad & 2x + 5 = 25 \\ & \therefore 2x = 20 \\ & \therefore x = 10 \\ \text{b} \quad & 3x - 7 > 11 \\ & \therefore 3x > 18 \\ & \therefore x > 6 \\ \text{c} \quad & 5x + 16 = 20 \\ & \therefore 5x = 4 \\ & \therefore x = \frac{4}{5} \\ \text{d} \quad & \frac{x}{3} - 7 = 10 \\ & \therefore \frac{x}{3} = 17 \\ & \therefore x = 51 \\ \text{e} \quad & 6x + 11 < 4x - 9 \\ & \therefore 2x < -20 \\ & \therefore x < -10 \\ \text{f} \quad & \frac{3x - 2}{5} = 8 \\ & \therefore 3x - 2 = 40 \\ & \therefore 3x = 42 \\ & \therefore x = 14 \\ \text{g} \quad & 1 - 2x \geq 19 \\ & \therefore -2x \geq 18 \\ & \therefore 2x \leq -18 \\ & \therefore x \leq -9 \\ \text{h} \quad & \frac{1}{2}x + 1 = \frac{2}{3}x - 2 \\ & \therefore \frac{3}{6}x - \frac{4}{6}x = -3 \\ & \therefore -\frac{1}{6}x = -3 \\ & \therefore x = 18 \\ \text{i} \quad & \frac{2}{3} - \frac{3x}{4} = \frac{1}{2}(2x - 1) \quad \text{Multiplying each term by} \\ & \quad \quad \quad \text{the LCD of 12 gives} \\ & \therefore 8 - 9x = 6(2x - 1) \\ & \therefore 8 - 9x = 12x - 6 \\ & \therefore 14 = 21x \quad \text{i.e., } x = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} 2 \quad \text{a} \quad & x + 2y = 9 \quad \dots (1) \\ & x - y = 3 \quad \dots (2) \\ & \text{Multiplying (2) by 2 gives} \\ & \quad x + 2y = 9 \\ & \quad 2x - 2y = 6 \\ & \hline & \therefore 3x = 15 \quad \{\text{adding}\} \\ & \therefore x = 5 \\ & \text{Substituting } x = 5 \text{ into (2) gives} \\ & \quad 5 - y = 3 \\ & \therefore y = 2 \\ & \therefore x = 5 \text{ and } y = 2 \\ \text{b} \quad & 2x + 5y = 28 \quad \dots (1) \\ & x - 2y = 2 \quad \dots (2) \\ & \text{Multiplying (2) by } -2 \text{ gives} \\ & \quad 2x + 5y = 28 \\ & \quad -2x + 4y = -4 \\ & \hline & \therefore 9y = 24 \quad \{\text{adding}\} \\ & \therefore y = \frac{24}{9} = \frac{8}{3} \\ & \text{Substituting } y = \frac{8}{3} \text{ into (2) gives} \\ & \quad x - 2\left(\frac{8}{3}\right) = 2 \quad \therefore x - \frac{16}{3} = 2 \quad \text{and so } x = \frac{22}{3} \\ & \therefore x = \frac{22}{3} \text{ and } y = \frac{8}{3} \\ \text{c} \quad & 7x + 2y = -4 \quad \dots (1) \\ & 3x + 4y = 14 \quad \dots (2) \\ & \text{Multiplying (1) by } -2 \text{ gives} \\ & \quad -14x - 4y = 8 \\ & \quad 3x + 4y = 14 \\ & \hline & \therefore -11x = 22 \quad \{\text{adding}\} \\ & \therefore x = -2 \\ & \text{Substituting } x = -2 \text{ into (2) gives} \\ & \quad 3(-2) + 4y = 14 \\ & \therefore -6 + 4y = 14 \\ & \therefore 4y = 20 \quad \text{and } \therefore y = 5 \\ & \therefore x = -2 \text{ and } y = 5 \\ \text{d} \quad & 5x - 4y = 27 \quad \dots (1) \\ & 3x + 2y = 9 \quad \dots (2) \\ & \text{Multiplying (2) by 2 gives} \\ & \quad 5x - 4y = 27 \\ & \quad 6x + 4y = 18 \\ & \hline & \therefore 11x = 45 \quad \{\text{adding}\} \\ & \therefore x = \frac{45}{11} \\ & \text{Substituting } x = \frac{45}{11} \text{ into (1) gives} \\ & \quad 5\left(\frac{45}{11}\right) - 4y = 27 \quad \therefore \frac{225}{11} - 27 = 4y \\ & \therefore 4y = -\frac{72}{11} \quad \text{and } \therefore y = -\frac{18}{11} \\ & \therefore x = \frac{45}{11} \text{ and } y = -\frac{18}{11} \end{aligned}$$

$$\mathbf{e} \quad x + 2y = 5 \quad \dots (1)$$

$$2x + 4y = 1 \quad \dots (2)$$

Multiplying (1) by -2 gives

$$-2x - 4y = -10$$

$$2x + 4y = 1$$

$$\hline \therefore 0 = -9 \quad \{\text{adding}\}$$

which is absurd

$\therefore$ there are no solutions

$$\mathbf{f} \quad \frac{x}{2} + \frac{y}{3} = 5 \quad \dots (1)$$

$$\frac{x}{3} + \frac{y}{4} = 1 \quad \dots (2)$$

Multiplying (1) by 18 and (2) by -24 gives

$$9x + 6y = 90 \quad \dots (3)$$

$$-8x - 6y = -24$$

$$\hline \therefore x = 66 \quad \{\text{adding}\}$$

Substituting $x = 66$ into (3) gives

$$9 \times 66 + 6y = 90$$

$$\therefore 6y = 90 - 594 = -504$$

$$\therefore y = -84$$

$$\therefore x = 66 \text{ and } y = -84$$

EXERCISE F

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & 5 - (-11) \\ &= 5 + 11 \\ &= 16 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & |5| - |-11| \\ &= 5 - 11 \\ &= -6 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & |5 - (-11)| \\ &= |5 + 11| \\ &= |16| \\ &= 16 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & |(-2)^2 + 11(-2)| \\ &= |4 - 22| \\ &= |-18| \\ &= 18 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & |-6| - |-8| \\ &= 6 - 8 \\ &= -2 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & |-6 - (-8)| \\ &= |-6 + 8| \\ &= |2| \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad & |a| = |-2| \\ &= 2 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & |b| = |3| \\ &= 3 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & |a| |b| = |-2| |3| \\ &= 2 \times 3 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & |ab| = |-2 \times 3| \\ &= |-6| \\ &= 6 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & |a - b| = |-2 - 3| \\ &= |-5| \\ &= 5 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & |a| - |b| = |-2| - |3| \\ &= 2 - 3 \\ &= -1 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & |a + b| = |-2 + 3| \\ &= |1| \\ &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & |a| + |b| = |-2| + |3| \\ &= 2 + 3 \\ &= 5 \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & |a|^2 = |-2|^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

$$\mathbf{j} \quad a^2 = (-2)^2 = 4$$

$$\mathbf{k} \quad \left| \frac{c}{a} \right| = \left| \frac{-4}{-2} \right| = |2| = 2$$

$$\mathbf{l} \quad \frac{|c|}{|a|} = \frac{|-4|}{|-2|} = \frac{4}{2} = 2$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & |x| = 3 \\ \therefore x &= \pm 3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & |x| = -5 \\ \text{but } |x| &\geq 0 \text{ for all } x \\ &(\text{property of modulus}) \\ \therefore &\text{no solution} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & |x| = 0 \\ \therefore x &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & |x - 1| = 3 \\ \therefore x - 1 &= \pm 3 \\ \therefore x &= 1 \pm 3 \\ \therefore x &= -2 \text{ or } 4 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & |3 - x| = 4 \\ \therefore 3 - x &= \pm 4 \\ \therefore -x &= -3 \pm 4 \\ \therefore x &= 3 \mp 4 \\ \therefore x &= -1 \text{ or } 7 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & |x + 5| = -1 \\ \text{but } |x + 5| &\geq 0 \text{ for all } x \\ &(\text{property of modulus}) \\ \therefore &\text{no solution} \end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad |3x - 2| &= 1 \\ \therefore 3x - 2 &= \pm 1 \\ \therefore 3x &= 2 \pm 1 \\ \therefore 3x &= 3 \text{ or } 1 \\ \therefore x &= \frac{1}{3} \text{ or } 1\end{aligned}$$

$$\begin{aligned}\mathbf{h} \quad |3 - 2x| &= 3 \\ \therefore 3 - 2x &= \pm 3 \\ \therefore 2x &= 3 \mp 3 \\ \therefore 2x &= 0 \text{ or } 6 \\ \therefore x &= 0 \text{ or } 3\end{aligned}$$

$$\begin{aligned}\mathbf{i} \quad |2 - 5x| &= 12 \\ \therefore 2 - 5x &= \pm 12 \\ \therefore 5x &= 2 \mp 12 \\ \therefore 5x &= -10 \text{ or } 14 \\ \therefore x &= -2 \text{ or } \frac{14}{5}\end{aligned}$$

EXERCISE G

$$\begin{aligned}\mathbf{1} \quad \mathbf{a} \quad (2x + 3)(x + 1) &= 2x^2 + 2x + 3x + 3 \\ &= 2x^2 + 5x + 3 \\ \mathbf{d} \quad (x + 2)(3x - 5) &= 3x^2 - 5x + 6x - 10 \\ &= 3x^2 + x - 10 \\ \mathbf{g} \quad (3x + 4)(5x - 3) &= 15x^2 - 9x + 20x - 12 \\ &= 15x^2 + 11x - 12 \\ \mathbf{j} \quad (5 - 2x)(3 - 2x) &= 15 - 10x - 6x + 4x^2 \\ &= 4x^2 - 16x + 15\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad (3x + 4)(x + 2) &= 3x^2 + 6x + 4x + 8 \\ &= 3x^2 + 10x + 8 \\ \mathbf{e} \quad (7 - 2x)(2 + 3x) &= 14 + 21x - 4x - 6x^2 \\ &= -6x^2 + 17x + 14 \\ \mathbf{h} \quad (1 - 3x)(2 - 5x) &= 2 - 5x - 6x + 15x^2 \\ &= 15x^2 - 11x + 2 \\ \mathbf{k} \quad -(x + 1)(x + 2) &= -(x^2 + 2x + x + 2) \\ &= -(x^2 + 3x + 2) \\ &= -x^2 - 3x - 2\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad (5x - 2)(2x + 1) &= 10x^2 + 5x - 4x - 2 \\ &= 10x^2 + x - 2 \\ \mathbf{f} \quad (1 - 3x)(5 + 2x) &= 5 + 2x - 15x - 6x^2 \\ &= -6x^2 - 13x + 5 \\ \mathbf{i} \quad (7 - x)(3 - 2x) &= 21 - 14x - 3x + 2x^2 \\ &= 2x^2 - 17x + 21 \\ \mathbf{l} \quad -2(x - 1)(2x + 3) &= -2(2x^2 + 3x - 2x - 3) \\ &= -2(2x^2 + x - 3) \\ &= -4x^2 - 2x + 6\end{aligned}$$

$$\begin{aligned}\mathbf{2} \quad \mathbf{a} \quad (x + 6)(x - 6) &= x^2 - 6^2 \\ &= x^2 - 36 \\ \mathbf{d} \quad (3x - 2)(3x + 2) &= (3x)^2 - 2^2 \\ &= 9x^2 - 4 \\ \mathbf{g} \quad (3 - x)(3 + x) &= 3^2 - x^2 \\ &= 9 - x^2 \\ \mathbf{j} \quad (x + \sqrt{2})(x - \sqrt{2}) &= x^2 - (\sqrt{2})^2 \\ &= x^2 - 2\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad (x + 8)(x - 8) &= x^2 - 8^2 \\ &= x^2 - 64 \\ \mathbf{e} \quad (4x + 5)(4x - 5) &= (4x)^2 - 5^2 \\ &= 16x^2 - 25 \\ \mathbf{h} \quad (7 - x)(7 + x) &= 7^2 - x^2 \\ &= 49 - x^2 \\ \mathbf{k} \quad (x + \sqrt{5})(x - \sqrt{5}) &= x^2 - (\sqrt{5})^2 \\ &= x^2 - 5\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad (2x - 1)(2x + 1) &= (2x)^2 - 1^2 \\ &= 4x^2 - 1 \\ \mathbf{f} \quad (5x - 3)(5x + 3) &= (5x)^2 - 3^2 \\ &= 25x^2 - 9 \\ \mathbf{i} \quad (7 + 2x)(7 - 2x) &= 7^2 - (2x)^2 \\ &= 49 - 4x^2 \\ \mathbf{l} \quad (2x - \sqrt{3})(2x + \sqrt{3}) &= (2x)^2 - (\sqrt{3})^2 \\ &= 4x^2 - 3\end{aligned}$$

$$\begin{aligned}\mathbf{3} \quad \mathbf{a} \quad (x + 5)^2 &= x^2 + 2(x)(5) + 5^2 \\ &= x^2 + 10x + 25 \\ \mathbf{d} \quad (x - 6)^2 &= x^2 - 2(x)(6) + 6^2 \\ &= x^2 - 12x + 36 \\ \mathbf{g} \quad (11 - x)^2 &= 11^2 - 2(11)(x) + x^2 \\ &= x^2 - 22x + 121 \\ \mathbf{j} \quad (3x + 2)^2 &= (3x)^2 + 2(3x)(2) + 2^2 \\ &= 9x^2 + 12x + 4\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad (x + 7)^2 &= x^2 + 2(x)(7) + 7^2 \\ &= x^2 + 14x + 49 \\ \mathbf{e} \quad (3 + x)^2 &= 3^2 + 2(3)(x) + x^2 \\ &= x^2 + 6x + 9 \\ \mathbf{h} \quad (10 - x)^2 &= 10^2 - 2(10)(x) + x^2 \\ &= x^2 - 20x + 100 \\ \mathbf{k} \quad (5 - 2x)^2 &= 5^2 - 2(5)(2x) + (2x)^2 \\ &= 4x^2 - 20x + 25\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad (x - 2)^2 &= x^2 - 2(x)(2) + 2^2 \\ &= x^2 - 4x + 4 \\ \mathbf{f} \quad (5 + x)^2 &= 5^2 + 2(5)(x) + x^2 \\ &= x^2 + 10x + 25 \\ \mathbf{i} \quad (2x + 7)^2 &= (2x)^2 + 2(2x)(7) + 7^2 \\ &= 4x^2 + 28x + 49 \\ \mathbf{l} \quad (7 - 3x)^2 &= 7^2 - 2(7)(3x) + (3x)^2 \\ &= 9x^2 - 42x + 49\end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad y &= 2(x+2)(x+3) \\
 &= 2(x^2 + 3x + 2x + 6) \\
 &= 2(x^2 + 5x + 6) \\
 &= 2x^2 + 10x + 12
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad y &= -(x+1)(x-7) \\
 &= -(x^2 - 7x + x - 7) \\
 &= -(x^2 - 6x - 7) \\
 &= -x^2 + 6x + 7
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad y &= 4(x-1)(x-5) \\
 &= 4(x^2 - 5x - x + 5) \\
 &= 4(x^2 - 6x + 5) \\
 &= 4x^2 - 24x + 20
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad y &= -5(x-1)(x-6) \\
 &= -5(x^2 - 6x - x + 6) \\
 &= -5(x^2 - 7x + 6) \\
 &= -5x^2 + 35x - 30
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad y &= -\frac{5}{2}(x-4)^2 \\
 &= -\frac{5}{2}(x^2 - 2(x)(4) + 4^2) \\
 &= -\frac{5}{2}(x^2 - 8x + 16) \\
 &= -\frac{5}{2}x^2 + 20x - 40
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad &1 + 2(x+3)^2 \\
 &= 1 + 2(x^2 + 2(x)(3) + 3^2) \\
 &= 1 + 2(x^2 + 6x + 9) \\
 &= 1 + 2x^2 + 12x + 18 \\
 &= 2x^2 + 12x + 19
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad &3 - (3-x)^2 \\
 &= 3 - (9 - 2(3)(x) + x^2) \\
 &= 3 - (x^2 - 6x + 9) \\
 &= 3 - x^2 + 6x - 9 \\
 &= -x^2 + 6x - 6
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad &1 + 2(4-x)^2 \\
 &= 1 + 2(4^2 - 2(4)(x) + x^2) \\
 &= 1 + 2(x^2 - 8x + 16) \\
 &= 1 + 2x^2 - 16x + 32 \\
 &= 2x^2 - 16x + 33
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad &(x+2)^2 - (x+1)(x-4) \\
 &= x^2 + 2(x)(2) + 2^2 - (x^2 - 4x + x - 4) \\
 &= x^2 + 4x + 4 - (x^2 - 3x - 4) \\
 &= x^2 + 4x + 4 - x^2 + 3x + 4 \\
 &= 7x + 8
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad y &= 3(x-1)^2 + 4 \\
 &= 3(x^2 - 2(x)(1) + 1^2) + 4 \\
 &= 3(x^2 - 2x + 1) + 4 \\
 &= 3x^2 - 6x + 3 + 4 \\
 &= 3x^2 - 6x + 7
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad y &= -(x+2)^2 - 11 \\
 &= -(x^2 + 2(x)(2) + 2^2) - 11 \\
 &= -(x^2 + 4x + 4) - 11 \\
 &= -x^2 - 4x - 4 - 11 \\
 &= -x^2 - 4x - 15
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad y &= -\frac{1}{2}(x+4)^2 - 6 \\
 &= -\frac{1}{2}(x^2 + 2(x)(4) + 4^2) - 6 \\
 &= -\frac{1}{2}(x^2 + 8x + 16) - 6 \\
 &= -\frac{1}{2}x^2 - 4x - 8 - 6 \\
 &= -\frac{1}{2}x^2 - 4x - 14
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad y &= \frac{1}{2}(x+2)^2 - 6 \\
 &= \frac{1}{2}(x^2 + 2(x)(2) + 2^2) - 6 \\
 &= \frac{1}{2}(x^2 + 4x + 4) - 6 \\
 &= \frac{1}{2}x^2 + 2x + 2 - 6 \\
 &= \frac{1}{2}x^2 + 2x - 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad &2 + 3(x-2)(x+3) \\
 &= 2 + 3(x^2 + 3x - 2x - 6) \\
 &= 2 + 3(x^2 + x - 6) \\
 &= 2 + 3x^2 + 3x - 18 \\
 &= 3x^2 + 3x - 16
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad &5 - (x+5)(x-4) \\
 &= 5 - (x^2 - 4x + 5x - 20) \\
 &= 5 - (x^2 + x - 20) \\
 &= 5 - x^2 - x + 20 \\
 &= -x^2 - x + 25
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad &x^2 - 3x - (x+2)(x-2) \\
 &= x^2 - 3x - (x^2 - 2^2) \\
 &= x^2 - 3x - x^2 + 4 \\
 &= -3x + 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad &(2x+3)^2 + 3(x+1)^2 \\
 &= (2x)^2 + 2(2x)(3) + 3^2 + 3(x^2 + 2(x)(1) + 1^2) \\
 &= 4x^2 + 12x + 9 + 3(x^2 + 2x + 1) \\
 &= 4x^2 + 12x + 9 + 3x^2 + 6x + 3 \\
 &= 7x^2 + 18x + 12
 \end{aligned}$$

$$\begin{array}{ll}
 \text{i} & x^2 + 3x - 2(x-4)^2 \\
 & = x^2 + 3x - 2(x^2 - 2(x)(4) + 4^2) \\
 & = x^2 + 3x - 2(x^2 - 8x + 16) \\
 & = x^2 + 3x - 2x^2 + 16x - 32 \\
 & = -x^2 + 19x - 32 \\
 \text{j} & (3x-2)^2 - 2(x+1)^2 \\
 & = (3x)^2 - 2(3x)(2) + 2^2 - 2(x^2 + 2(x)(1) + 1^2) \\
 & = 9x^2 - 12x + 4 - 2(x^2 + 2x + 1) \\
 & = 9x^2 - 12x + 4 - 2x^2 - 4x - 2 \\
 & = 7x^2 - 16x + 2
 \end{array}$$

EXERCISE H

$$\begin{array}{lll}
 \text{1 a} & 3x^2 + 9x & \text{b} \quad 2x^2 + 7x \\
 & = 3x(x+3) & = x(2x+7) \\
 \text{d} & 6x^2 - 15x & \text{e} \quad 9x^2 - 25 \\
 & = 3x(2x-5) & = (3x)^2 - 5^2 \\
 & & = (3x+5)(3x-5) \\
 \text{g} & 2x^2 - 8 & \text{h} \quad 3x^2 - 9 \\
 & = 2(x^2 - 4) & = 3(x^2 - 3) \\
 & = 2(x^2 - 2^2) & = 3(x^2 - (\sqrt{3})^2) \\
 & = 2(x+2)(x-2) & = 3(x+\sqrt{3})(x-\sqrt{3}) \\
 \text{j} & x^2 - 8x + 16 & \text{k} \quad x^2 - 10x + 25 \\
 & = x^2 - 2(x)(4) + 4^2 & = x^2 - 2(x)(5) + 5^2 \\
 & = (x-4)^2 & = (x-5)^2 \\
 \text{m} & 16x^2 + 40x + 25 & \text{n} \quad 9x^2 + 12x + 4 \\
 & = (4x)^2 + 2(4x)(5) + 5^2 & = (3x)^2 + 2(3x)(2) + 2^2 \\
 & = (4x+5)^2 & = (3x+2)^2 \\
 \text{o} & x^2 - 22x + 121 & \\
 & = x^2 - 2(x)(11) + 11^2 & = (x-11)^2 \\
 \text{2 a} & x^2 + 9x + 8 & \text{b} \quad x^2 + 7x + 12 \\
 & = (x+1)(x+8) & = (x+3)(x+4) \\
 & \{\text{as sum} = 9, \text{ product} = 8\} & \{\text{as sum} = 7, \text{ product} = 12\} \\
 \text{c} & x^2 - 7x - 18 & \text{d} \quad x^2 + 4x - 21 \\
 & = (x-9)(x+2) & = (x+7)(x-3) \\
 & \{\text{as sum} = -7, \text{ product} = -18\} & \{\text{as sum} = 4, \text{ product} = -21\} \\
 \text{e} & x^2 - 9x + 18 & \text{f} \quad x^2 + x - 6 \\
 & = (x-6)(x-3) & = (x+3)(x-2) \\
 & \{\text{as sum} = -9, \text{ product} = 18\} & \{\text{as sum} = 1, \text{ product} = -6\} \\
 \text{g} & -x^2 + x + 2 & \text{h} \quad 3x^2 - 42x + 99 \\
 & = -(x^2 - x - 2) & = 3(x^2 - 14x + 33) \\
 & = -(x-2)(x+1) & = 3(x-3)(x-11) \\
 & \{\text{as sum} = -1, \text{ product} = -2\} & \{\text{as sum} = -14, \text{ product} = 33\} \\
 \text{i} & -2x^2 - 4x - 2 & \text{j} \quad 2x^2 + 6x - 20 \\
 & = -2(x^2 + 2x + 1) & = 2(x^2 + 3x - 10) \\
 & = -2(x^2 + 2(x)(1) + 1^2) & = 2(x+5)(x-2) \\
 & = -2(x+1)^2 & \{\text{as sum} = 3, \text{ product} = -10\}
 \end{array}$$

$$\mathbf{k} \quad 2x^2 - 10x - 48$$

$$= 2(x^2 - 5x - 24)$$

$$= 2(x - 8)(x + 3)$$

$$\{\text{as sum} = -5, \text{ product} = -24\}$$

$$\mathbf{l} \quad -2x^2 + 14x - 12$$

$$= -2(x^2 - 7x + 6)$$

$$= -2(x - 6)(x - 1)$$

$$\{\text{as sum} = -7, \text{ product} = 6\}$$

$$\mathbf{m} \quad -3x^2 + 6x - 3$$

$$= -3(x^2 - 2x + 1)$$

$$= -3(x^2 - 2(x)(1) + 1^2)$$

$$= -3(x - 1)^2$$

$$\mathbf{n} \quad -x^2 - 2x - 1$$

$$= -(x^2 + 2x + 1)$$

$$= -(x^2 + 2(x)(1) + 1^2)$$

$$= -(x + 1)^2$$

$$\bullet \quad -5x^2 + 10x + 40$$

$$= -5(x^2 - 2x - 8)$$

$$= -5(x - 4)(x + 2)$$

$$\{\text{as sum} = -2, \text{ prod.} = -8\}$$

$$\mathbf{3} \quad \mathbf{a} \quad 2x^2 + 5x - 12$$

$$= 2x^2 + 8x - 3x - 12$$

$$= 2x(x + 4) - 3(x + 4)$$

$$= (2x - 3)(x + 4)$$

$$\text{has } ac = 2 \times -12 = -24$$

Factors of -24 which add to 5 are 8 and -3 .

$$\mathbf{b} \quad 3x^2 - 5x - 2$$

$$= 3x^2 - 6x + x - 2$$

$$= 3x(x - 2) + (x - 2)$$

$$= (3x + 1)(x - 2)$$

$$\text{has } ac = 3 \times -2 = -6$$

Factors of -6 which add to -5 are -6 and 1 .

$$\mathbf{c} \quad 7x^2 - 9x + 2$$

$$= 7x^2 - 7x - 2x + 2$$

$$= 7x(x - 1) - 2(x - 1)$$

$$= (7x - 2)(x - 1)$$

$$\text{has } ac = 7 \times 2 = 14$$

Factors of 14 which add to -9 are -7 and -2 .

$$\mathbf{d} \quad 6x^2 - x - 2$$

$$= 6x^2 + 3x - 4x - 2$$

$$= 3x(2x + 1) - 2(2x + 1)$$

$$= (3x - 2)(2x + 1)$$

$$\text{has } ac = 6 \times -2 = -12$$

Factors of -12 which add to -1 are 3 and -4 .

$$\mathbf{e} \quad 4x^2 - 4x - 3$$

$$= 4x^2 + 2x - 6x - 3$$

$$= 2x(2x + 1) - 3(2x + 1)$$

$$= (2x - 3)(2x + 1)$$

$$\text{has } ac = 4 \times -3 = -12$$

Factors of -12 which add to -4 are -6 and 2 .

$$\mathbf{f} \quad 10x^2 - x - 3$$

$$= 10x^2 + 5x - 6x - 3$$

$$= 5x(2x + 1) - 3(2x + 1)$$

$$= (5x - 3)(2x + 1)$$

$$\text{has } ac = 10 \times -3 = -30$$

Factors of -30 which add to -1 are -6 and 5 .

$$\mathbf{g} \quad 2x^2 - 11x - 6$$

$$= 2x^2 - 12x + x - 6$$

$$= 2x(x - 6) + (x - 6)$$

$$= (2x + 1)(x - 6)$$

$$\text{has } ac = 2 \times -6 = -12$$

Factors of -12 which add to -11 are -12 and 1 .

$$\mathbf{h} \quad 3x^2 - 5x - 28$$

$$= 3x^2 - 12x + 7x - 28$$

$$= 3x(x - 4) + 7(x - 4)$$

$$= (3x + 7)(x - 4)$$

$$\text{has } ac = 3 \times -28 = -84$$

Factors of -84 which add to -5 are -12 and 7 .

$$\begin{aligned}
 \mathbf{i} \quad & 8x^2 + 2x - 3 \\
 &= 8x^2 - 4x + 6x - 3 \\
 &= 4x(2x - 1) + 3(2x - 1) \\
 &= (4x + 3)(2x - 1)
 \end{aligned}$$

$$\text{has } ac = 8 \times -3 = -24$$

Factors of -24 which add to 2 are 6 and -4 .

$$\begin{aligned}
 \mathbf{j} \quad & 10x^2 - 9x - 9 \\
 &= 10x^2 - 15x + 6x - 9 \\
 &= 5x(2x - 3) + 3(2x - 3) \\
 &= (5x + 3)(2x - 3)
 \end{aligned}$$

$$\text{has } ac = 10 \times -9 = -90$$

Factors of -90 which add to -9 are -15 and 6 .

$$\begin{aligned}
 \mathbf{k} \quad & 3x^2 + 23x - 8 \\
 &= 3x^2 - x + 24x - 8 \\
 &= x(3x - 1) + 8(3x - 1) \\
 &= (x + 8)(3x - 1)
 \end{aligned}$$

$$\text{has } ac = 3 \times -8 = -24$$

Factors of -24 which add to 23 are 24 and -1 .

$$\begin{aligned}
 \mathbf{l} \quad & 6x^2 + 7x + 2 \\
 &= 6x^2 + 3x + 4x + 2 \\
 &= 3x(2x + 1) + 2(2x + 1) \\
 &= (3x + 2)(2x + 1)
 \end{aligned}$$

$$\text{has } ac = 6 \times 2 = 12$$

Factors of 12 which add to 7 are 4 and 3 .

$$\begin{aligned}
 \mathbf{m} \quad & -4x^2 - 2x + 6 \\
 &= -2(2x^2 + x - 3) \\
 &= -2(2x^2 - 2x + 3x - 3) \\
 &= -2[2x(x - 1) + 3(x - 1)] \\
 &= -2(2x + 3)(x - 1)
 \end{aligned}$$

$$\text{has } ac = 2 \times -3 = -6$$

Factors of -6 which add to 1 are 3 and -2 .

$$\begin{aligned}
 \mathbf{n} \quad & 12x^2 - 16x - 3 \\
 &= 12x^2 - 18x + 2x - 3 \\
 &= 6x(2x - 3) + (2x - 3) \\
 &= (6x + 1)(2x - 3)
 \end{aligned}$$

$$\text{has } ac = 12 \times -3 = -36$$

Factors of -36 which add to -16 are -18 and 2 .

$$\begin{aligned}
 \mathbf{o} \quad & -6x^2 - 9x + 42 \\
 &= -3(2x^2 + 3x - 14) \\
 &= -3(2x^2 - 4x + 7x - 14) \\
 &= -3[2x(x - 2) + 7(x - 2)] \\
 &= -3(2x + 7)(x - 2)
 \end{aligned}$$

$$\text{has } ac = 2 \times -14 = -28$$

Factors of -28 which add to 3 are 7 and -4 .

$$\begin{aligned}
 \mathbf{p} \quad & 21x - 10 - 9x^2 \\
 &= -(9x^2 - 21x + 10) \\
 &= -(9x^2 - 6x - 15x + 10) \\
 &= -[3x(3x - 2) - 5(3x - 2)] \\
 &= -(3x - 5)(3x - 2)
 \end{aligned}$$

$$\text{has } ac = 9 \times 10 = 90$$

Factors of 90 which add to -21 are -6 and -15 .

$$\begin{aligned}
 \mathbf{q} \quad & 8x^2 - 6x - 27 \\
 &= 8x^2 + 12x - 18x - 27 \\
 &= 4x(2x + 3) - 9(2x + 3) \\
 &= (4x - 9)(2x + 3)
 \end{aligned}$$

$$\text{has } ac = 8 \times -27 = -216$$

Factors of -216 which add to -6 are -18 and 12 .

$$\begin{aligned}
 \mathbf{r} \quad & 12x^2 + 13x + 3 \\
 &= 12x^2 + 4x + 9x + 3 \\
 &= 4x(3x + 1) + 3(3x + 1) \\
 &= (4x + 3)(3x + 1)
 \end{aligned}$$

$$\text{has } ac = 12 \times 3 = 36$$

Factors of 36 which add to 13 are 9 and 4 .

$$\begin{aligned}
 \mathbf{s} \quad & 12x^2 + 20x + 3 \\
 & = 12x^2 + 2x + 18x + 3 \\
 & = 2x(6x + 1) + 3(6x + 1) \\
 & = (2x + 3)(6x + 1)
 \end{aligned}$$

$$\text{has } ac = 12 \times 3 = 36$$

Factors of 36 which add to 20 are 2 and 18.

$$\begin{aligned}
 \mathbf{t} \quad & 15x^2 - 22x + 8 \\
 & = 15x^2 - 10x - 12x + 8 \\
 & = 5x(3x - 2) - 4(3x - 2) \\
 & = (5x - 4)(3x - 2)
 \end{aligned}$$

$$\text{has } ac = 15 \times 8 = 120$$

Factors of 120 which add to -22 are -10 and -12 .

$$\begin{aligned}
 \mathbf{u} \quad & 14x^2 - 11x - 15 \\
 & = 14x^2 - 21x + 10x - 15 \\
 & = 7x(2x - 3) + 5(2x - 3) \\
 & = (7x + 5)(2x - 3)
 \end{aligned}$$

$$\text{has } ac = 14 \times -15 = -210$$

Factors of -210 which add to -11 are -21 and 10 .

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & 3(x + 4) + 2(x + 4)(x - 1) \\
 & = (x + 4)[3 + 2(x - 1)] \\
 & = (x + 4)(3 + 2x - 2) \\
 & = (x + 4)(2x + 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 8(2 - x) - 3(x + 1)(2 - x) \\
 & = (2 - x)[8 - 3(x + 1)] \\
 & = (2 - x)(8 - 3x - 3) \\
 & = (2 - x)(5 - 3x)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 6(x + 2)^2 + 9(x + 2) \\
 & = (x + 2)[6(x + 2) + 9] \\
 & = (x + 2)(6x + 12 + 9) \\
 & = (x + 2)(6x + 21) \\
 & = (x + 2) \times 3(2x + 7) \\
 & = 3(x + 2)(2x + 7)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & 4(x + 5) + 8(x + 5)^2 \\
 & = (x + 5)[4 + 8(x + 5)] \\
 & = (x + 5)(4 + 8x + 40) \\
 & = (x + 5)(8x + 44) \\
 & = (x + 5) \times 4(2x + 11) \\
 & = 4(x + 5)(2x + 11)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & (x + 2)(x + 3) - (x + 3)(2 - x) \\
 & = (x + 3)[(x + 2) - (2 - x)] \\
 & = (x + 3)(x + 2 - 2 + x) \\
 & = (x + 3)(2x) \\
 & = 2x(x + 3)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & (x + 3)^2 + 2(x + 3) - x(x + 3) \\
 & = (x + 3)[(x + 3) + 2 - x] \\
 & = (x + 3)(x + 3 + 2 - x) \\
 & = (x + 3)(5) \\
 & = 5(x + 3)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & 5(x - 2) - 3(2 - x)(x + 7) \\
 & = 5(x - 2) + 3(x - 2)(x + 7) \\
 & = (x - 2)[5 + 3(x + 7)] \\
 & = (x - 2)(5 + 3x + 21) \\
 & = (x - 2)(3x + 26)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & 3(1 - x) + 2(x + 1)(x - 1) \\
 & = -3(x - 1) + 2(x + 1)(x - 1) \\
 & = (x - 1)[-3 + 2(x + 1)] \\
 & = (x - 1)(-3 + 2x + 2) \\
 & = (x - 1)(2x - 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad & (x + 3)^2 - 16 \\
 & = (x + 3)^2 - 4^2 \\
 & = (x + 3 + 4)(x + 3 - 4) \\
 & = (x + 7)(x - 1) \\
 \mathbf{c} \quad & (x + 4)^2 - (x - 2)^2 \\
 & = [(x + 4) + (x - 2)][(x + 4) - (x - 2)] \\
 & = (x + 4 + x - 2)(x + 4 - x + 2) \\
 & = (2x + 2)(6) \\
 & = 2(x + 1)(6) \\
 & = 12(x + 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 4 - (1 - x)^2 \\
 & = 2^2 - (1 - x)^2 \\
 & = [2 + (1 - x)][2 - (1 - x)] \\
 & = (2 + 1 - x)(2 - 1 + x) \\
 & = (3 - x)(x + 1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & 16 - 4(x + 2)^2 \\
 & = 4[4 - (x + 2)^2] \\
 & = 4[2 + (x + 2)][2 - (x + 2)] \\
 & = 4(x + 4)(-x) \\
 & = -4x(x + 4)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & (2x+3)^2 - (x-1)^2 \\
 &= [(2x+3) + (x-1)][(2x+3) - (x-1)] \\
 &= (2x+3+x-1)(2x+3-x+1) \\
 &= (3x+2)(x+4)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & 3x^2 - 3(x+2)^2 \\
 &= 3[x^2 - (x+2)^2] \\
 &= 3[x + (x+2)][x - (x+2)] \\
 &= 3(x+x+2)(x-x-2) \\
 &= 3(2x+2)(-2) \\
 &= -6(2x+2) \\
 &= -12(x+1)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & 12x^2 - 27(3+x)^2 \\
 &= 3[4x^2 - 9(3+x)^2] \\
 &= 3[(2x)^2 - 3^2(3+x)^2] \\
 &= 3[(2x)^2 - (3(3+x))^2] \\
 &= 3[2x + 3(3+x)][2x - 3(3+x)] \\
 &= 3[(2x+9+3x)(2x-9-3x)] \\
 &= 3(5x+9)(-x-9) \\
 &= -3(5x+9)(x+9)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & (x+h)^2 - x^2 \\
 &= [(x+h) + x][(x+h) - x] \\
 &= (x+h+x)(x+h-x) \\
 &= (2x+h)(h) \\
 &= h(2x+h)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & 5x^2 - 20(2-x)^2 \\
 &= 5[x^2 - 4(2-x)^2] \\
 &= 5[x^2 - 2^2(2-x)^2] \\
 &= 5[x^2 - (2(2-x))^2] \\
 &= 5[x + 2(2-x)][x - 2(2-x)] \\
 &= 5(x+4-2x)(x-4+2x) \\
 &= 5(-x+4)(3x-4) \\
 &= -5(x-4)(3x-4)
 \end{aligned}$$

EXERCISE I

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & a+x=b \\
 \therefore x &= b-a
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & ax=b \\
 \therefore x &= \frac{b}{a}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 2x+a=d \\
 \therefore 2x &= d-a \\
 \therefore x &= \frac{d-a}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & c+x=t \\
 \therefore x &= t-c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & 5x+2y=20 \\
 \therefore 5x &= 20-2y \\
 \therefore x &= \frac{20-2y}{5}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & 2x+3y=12 \\
 \therefore 2x &= 12-3y \\
 \therefore x &= \frac{12-3y}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & 7x+3y=d \\
 \therefore 7x &= d-3y \\
 \therefore x &= \frac{d-3y}{7}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & ax+by=c \\
 \therefore ax &= c-by \\
 \therefore x &= \frac{c-by}{a}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & y=mx+c \\
 \therefore mx &= y-c \\
 \therefore x &= \frac{y-c}{m}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & az = \frac{b}{c} \\
 \therefore z &= \frac{b}{ac}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{a}{z} = d \\
 \therefore \frac{z}{a} &= \frac{1}{d} \\
 \therefore z &= \frac{a}{d}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{3}{d} = \frac{2}{z} \\
 \therefore 3z &= 2d \\
 \therefore z &= \frac{2d}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & F=ma \\
 \therefore a &= \frac{F}{m}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & C=2\pi r \\
 \therefore r &= \frac{C}{2\pi}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & V=ldh \\
 \therefore d &= \frac{V}{lh}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & A = \frac{b}{K} \\
 \therefore KA &= b \\
 \therefore K &= \frac{b}{A}
 \end{aligned}$$

- 4 a** $A = \pi r^2$
 $\therefore \frac{A}{\pi} = r^2$
 $\therefore r = \sqrt{\frac{A}{\pi}}$
- b** $N = \frac{x^5}{a}$
 $\therefore aN = x^5$
 $\therefore x = \sqrt[5]{aN}$
- c** $V = \frac{4}{3}\pi r^3$
 $\therefore \frac{3}{4}V = \pi r^3$
 $\therefore \frac{3V}{4\pi} = r^3$
 $\therefore r = \sqrt[3]{\frac{3V}{4\pi}}$
- d** $D = \frac{n}{x^3}$
 $\therefore Dx^3 = n$
 $\therefore x^3 = \frac{n}{D}$
 $\therefore x = \sqrt[3]{\frac{n}{D}}$
- 5 a** $d = \frac{\sqrt{a}}{n}$
 $\therefore dn = \sqrt{a}$
 $\therefore a = (dn)^2$
 $\therefore a = d^2n^2$
- b** $T = \frac{1}{5}\sqrt{l}$
 $\therefore 5T = \sqrt{l}$
 $\therefore l = (5T)^2$
 $\therefore l = 25T^2$
- c** $c = \sqrt{a^2 - b^2}$
 $\therefore c^2 = a^2 - b^2$
 $\therefore a^2 = c^2 + b^2$
 $\therefore a = \pm\sqrt{b^2 + c^2}$
- d** $T = 2\pi\sqrt{\frac{l}{g}}$
 $\therefore \frac{T}{2\pi} = \sqrt{\frac{l}{g}}$
 $\therefore \left(\frac{T}{2\pi}\right)^2 = \frac{l}{g}$
 $\therefore \frac{T^2}{4\pi^2} = \frac{l}{g}$
 $\therefore l = \frac{gT^2}{4\pi^2}$
- e** $P = 2(a + b)$
 $\therefore \frac{P}{2} = a + b$
 $\therefore a = \frac{P}{2} - b$
- f** $A = \pi r^2 + 2\pi rh$
 $\therefore A - \pi r^2 = 2\pi rh$
 $\therefore h = \frac{A - \pi r^2}{2\pi r}$
- g** $I = \frac{E}{R + r}$
 $\therefore E = I(R + r)$
 $\therefore \frac{E}{I} = R + r$
 $\therefore r = \frac{E}{I} - R$
- h** $A = \frac{B}{p - q}$
 $\therefore A(p - q) = B$
 $\therefore p - q = \frac{B}{A}$
 $\therefore q = p - \frac{B}{A}$
- 6 a** $k = \frac{d^2}{2ab}$
 $\therefore d^2 = k(2ab)$
 $\therefore \frac{d^2}{k} = 2ab$
 $\therefore a = \frac{d^2}{2kb}$
- b** When $k = 112$, $d = 24$, $b = 2$,
 $a = \frac{24^2}{2 \times 112 \times 2}$
 $= \frac{576}{448}$
 $\div 1.29$
- 7 a** $V = \frac{4}{3}\pi r^3$
 $\therefore \frac{3}{4}V = \pi r^3$
 $\therefore r^3 = \frac{3V}{4\pi}$
 $\therefore r = \sqrt[3]{\frac{3V}{4\pi}}$
- b** When $V = 40$,
 $r = \sqrt[3]{\frac{3 \times 40}{4\pi}}$
 $\div 2.122 \text{ cm}$
- 8 a** $S = \frac{1}{2}at^2$
 $\therefore t^2 = \frac{2S}{a}$
 $\therefore t = \sqrt{\frac{2S}{a}}$
- b** $10 \text{ m} \equiv 10 \times 100 \text{ cm} = 1000 \text{ cm}$
i.e., we need to find t when $a = 8$,
 $S = 1000$
 $t = \sqrt{\frac{2 \times 1000}{8}}$
 $\therefore t \div 15.81 \text{ seconds}$

$$9 \quad \mathbf{a} \quad \frac{1}{f} = \frac{1}{u} + \frac{1}{v} \quad \therefore \quad \frac{1}{f} - \frac{1}{u} = \frac{1}{v}$$

$$\therefore \quad \frac{u-f}{uf} = \frac{1}{v} \quad \text{and so} \quad \therefore \quad v = \frac{uf}{u-f}$$

$$\mathbf{b} \quad \mathbf{i} \quad \text{When } u = 50, \quad f = 8$$

$$\text{so } v = \frac{50 \times 8}{50 - 8}$$

$$\therefore v \doteq 9.52 \text{ cm}$$

$$\mathbf{ii} \quad \text{When } u = 30, \quad f = 8$$

$$\text{so } v = \frac{30 \times 8}{30 - 8}$$

$$\therefore v \doteq 10.9 \text{ cm}$$

10 **a**

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

$$\therefore m \sqrt{1 - \left(\frac{v}{c}\right)^2} = m_0$$

$$\therefore \sqrt{1 - \left(\frac{v}{c}\right)^2} = \frac{m_0}{m}$$

$$\therefore 1 - \left(\frac{v}{c}\right)^2 = \left(\frac{m_0}{m}\right)^2$$

$$\therefore \left(\frac{v}{c}\right)^2 = 1 - \frac{m_0^2}{m^2}$$

$$\therefore \frac{v}{c} = \sqrt{\frac{m^2 - m_0^2}{m^2}}$$

$$\therefore v = \frac{c}{m} \sqrt{m^2 - m_0^2}$$

b When $m = 3m_0$,

$$v = \frac{c}{3m_0} \sqrt{(3m_0)^2 - m_0^2}$$

$$= \frac{c}{3m_0} \sqrt{9m_0^2 - m_0^2}$$

$$= \frac{c}{3m_0} \sqrt{8m_0^2}$$

$$= \frac{c}{3m_0} \sqrt{8} m_0$$

$$= \frac{\sqrt{8}}{3} c$$

c When $m = 30m_0$, $c = 3 \times 10^8$,

$$v = \frac{3 \times 10^8}{30m_0} \sqrt{(30m_0)^2 - m_0^2}$$

$$= \frac{30 \times 10^7}{30m_0} \sqrt{900m_0^2 - m_0^2}$$

$$= \frac{10^7}{m_0} \sqrt{899m_0^2}$$

$$= \frac{10^7}{m_0} \times \sqrt{899} \times m_0$$

$$= 10^7 \times \sqrt{899}$$

$$\doteq 2.998 \times 10^8 \text{ m/s}$$

EXERCISE J

1 **a** $3 + \frac{x}{5}$

$$= 3 \times \frac{5}{5} + \frac{x}{5}$$

$$= \frac{15}{5} + \frac{x}{5}$$

$$= \frac{x+15}{5}$$

b $1 + \frac{3}{x}$

$$= 1 \times \frac{x}{x} + \frac{3}{x}$$

$$= \frac{x}{x} + \frac{3}{x}$$

$$= \frac{x+3}{x}$$

c $3 + \frac{x-2}{2}$

$$= 3 \times \frac{2}{2} + \frac{x-2}{2}$$

$$= \frac{6}{2} + \frac{x-2}{2}$$

$$= \frac{6+x-2}{2}$$

$$= \frac{x+4}{2}$$

$$\mathbf{d} \quad 3 - \frac{x-2}{4}$$

$$= 3 \times \frac{4}{4} - \frac{x-2}{4}$$

$$= \frac{12}{4} - \frac{x-2}{4}$$

$$= \frac{12-x+2}{4}$$

$$= \frac{14-x}{4}$$

$$\mathbf{e} \quad \frac{2+x}{3} + \frac{x-4}{5}$$

$$= \frac{2+x}{3} \times \frac{5}{5} + \frac{x-4}{5} \times \frac{3}{3}$$

$$= \frac{5(2+x)}{15} + \frac{3(x-4)}{15}$$

$$= \frac{10+5x+3x-12}{15}$$

$$= \frac{8x-2}{15}$$

$$\mathbf{f} \quad \frac{2x+5}{4} - \frac{x-1}{6}$$

$$= \frac{2x+5}{4} \times \frac{3}{3} - \frac{x-1}{6} \times \frac{2}{2}$$

$$= \frac{3(2x+5)}{12} - \frac{2(x-1)}{12}$$

$$= \frac{6x+15-2x+2}{12}$$

$$= \frac{4x+17}{12}$$

$$\mathbf{2} \quad \mathbf{a} \quad 1 + \frac{3}{x+2}$$

$$= 1 \times \frac{x+2}{x+2} + \frac{3}{x+2}$$

$$= \frac{x+2+3}{x+2}$$

$$= \frac{x+5}{x+2}$$

$$\mathbf{b} \quad -2 + \frac{3}{x-4}$$

$$= -2 \times \frac{x-4}{x-4} + \frac{3}{x-4}$$

$$= \frac{-2(x-4)+3}{x-4}$$

$$= \frac{-2x+8+3}{x-4}$$

$$= \frac{11-2x}{x-4}$$

$$\mathbf{c} \quad -3 - \frac{2}{x-1}$$

$$= -3 \times \frac{x-1}{x-1} - \frac{2}{x-1}$$

$$= \frac{-3(x-1)-2}{x-1}$$

$$= \frac{-3x+3-2}{x-1}$$

$$= \frac{1-3x}{x-1}$$

$$\mathbf{d} \quad \frac{2x-1}{x+1} + 3$$

$$= \frac{2x-1}{x+1} + 3 \times \frac{x+1}{x+1}$$

$$= \frac{2x-1+3(x+1)}{x+1}$$

$$= \frac{2x-1+3x+3}{x+1}$$

$$= \frac{5x+2}{x+1}$$

$$\mathbf{e} \quad 3 - \frac{x}{x+1}$$

$$= 3 \times \frac{x+1}{x+1} - \frac{x}{x+1}$$

$$= \frac{3(x+1)-x}{x+1}$$

$$= \frac{3x+3-x}{x+1}$$

$$= \frac{2x+3}{x+1}$$

$$\mathbf{f} \quad -1 + \frac{4}{1-x}$$

$$= -1 \times \frac{1-x}{1-x} + \frac{4}{1-x}$$

$$= \frac{-(1-x)+4}{1-x}$$

$$= \frac{x-1+4}{1-x}$$

$$= \frac{x+3}{1-x}$$

$$\mathbf{3} \quad \mathbf{a} \quad \frac{3x}{2x-5} + \frac{2x+5}{x-2}$$

$$= \frac{3x}{2x-5} \times \frac{x-2}{x-2} + \frac{2x+5}{x-2} \times \frac{2x-5}{2x-5}$$

$$= \frac{3x(x-2) + (2x+5)(2x-5)}{(2x-5)(x-2)}$$

$$= \frac{3x^2 - 6x + (4x^2 - 5^2)}{(2x-5)(x-2)}$$

$$= \frac{3x^2 - 6x + 4x^2 - 25}{(2x-5)(x-2)}$$

$$= \frac{7x^2 - 6x - 25}{(2x-5)(x-2)}$$

$$\mathbf{b} \quad \frac{1}{x-2} - \frac{1}{x-3}$$

$$= \frac{1}{x-2} \times \frac{x-3}{x-3} - \frac{1}{x-3} \times \frac{x-2}{x-2}$$

$$= \frac{(x-3) - (x-2)}{(x-2)(x-3)}$$

$$= \frac{x-3-x+2}{(x-2)(x-3)}$$

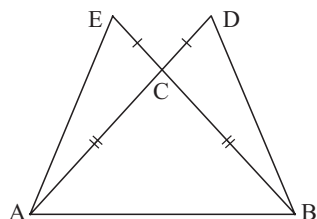
$$= -\frac{1}{(x-2)(x-3)}$$

$$\begin{aligned}
 \text{c} \quad & \frac{5x}{x-4} + \frac{3x-2}{x+4} \\
 &= \frac{5x}{x-4} \times \frac{x+4}{x+4} + \frac{3x-2}{x+4} \times \frac{x-4}{x-4} \\
 &= \frac{5x(x+4) + (3x-2)(x-4)}{(x+4)(x-4)} \\
 &= \frac{5x^2 + 20x + 3x^2 - 12x - 2x + 8}{(x+4)(x-4)} \\
 &= \frac{8x^2 + 6x + 8}{(x+4)(x-4)}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & \frac{2x+1}{x-3} - \frac{x+4}{2x+1} \\
 &= \frac{2x+1}{x-3} \times \frac{2x+1}{2x+1} - \frac{x+4}{2x+1} \times \frac{x-3}{x-3} \\
 &= \frac{(2x+1)^2 - (x+4)(x-3)}{(x-3)(2x+1)} \\
 &= \frac{(2x)^2 + 2(2x)(1) + 1^2 - (x^2 - 3x + 4x - 12)}{(x-3)(2x+1)} \\
 &= \frac{4x^2 + 4x + 1 - (x^2 + x - 12)}{(x-3)(2x+1)} \\
 &= \frac{4x^2 + 4x + 1 - x^2 - x + 12}{(x-3)(2x+1)} \\
 &= \frac{3x^2 + 3x + 13}{(x-3)(2x+1)}
 \end{aligned}$$

EXERCISE K.1

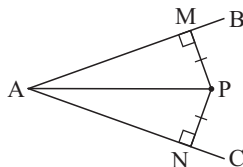
1



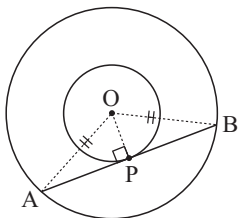
$\angle ECA = \angle DCB$ {opposite angles}
 Also, $EC = CD$ and $AC = BC$ {given}
 $\therefore \Delta s AEC$ and BDC are congruent (SAS)
 $\therefore AE = BD$

2 $MP = NP$ {given}

AP is common to both ΔAMP and ΔANP
 $\therefore \Delta s AMP$ and ANP are congruent (RHS)
 $\therefore \angle MAP = \angle NAP$ {corresponding angles}
 i.e., AP bisects $\angle BAC$
 $\therefore P$ lies on the bisector of $\angle BAC$



3

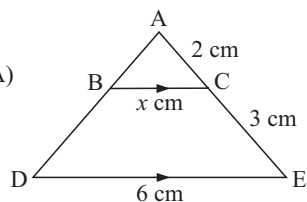


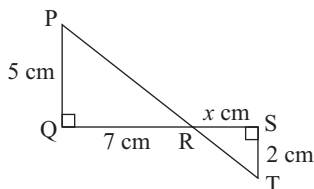
$OA = OB$ {both radii of circle}
 $\angle OPA = \angle OPB = 90^\circ$
 {since AB is a tangent to inner circle}
 OP is common to both ΔAOP and ΔBOP
 $\therefore \Delta s AOP$ and BOP are congruent (RHS)
 $\therefore AP = BP$ {corresponding sides}
 $\therefore P$ is the midpoint of AB

EXERCISE K.2

1 a $\angle ABC = \angle ADE$ {corresponding angles}
 $\angle ACB = \angle AED$ {corresponding angles}
 i.e., $\Delta s ABC$ and ADE are equiangular (AAA)
 and hence similar.

$$\begin{aligned}
 \therefore \frac{x}{2} &= \frac{6}{2+3} = \frac{6}{5} \\
 \therefore x &= \frac{12}{5} = 2.4
 \end{aligned}$$



b

$$\angle PRQ = \angle TRS \quad \{\text{opposite angles}\}$$

$\therefore \triangle s PQR$ and TSR are equiangular (AAA)
and hence similar.

$$\therefore \frac{5}{7} = \frac{2}{x}$$

$$\therefore 5x = 14$$

$$\therefore x = \frac{14}{5} = 2.8$$

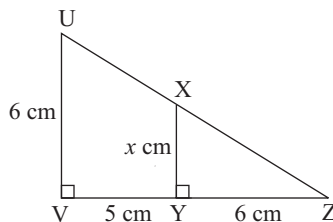
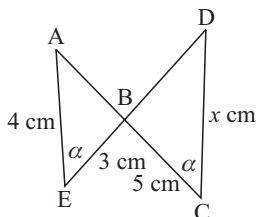
c $\angle VUZ = \angle YXZ$ {corresponding angles}

$\therefore \triangle s UVZ$ and XYZ are equiangular (AAA)
and hence similar.

$$\therefore \frac{x}{6} = \frac{6}{5+6}$$

$$\therefore \frac{x}{6} = \frac{6}{11}$$

$$\therefore x = \frac{36}{11} = 3\frac{3}{11}$$

**d**

$$\angle ABE = \angle DBC \quad \{\text{opposite angles}\}$$

$\therefore \triangle s ABE$ and DBC are equiangular (AAA)
and hence similar.

$$\therefore \frac{x}{5} = \frac{4}{3}$$

$$\therefore x = \frac{20}{3} = 6\frac{2}{3}$$

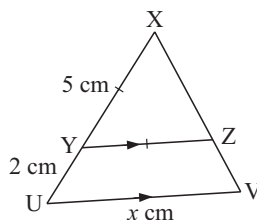
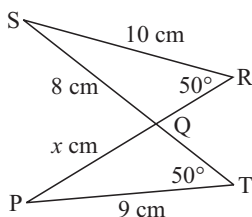
e $\angle XYZ = \angle XUV$ {corresponding angles}

$$\angle XZY = \angle XVU \quad \{\text{corresponding angles}\}$$

$\therefore \triangle s XYZ$ and XUV are equiangular (AAA)
and hence similar.

$$\therefore \frac{x}{2+5} = \frac{5}{5}$$

$$\therefore \frac{x}{7} = 1 \quad \text{and so } x = 7$$

**f**

$$\angle SQR = \angle PQT \quad \{\text{opposite angles}\}$$

$\therefore \triangle s SQR$ and PQT are equiangular (AAA)
and hence similar.

$$\therefore \frac{x}{9} = \frac{8}{10}$$

$$\therefore x = \frac{72}{10} = 7.2$$

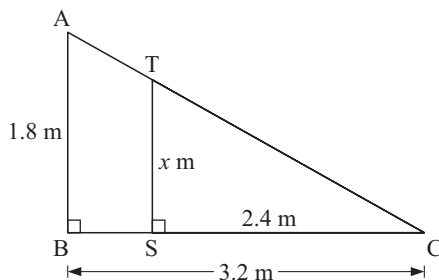
2 $\angle BAC = \angle STC$ {corresponding angles}

$\therefore \triangle s ABC$ and TSC are equiangular (AAA)
and hence similar.

$$\therefore \frac{x}{2.4} = \frac{1.8}{3.2}$$

$$\therefore x = \frac{2.4 \times 1.8}{3.2} = 1.35$$

i.e., the son is 1.35 m tall



EXERCISE L

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad AB &= \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2} \\ &= \sqrt{(4 - 1)^2 + (5 - 3)^2} \\ &= \sqrt{9 + 4} \\ &= \sqrt{13} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad OC &= \sqrt{(x_C - x_O)^2 + (y_C - y_O)^2} \\ &= \sqrt{(3 - 0)^2 + (-5 - 0)^2} \\ &= \sqrt{9 + 25} \\ &= \sqrt{34} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad PQ &= \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2} \\ &= \sqrt{(1 - 5)^2 + (4 - 2)^2} \\ &= \sqrt{16 + 4} \\ &= \sqrt{20} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad ST &= \sqrt{(x_T - x_S)^2 + (y_T - y_S)^2} \\ &= \sqrt{(-1 - 0)^2 + (0 - -3)^2} \\ &= \sqrt{1 + 9} \\ &= \sqrt{10} \text{ units} \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad x_M = \frac{x_A + x_B}{2} = \frac{3 + 1}{2} = 2 \quad \text{and} \quad y_M = \frac{y_A + y_B}{2} = \frac{6 + 0}{2} = 3$$

$\therefore$ the midpoint is at $(2, 3)$

$$\mathbf{b} \quad x_M = \frac{x_A + x_B}{2} = \frac{5 - 1}{2} = 2 \quad \text{and} \quad y_M = \frac{y_A + y_B}{2} = \frac{2 - 4}{2} = -1$$

$\therefore$ the midpoint is at $(2, -1)$

$$\mathbf{c} \quad x_M = \frac{x_A + x_B}{2} = \frac{7 + 0}{2} = \frac{7}{2} \quad \text{and} \quad y_M = \frac{y_A + y_B}{2} = \frac{0 + 3}{2} = \frac{3}{2}$$

$\therefore$ the midpoint is at $(\frac{7}{2}, \frac{3}{2})$

$$\mathbf{d} \quad x_M = \frac{x_A + x_B}{2} = \frac{5 - 1}{2} = 2 \quad \text{and} \quad y_M = \frac{y_A + y_B}{2} = \frac{-2 - 3}{2} = -\frac{5}{2}$$

$\therefore$ the midpoint is at $(2, -\frac{5}{2})$

$$\mathbf{3} \quad \mathbf{a} \quad \begin{array}{c} \text{slope} = \frac{\text{y-step}}{\text{x-step}} \\ = \frac{2}{3} \end{array}$$

$$\mathbf{b} \quad \begin{array}{c} \text{slope} = \frac{\text{y-step}}{\text{x-step}} \\ = \frac{2}{-5} \\ = -\frac{2}{5} \end{array}$$

$$\mathbf{c} \quad \begin{array}{c} \text{slope} = \frac{\text{y-step}}{\text{x-step}} \\ = \frac{3}{3} \\ = 1 \end{array}$$

$$\mathbf{d} \quad \begin{array}{c} \text{slope} = \frac{\text{y-step}}{\text{x-step}} \\ = \frac{0}{7} \\ = 0 \end{array}$$

$$\mathbf{e} \quad \begin{array}{c} \text{slope} = \frac{\text{y-step}}{\text{x-step}} \\ = \frac{4}{-1} \\ = -4 \end{array}$$

$$\mathbf{f} \quad \begin{array}{c} \text{slope} = \frac{\text{y-step}}{\text{x-step}} \\ = \frac{3}{0} \\ \text{which is undefined} \end{array}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{7 - 3}{4 - 2} \\ &= 2 \end{aligned}$$

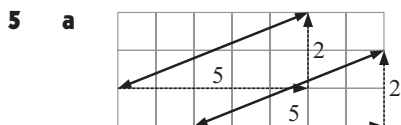
$$\begin{aligned} \mathbf{b} \quad \text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{8 - 2}{5 - 3} \\ &= 3 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5 - 2}{-1 - (-1)} \\ &= \frac{3}{0} \\ &\text{which is undefined} \end{aligned}$$

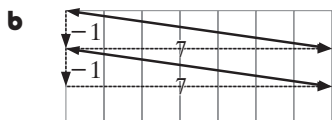
$$\begin{aligned} \text{d} \quad \text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-3 - (-3)}{-1 - 4} \\ &= \frac{0}{-5} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{e} \quad \text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4 - 0}{-1 - 0} \\ &= -4 \end{aligned}$$

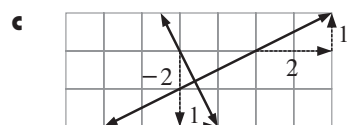
$$\begin{aligned} \text{f} \quad \text{gradient} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2 - (-1)}{-1 - 3} \\ &= \frac{-1}{-4} \\ &= \frac{1}{4} \end{aligned}$$



The lines have the same slope ($\frac{2}{5}$), so are parallel.

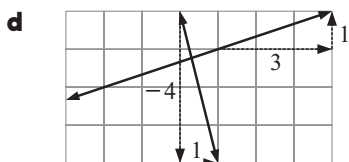


The lines have the same slope ($-\frac{1}{7}$), so are parallel.



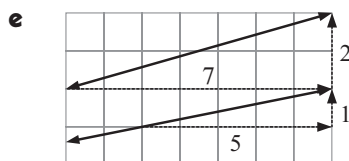
The two lines have slopes $\frac{1}{2}$ and $-\frac{2}{1}$, so the product of the slopes is -1 .

Hence the lines are perpendicular.



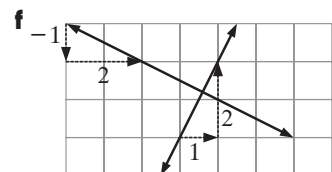
The two lines have slopes $\frac{1}{3}$ and $-\frac{4}{1}$, so the product of the slopes is $-\frac{4}{3}$.

Hence the lines are neither parallel nor perpendicular.



The two lines have slopes $\frac{2}{7}$ and $\frac{1}{5}$, so the product of the slopes is $\frac{2}{35}$.

Hence they are neither parallel nor perpendicular.



The two lines have slopes $-\frac{1}{2}$ and $\frac{2}{1}$, so the product of the slopes is -1 .

Hence the lines are perpendicular.

6 a $\text{gradient} = -\frac{1}{(\frac{3}{4})} = -\frac{4}{3}$

b $\text{gradient} = -\frac{1}{(\frac{11}{3})} = -\frac{3}{11}$

c $\text{gradient} = -\frac{1}{4}$

d $\text{gradient} = -\frac{1}{(-\frac{1}{3})} = 3$

e $\text{gradient} = -\frac{1}{(-5)} = \frac{1}{5}$

f $\text{gradient} = -\frac{1}{0}$
which is undefined

7 a The equation of the line is

$$\begin{aligned} \frac{y-1}{x-4} &= 2 \\ \therefore y-1 &= 2(x-4) \\ \therefore y-1 &= 2x-8 \\ \therefore y &= 2x-7 \end{aligned}$$

b The equation of the line is

$$\begin{aligned} \frac{y-2}{x-1} &= -2 \\ \therefore y-2 &= -2(x-1) \\ \therefore y-2 &= -2x+2 \\ \therefore y &= -2x+4 \end{aligned}$$

- c**
- The equation of the line is

$$\begin{aligned}\frac{y-0}{x-5} &= 3 \\ \therefore y &= 3(x-5) \\ \therefore y &= 3x-15\end{aligned}$$

- e**
- The equation of the line is

$$\begin{aligned}\frac{y-5}{x-1} &= -4 \\ \therefore y-5 &= -4(x-1) \\ \therefore y-5 &= -4x+4 \\ \therefore y &= -4x+9\end{aligned}$$

- 8 a**
- The equation of the line is

$$\begin{aligned}\frac{y-1}{x-2} &= \frac{3}{2} \\ \therefore 2(y-1) &= 3(x-2) \\ \therefore 2y-2 &= 3x-6 \\ \therefore 3x-2y &= 4\end{aligned}$$

- c**
- The equation of the line is

$$\begin{aligned}\frac{y-0}{x-4} &= \frac{1}{3} \\ \therefore 3y &= x-4 \\ \therefore x-3y &= 4\end{aligned}$$

- e**
- The equation of the line is

$$\begin{aligned}\frac{y-(-3)}{x-(-1)} &= 3 \\ \therefore y+3 &= 3(x+1) \\ \therefore y+3 &= 3x+3 \\ \therefore 3x-y &= 0\end{aligned}$$

- 9 a** The slope of the line is $\frac{2-1}{3-0} = \frac{1}{3}$ $\therefore$ its equation is $\frac{y-1}{x-0} = \frac{1}{3}$ $\therefore 3(y-1) = x$
 $\therefore 3y-3 = x$
 $\therefore x-3y = -3$

- b** The slope of the line is $\frac{-1-4}{0-1} = 5$ $\therefore$ its equation is $\frac{y-(-1)}{x-0} = 5$
 $\therefore y+1 = 5x$
 $\therefore 5x-y = 1$

- c** The slope of the line is $\frac{-4-(-1)}{-1-2} = \frac{-3}{-3} = 1$ $\therefore$ its equation is $\frac{y-(-1)}{x-2} = 1$
 $\therefore y+1 = x-2$
 $\therefore x-y = 3$

- d** The slope of the line is $\frac{2-(-2)}{5-0} = \frac{4}{5}$ $\therefore$ its equation is $\frac{y-(-2)}{x-0} = \frac{4}{5}$
 $\therefore 5(y+2) = 4x$
 $\therefore 5y+10 = 4x$
 $\therefore 4x-5y = 10$

- d**
- The equation of the line is

$$\begin{aligned}\frac{y-7}{x-(-1)} &= -3 \\ \therefore y-7 &= -3(x+1) \\ \therefore y-7 &= -3x-3 \\ \therefore y &= -3x+4\end{aligned}$$

- f**
- The equation of the line is

$$\begin{aligned}\frac{y-7}{x-2} &= 1 \\ \therefore y-7 &= x-2 \\ \therefore y &= x+5\end{aligned}$$

- b**
- The equation of the line is

$$\begin{aligned}\frac{y-4}{x-1} &= -\frac{3}{2} \\ \therefore -2(y-4) &= 3(x-1) \\ \therefore -2y+8 &= 3x-3 \\ \therefore 3x+2y &= 11\end{aligned}$$

- d**
- The equation of the line is

$$\begin{aligned}\frac{y-6}{x-0} &= -4 \\ \therefore y-6 &= -4x \\ \therefore 4x+y &= 6\end{aligned}$$

- f**
- The equation of the line is

$$\begin{aligned}\frac{y-(-2)}{x-4} &= -\frac{4}{9} \\ \therefore -9(y+2) &= 4(x-4) \\ \therefore -9y-18 &= 4x-16 \\ \therefore 4x+9y &= -2\end{aligned}$$

- e** The slope of the line is $\frac{2-0}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$ $\therefore$ its equation is $\frac{y-0}{x-(-1)} = \frac{1}{2}$
 $\therefore 2y = x + 1$
 $\therefore x - 2y = -1$
- f** The slope of the line is $\frac{-3-(-1)}{2-(-1)} = \frac{-2}{3}$ $\therefore$ its equation is $\frac{y-(-1)}{x-(-1)} = -\frac{2}{3}$
 $\therefore -3(y+1) = 2(x+1)$
 $\therefore -3y-3 = 2x+2$
 $\therefore 2x+3y = -5$
- 10 a** Since both points on the line have y -coordinate -2 , it must be horizontal.
 $\therefore$ its equation is $y = -2$.
- b** Since both points on the line have x -coordinate 6 , it must be vertical.
 $\therefore$ its equation is $x = 6$.
- c** Since both points on the line have x -coordinate -3 , it must be vertical.
 $\therefore$ its equation is $x = -3$.

- 11 a** $2x - 3y = 6$
 $\therefore 3y = 2x - 6$
 $\therefore y = \frac{2}{3}x - 2$
 $\therefore$ slope $= \frac{2}{3}$, y -intercept $= -2$
When $y = 0$, $2x = 6$
 $\therefore x$ -intercept $= 3$
- b** $4x + 5y = 20$
 $\therefore 5y = -4x + 20$
 $\therefore y = -\frac{4}{5}x + 4$
 $\therefore$ slope $= -\frac{4}{5}$, y -intercept $= 4$
When $y = 0$, $4x = 20$
 $\therefore x$ -intercept $= 5$
- c** $y = -2x + 5$
 $\therefore$ slope $= -2$, y -intercept $= 5$
When $y = 0$, $-2x + 5 = 0 \therefore x = \frac{5}{2}$
i.e., the x -intercept $= \frac{5}{2}$
- d** $x = 8$
 $\therefore$ the line is vertical
 $\therefore$ slope is undefined,
no y -intercept, x -intercept $= 8$
- e** $y = 5$
 $\therefore$ the line is horizontal
 $\therefore$ slope $= 0$,
 y -intercept $= 5$, no x -intercept
- f** $x + y = 11$
 $\therefore y = 11 - x$
 $\therefore$ slope $= -1$, y -intercept $= 11$
When $y = 0$, $x = 11$
 $\therefore x$ -intercept $= 11$
- g** $4x + y = 8$
 $\therefore y = -4x + 8$
 $\therefore$ slope $= -4$, y -intercept $= 8$
When $y = 0$, $4x = 8$
 $\therefore x$ -intercept $= 2$
- h** $x - 3y = 12$
 $\therefore 3y = x - 12$
 $\therefore y = \frac{1}{3}x - 4$
 $\therefore$ slope $= \frac{1}{3}$, y -intercept $= -4$
When $y = 0$, $x = 12$
 $\therefore x$ -intercept $= 12$

Summary of results:

	Equation	Slope	x -int.	y -int.
a	$2x - 3y = 6$	$\frac{2}{3}$	3	-2
b	$4x + 5y = 20$	$-\frac{4}{5}$	5	4
c	$y = -2x + 5$	-2	$\frac{5}{2}$	5
d	$x = 8$	undefined	8	none

	Equation	Slope	x -int.	y -int.
e	$y = 5$	0	none	5
f	$x + y = 11$	-1	11	11
g	$4x + y = 8$	-4	2	8
h	$x - 3y = 12$	$\frac{1}{3}$	12	-4

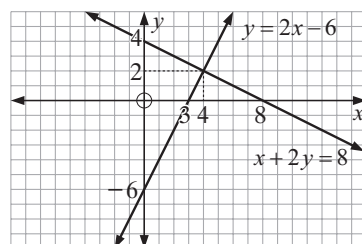
- 12 a** Substituting $(3, 4)$ into $3x - 2y = 1$ gives $3(3) - 2(4) = 1$
 i.e., $1 = 1$ which is true
 $\therefore (3, 4)$ lies on the line
- b** Substituting $(-2, 5)$ into $5x + 3y = -5$ gives $5(-2) + 3(5) = -5$
 i.e., $5 = -5$ which is false
 $\therefore (-2, 5)$ does not lie on the line
- c** Substituting $(6, -\frac{1}{2})$ into $3x - 8y = 22$ gives $3(6) - 8(-\frac{1}{2}) = 22$
 i.e., $22 = 22$ which is true
 $\therefore (6, -\frac{1}{2})$ lies on the line

- 13 a** For $x + 2y = 8$,
 when $x = 0$, $y = 4$
 when $y = 0$, $x = 8$

x	0	8
y	4	0

- For $y = 2x - 6$,
 when $x = 0$, $y = -6$
 when $y = 0$, $x = 3$

x	0	3
y	-6	0



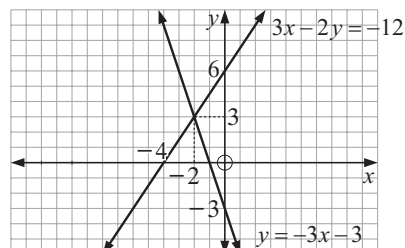
The lines meet at $(4, 2)$.

- b** For $y = -3x - 3$,
 when $x = 0$, $y = -3$
 when $y = 0$, $x = -1$

x	0	-1
y	-3	0

- For $3x - 2y = -12$,
 when $x = 0$, $y = 6$
 when $y = 0$, $x = -4$

x	0	-4
y	6	0



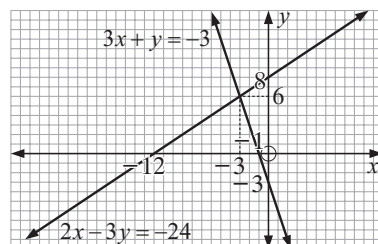
The lines meet at $(-2, 3)$.

- c** For $3x + y = -3$,
 when $x = 0$, $y = -3$
 when $y = 0$, $x = -1$

x	0	-1
y	-3	0

- For $2x - 3y = -24$,
 when $x = 0$, $y = 8$
 when $y = 0$, $x = -12$

x	0	-12
y	8	0



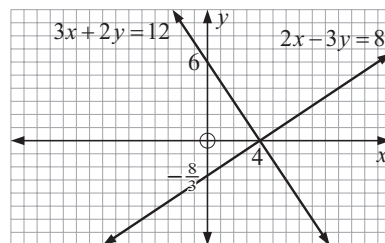
The lines meet at $(-3, 6)$.

- d** For $2x - 3y = 8$,
 when $x = 0$, $y = -\frac{8}{3}$
 when $y = 0$, $x = 4$

x	0	4
y	$-\frac{8}{3}$	0

- For $3x + 2y = 12$,
 when $x = 0$, $y = 6$
 when $y = 0$, $x = 4$

x	0	4
y	6	0



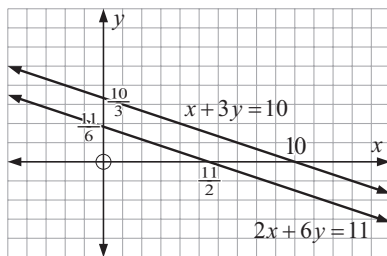
The lines meet at $(4, 0)$.

- e** For $x + 3y = 10$,
 when $x = 0$, $y = \frac{10}{3}$
 when $y = 0$, $x = 10$

x	0	10
y	$\frac{10}{3}$	0

- For $2x + 6y = 11$,
 when $x = 0$, $y = \frac{11}{6}$
 when $y = 0$, $x = \frac{11}{2}$

x	0	$\frac{11}{2}$
y	$\frac{11}{6}$	0



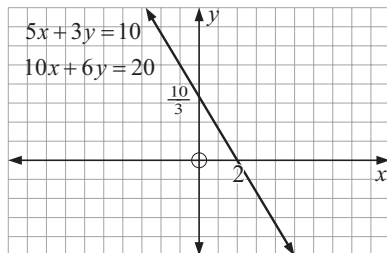
The lines are parallel, so never meet.

- f** For $5x + 3y = 10$,
 when $x = 0$, $y = \frac{10}{3}$
 when $y = 0$, $x = 2$

x	0	2
y	$\frac{10}{3}$	0

- For $10x + 6y = 20$,
 when $x = 0$, $y = \frac{20}{6} = \frac{10}{3}$
 when $y = 0$, $x = 2$

x	0	2
y	$\frac{10}{3}$	0



The lines are coincident.

- 14** **a** Since the line is horizontal, its equation is $y = -4$.
b Since the line is vertical, its equation is $x = 5$.
c Since the line is vertical, its equation is $x = -1$.
d Since the line is horizontal, its equation is $y = 2$.
e The x -axis corresponds to $y = 0$.
f The y -axis corresponds to $x = 0$.

- 15** **a** The line has equation $\frac{y-4}{x-(-1)} = \frac{3}{4}$ $\therefore 4(y-4) = 3(x+1)$
 $\therefore 4y - 16 = 3x + 3$
 $\therefore 3x - 4y = -19$

- b** The line has slope $= \frac{0 - (-5)}{7 - 2} = \frac{5}{5} = 1$ $\therefore$ its equation is $\frac{y-0}{x-7} = 1$
 $\therefore y = x - 7$
 $\therefore x - y = 7$

- c** $y = 3x - 2$ has slope 3, so this line has slope 3 also.

It passes through $(0, 0)$, so its equation is $\frac{y-0}{x-0} = 3$ i.e., $y = 3x$

- d** Now a line parallel to $2x + 3y = 8$ has equation $2x + 3y = k$, where k is a constant.
 Since $(-1, 7)$ lies on the line, $2(-1) + 3(7) = k$
 $\therefore k = 19$
 $\therefore$ the line is $2x + 3y = 19$

- e** $y = -2x + 5$ has slope -2 $\therefore$ lines perpendicular to it have slope $\frac{1}{2}$.

But this line must pass through $(3, -1)$. $\therefore \frac{y-(-1)}{x-3} = \frac{1}{2}$
 $\therefore 2(y+1) = x-3$
 $\therefore 2y+2 = x-3$
 $\therefore x-2y = 5$

- f** If $3x - y = 11$, then $y = 3x - 11$
 $\therefore$ this line has slope 3 $\therefore$ lines perpendicular to it have slope $-\frac{1}{3}$.

But this line must pass through $(-2, 5)$. $\therefore \frac{y-5}{x-(-2)} = -\frac{1}{3}$
 $\therefore -3(y-5) = x+2$
 $\therefore -3y+15 = x+2$
 $\therefore x+3y = 13$

- 16 a** Keach Avenue passes through $(5, 11)$ and $(13, 12)$

$\therefore$ its line has slope $\frac{12-11}{13-5} = \frac{1}{8}$
 $\therefore$ its equation is $\frac{y-12}{x-13} = \frac{1}{8}$ $\therefore 8(y-12) = x-13$
 $\therefore 8y-96 = x-13$
 $\therefore x-8y = -83$

- b** Peacock Street is perpendicular to Keach Avenue, so its line has slope -8 .

But this line also passes through $(3, 17)$. $\therefore \frac{y-17}{x-3} = -8$
 $\therefore y-17 = -8(x-3)$
 $\therefore y-17 = -8x+24$
 $\therefore 8x+y = 41$

- c** Diagonal Road runs from $(5, 11)$ to $(7, 20)$, but ends at these points.

Hence there is a restricted domain $5 \leq x \leq 7$.

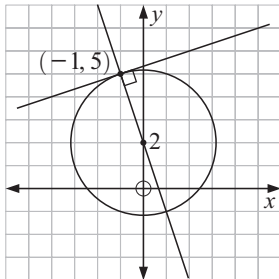
Its slope is $\frac{20-11}{7-5} = \frac{9}{2}$
 $\therefore$ its equation is $\frac{y-11}{x-5} = \frac{9}{2}$ $\therefore 2(y-11) = 9(x-5)$
 $\therefore 2y-22 = 9x-45$
 $\therefore 9x-2y = 23, 5 \leq x \leq 7$

- d** Plunkit Street has $x = 8$, so it meets Keach Avenue when $8-8y = -83$ {using **a**}

$\therefore 8y = 91$
 $\therefore y = \frac{91}{8}$

$\therefore$ they intersect at $(8, \frac{91}{8})$

- 17 a**

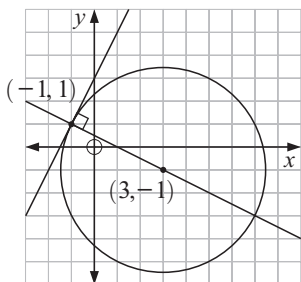


The line through $(-1, 5)$ and $(0, 2)$

has slope $= \frac{5-2}{-1-0} = -3$

$\therefore$ the slope of the tangent is $\frac{1}{3}$

$\therefore$ its equation is $\frac{y-5}{x-(-1)} = \frac{1}{3}$
 $\therefore 3(y-5) = x+1$
 $\therefore 3y-15 = x+1$
 $\therefore x-3y = -16$

b

The line through $(-1, 1)$ and $(3, -1)$

$$\text{has slope} = \frac{-1 - 1}{3 - (-1)} = -\frac{1}{2}$$

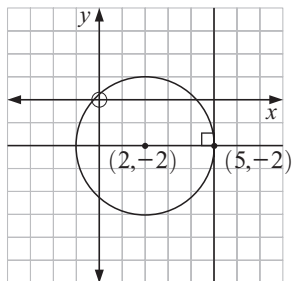
$\therefore$ the slope of the tangent is 2

$$\therefore \text{ its equation is } \frac{y - 1}{x - (-1)} = 2$$

$$\therefore y - 1 = 2(x + 1)$$

$$\therefore y - 1 = 2x + 2$$

$$\therefore 2x - y = -3$$

c

The line through $(2, -2)$ and $(5, -2)$

$$\text{has slope} = \frac{-2 - (-2)}{5 - 2} = 0$$

i.e., it is horizontal

$\therefore$ the tangent must be vertical

$\therefore$ its equation is $x = 5$

18 Suppose the outlet is at $T(x, 8)$.

$$\text{Then } AT = BT, \text{ so } \sqrt{(x - 5)^2 + (8 - 5)^2} = \sqrt{(x - 7)^2 + (8 - 10)^2}$$

$$\therefore (x - 5)^2 + (8 - 5)^2 = (x - 7)^2 + (8 - 10)^2$$

$$\therefore x^2 - 10x + 25 + 9 = x^2 - 14x + 49 + 4$$

$$\therefore -10x + 34 = -14x + 53$$

$$\therefore 4x = 19$$

$$\therefore x = \frac{19}{4} \quad \therefore \text{ the outlet is at } (4\frac{3}{4}, 8)$$

19 a Suppose C is at $(x, 7)$

$$\text{Then } AC = BC, \text{ so } \sqrt{(x - 2)^2 + (7 - 3)^2} = \sqrt{(x - 5)^2 + (7 - 4)^2}$$

$$\therefore (x - 2)^2 + (7 - 3)^2 = (x - 5)^2 + (7 - 4)^2$$

$$\therefore x^2 - 4x + 4 + 16 = x^2 - 10x + 25 + 9$$

$$\therefore -4x + 20 = -10x + 34$$

$$\therefore 6x = 14$$

$$\therefore x = \frac{7}{3} \quad \therefore \text{ the pumping station is at } (\frac{7}{3}, 7)$$

b The length of each pipe = $\sqrt{(x - 2)^2 + (7 - 3)^2}$

$$= \sqrt{(\frac{7}{3} - 2)^2 + 16}$$

$$= \sqrt{16\frac{1}{9}} \quad \therefore \text{ the total length} = 2\sqrt{16\frac{1}{9}} \div 8.03 \text{ km}$$

c Now CD is horizontal, so BD is vertical $\therefore$ the x -coordinate of D is 5

$\therefore$ D is at $(5, 7)$

$$\text{Then } AB + BD = \sqrt{(5 - 2)^2 + (4 - 3)^2} + \sqrt{(5 - 5)^2 + (7 - 4)^2}$$

$$= \sqrt{9 + 1} + \sqrt{0 + 9}$$

$$= \sqrt{10} + 3$$

$$\div 6.162 \text{ km} \quad \text{Hence yes, it would be much cheaper.}$$

- 20** Suppose Jason's girlfriend lives at $G(x, 8)$

$$\text{Then } GJ = \sqrt{(x-4)^2 + (8-1)^2} = 11.73$$

$$\therefore (x-4)^2 + 49 = 11.73^2$$

$$\therefore (x-4)^2 = 88.5929$$

$$\therefore x-4 \div \pm 9.412$$

$$\therefore x \div -5.412 \text{ or } 13.412$$

i.e., Jason's girlfriend lives at $(-5.412, 8)$ or $(13.41, 8)$

- 21** If the centre is $C(a, b)$ then PC is always r units.

$$\therefore \sqrt{(x-a)^2 + (y-b)^2} = r \quad \{\text{distance formula}\}$$

$$\therefore (x-a)^2 + (y-b)^2 = r^2$$

22 a $(x-4)^2 + (y-3)^2 = 5^2$

i.e., $(x-4)^2 + (y-3)^2 = 25$

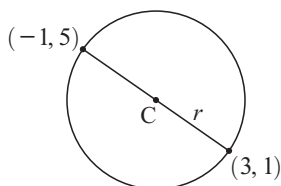
b $(x-(-1))^2 + (y-5)^2 = 2^2$

i.e., $(x+1)^2 + (y-5)^2 = 4$

c $(x-0)^2 + (y-0)^2 = 10^2$

i.e., $x^2 + y^2 = 100$

d



$$C \text{ is } \left(\frac{-1+3}{2}, \frac{5+1}{2} \right) \text{ i.e., } (1, 3)$$

$$\text{and } r = \sqrt{(3-1)^2 + (1-3)^2}$$

$$= \sqrt{4+4}$$

$$= \sqrt{8}$$

$$\therefore \text{equation is } (x-1)^2 + (y-3)^2 = (\sqrt{8})^2$$

$$\text{i.e., } (x-1)^2 + (y-3)^2 = 8$$

- 23 a** The centre is $(1, 3)$ and radius $\sqrt{4} = 2$ units.

- b** The centre is $(0, -2)$ and radius $\sqrt{16} = 4$ units.

- c** The centre is $(0, 0)$ and radius $\sqrt{7}$ units.

- 24 a** $(x-2)^2 + (y+3)^2 = 20$ has centre $(2, -3)$ and radius $\sqrt{20}$ units.

b If $x = 4$ and $y = 1$, LHS $= (4-2)^2 + (1+3)^2$

$$= 2^2 + 4^2$$

$$= 20$$

$$= \text{RHS } \checkmark$$

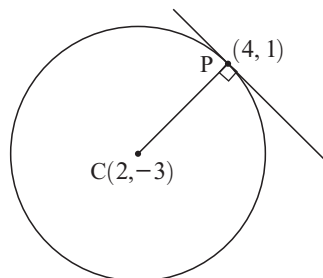
c Slope of radius CP is $\frac{1-(-3)}{4-2} = \frac{4}{2} = \frac{2}{1}$

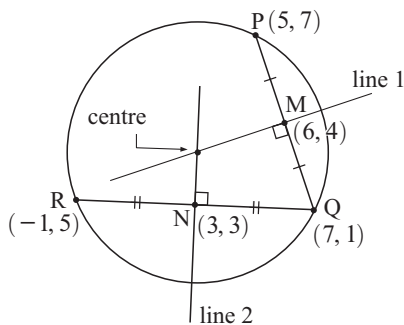
$$\therefore \text{slope of tangent is } -\frac{1}{2}$$

$$\therefore \text{equation is } \frac{y-1}{x-4} = -\frac{1}{2}$$

$$\text{i.e., } 2y-2 = -x+4$$

$$\text{i.e., } x+2y = 6$$



25

$$M \text{ is } \left(\frac{5+7}{2}, \frac{7+1}{2} \right) \text{ i.e., } (6, 4)$$

$$N \text{ is } \left(\frac{7+(-1)}{2}, \frac{1+5}{2} \right) \text{ i.e., } (3, 3)$$

$$\text{Slope of PQ} = \frac{1-7}{7-5} = -\frac{6}{2} = -3$$

$$\therefore \text{ slope of line 1} = \frac{1}{3} \quad \{\text{negative reciprocal}\}$$

$$\therefore \text{ equation of line 1 is } \frac{y-4}{x-6} = \frac{1}{3}$$

$$\text{i.e., } y-4 = \frac{1}{3}x-2$$

$$\text{i.e., } y = \frac{1}{3}x+2 \quad \dots\dots (1)$$

$$\text{Slope of QR} = \frac{5-1}{-1-7} = \frac{4}{-8} = -\frac{1}{2}$$

$$\therefore \text{ slope of line 2 is } 2 \quad \{\text{negative reciprocal}\}$$

$$\therefore \text{ equation of line 2 is } \frac{y-3}{x-3} = 2$$

$$\text{i.e., } y-3 = 2x-6$$

$$\text{i.e., } y = 2x-3 \quad \dots\dots (2)$$

We now solve (1) and (2) simultaneously

$$\text{Now } \frac{1}{3}x+2 = 2x-3$$

$$\therefore x+6 = 6x-9$$

$$\therefore 15 = 5x$$

$$\therefore x = 3 \quad \text{and so } y = 2(3)-3 = 3$$

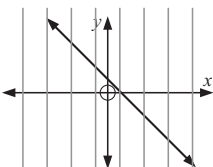
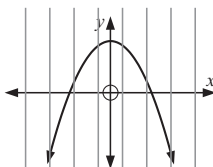
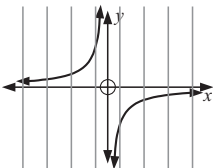
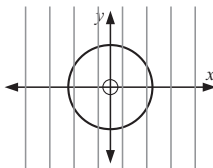
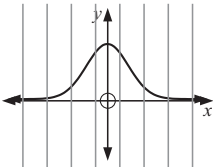
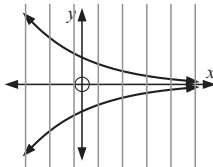
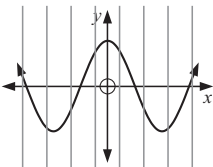
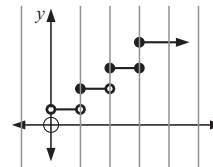
$$\therefore \text{ the centre is at } (3, 3).$$

Chapter 1

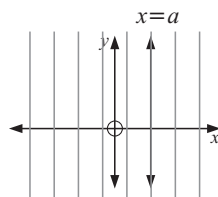
FUNCTIONS

EXERCISE 1A

- 1**
- a** $(1, 3), (2, 4), (3, 5), (4, 6)$ is a function since no two ordered pairs have the same x -coordinate.
 - b** $(1, 3), (3, 2), (1, 7), (-1, 4)$ is not a function since two of the ordered pairs, $(1, 3)$ and $(1, 7)$, have the same x -coordinate of 1.
 - c** $(2, -1), (2, 0), (2, 3), (2, 11)$ is not a function since each ordered pair has the same x -coordinate of 2.
 - d** $(7, 6), (5, 6), (3, 6), (-4, 6)$ is a function since no two ordered pairs have the same x -coordinate.
 - e** $(0, 0), (1, 0), (3, 0), (5, 0)$ is a function since no two ordered pairs have the same x -coordinate.
 - f** $(0, 0), (0, -2), (0, 2), (0, 4)$ is not a function since each ordered pair has the same x -coordinate of 0.

- 2**
- a**  i.e., each line cuts the graph no more than once $\therefore$ it is a function
 - b**  i.e., each line cuts the graph no more than once $\therefore$ it is a function
 - c**  i.e., each line cuts the graph no more than once $\therefore$ it is a function
 - d**  i.e., the lines cut the graph more than once $\therefore$ it is not a function
 - e**  i.e., each line cuts the graph no more than once $\therefore$ it is a function
 - f**  i.e., the lines cut the graph more than once $\therefore$ it is not a function
 - g**  i.e., each line cuts the graph no more than once $\therefore$ it is a function
 - h**  i.e., one line cuts the graph more than once $\therefore$ it is not a function

- 3** The graph of a straight line is not a function if the graph is a vertical line, i.e., $x = a$ for all a .
The vertical line through $x = a$ cuts the graph at every point $\therefore$ it is not a function.



- 4** $x^2 + y^2 = 9$ is the equation of a circle, centre $(0, 0)$ and radius 3.

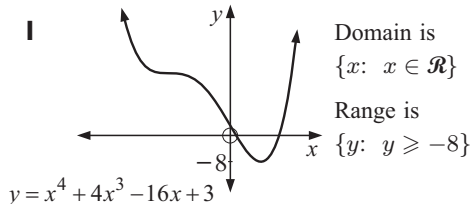
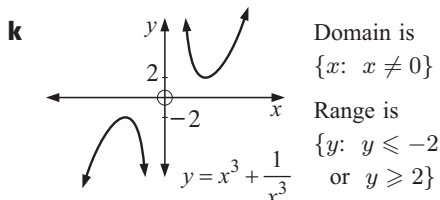
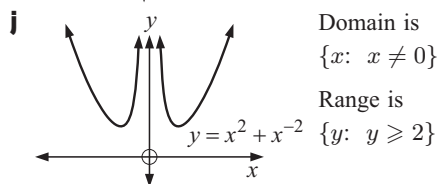
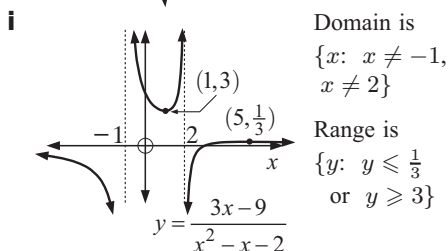
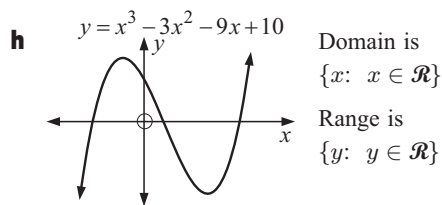
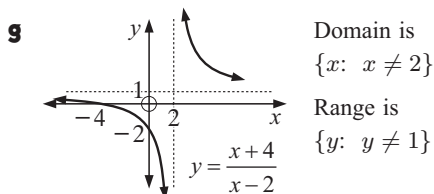
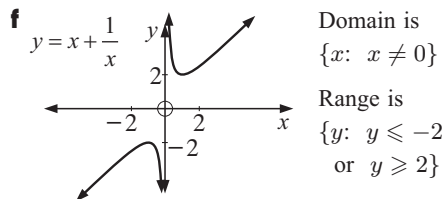
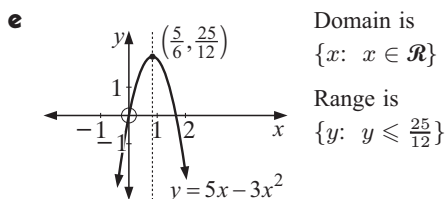
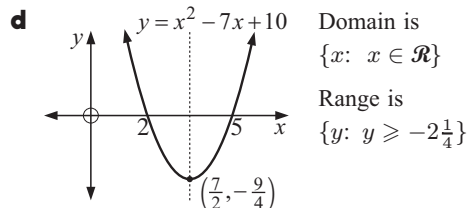
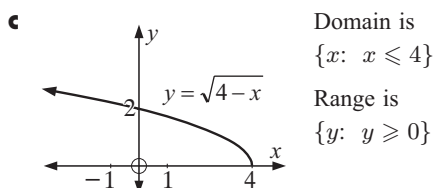
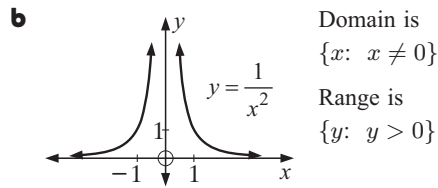
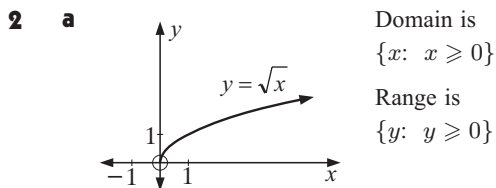
Now $x^2 + y^2 = 9$

$\therefore y^2 = 9 - x^2$

$\therefore y = \pm\sqrt{9 - x^2}$ Hence y has two real values for any value of x where $-3 < x < 3$.

EXERCISE 1B

- 1** **a** Domain is $\{x: -1 < x \leq 5\}$ **b** Domain is $\{x: x \neq 2\}$
 Range is $\{y: 1 < y \leq 3\}$ Range is $\{y: y \neq -1\}$
- c** Domain is $\{x: x \in \mathcal{R}\}$ **d** Domain is $\{x: x \in \mathcal{R}\}$
 Range is $\{y: 0 < y \leq 2\}$ Range is $\{y: y \geq -1\}$
- e** Domain is $\{x: x \geq -4\}$ **f** Domain is $\{x: x \neq \pm 2\}$
 Range is $\{y: y \geq -3\}$ Range is $\{y: y \leq -1 \text{ or } y > 0\}$



EXERCISE 1C

$$1 \quad \mathbf{a} \quad f(0) = 3(0) + 2 = 2 \quad \mathbf{b} \quad f(2) = 3(2) + 2 = 8 \quad \mathbf{c} \quad f(-1) = 3(-1) + 2 = -1$$

$$\mathbf{d} \quad f(-5) = 3(-5) + 2 = -13 \quad \mathbf{e} \quad f(-\frac{1}{3}) = 3(-\frac{1}{3}) + 2 = 1$$

$$2 \quad \mathbf{a} \quad f(0) = 3(0) - 0^2 + 2 = 2 \quad \mathbf{b} \quad f(3) = 3(3) - 3^2 + 2 = 9 - 9 + 2 = 2 \quad \mathbf{c} \quad f(-3) = 3(-3) - (-3)^2 + 2 = -9 - 9 + 2 = -16$$

$$\mathbf{d} \quad f(-7) = 3(-7) - (-7)^2 + 2 = -21 - 49 + 2 = -68 \quad \mathbf{e} \quad f(\frac{3}{2}) = 3(\frac{3}{2}) - (\frac{3}{2})^2 + 2 = \frac{9}{2} - \frac{9}{4} + 2 = \frac{17}{4}$$

$$3 \quad \mathbf{a} \quad f(a) = 7 - 3a \quad \mathbf{b} \quad f(-a) = 7 - 3(-a) = 7 + 3a \quad \mathbf{c} \quad f(a+3) = 7 - 3(a+3) = 7 - 3a - 9 = -3a - 2$$

$$\mathbf{d} \quad f(b-1) = 7 - 3(b-1) = 7 - 3b + 3 = 10 - 3b \quad \mathbf{e} \quad f(x+2) = 7 - 3(x+2) = 7 - 3x - 6 = 1 - 3x$$

$$4 \quad \mathbf{a} \quad F(x+4) = 2(x+4)^2 + 3(x+4) - 1 = 2(x^2 + 8x + 16) + 3x + 12 - 1 = 2x^2 + 16x + 32 + 3x + 11 = 2x^2 + 19x + 43$$

$$\mathbf{b} \quad F(2-x) = 2(2-x)^2 + 3(2-x) - 1 = 2(4 - 4x + x^2) + 6 - 3x - 1 = 8 - 8x + 2x^2 + 5 - 3x = 2x^2 - 11x + 13$$

$$\mathbf{c} \quad F(-x) = 2(-x)^2 + 3(-x) - 1 = 2x^2 - 3x - 1$$

$$\mathbf{d} \quad F(x^2) = 2(x^2)^2 + 3(x^2) - 1 = 2x^4 + 3x^2 - 1$$

$$\mathbf{e} \quad F(x^2 - 1) = 2(x^2 - 1)^2 + 3(x^2 - 1) - 1 = 2(x^4 - 2x^2 + 1) + 3x^2 - 3 - 1 = 2x^4 - 4x^2 + 2 + 3x^2 - 4 = 2x^4 - x^2 - 2$$

$$5 \quad \mathbf{a} \quad \mathbf{i} \quad G(2) = \frac{2(2) + 3}{2 - 4} = \frac{7}{-2} = -\frac{7}{2} \quad \mathbf{ii} \quad G(0) = \frac{2(0) + 3}{0 - 4} = \frac{3}{-4} = -\frac{3}{4} \quad \mathbf{iii} \quad G(-\frac{1}{2}) = \frac{2(-\frac{1}{2}) + 3}{-\frac{1}{2} - 4} = \frac{-1 + 3}{-\frac{9}{2}} = \frac{2}{(-\frac{9}{2})} = -\frac{4}{9}$$

$$\mathbf{b} \quad G(x) = \frac{2x + 3}{x - 4} \text{ is undefined when } x - 4 = 0 \text{ i.e., when } x = 4$$

So, when $x = 4$, $G(x)$ does not exist.

$$\mathbf{c} \quad G(x+2) = \frac{2(x+2) + 3}{(x+2) - 4} = \frac{2x + 4 + 3}{x + 2 - 4} = \frac{2x + 7}{x - 2}$$

$$\mathbf{d} \quad G(x) = -3 \text{ i.e., } \frac{2x + 3}{x - 4} = -3 \quad \therefore 2x + 3 = -3(x - 4)$$

$$\therefore 2x + 3 = -3x + 12$$

$$\therefore 5x = 9 \text{ and so } x = \frac{9}{5}$$

- 6** f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x .

7 a $V(4) = 9650 - 860(4)$
 $= 9650 - 3440$
 $= 6210$

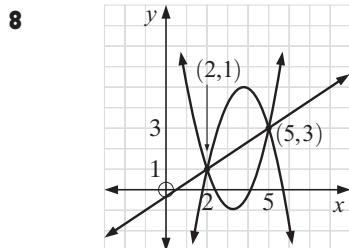
i.e., the value of the photocopier 4 years after purchase is 6210 Yen.

c Original purchase price is when $t = 0$,
 i.e., $V(0) = 9650 - 860(0)$
 $= 9650$

b If $V(t) = 5780$,
 then $9650 - 860t = 5780$
 $\therefore 860t = 3870$
 $\therefore t = 4.5$

i.e., the value of the photocopier $4\frac{1}{2}$ years after purchase is 5780 Yen.

i.e., the original purchase price was 9650 Yen.



First sketch the linear function which passes through the two points (2, 1) and (5, 3).

Then sketch two quadratic functions which also pass through the two points.

9 $f(x) = ax + b$ where $f(2) = 1$ and $f(-3) = 11$

i.e., $a(2) + b = 1$ and $a(-3) + b = 11$

$\therefore 2a + b = 1$ $\therefore -3a + b = 11$

$\therefore b = 1 - 2a$ (1) $\therefore b = 11 + 3a$ (2)

Solving (1) and (2) simultaneously, $1 - 2a = 11 + 3a$

$\therefore 5a = -10$

$\therefore a = -2$

Substituting $a = -2$ into (1) gives $b = 1 - 2(-2) = 5$ i.e., $a = -2$, $b = 5$

Hence $f(x) = -2x + 5$

10 $T(x) = ax^2 + bx + c$ where $T(0) = -4$, $T(1) = -2$ and $T(2) = 6$

i.e., $a(0)^2 + b(0) + c = -4$

$\therefore c = -4$

Also, $a(1)^2 + b(1) + c = -2$ and $a(2)^2 + b(2) + c = 6$

$\therefore a + b + c = -2$ and $\therefore 4a + 2b + c = 6$

Substituting $c = -4$ into both equations gives

$a + b + (-4) = -2$ and $4a + 2b + (-4) = 6$

$\therefore a + b = 2$ $\therefore 4a + 2b = 10$ (2)

$\therefore a = 2 - b$ (1)

Substituting (1) into (2) gives $4(2 - b) + 2b = 10$ $\therefore 8 - 4b + 2b = 10$

$\therefore -2b = 2$

$\therefore b = -1$

Now, substituting $b = -1$ into (1) gives $a = 2 - (-1) = 3$ i.e., $a = 3$, $b = -1$, $c = -4$

EXERCISE 1D

1 a $(f \circ g)(x)$ $= f(g(x))$ $= f(1 - x)$ $= 2(1 - x) + 3$ $= 2 - 2x + 3$ $= 5 - 2x$	b $(g \circ f)(x)$ $= g(f(x))$ $= g(2x + 3)$ $= 1 - (2x + 3)$ $= 1 - 2x - 3$ $= -2x - 2$	c $(f \circ g)(-3)$ $= f(g(-3))$ $= f(1 - (-3))$ $= f(4)$ $= 2(4) + 3$ $= 11$
--	--	---

2 $(f \circ g)(x) = f(g(x))$ $= f(2 - x)$ $= (2 - x)^2$ Domain is $\{x: x \in \mathcal{R}\}$ Range is $\{y: y \geq 0\}$	$(g \circ f)(x) = g(f(x))$ $= g(x^2)$ $= 2 - x^2$ Domain is $\{x: x \in \mathcal{R}\}$ Range is $\{y: y \geq 2\}$
--	---

3 a $(f \circ g)(x)$ $= f(g(x))$ $= f(3 - x)$ $= (3 - x)^2 + 1$ $= 9 - 6x + x^2 + 1$ $= x^2 - 6x + 10$	b $(g \circ f)(x)$ $= g(f(x))$ $= g(x^2 + 1)$ $= 3 - (x^2 + 1)$ $= 3 - x^2 - 1$ $= -x^2 + 2$	c $(g \circ f)(x) = f(x)$ i.e., $-x^2 + 2 = f(x)$ {from b } i.e., $-x^2 + 2 = x^2 + 1$ $\therefore 2x^2 = 1$ $\therefore x^2 = \frac{1}{2}$ $\therefore x = \pm \frac{1}{\sqrt{2}}$
--	--	--

4 a $ax + b = cx + d$ is true for all x {given}

Let $x = 0$, Let $x = 1$, $\therefore a(1) + b = c(1) + d$

$\therefore a(0) + b = c(0) + d$ $\therefore a + b = c + d$

$\therefore b = d$ (1) but $b = d$ (from (1))

b $(f \circ g)(x) = x$ for all x {given} $\therefore a + d = c + d$

$\therefore f(g(x)) = x$ $\therefore a = c$

$\therefore f(ax + b) = x$

$\therefore 2(ax + b) + 3 = x$

$\therefore 2ax + 2b + 3 = x$ for all x

When $x = 0$, When $x = 1$, $2a(1) + 2b + 3 = 1$

$2a(0) + 2b + 3 = 0$ $\therefore 2a + 2b = -2$

$\therefore 2b = -3$ $\therefore 2a = -2 - 2(-\frac{3}{2}) = 1$

$\therefore b = -\frac{3}{2}$ $\therefore a = \frac{1}{2}$

i.e., $a = \frac{1}{2}$ and $b = -\frac{3}{2}$ as required.

c If $(g \circ f)(x) = x$ then $g(f(x)) = x$ i.e., $g(2x + 3) = x$ i.e., $a(2x + 3) + b = x$ $\therefore 2ax + 3a + b = x$	When $x = 0$, $2a(0) + 3a + b = 0$ $\therefore 3a = -b$ $\therefore a = -\frac{1}{3}b$ (1)
---	--

When $x = 1$, $2a(1) + 3a + b = 1$

$\therefore 2a + 3a + b = 1$

$\therefore 5a + b = 1$

$\therefore 5(-\frac{1}{3}b) + b = 1$ (from (1))

$\therefore -\frac{2}{3}b = 1$

$\therefore b = -\frac{3}{2}$

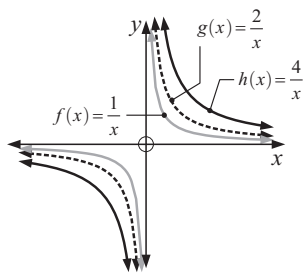
Substituting $b = -\frac{3}{2}$ into (1) gives

$a = -\frac{1}{3}(-\frac{3}{2}) = \frac{1}{2}$

i.e., $a = \frac{1}{2}$ and $b = -\frac{3}{2}$ as required.

$\therefore$ the result in **b** is also true if

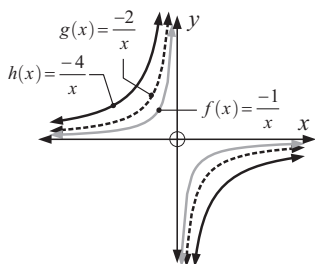
$(g \circ f)(x) = x$ for all x .

EXERCISE 1E**1**

$f(x)$, $g(x)$ and $h(x)$ are all reciprocal functions which are all asymptotic about the x - and y -axes.

The graphs all lie in the 1st and 3rd quadrants.

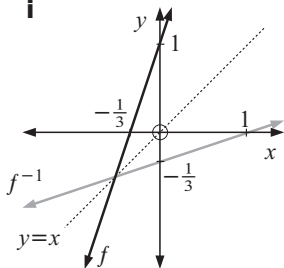
The smaller the numerator, the closer is the graph to the axes. Thus the graph of $f(x) = \frac{1}{x}$ is closer to the axes than $g(x) = \frac{2}{x}$ for corresponding values of x , and $g(x) = \frac{2}{x}$ is closer to the axes than $h(x) = \frac{4}{x}$.

2

$f(x)$, $g(x)$ and $h(x)$ are all reciprocal functions which are all asymptotic about the x - and y -axes.

The graphs all lie in the 2nd and 4th quadrants.

The smaller the numerator, the closer is the graph to the axes. Thus the graph of $f(x) = -\frac{1}{x}$ is closer to the axes than $g(x) = -\frac{2}{x}$ for corresponding values of x , and $g(x) = -\frac{2}{x}$ is closer to the axes than $h(x) = -\frac{4}{x}$.

EXERCISE 1F**1 a i**

ii $f(x)$ passes through $(0, 1)$ and $(-\frac{1}{3}, 0)$

$\therefore f^{-1}(x)$ passes through $(1, 0)$ and $(0, -\frac{1}{3})$

$f^{-1}(x)$ has slope $\frac{-\frac{1}{3} - 0}{0 - 1} = \frac{-\frac{1}{3}}{-1} = \frac{1}{3}$

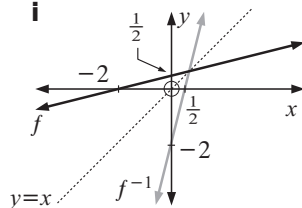
So, its equation is $\frac{y - 0}{x - 1} = \frac{1}{3}$

i.e., $y = \frac{x - 1}{3}$

i.e., $f^{-1}(x) = \frac{x - 1}{3}$

iii f is $y = 3x + 1$
 so f^{-1} is $x = 3y + 1$
 $\therefore x - 1 = 3y$
 $\therefore y = \frac{x - 1}{3}$

i.e., $f^{-1}(x) = \frac{x - 1}{3}$

b i

ii $f(x)$ passes through $(0, \frac{1}{2})$ and $(-2, 0)$

$\therefore f^{-1}(x)$ passes through $(\frac{1}{2}, 0)$ and $(0, -2)$

$f^{-1}(x)$ has slope $\frac{-2 - 0}{0 - \frac{1}{2}} = \frac{-2}{-\frac{1}{2}} = 4$

So, its equation is $\frac{y - 0}{x - \frac{1}{2}} = 4$ i.e., $y = 4x - 2$
 i.e., $f^{-1}(x) = 4x - 2$

iii f is $y = \frac{x + 2}{4}$
 so f^{-1} is $x = \frac{y + 2}{4}$

$\therefore 4x = y + 2$
 $\therefore y = 4x - 2$
 i.e., $f^{-1}(x) = 4x - 2$

2 a i f is $y = 2x + 5$
 so f^{-1} is $x = 2y + 5$
 $\therefore x - 5 = 2y$
 $\therefore y = \frac{x - 5}{2}$
 i.e., $f^{-1}(x) = \frac{x - 5}{2}$

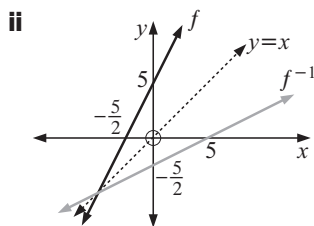
iii $(f^{-1} \circ f)(x)$ and
 $= f^{-1}(2x + 5)$
 $= \frac{2x + 5 - 5}{2}$
 $= \frac{2x}{2}$
 $= x$

b i f is $y = \frac{3 - 2x}{4}$
 so f^{-1} is $x = \frac{3 - 2y}{4}$
 $\therefore 4x = 3 - 2y$
 $\therefore 4x - 3 = -2y$
 $\therefore y = -2x + \frac{3}{2}$
 i.e., $f^{-1}(x) = -2x + \frac{3}{2}$

iii $(f^{-1} \circ f)(x)$ and
 $= f^{-1}(f(x))$
 $= f^{-1}\left(\frac{3 - 2x}{4}\right)$
 $= -2\left(\frac{3 - 2x}{4}\right) + \frac{3}{2}$
 $= \frac{3 - 2x}{-2} + \frac{3}{2}$
 $= -\frac{3}{2} + x + \frac{3}{2}$
 $= x$

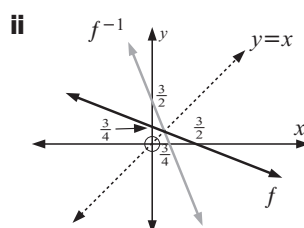
c i f is $y = x + 3$
 so f^{-1} is $x = y + 3$
 $\therefore y = x - 3$
 i.e., $f^{-1}(x) = x - 3$

iii $(f^{-1} \circ f)(x)$ and
 $= f^{-1}(f(x))$
 $= f^{-1}(x + 3)$
 $= (x + 3) - 3$
 $= x$



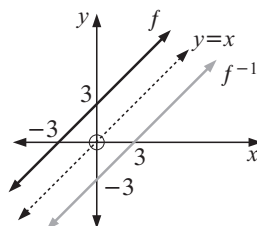
$f(x)$ passes through $(0, 5)$ and $(-\frac{5}{2}, 0)$
 $\therefore f^{-1}(x)$ passes through $(5, 0)$ and $(0, -\frac{5}{2})$

$(f \circ f^{-1})(x)$
 $= f(f^{-1}(x))$
 $= f\left(\frac{x - 5}{2}\right)$
 $= 2\left(\frac{x - 5}{2}\right) + 5$
 $= x - 5 + 5$
 $= x$



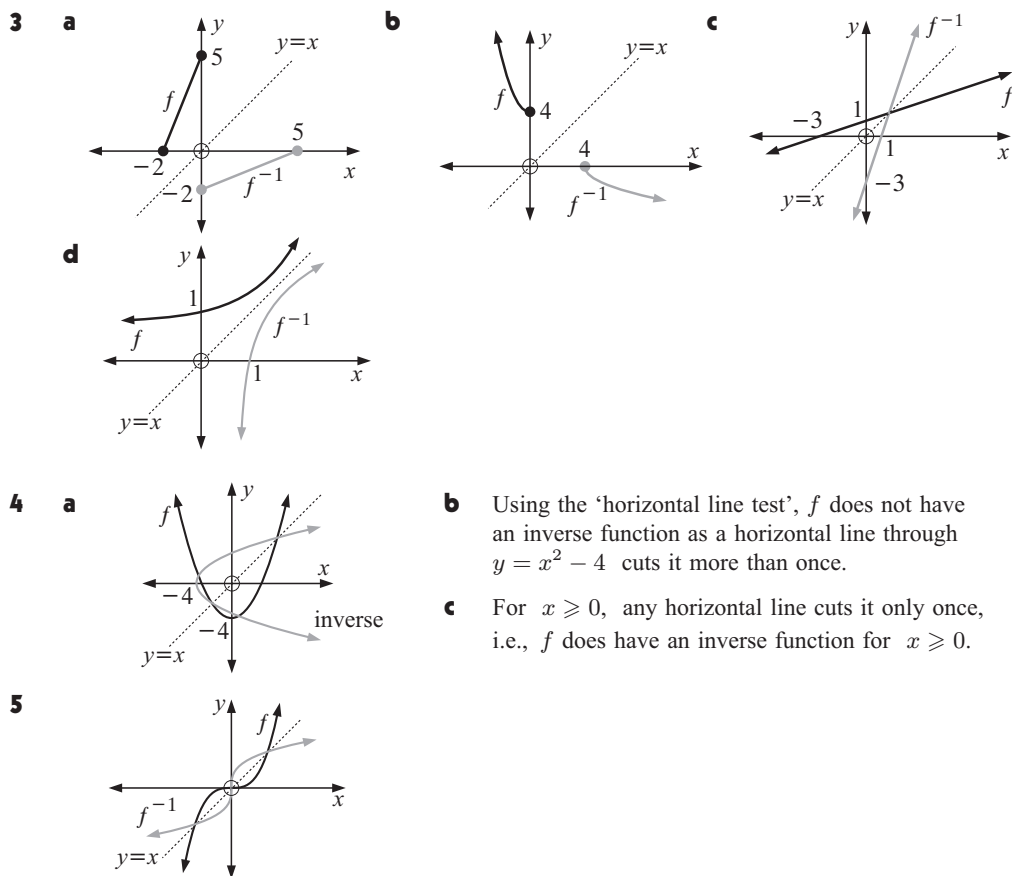
$f(x)$ passes through $(0, \frac{3}{4})$ and $(\frac{3}{2}, 0)$
 $\therefore f^{-1}(x)$ passes through $(\frac{3}{4}, 0)$ and $(0, \frac{3}{2})$

$(f \circ f^{-1})(x)$
 $= f(f^{-1}(x))$
 $= f(-2x + \frac{3}{2})$
 $= \frac{3 - 2(-2x + \frac{3}{2})}{4}$
 $= \frac{3 + 4x - 3}{4}$
 $= \frac{4x}{4}$
 $= x$

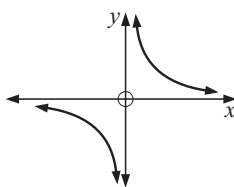


$f(x)$ passes through $(0, 3)$ and $(-3, 0)$
 $\therefore f^{-1}(x)$ passes through $(3, 0)$ and $(0, -3)$

$(f \circ f^{-1})(x)$
 $= f(f^{-1}(x))$
 $= f(x - 3)$
 $= (x - 3) + 3$
 $= x$

**EXERCISE 1G**

- 1 a** $f(x) = \frac{1}{x}$ has graph



No vertical line cuts the graph more than once, so it is a function.

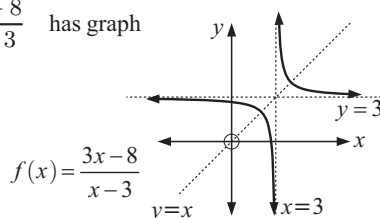
No horizontal line cuts the graph more than once.

Hence, $f(x) = \frac{1}{x}$, $x \neq 0$ has an inverse function.

- b** $f(x) = \frac{1}{x}$ i.e., $y = \frac{1}{x}$ has an inverse function $x = \frac{1}{y}$ and so $y = \frac{1}{x}$

So, $f^{-1}(x) = \frac{1}{x}$ It is a self-inverse function.

- 2 a** $f(x) = \frac{3x-8}{x-3}$ has graph



The vertical line test shows it to be a function.

The horizontal line test shows it has an inverse function.

Symmetry about $y = x$ shows it is a self-inverse function.

$$\text{b } f(x) = \frac{3x-8}{x-3} \text{ i.e., } y = \frac{3x-8}{x-3} \text{ has an inverse function } x = \frac{3y-8}{y-3}$$

$$\therefore x(y-3) = 3y-8$$

$$\therefore xy-3x = 3y-8$$

$$\therefore y(x-3) = 3x-8$$

$$\therefore y = \frac{3x-8}{x-3}$$

i.e., $f(x) = f^{-1}(x)$ $\therefore$ it is a self-inverse function.

- 3 a** If $y = f(x)$ has an inverse function, then the inverse function must also be a function. Thus, it must satisfy the ‘vertical line test’, i.e., no vertical line can cut it more than once. This condition for the inverse function cannot be satisfied if the original function does not satisfy the ‘horizontal line test’. Thus, the ‘horizontal line test’ is a valid test for the existence of an inverse function.

b i This graph satisfies the ‘horizontal line test’ and therefore has an inverse function.

ii, iii These graphs both fail the ‘horizontal line test’ so neither of these have inverse functions.

c ii Domain $\{x: x \geq 1\}$ (or $\{x: x \leq 1\}$) **iii** Domain $\{x: x \geq 1\}$ (or $\{x: x \leq -2\}$)

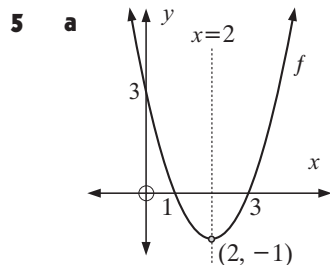
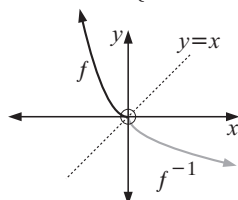
4 a f is $y = x^2, x \leq 0$

so f^{-1} is $x = y^2, y \leq 0$

$$\therefore y = -\sqrt{x}$$

i.e., $f^{-1}(x) = -\sqrt{x}$

b



$f: x \rightarrow x^2 - 4x + 3$ satisfies the ‘vertical line test’ so is therefore a function. It does not however satisfy the horizontal line test as any horizontal line above the vertex cuts the graph twice. Therefore it does not have an inverse function.

b For $x \geq 2$, all horizontal lines cut the graph no more than once. Therefore f has an inverse function for $x \geq 2$.

c f is $y = x^2 - 4x + 3, x \geq 2$
so f^{-1} is $x = y^2 - 4y + 3, y \geq 2$

$$\text{i.e., } x = (y-2)^2 - 4 + 3, y \geq 2$$

$$= (y-2)^2 - 1, y \geq 2$$

$$\therefore x+1 = (y-2)^2, y \geq 2$$

$$\therefore y-2 = \sqrt{x+1}, y \geq 2$$

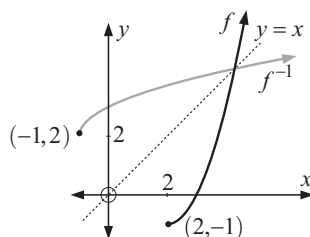
$$\therefore y = 2 + \sqrt{1+x}, y \geq 2$$

i.e., $f^{-1}(x) = 2 + \sqrt{1+x}$

as required

d i domain of f is $\{x: x \geq 2\}$,
range is $\{y: y \geq -1\}$

ii domain of f^{-1} is $\{x: x \geq -1\}$,
range is $\{y: y \geq 2\}$



e $f \circ f^{-1} = f(f^{-1})$

$$= (2 + \sqrt{1+x})^2 - 4(2 + \sqrt{1+x}) + 3$$

$$= 4 + 4\sqrt{1+x} + 1 + x - 8 - 4\sqrt{1+x} + 3$$

$$= x$$

$$f^{-1} \circ f = f^{-1}(f)$$

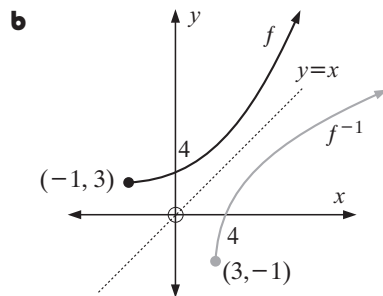
$$= 2 + \sqrt{1+x^2-4x+3}$$

$$= 2 + \sqrt{(x-2)^2}$$

$$= 2 + x - 2$$

$$= x$$

- 6 a** f is $y = (x+1)^2 + 3, x \geq -1$
 so f^{-1} is $x = (y+1)^2 + 3, y \geq -1$
 i.e., $x - 3 = (y+1)^2, y \geq -1$
 $\therefore y + 1 = \sqrt{x-3}, y \geq -1, x \geq 3$
 $\therefore y = \sqrt{x-3} - 1, y \geq -1, x \geq 3$
- c i** Domain $\{x: x \geq -1\}$, Range $\{y: y \geq 3\}$
ii Domain $\{x: x \geq 3\}$, Range $\{y: y \geq -1\}$



- 7 a** g is $y = \frac{8-x}{2}$
 so g^{-1} is $x = \frac{8-y}{2}$
 $\therefore 2x = 8 - y$
 $\therefore y = 8 - 2x$
 i.e., $g^{-1}(x) = 8 - 2x$
 Now $g^{-1}(-1) = 8 - 2(-1) = 10$

- b** $(f \circ g^{-1})(x) = 9$
 $\therefore f(g^{-1}(x)) = 9$
 $\therefore f(8 - 2x) = 9$
 $\therefore 2(8 - 2x) + 5 = 9$
 $\therefore 16 - 4x + 5 = 9$
 $\therefore -4x = -12$
 $\therefore x = 3$

- 8 a i** f is $y = 5^x$
 so, $f(2) = 5^2 = 25$
- ii** g is $y = \sqrt{x}$
 so g^{-1} is $x = \sqrt{y}$
 $\therefore y = x^2$
 i.e., $g^{-1}(x) = x^2, x \geq 0$
 $\therefore g^{-1}(4) = 4^2$
 $\therefore g^{-1}(4) = 16$

- b** $(g^{-1} \circ f)(x) = 25$
 $\therefore g^{-1}(f(x)) = 25$
 $\therefore g^{-1}(5^x) = 25$
 $\therefore (5^x)^2 = 25 \quad \{\text{as } g^{-1}(x) = x^2, x \geq 0\}$

and so $5^{2x} = 5^2$
 $\therefore 2x = 2$
 $\therefore x = 1$

- 9** Show: $(f^{-1} \circ g^{-1})(x) = (g \circ f)^{-1}(x)$

f is $y = 2x$ g is $y = 4x - 3$
 so f^{-1} is $x = 2y$ so g^{-1} is $x = 4y - 3$
 $\therefore y = \frac{x}{2}$ $\therefore 4y = x + 3$
 i.e., $f^{-1}(x) = \frac{x}{2}$ $\therefore y = \frac{x+3}{4}$
 i.e., $g^{-1}(x) = \frac{x+3}{4}$

Now, $(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x))$
 $= f^{-1}\left(\frac{x+3}{4}\right)$
 $= \frac{\left(\frac{x+3}{4}\right)}{2}$

$\therefore (f^{-1} \circ g^{-1})(x) = \frac{x+3}{8} = (g \circ f)^{-1}(x)$ as required

$(g \circ f)(x) = g(f(x))$
 $= g(2x)$
 $= 4(2x) - 3$
 i.e., $(g \circ f)(x) = 8x - 3$
 i.e., $g \circ f$ is $y = 8x - 3$
 so $(g \circ f)^{-1}$ is $x = 8y - 3$
 $\therefore y = \frac{x+3}{8}$
 i.e., $(g \circ f)^{-1}(x) = \frac{x+3}{8}$

10 a f is $y = 2x$ so f^{-1} is $x = 2y$ $\therefore y = \frac{x}{2}$ i.e., $f^{-1}(x) = \frac{x}{2} \neq 2x$ So, $f^{-1}(x) \neq f(x)$	b f is $y = x$ so f^{-1} is $x = y$ $\therefore y = x$ i.e., $f^{-1}(x) = x$ So, $f^{-1}(x) = f(x)$	c f is $y = -x$ so f^{-1} is $x = -y$ $\therefore y = -x$ i.e., $f^{-1}(x) = -x$ So, $f^{-1}(x) = f(x)$
d f is $y = \frac{2}{x}$ so f^{-1} is $x = \frac{2}{y}$ $\therefore y = \frac{2}{x}$ i.e., $f^{-1}(x) = \frac{2}{x}$ So, $f^{-1}(x) = f(x)$	e f is $y = -\frac{6}{x}$ so f^{-1} is $x = -\frac{6}{y}$ $\therefore y = -\frac{6}{x}$ i.e., $f^{-1}(x) = -\frac{6}{x}$ So, $f^{-1}(x) = f(x)$	

i.e., $f^{-1}(x) = f(x)$ is true for parts **b, c, d** and **e**.

11 a f is $y = 3x + 1$
 so f^{-1} is $x = 3y + 1$
 $\therefore y = \frac{x-1}{3}$ i.e., $f^{-1}(x) = \frac{x-1}{3}$

$(f \circ f^{-1})(x) = f(f^{-1}(x))$ $= f\left(\frac{x-1}{3}\right)$ $= 3\left(\frac{x-1}{3}\right) + 1$ $= x - 1 + 1$	$(f^{-1} \circ f)(x) = f^{-1}(f(x))$ $= f^{-1}(3x + 1)$ $= \frac{3x + 1 - 1}{3}$ $= \frac{3x}{3}$
---	--

i.e., $(f \circ f^{-1})(x) = x$ i.e., $(f^{-1} \circ f)(x) = x = (f \circ f^{-1})(x)$ as required

b f is $y = \frac{x+3}{4}$ so f^{-1} is $x = \frac{y+3}{4}$
 $\therefore 4x = y + 3$
 $\therefore y = 4x - 3$
 i.e., $f^{-1}(x) = 4x - 3$

$(f \circ f^{-1})(x) = f(f^{-1}(x))$ $= f(4x - 3)$ $= \frac{4x - 3 + 3}{4}$ $= \frac{4x}{4}$	$(f^{-1} \circ f)(x) = f^{-1}(f(x))$ $= f^{-1}\left(\frac{x+3}{4}\right)$ $= 4\left(\frac{x+3}{4}\right) - 3$ $= x + 3 - 3$
---	--

i.e., $(f \circ f^{-1})(x) = x$ i.e., $(f^{-1} \circ f)(x) = x = (f \circ f^{-1})(x)$ as required

c f is $y = \sqrt{x}$ for $x \geq 0$ so f^{-1} is $x = \sqrt{y}$

$$\therefore y = x^2$$

$$\text{i.e., } f^{-1}(x) = x^2 \text{ for } x \geq 0$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) \qquad (f^{-1} \circ f)(x) = f^{-1}(f(x))$$

$$= f(x^2) \qquad = f^{-1}(\sqrt{x})$$

$$= \sqrt{x^2} \qquad = (\sqrt{x})^2$$

$$\text{i.e., } (f \circ f^{-1})(x) = x \qquad \text{i.e., } (f^{-1} \circ f)(x) = x = (f \circ f^{-1})(x) \text{ as required}$$

12 a $f(x)$ passes through $A(x, f(x))$, so $f^{-1}(x)$ passes through $B(f(x), x)$

b Substitute the coordinates of $B(f(x), x)$ into $y = f^{-1}(x)$:

$$\text{i.e., } x = f^{-1}(f(x)) = (f^{-1} \circ f)(x)$$

c B has coordinates $(x, f^{-1}(x))$ since it lies on $y = f^{-1}(x)$,

so A has coordinates $(f^{-1}(x), x)$ as $f(x)$ is the inverse of $f^{-1}(x)$.

Substitute the coordinates of $A(f^{-1}(x), x)$ into $y = f(x)$:

$$\text{i.e., } x = f(f^{-1}(x))$$

$$\text{i.e., } f(f^{-1}(x)) = x \text{ as required}$$

REVIEW SET 1A

$$\begin{array}{lll} \mathbf{1} \quad \mathbf{a} & f(x) = 2x - x^2 & \mathbf{b} \quad f(-3) = 2(-3) - (-3)^2 \quad \mathbf{c} \quad f(-\tfrac{1}{2}) = 2(-\tfrac{1}{2}) - (-\tfrac{1}{2})^2 \\ & f(2) = 2(2) - 2^2 & = -6 - 9 \qquad = -1 - \tfrac{1}{4} \\ & = 0 & = -15 \qquad = -\tfrac{5}{4} \end{array}$$

- 2 a**
- i** range is $\{y: y \geq -5\}$, domain is $\{x: x \in \mathcal{R}\}$
 - ii** x -intercepts are -1 and 5 ; y -intercept is $-\frac{25}{9}$
 - iii** The graph passes the 'vertical line test' so is therefore a function.
 - iv** No, as it fails the horizontal line test.
- b**
- i** range is $\{y: y = 1 \text{ or } -3\}$, domain is $\{x: x \in \mathcal{R}\}$
 - ii** there are no x -intercepts; y -intercept is 1
 - iii** The graph passes the 'vertical line test' so is therefore a function.
 - iv** No, as it fails the horizontal line test.

- 3 a** domain is $\{x: x \geq -2\}$, range is $\{y: 1 \leq y < 3\}$
- b** domain is $\{x: x \in \mathcal{R}\}$, range is $\{y: y = -1, 1 \text{ or } 2\}$

$$\begin{array}{ll} \mathbf{4} \quad \mathbf{a} & h(x) = 7 - 3x \\ & h(2x - 1) = 7 - 3(2x - 1) \\ & = 7 - 6x + 3 \\ & = 10 - 6x \\ \mathbf{b} & h(2x - 1) = -2 \\ \therefore & 7 - 3(2x - 1) = -2 \\ \therefore & 7 - 6x + 3 = -2 \\ \therefore & -6x = -12 \\ \therefore & x = 2 \end{array}$$

5 $f(x) = ax^2 + bx + c$, where $f(0) = 5$, $f(-2) = 21$ and $f(3) = -4$

When $f(0) = 5$,

$$5 = a(0)^2 + b(0) + c$$

$$\therefore 5 = c$$

$$\therefore c = 5 \quad \dots (1)$$

When $f(-2) = 21$,

$$21 = a(-2)^2 + b(-2) + c$$

$$= 4a - 2b + c$$

$$= 4a - 2b + 5 \quad \{\text{using (1)}\}$$

$$\therefore 4a - 2b = 16$$

$$\therefore 2a - b = 8 \quad \text{and so } b = 2a - 8 \quad \dots (2)$$

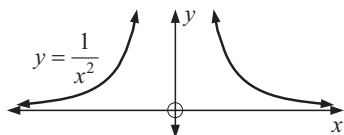
$$\begin{aligned}
 \text{When } f(3) &= -4, & -4 &= a(3)^2 + b(3) + c \\
 \therefore -4 &= 9a + 3b + c \\
 \therefore -4 &= 9a + 3b + 5 \quad \{\text{using (1)}\} \\
 \therefore -4 &= 9a + 3(2a - 8) + 5 \quad \{\text{using (2)}\} \\
 \therefore -9 &= 9a + 6a - 24 \\
 \therefore 15 &= 15a \quad \text{and so } a = 1
 \end{aligned}$$

Now, substituting $a = 1$ into (2) gives $b = 2(1) - 8 = -6$

i.e., $a = 1$, $b = -6$, $c = 5$

6 a $f(x) = \frac{1}{x^2}$ is meaningless when $x^2 = 0$
i.e., when $x = 0$

b



c domain of $f(x)$ is $\{x: x \neq 0\}$
range of $f(x)$ is $\{y: y > 0\}$

7 a $f(g(x)) = f(x^2 + 2)$
 $= 2(x^2 + 2) - 3$
 $= 2x^2 + 4 - 3$
 $= 2x^2 + 1$

b $g(f(x)) = g(2x - 3)$
 $= (2x - 3)^2 + 2$
 $= 4x^2 - 12x + 9 + 2$
 $= 4x^2 - 12x + 11$

8 a i $(f \circ g)(x) = f(g(x))$
 $= f(\sqrt{x})$
 $= 1 - 2\sqrt{x}$

ii $(g \circ f)(x) = g(f(x))$
 $= g(1 - 2x)$
 $= \sqrt{1 - 2x}$

b The domain of $f \circ g$ is $\{x: x \geq 0\}$ for $\sqrt{x}$ to be defined.
The range of $f \circ g$ is $\{y: y \leq 1\}$.

The domain of $g \circ f$ is obtained by noticing that $1 - 2x$ must be ≥ 0 .

$$\begin{aligned}
 \therefore 2x &\leq 1 \\
 \text{i.e., } x &\leq \frac{1}{2} \\
 \text{i.e., } \{x: x &\leq \frac{1}{2}\} \\
 \text{i.e., } x &\in] -\infty, \frac{1}{2}]
 \end{aligned}$$

The range of $g \circ f$ is $\{y: y \geq 0\}$ i.e., $y \in [0, \infty[$

9 a $f(g(x)) = \sqrt{1 - x^2}$
 $= f(1 - x^2)$
i.e., $f(x) = \sqrt{x}$,
 $g(x) = 1 - x^2$

b $g(f(x)) = \left(\frac{x-2}{x+1}\right)^2 = g\left(\frac{x-2}{x+1}\right)$
i.e., $g(x) = x^2$,
 $f(x) = \frac{x-2}{x+1}$

REVIEW SET 1B

1 $g(x) = x^2 - 3x$

a $g(x+1) = (x+1)^2 - 3(x+1)$
 $= x^2 + 2x + 1 - 3x - 3$
 $= x^2 - x - 2$

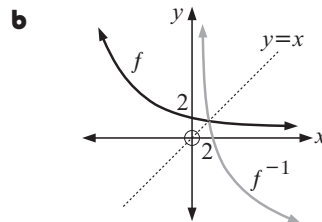
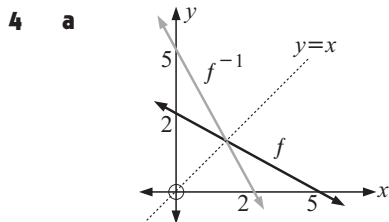
b $g(x^2 - 2) = (x^2 - 2)^2 - 3(x^2 - 2)$
 $= x^4 - 4x^2 + 4 - 3x^2 + 6$
 $= x^4 - 7x^2 + 10$

2 a $f(x) = 7 - 4x$
 i.e., $y = 7 - 4x$
 so $f^{-1}(x)$ is $x = 7 - 4y$
 $\therefore y = \frac{7-x}{4}$
 i.e., $f^{-1}(x) = \frac{7-x}{4}$

b $f(x) = \frac{3+2x}{5}$
 i.e., $y = \frac{3+2x}{5}$
 so $f^{-1}(x)$ is $x = \frac{3+2y}{5}$
 $\therefore 5x = 3 + 2y$
 $\therefore y = \frac{5x-3}{2}$ i.e., $f^{-1}(x) = \frac{5x-3}{2}$

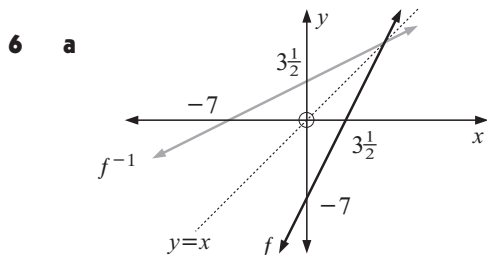
3 a $y = (x-1)(x-5)$
 i.e., x -intercepts are $x = 1$ and 5
 $\therefore$ vertex is at $x = 3$, $y = (3-1)(3-5) = 2 \times (-2) = -4$
 i.e., vertex is at $(3, -4)$
 domain is $\{x: x \in \mathcal{R}\}$, range is $\{y: y \geq -4\}$ i.e., $y \in [-4, \infty[$

b From the graph, domain is $\{x: x \neq 0, 2\}$, range is $\{y: y \leq -1 \text{ or } y > 0\}$



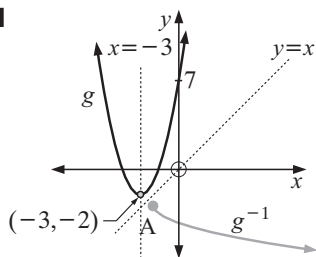
5 a $f(x) = 4x + 2$
 i.e., $y = 4x + 2$
 so $f^{-1}(x)$ is $x = 4y + 2$
 $\therefore y = \frac{x-2}{4}$
 i.e., $f^{-1}(x) = \frac{x-2}{4}$

b $f(x) = \frac{3-5x}{4}$ i.e., $y = \frac{3-5x}{4}$
 so $f^{-1}(x)$ is $x = \frac{3-5y}{4}$
 $\therefore 4x = 3 - 5y$
 $\therefore y = \frac{3-4x}{5}$
 i.e., $f^{-1}(x) = \frac{3-4x}{5}$



b $f(x) = 2x - 7$
 i.e., $y = 2x - 7$
 so $f^{-1}(x)$ is $x = 2y - 7$
 $\therefore y = \frac{x+7}{2}$
 i.e., $f^{-1}(x) = \frac{x+7}{2}$

c $f \circ f^{-1}$ and $f^{-1} \circ f$
 $= f(f^{-1}(x))$ $= f^{-1}(f(x))$
 $= f\left(\frac{x+7}{2}\right)$ $= f^{-1}(2x-7)$
 $= 2\left(\frac{x+7}{2}\right) - 7$ $= \frac{2x-7+7}{2}$
 $= x+7-7$ $= \frac{2x}{2}$
 $= x$ $= x$ So $f \circ f^{-1} = f^{-1} \circ f = e$

7 a, d


b If $x \leq -3$, we have the graph to the left of $x = -3$, and any horizontal line through the graph cuts it no more than once. Therefore it has an inverse function.

c

$$\begin{aligned}
 g(x) &= x^2 + 6x + 7, \quad x \leq -3 \\
 \text{i.e., } y &= x^2 + 6x + 7, \quad x \leq -3 \\
 \text{so } g^{-1}(x) \text{ is } x &= y^2 + 6y + 7, \quad y \leq -3 \\
 &= (y+3)^2 - 9 + 7 \\
 \therefore x+2 &= (y+3)^2 \\
 \therefore y+3 &= \pm\sqrt{x+2} \\
 \therefore y &= -3 \pm \sqrt{x+2} \\
 \text{but } y &\leq -3, \text{ so } y = -3 - \sqrt{x+2}
 \end{aligned}$$

e The range of g is $\{y: y \geq -2\}$, so the domain of g^{-1} is $\{x: x \geq -2\}$ and the range of g^{-1} is $\{y: y \leq -3\}$

8 a

$$\begin{aligned}
 h(x) &= (x-4)^2 + 3, \quad x \geq 4 \\
 \text{i.e., } y &= (x-4)^2 + 3, \quad x \geq 4 \\
 \text{so } h^{-1}(x) \text{ is } x &= (y-4)^2 + 3, \quad y \geq 4 \\
 \therefore x-3 &= (y-4)^2 \\
 \therefore y-4 &= \pm\sqrt{x-3} \\
 \therefore y &= 4 \pm \sqrt{x-3} \\
 \text{but } y &\geq 4, \text{ so } y = 4 + \sqrt{x-3} \\
 \text{i.e., } h^{-1}(x) &= 4 + \sqrt{x-3}, \quad x \geq 3
 \end{aligned}$$

b

$$\begin{aligned}
 h \circ h^{-1} &= h(h^{-1}(x)) \\
 &= h(4 + \sqrt{x-3}) \\
 &= (4 + \sqrt{x-3} - 4)^2 + 3 \\
 &= (\sqrt{x-3})^2 + 3 \\
 &= x - 3 + 3 \\
 &= x \\
 h^{-1} \circ h &= h^{-1}(h(x)) \\
 &= h^{-1}((x-4)^2 + 3) \\
 &= 4 + \sqrt{(x-4)^2 + 3 - 3} \\
 &= 4 + \sqrt{(x-4)^2} \\
 &= 4 + x - 4 \quad \text{as } x \geq 4 \\
 &= x
 \end{aligned}$$

9

$$\begin{aligned}
 f(x) &= 3x + 6 \\
 \text{i.e., } y &= 3x + 6 \\
 \text{so } f^{-1}(x) \text{ is } x &= 3y + 6 \\
 \therefore y &= \frac{x-6}{3} \quad \text{so } h^{-1}(x) \text{ is } x = \frac{y}{3} \\
 \text{i.e., } f^{-1}(x) &= \frac{x-6}{3} \quad \therefore y = 3x \\
 \text{i.e., } h^{-1}(x) &= 3x
 \end{aligned}$$

Now $(f^{-1} \circ h^{-1})(x) = f^{-1}(h^{-1}(x))$

$$\begin{aligned}
 &= f^{-1}(3x) \\
 &= \frac{3x-6}{3} \\
 &= x-2
 \end{aligned}$$

$$\begin{aligned}
 (h \circ f)(x) &= h(f(x)) \\
 &= h(3x+6) \\
 &= \frac{3x+6}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{i.e., } y &= x+2 \\
 \text{i.e., } (h \circ f)^{-1}(x) \text{ is } x &= y+2 \\
 \therefore y &= x-2 \\
 \text{i.e., } (h \circ f)^{-1}(x) &= x-2
 \end{aligned}$$

i.e., $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$ as required

Chapter 2

SEQUENCES AND SERIES

EXERCISE 2A

- 1** **a** 4, 13, 22, 31, **b** 45, 39, 33, 27, **c** 2, 6, 18, 54, **d** 96, 48, 24, 12,
- 2** **a** The sequence starts at 8 and each term is 8 more than the previous term. The next two terms are 40 and 48.
b The sequence starts at 2 and each term is 3 more than the previous term. The next two terms are 14 and 17.
c The sequence starts at 36 and each term is 5 less than the previous term. The next two terms are 16 and 11.
d The sequence starts at 96 and each term is 7 less than the previous term. The next two terms are 68 and 61.
e The sequence starts at 1 and each term is 4 times the previous term. The next two terms are 256 and 1024.
f The sequence starts at 2 and each term is 3 times the previous term. The next two terms are 162 and 486.
g The sequence starts at 480 and each term is half the previous term. The next two terms are 30 and 15.
h The sequence starts at 243 and each term is one third of the previous term. The next two terms are 3 and 1.
i The sequence starts at 50 000 and each term is one fifth of the previous term. The next two terms are 80 and 16.
- 3** **a** Each term is the square of the number of the term. The next three terms are 25, 36 and 49.
b Each term is the cube of the number of the term. The next three terms are 125, 216 and 343.
c Each term is $n \times (n + 1)$ where n is the number of the term. The next three terms are 30, 42 and 56.

EXERCISE 2B

- 1** **a** $\{2n\}$ generates the sequence 2, 4, 6, 8, 10, (letting $n = 1, 2, 3, 4, 5, \dots$)
b $\{2n + 2\}$ generates the sequence 4, 6, 8, 10, 12, (letting $n = 1, 2, 3, 4, 5, \dots$)
c $\{2n - 1\}$ generates the sequence 1, 3, 5, 7, 9, (letting $n = 1, 2, 3, 4, 5, \dots$)
d $\{2n - 3\}$ generates the sequence -1, 1, 3, 5, 7, (letting $n = 1, 2, 3, 4, 5, \dots$)
e $\{2n + 3\}$ generates the sequence 5, 7, 9, 11, 13, (letting $n = 1, 2, 3, 4, 5, \dots$)
f $\{2n + 11\}$ generates the sequence 13, 15, 17, 19, 21, (letting $n = 1, 2, 3, 4, 5, \dots$)
g $\{3n + 1\}$ generates the sequence 4, 7, 10, 13, 16, (letting $n = 1, 2, 3, 4, 5, \dots$)
h $\{4n - 3\}$ generates the sequence 1, 5, 9, 13, 17, (letting $n = 1, 2, 3, 4, 5, \dots$)
- 2** **a** $\{2^n\}$ generates the sequence 2, 4, 8, 16, 32, (letting $n = 1, 2, 3, 4, 5, \dots$)
b $\{3 \times 2^n\}$ generates the sequence 6, 12, 24, 48, 96, (letting $n = 1, 2, 3, 4, 5, \dots$)
c $\{6 \times (\frac{1}{2})^n\}$ generates the sequence 3, $\frac{3}{2}$, $\frac{3}{4}$, $\frac{3}{8}$, $\frac{3}{16}$, (letting $n = 1, 2, 3, 4, 5, \dots$)
d $\{(-2)^n\}$ generates the sequence -2, 4, -8, 16, -32, (letting $n = 1, 2, 3, 4, 5, \dots$)
- 3** $\{15 - (-2)^n\}$ generates the sequence with first five terms:
$$t_1 = 15 - (-2)^1 = 17, \quad t_2 = 15 - (-2)^2 = 11, \quad t_3 = 15 - (-2)^3 = 23,$$
$$t_4 = 15 - (-2)^4 = -1, \quad t_5 = 15 - (-2)^5 = 47$$

EXERCISE 2C

- 1 a** $17 - 6 = 11$ So, assuming that the pattern continues, consecutive terms differ by 11.
 $28 - 17 = 11$ $\therefore$ the sequence is arithmetic with $u_1 = 6$, $d = 11$.
 $39 - 28 = 11$
 $50 - 39 = 11$
- b** $u_n = u_1 + (n-1)d$ **c** $u_{50} = 11(50) - 5$ **d** Let $u_n = 325 = 11n - 5$
 $= 6 + (n-1)11$ $= 545$ $\therefore 330 = 11n$
i.e., $u_n = 11n - 5$ $\therefore n = 30$
So, 325 is a member, i.e., u_{30} .
- e** Let $u_n = 761 = 11n - 5$
 $\therefore 766 = 11n$
 $\therefore n = 69\frac{7}{11}$, but n is an integer, so 761 is not a member of the sequence.
- 2 a** $83 - 87 = -4$ So, assuming that the pattern continues, consecutive terms
 $79 - 83 = -4$ differ by -4 .
 $75 - 79 = -4$ $\therefore$ the sequence is arithmetic with $u_1 = 87$, $d = -4$.
- b** $u_n = u_1 + (n-1)d$ **c** $u_{40} = 91 - 4(40)$ **d** Let $u_n = -143 = 91 - 4n$
 $= 87 + (n-1)(-4)$ $= 91 - 160$ $\therefore 4n = 234$
 $= 87 - 4n + 4$ $= -69$ $\therefore n = 58\frac{1}{2}$
i.e., $u_n = 91 - 4n$ but n is an integer, so
 -143 is not a member
of the sequence.
- 3 a** $u_n = 3n - 2$ $u_1 = 3(1) - 2 = 1$ $u_{n+1} = 3(n+1) - 2 = 3n + 1$
 $u_{n+1} - u_n = (3n+1) - (3n-2)$ So, assuming that the pattern continues,
 $= 3$, a constant consecutive terms differ by 3.
 $\therefore$ the sequence is arithmetic with $u_1 = 1$
and $d = 3$.
- b** $u_1 = 1$, $d = 3$ **c** $u_{57} = 3(57) - 2 = 169$
- d** Let $u_n = 450 = 3n - 2$ i.e., $3n = 452$ and so $n = 150\frac{2}{3}$
So, try the two values on either side of $n = 150\frac{2}{3}$, i.e., for $n = 150$ and $n = 151$:
 $u_{150} = 3(150) - 2$ and $u_{151} = 3(151) - 2$
 $= 448$ $= 451$
i.e., $u_{151} = 451$ is the least term which is greater than 450.
- 4 a** $u_n = \frac{71-7n}{2} = 35\frac{1}{2} - \frac{7}{2}n$ $u_1 = \frac{71-7(1)}{2} = 32$
 $u_{n+1} = \frac{71-7(n+1)}{2} = \frac{71-7n-7}{2} = \frac{64-7n}{2} = 32 - \frac{7}{2}n$
 $u_{n+1} - u_n = (32 - \frac{7}{2}n) - (35\frac{1}{2} - \frac{7}{2}n) = -\frac{7}{2}$ a constant
So, assuming that the pattern continues, consecutive terms differ by $-\frac{7}{2}$.
 $\therefore$ the sequence is arithmetic with $u_1 = 32$, $d = -\frac{7}{2}$.
- b** $u_1 = 32$, $d = -\frac{7}{2}$ **c** $u_{75} = \frac{71-7(75)}{2} = -227$

$$\text{d Let } u_n = -200 = \frac{71 - 7n}{2} \quad \text{i.e., } -400 = 71 - 7n \quad \therefore 7n = 471$$

$$\therefore n = 67\frac{2}{7}$$

So, try the two values on either side of $n = 67\frac{2}{7}$, i.e., for $n = 67$ and $n = 68$:

$$u_{67} = \frac{71 - 7(67)}{2} = -199 \quad \text{and} \quad u_{68} = \frac{71 - 7(68)}{2} = -202\frac{1}{2}$$

i.e., terms of the sequence are less than -200 for $n \geq 68$.

5 a The terms are consecutive,

$$\therefore k - 32 = 3 - k$$

{equating common differences}

$$\therefore 2k = 35 \quad \text{and so} \quad k = 17\frac{1}{2}$$

b The terms are consecutive,

$$\therefore (2k + 1) - (k + 1) = 13 - (2k + 1)$$

$$\therefore k = 12 - 2k$$

$$\therefore 3k = 12 \quad \text{and so} \quad k = 4$$

c The terms are consecutive,

$$k - 5 = k^2 - 8 - k$$

{equating common differences}

$$\therefore k^2 - 2k - 3 = 0$$

$$\therefore (k - 3)(k + 1) = 0$$

$$\therefore k = -1 \text{ or } 3$$

6 a $u_7 = 41 \quad \therefore u_1 + 6d = 41 \quad \dots (1)$

$u_{13} = 77 \quad \therefore u_1 + 12d = 77 \quad \dots (2)$

Solving simultaneously,

$$-u_1 - 6d = -41$$

$$u_1 + 12d = 77$$

$$\hline \therefore 6d = 36 \quad \text{{adding the equations}}$$

$$\therefore d = 6$$

So in (1), $u_1 + 6(6) = 41$

$$\therefore u_1 + 36 = 41$$

$$\therefore u_1 = 5$$

Now $u_n = u_1 + (n - 1)d$

$$\therefore u_n = 5 + (n - 1)6$$

$$\therefore u_n = 6n - 1$$

c $u_7 = 1 \quad \therefore u_1 + 6d = 1 \quad \dots (1)$

$u_{15} = -39 \quad \therefore u_1 + 14d = -39 \quad \dots (2)$

Solving simultaneously,

$$-u_1 - 6d = -1$$

$$u_1 + 14d = -39$$

$$\hline \therefore 8d = -40 \quad \text{{adding the equations}}$$

$$\therefore d = -5$$

So in (1), $u_1 + 6(-5) = 1$

$$\therefore u_1 - 30 = 1$$

$$\therefore u_1 = 31$$

Now $u_n = u_1 + (n - 1)d$

$$\therefore u_n = 31 + (n - 1)(-5)$$

$$\therefore u_n = 31 - 5n + 5$$

$$\therefore u_n = -5n + 36$$

b $u_5 = -2 \quad \therefore u_1 + 4d = -2 \quad \dots (1)$

$u_{12} = -12\frac{1}{2} \quad \therefore u_1 + 11d = -12\frac{1}{2} \quad \dots (2)$

Solving simultaneously,

$$-u_1 - 4d = 2$$

$$u_1 + 11d = -12\frac{1}{2}$$

$$\hline \therefore 7d = -10\frac{1}{2} \quad \text{{adding the equations}}$$

$$\therefore d = -\frac{3}{2}$$

So in (1), $u_1 + 4(-\frac{3}{2}) = -2$

$$\therefore u_1 = 4$$

Now $u_n = u_1 + (n - 1)d$

$$\therefore u_n = 4 + (n - 1)(-\frac{3}{2})$$

$$\therefore u_n = -\frac{3}{2}n + \frac{11}{2}$$

d $u_{11} = -16 \quad \therefore u_1 + 10d = -16 \quad \dots (1)$

$u_8 = -11\frac{1}{2} \quad \therefore u_1 + 7d = -11\frac{1}{2} \quad \dots (2)$

Solving simultaneously,

$$-u_1 - 10d = 16$$

$$u_1 + 7d = -11\frac{1}{2}$$

$$\hline \therefore -3d = 4\frac{1}{2} \quad \text{{adding the equations}}$$

$$\therefore d = -\frac{3}{2}$$

So in (1), $u_1 + 10(-\frac{3}{2}) = -16$

$$\therefore u_1 - 15 = -16$$

$$\therefore u_1 = -1$$

Now $u_n = u_1 + (n - 1)d$

$$\therefore u_n = -1 + (n - 1)(-\frac{3}{2})$$

$$\therefore u_n = -\frac{3}{2}n + \frac{1}{2}$$

- 7 a** Let the numbers be $5, 5 + d, 5 + 2d, 5 + 3d, 10$.

$$\text{Then } 5 + 4d = 10$$

$$\therefore 4d = 5$$

$$\therefore d = \frac{5}{4} = 1\frac{1}{4}$$

So the numbers are $5, 6\frac{1}{4}, 7\frac{1}{2}, 8\frac{3}{4}, 10$.

- b** Let the numbers be $-1, -1 + d, -1 + 2d, -1 + 3d, -1 + 4d, -1 + 5d, -1 + 6d, 32$.

$$\text{Then } -1 + 7d = 32$$

$$\therefore 7d = 33$$

$$\therefore d = \frac{33}{7} = 4\frac{5}{7}$$

So the numbers are $-1, 3\frac{5}{7}, 8\frac{3}{7}, 13\frac{1}{7}, 17\frac{6}{7}, 22\frac{4}{7}, 27\frac{2}{7}, 32$.

- 8 a** $u_1 = 36, \quad 35\frac{1}{3} - 36 = -\frac{2}{3}$

$$34\frac{2}{3} - 35\frac{1}{3} = -\frac{2}{3}, \quad \text{so } d = -\frac{2}{3}$$

b $u_n = u_1 + (n-1)d$

$$\therefore -30 = 36 + (n-1)\left(-\frac{2}{3}\right) \quad \{\text{letting } u_n = -30, \text{ the last term of the sequence}\}$$

$$\therefore -66 = -\frac{2}{3}n + \frac{2}{3}$$

$$\therefore \frac{2}{3}n = 66\frac{2}{3}$$

$$\therefore n = 100$$

i.e., the sequence has 100 terms

- 9** $u_1 = 23, \quad 36 - 23 = 13$

$$49 - 36 = 13$$

$$62 - 49 = 13, \quad \text{so } d = 13$$

$$\therefore u_n = u_1 + (n-1)d$$

$$\text{i.e., } u_n = 23 + (n-1)13$$

$$= 23 + 13n - 13$$

$$\therefore u_n = 13n + 10$$

$$\text{Let } u_n = 100\,000 = 13n + 10$$

$$\therefore 99\,990 = 13n$$

$$\therefore n = 7691\frac{7}{13}$$

So, try the two values on either side of $n = 7691\frac{7}{13}$, i.e., for $n = 7691$ and $n = 7692$:

$$\text{i.e., } u_{7691} = 13(7691) + 10 \quad \text{and} \quad u_{7692} = 13(7692) + 10$$

$$= 99\,993$$

$$= 100\,006$$

i.e., the first term to exceed 100 000 is $u_{7692} = 100\,006$.

EXERCISE 2D

- 1 a** $\frac{6}{2} = 3 \therefore r = 3, \quad u_1 = 2 \therefore b = 6 \times 3 = 18 \quad \text{and} \quad c = 18 \times 3 = 54$

b $\frac{5}{10} = \frac{1}{2} \therefore r = \frac{1}{2}, \quad u_1 = 10 \therefore b = 5 \times \frac{1}{2} = 2\frac{1}{2} \quad \text{and} \quad c = 2\frac{1}{2} \times \frac{1}{2} = 1\frac{1}{4}$

c $\frac{-6}{12} = -\frac{1}{2} \therefore r = -\frac{1}{2}, \quad u_1 = 12 \therefore b = -6 \times -\frac{1}{2} = 3 \quad \text{and} \quad c = 3 \times -\frac{1}{2} = -1\frac{1}{2}$

- 2 a** $\frac{10}{5} = \frac{20}{10} = \frac{40}{20} = 2$

So, assuming the pattern continues, consecutive terms have a common ratio of 2.

$\therefore$ the sequence is geometric with $u_1 = 5$ and $r = 2$.

$$\begin{aligned} \mathbf{b} \quad u_n &= u_1 r^{n-1} \\ \therefore u_n &= 5 \times 2^{n-1} \\ \text{so } u_{15} &= 5 \times 2^{14} \\ &= 81\,920 \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad \frac{-6}{12} &= -\frac{1}{2} & \text{So, assuming the pattern continues, consecutive terms have a} \\ & & \text{common ratio of } -\frac{1}{2}. \\ \frac{3}{-6} &= -\frac{1}{2} & \therefore \text{the sequence is geometric with } u_1 = 12 \text{ and } r = -\frac{1}{2}. \\ \frac{-1.5}{3} &= \frac{(-\frac{3}{2})}{3} = -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad u_n &= u_1 r^{n-1} & \text{so } u_{13} &= 12 \times \left(-\frac{1}{2}\right)^{13-1} \\ \therefore u_n &= 12 \times \left(-\frac{1}{2}\right)^{n-1} & &= 12 \times \left(-\frac{1}{2}\right)^{12} \\ & & &= 12 \times \frac{1}{4096} \\ & & &= 3 \times \frac{1}{1024} \\ & & &= \frac{3}{1024} \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \frac{-6}{8} &= -\frac{3}{4} & \text{So, assuming the pattern continues, consecutive terms have a} \\ & & \text{common ratio of } -\frac{3}{4}. \\ \frac{4.5}{-6} &= -\frac{(\frac{9}{2})}{6} = -\frac{3}{4} & \therefore \text{the sequence is geometric with } u_1 = 8 \text{ and } r = -\frac{3}{4}. \\ \frac{-3.375}{4.5} &= \frac{(-\frac{27}{8})}{(\frac{9}{2})} = -\frac{3}{4} \end{aligned}$$

$$u_n = u_1 r^{n-1} = 8 \times \left(-\frac{3}{4}\right)^{n-1} \quad \text{i.e., } u_{10} = 8 \times \left(-\frac{3}{4}\right)^9 \div -0.600\,677\,49$$

$$\begin{aligned} \mathbf{5} \quad \frac{4\sqrt{2}}{8} &= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} & \text{So, assuming the pattern continues, consecutive terms have a} \\ & & \text{common ratio of } \frac{1}{\sqrt{2}}. \\ \frac{4}{4\sqrt{2}} &= \frac{1}{\sqrt{2}} & \therefore \text{the sequence is geometric with } u_1 = 8 \text{ and } r = \frac{1}{\sqrt{2}}. \\ \frac{2\sqrt{2}}{4} &= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} u_n &= u_1 r^{n-1} = 8 \left(\frac{1}{\sqrt{2}}\right)^{n-1} = 2^3 \times \left(2^{-\frac{1}{2}}\right)^{n-1} = 2^3 \times 2^{-\frac{1}{2}n + \frac{1}{2}} \\ \text{i.e., } u_n &= 2^{\frac{7}{2} - \frac{n}{2}} \end{aligned}$$

6 a Since the terms are geometric,

$$\frac{k}{7} = \frac{28}{k} \quad \therefore k^2 = 196$$

$$\therefore k = \pm 14$$

c Since the terms are geometric,

$$\frac{k+8}{k} = \frac{9k}{k+8}$$

$$\therefore (k+8)^2 = 9k^2$$

$$\therefore k^2 + 16k + 64 = 9k^2$$

$$\therefore 8k^2 - 16k - 64 = 0$$

$$\therefore k^2 - 2k - 8 = 0$$

$$\therefore (k+2)(k-4) = 0 \quad \text{and so } k = -2 \text{ or } 4$$

b Since the terms are geometric,

$$\frac{3k}{k} = \frac{20-k}{3k} = 3$$

$$\therefore 20 - k = 9k$$

$$\therefore 20 = 10k$$

$$\therefore k = 2$$

$$7 \quad a \quad u_4 = 24 \quad \therefore u_1 \times r^3 = 24 \quad \dots (1)$$

$$u_7 = 192 \quad \therefore u_1 \times r^6 = 192 \quad \dots (2)$$

$$\text{So, } \frac{u_1 r^6}{u_1 r^3} = \frac{192}{24} \quad \{(2) \div (1)\}$$

$$\therefore r^3 = 8 \quad \therefore r = 2$$

$$\text{So in (1), } u_1 \times 2^3 = 24$$

$$\therefore u_1 \times 8 = 24$$

$$\therefore u_1 = 3$$

$$\therefore u_n = 3 \times 2^{n-1}$$

$$c \quad u_7 = 24 \quad \therefore u_1 \times r^6 = 24 \quad \dots (1)$$

$$u_{15} = 384 \quad \therefore u_1 \times r^{14} = 384 \quad \dots (2)$$

$$\text{So, } \frac{u_1 r^{14}}{u_1 r^6} = \frac{384}{24} \quad \{(2) \div (1)\}$$

$$\therefore r^8 = 16 \quad \therefore r = \sqrt{2}$$

$$\text{So in (1), } u_1 \times (\sqrt{2})^6 = 24$$

$$\therefore u_1 \times 8 = 24$$

$$\therefore u_1 = 3$$

$$\text{Now } u_n = u_1 r^{n-1}$$

$$\therefore u_n = 3 \times (\sqrt{2})^{n-1}$$

$$b \quad u_3 = 8 \quad \therefore u_1 \times r^2 = 8 \quad \dots (1)$$

$$u_6 = -1 \quad \therefore u_1 \times r^5 = -1 \quad \dots (2)$$

$$\text{So, } \frac{u_1 r^5}{u_1 r^2} = -\frac{1}{8} \quad \{(2) \div (1)\}$$

$$\therefore r^3 = -\frac{1}{8} \quad \therefore r = -\frac{1}{2}$$

$$\text{So in (1), } u_1 \times \left(-\frac{1}{2}\right)^2 = 8$$

$$\therefore u_1 \times \frac{1}{4} = 8$$

$$\therefore u_1 = 32$$

$$\therefore u_n = 32 \times \left(-\frac{1}{2}\right)^{n-1}$$

$$d \quad u_3 = 5 \quad \therefore u_1 \times r^2 = 5 \quad \dots (1)$$

$$u_7 = \frac{5}{4} \quad \therefore u_1 \times r^6 = \frac{5}{4} \quad \dots (2)$$

$$\text{So, } \frac{u_1 r^6}{u_1 r^2} = \frac{\left(\frac{5}{4}\right)}{5} \quad \{(2) \div (1)\}$$

$$\therefore r^4 = \frac{1}{4} \quad \therefore r = \frac{1}{\sqrt{2}}$$

$$\text{So in (1), } u_1 \times \left(\frac{1}{\sqrt{2}}\right)^2 = 5$$

$$\therefore u_1 \times \frac{1}{2} = 5$$

$$\therefore u_1 = 10$$

$$\text{Now } u_n = u_1 r^{n-1}$$

$$\therefore u_n = 10 \times \left(\frac{1}{\sqrt{2}}\right)^{n-1}$$

$$\therefore u_n = 10 \times (\sqrt{2})^{1-n}$$

$$8 \quad a \quad 2, 6, 18, 54, \dots \quad \text{i.e., } u_1 = 2 \quad \text{and } r = 3$$

$$u_n = u_1 r^{n-1} \quad \therefore u_n = 2 \times 3^{n-1}$$

$$\text{Let } u_n = 10\,000 = 2 \times 3^{n-1} \quad \text{i.e., } 5000 = 3^{n-1}$$

$$\therefore n \div 8.7527 \quad \{\text{using technology}\}$$

So, try the two values on either side of $n = 8.7527$, i.e., for $n = 8$ and $n = 9$:

$$\begin{aligned} u_8 &= 2 \times 3^7 & \text{and} & & u_9 &= 2 \times 3^8 \\ &= 4374 & & & &= 13\,122 \end{aligned}$$

i.e., the first term to exceed 10 000 is $u_9 = 13\,122$.

$$b \quad 4, 4\sqrt{3}, 12, 12\sqrt{3}, \dots \quad \text{i.e., } u_1 = 4 \quad \text{and } r = \sqrt{3}$$

$$u_n = u_1 r^{n-1} \quad \therefore u_n = 4 \times (\sqrt{3})^{n-1}$$

$$\text{Let } u_n = 4800 \quad \text{i.e., } 1200 = (\sqrt{3})^{n-1}$$

$$\therefore n \div 13.91 \quad \{\text{using technology}\}$$

So, try the two values on either side of $n \div 13.91$, i.e., for $n = 13$ and $n = 14$:

$$\begin{aligned} u_{13} &= 4 \times (\sqrt{3})^{12} & \text{and} & & u_{14} &= 4 \times (\sqrt{3})^{13} \\ &= 2916 & & & &\div 5050.66 \end{aligned}$$

i.e., the first term to exceed 4800 is $u_{14} \div 5050.66$.

$$c \quad 12, 6, 3, 1.5, \dots \quad \text{i.e., } u_1 = 12 \quad \text{and } r = \frac{1}{2}$$

$$u_n = u_1 r^{n-1} \quad \therefore u_n = 12 \times \left(\frac{1}{2}\right)^{n-1}$$

$$\text{Let } 0.0001 = u_n = 12 \times \left(\frac{1}{2}\right)^{n-1}$$

$$\text{i.e., } 0.000\,008\bar{3} = \left(\frac{1}{2}\right)^{n-1}$$

$$\therefore n \div 17.87 \quad \{\text{using technology}\}$$

So, try the two values on either side of $n \div 17.87$, i.e., for $n = 17$ and $n = 18$:

$$u_{17} = 12 \times \left(\frac{1}{2}\right)^{16} \quad \text{and} \quad u_{18} = 12 \times \left(\frac{1}{2}\right)^{17}$$

$$\div 0.000\,1831$$

$$\div 0.000\,091\,55$$

i.e., the first term of the sequence which is less than 0.0001 is $u_{18} \div 0.000\,091\,55$.

9 a $u_{n+1} = u_1 \times r^n$, where $u_1 = 3000$, $r = 1.1$, $n = 3$

$$\text{i.e., } u_4 = 3000 \times (1.1)^3 \\ = 3993$$

So it amounts to \$3993.

b Interest = amount after 3 years – initial amount
= \$3993 – \$3000
= \$993

10 $u_{n+1} = u_1 \times r^n$, where $u_1 = 20\,000$, $r = 1.12$, $n = 4$

$$\text{i.e., } u_5 = 20\,000 \times (1.12)^4 \\ = 31\,470.39$$

$$\text{Interest} = 31\,470.39 - 20\,000 \text{ Euro} \\ = 11\,470.39 \text{ Euro}$$

11 a $u_{n+1} = u_1 \times r^n$, where $u_1 = 30\,000$, $r = 1.1$, $n = 4$

$$\text{i.e., } u_5 = 30\,000 \times (1.1)^4 \\ = 43\,923, \text{ i.e., the investment amounts to } 43\,923 \text{ Yen.}$$

b Interest = amount after 4 years – initial amount
= 43 923 – 30 000 Yen
= 13 923 Yen

12 $u_{n+1} = u_1 \times r^n$, where $u_1 = 80\,000$, $r = 1.09$, $n = 3$

$$\text{i.e., } u_4 = 80\,000 \times (1.09)^3 \\ = 103\,602.32$$

$$\text{Interest} = \text{amount after 3 years} - \text{initial amount} \\ = \$103\,602.32 - \$80\,000 \\ = \$23\,602.32$$

13 $u_{n+1} = u_1 \times r^n$, where $u_1 = 100\,000$, $r = 1 + \frac{0.08}{2} = 1.04$, $n = 10$

$$\text{i.e., } u_{11} = 100\,000 \times (1.04)^{10} \\ = 148\,024.43, \text{ i.e., it amounts to } 148\,024.43 \text{ Yen.}$$

14 $u_{n+1} = u_1 \times r^n$, where $u_1 = 45\,000$, $r = 1 + \frac{0.075}{4} = 1.01875$,

$$\text{i.e., } u_{10} = 45\,000 \times (1.01875)^7 \quad n = 9 \text{ (21 months = 7 'quarters')} \\ = 51\,249.06$$

i.e., it amounts to £51 249.06

- 15** $u_{n+1} = u_1 \times r^n$, where $u_{n+1} = 20\,000$, $r = 1.075$, $n = 4$
 $\therefore 20\,000 = u_1 \times (1.075)^4$
 $\therefore u_1 = 14\,976.01$ So, \$14 976.01 should be invested now.
- 16** $u_{n+1} = u_1 \times r^n$, where $u_{n+1} = 15\,000$, $r = 1.055$, $n = \frac{60}{12} = 5$
 $\therefore 15\,000 = u_1 \times (1.055)^5$
 $\therefore u_1 = 11\,477.02$ So, the initial investment required is £11 477.02.
- 17** $u_{n+1} = u_1 \times r^n$, where $u_{n+1} = 25\,000$, $r = 1 + \frac{0.08}{4} = 1.02$, $n = 3 \times 4 = 12$
 $\therefore 25\,000 = u_1 \times (1.02)^{12}$
 $\therefore u_1 = 19\,712.33$ i.e., should invest 19 712.33 Euro now.
- 18** $u_{n+1} = u_1 \times r^n$, where $u_{n+1} = 40\,000$, $r = 1 + \frac{0.09}{12} = 1.0075$, $n = 8 \times 12 = 96$
 $\therefore 40\,000 = u_1 \times (1.0075)^{96}$
 $\therefore u_1 = 19\,522.47$, i.e., initial investment should be 19 522.47 Yen.
- 19** $u_{n+1} = u_1 \times r^n$, where $u_1 = 500$, $r = 1.12$
- a i** $u_{11} = 500 \times (1.12)^{10}$ **ii** $u_{21} = 500 \times (1.12)^{20}$
 $\div 1552.92$ $\div 4823.15$
 $\div 1550$ ants $\div 4820$ ants
- b** For the population to reach 2000, $u_{n+1} = 500 \times (1.12)^n = 2000$
 $\therefore (1.12)^n = 4$
 $\therefore n \div 12.23$ {using technology}
 i.e., it will take approximately 12.2 weeks.
- 20** $u_{n+1} = u_1 \times r^n$, where $u_1 = 555$, $r = 0.955$
- a** $u_{16} = 555 \times (0.955)^{15}$
 $\div 278.19$ i.e., the population is approximately 278 animals in the year 2000.
- b** For the population to have declined to 50,
 $u_{n+1} = 555 \times (0.955)^n = 50$
 $\therefore (0.955)^n = 0.0900$
 $\therefore n \div 52.3$ {using technology}
 So, after approximately 52 years the population is 50, i.e., in the year 2037.

EXERCISE 2E.1

- 1 a i** 3, 11, 19, 27, so $u_1 = 3$, $d = 8$, i.e., $u_n = 3 + (n-1)8$
 $= 8n - 5$
 i.e., $S_n = 3 + 11 + 19 + 27 + \dots + (8n - 5)$
- ii** $S_5 = 3 + 11 + 19 + 27 + 35 = 95$
- b i** 42, 37, 32, 27, so $u_1 = 42$, $d = -5$, i.e., $u_n = 42 + (n-1)(-5)$
 $= 47 - 5n$
 i.e., $S_n = 42 + 37 + 32 + 27 + \dots + (47 - 5n)$
- ii** $S_5 = 42 + 37 + 32 + 27 + 22 = 160$
- c i** 12, 6, 3, $1\frac{1}{2}$, so $u_1 = 12$, $r = \frac{1}{2}$, i.e., $u_n = 12 \times (\frac{1}{2})^{n-1}$
 i.e., $S_n = 12 + 6 + 3 + 1\frac{1}{2} + \dots + 12(\frac{1}{2})^{n-1}$

$$\text{ii } S_5 = 12 + 6 + 3 + 1\frac{1}{2} + \frac{3}{4} = 23\frac{1}{4}$$

$$\text{d i } 2, 3, 4\frac{1}{2}, 6\frac{3}{4}, \dots \quad \text{so } u_1 = 2, \quad r = \frac{3}{2}, \quad \text{i.e., } u_n = 2 \times \left(\frac{3}{2}\right)^{n-1}$$

$$\text{i.e., } S_n = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + \dots + 2\left(\frac{3}{2}\right)^{n-1}$$

$$\text{ii } S_5 = 2 + 3 + 4\frac{1}{2} + 6\frac{3}{4} + 10\frac{1}{8} = 26\frac{3}{8}$$

$$\text{e i } 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \quad \text{so } u_1 = 1, \quad r = \frac{1}{2}, \quad \text{i.e., } u_n = 1 \times \left(\frac{1}{2}\right)^{n-1}$$

$$\text{i.e., } S_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + 2^{1-n}$$

$$\text{ii } S_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1\frac{15}{16}$$

$$\text{f i } 1, 8, 27, 64, \dots$$

$$\text{i.e., } S_n = 1 + 8 + 27 + 64 + \dots + n^3 \quad \{\text{since } 1 = 1^3, 8 = 2^3, 27 = 3^3, 64 = 4^3\}$$

$$\text{ii } S_5 = 1 + 8 + 27 + 64 + 125 = 225$$

EXERCISE 2E.2

- 1 a** The series is arithmetic with
 $u_1 = 3, \quad d = 4, \quad n = 20$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\text{So, } S_{20} = \frac{20}{2}(2 \times 3 + 19 \times 4)$$

$$= 10(6 + 76)$$

$$= 820$$

- c** The series is arithmetic with
 $u_1 = 100, \quad d = -7, \quad n = 40$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\text{So, } S_{40} = \frac{40}{2}(2 \times 100 + 39 \times (-7))$$

$$= 20(200 - 273)$$

$$= -1460$$

- 2 a** The series is arithmetic with
 $u_1 = 5, \quad d = 3, \quad u_n = 101$

$$\text{Since } u_n = 101,$$

$$\text{then } u_1 + (n-1)d = 101$$

$$\therefore 5 + 3(n-1) = 101$$

$$\therefore 5 + 3n - 3 = 101$$

$$\therefore 3n - 3 = 96$$

$$\therefore 3n = 99$$

$$\therefore n = 33$$

$$\text{So, } S_n = \frac{n}{2}(u_1 + u_n)$$

$$= \frac{33}{2}(5 + 101)$$

$$= 1749$$

- b** The series is arithmetic with

$$u_1 = \frac{1}{2}, \quad d = 2\frac{1}{2}, \quad n = 50$$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\text{So, } S_{50} = \frac{50}{2}\left(2 \times \frac{1}{2} + 49 \times \frac{5}{2}\right)$$

$$= 25(1 + 122\frac{1}{2})$$

$$= 3087\frac{1}{2}$$

- d** The series is arithmetic with

$$u_1 = 50, \quad d = -1\frac{1}{2}, \quad n = 80$$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\text{So, } S_{80} = \frac{80}{2}\left(2 \times 50 + 79 \times \left(-\frac{3}{2}\right)\right)$$

$$= 40(100 - \frac{237}{2})$$

$$= -740$$

- b** The series is arithmetic with

$$u_1 = 50, \quad d = -\frac{1}{2}, \quad u_n = -20$$

$$\text{Since } u_n = -20,$$

$$\text{then } u_1 + (n-1)d = -20$$

$$\therefore 50 + \left(-\frac{1}{2}\right)(n-1) = -20$$

$$\therefore -\frac{1}{2}n + \frac{1}{2} = -70$$

$$\therefore -\frac{1}{2}n = -\frac{141}{2}$$

$$\therefore n = 141$$

$$\text{So, } S_n = \frac{n}{2}(u_1 + u_n)$$

$$= \frac{141}{2}(50 + (-20))$$

$$= 2115$$

- c** The series is arithmetic with

$$u_1 = 8, \quad d = 2\frac{1}{2}, \quad u_n = 83$$

$$\text{Since } u_n = 83,$$

$$\text{then } u_1 + (n-1)d = 83$$

$$\therefore 8 + \frac{5}{2}(n-1) = 83$$

$$\therefore \frac{5}{2}n - \frac{5}{2} = 75$$

$$\therefore \frac{5}{2}n = \frac{155}{2}$$

$$\therefore n = 31$$

$$\text{So, } S_n = \frac{n}{2}(u_1 + u_n)$$

$$= \frac{31}{2}(8 + 83)$$

$$= 1410\frac{1}{2}$$

$$\mathbf{3} \quad u_1 = 5, \quad n = 7, \\ u_n = 53$$

$$S_n = \frac{n}{2}(u_1 + u_n)$$

$$= \frac{7}{2}(5 + 53)$$

$$= 203$$

$$\mathbf{4} \quad u_1 = 6, \quad n = 11, \\ u_n = -27$$

$$S_n = \frac{n}{2}(u_1 + u_n)$$

$$= \frac{11}{2}(6 + (-27))$$

$$= -115\frac{1}{2}$$

- 5** The total number of bricks can be expressed as an arithmetic series:

$$1 + 2 + 3 + 4 + \dots + n$$

We know that the total number of bricks is 171, i.e., $S_n = 171$.

Also, $u_1 = 1, \quad d = 1$ and we need to find n , the number of members (layers) of the series.

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) = 171$$

$$\therefore \frac{n}{2}(2 \times 1 + (n-1) \times 1) = 171$$

$$\therefore n(2 + n - 1) = 342$$

$$\therefore n(n+1) = 342$$

$$\therefore n^2 + n - 342 = 0$$

$$\therefore (n-18)(n+19) = 0$$

$$\therefore n = -19 \text{ or } 18$$

$$\text{but } n > 0, \therefore n = 18$$

So, there are 18 layers placed.

- 6** The total number of seats can be expressed as an arithmetic series:

$$22 + 23 + 24 + \dots + u_n$$

Row 1 has 22 seats, so $u_1 = 22$. Row 2 has 23 seats, so $d = 1$.

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$

$$\therefore S_n = \frac{n}{2}(2 \times 22 + 1(n-1))$$

$$= \frac{n}{2}(44 + n - 1) \quad \therefore S_n = \frac{n}{2}(n + 43) \quad \text{which is the total number of seats in } n \text{ rows.}$$

- a** Number of seats in row 44 = total no. of seats in every row – no. of seats in 43 rows

$$= S_{44} - S_{43}$$

$$= \frac{44}{2}(44 + 43) - \frac{43}{2}(43 + 43)$$

$$= 1914 - 1849$$

$$= 65$$

- b** Number of seats in a section = $S_{44} = 1914$ (from **a**)

- c** Number of seats in 25 sections = $S_{44} \times 25 = 1914 \times 25 = 47\,850$

- 7 a** The first 50 multiples of 11 can be expressed as an arithmetic series:

$$11 + 22 + 33 + \dots + u_{50} \quad \text{where } u_1 = 11, \quad d = 11, \quad n = 50$$

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) \quad \therefore S_{50} = \frac{50}{2}(2 \times 11 + 11(50-1))$$

$$= 25(22 + 539)$$

$$= 14\,025$$

- b** The multiples of 7 between 0 and 1000 can be expressed as an arithmetic series:

$$7 + 14 + 21 + 28 + \dots + u_n \quad \text{where } u_1 = 7, \quad d = 7$$

To find u_n , we need to find the largest multiple of 7 less than 1000, i.e., $\frac{1000}{7} \div 142.9$,
so $u_n = 142 \times 7 = 994$

$$\text{Now } u_n = u_1 + (n-1)d$$

$$\therefore 994 = 7 + 7(n-1)$$

$$\therefore 987 = 7n - 7$$

$$\therefore 7n = 994$$

$$\therefore n = 142$$

$$\text{So, } S_{142} = \frac{142}{2}(7 + 994) = 71\,071$$

- c** The integers between 1 and 100 which are not divisible by 3 can be expressed as:

$$1, 2, 4, 5, 7, 8, \dots, 100 \quad \text{where } u_1 = 1, \quad u_n = 100.$$

Alternatively, these integers can be expressed as two separate arithmetic series:

$$\text{i.e., } S_1 = 1 + 4 + 7 + \dots + 97 + 100 \quad \text{where } u_1 = 1, \quad d = 3, \quad u_n = 100$$

$$\text{and } S_2 = 2 + 5 + 8 + \dots + 95 + 98 \quad \text{where } u_1 = 2, \quad d = 3, \quad u_n = 98$$

$$\text{Now for } S_1, \quad u_n = u_1 + (n-1)d \quad \text{and for } S_2, \quad u_n = u_1 + (n-1)d$$

$$\therefore 100 = 1 + 3(n-1)$$

$$\therefore 98 = 2 + 3(n-1)$$

$$\therefore 99 = 3n - 3$$

$$\therefore 96 = 3n - 3$$

$$\therefore 3n = 102$$

$$\therefore 3n = 99$$

$$\therefore n = 34$$

$$\therefore n = 33$$

$$\text{i.e., } S_1 = \frac{34}{2}(1 + 100) = 1717 \quad \text{and } S_2 = \frac{33}{2}(2 + 98) = 1650$$

$$\text{i.e., total sum} = S_1 + S_2 = 1717 + 1650 = 3367$$

- 8** We need to show that $1 + 2 + 3 + 4 + \dots + n = \frac{n(n+1)}{2}$

The sum of the first n positive integers can be expressed as an arithmetic series:

$$1 + 2 + 3 + 4 + \dots + n, \quad \text{where } u_1 = 1, \quad d = 1, \quad u_n = n.$$

$$\text{So the sum of the series is } S_n = \frac{n}{2}(u_1 + u_n)$$

$$= \frac{n}{2}(1 + n) \quad \text{i.e., } S_n = \frac{n(n+1)}{2} \quad \text{as required.}$$

- 9** The series of odd numbers can be expressed as an arithmetic series:

$$1 + 3 + 5 + 7 + \dots \quad \text{where } u_1 = 1, \quad d = 2$$

a Now $u_n = u_1 + (n-1)d = 1 + 2(n-1)$

$$\text{i.e., } u_n = 2n - 1$$

- b** We need to show that S_n is n^2 .

The sum of the first n odd numbers can be expressed as an arithmetic series:

$$1 + 3 + 5 + 7 + \dots (2n-1) \quad \{\text{using } u_n = 2n-1 \text{ from a}\}$$

$$\text{So, } S_n = \frac{n}{2}(u_1 + u_n)$$

$$= \frac{n}{2}(1 + (2n-1))$$

$$= \frac{n}{2}(2n)$$

$$\text{i.e., } S_n = n^2 \quad \text{as required}$$

c $S_1 = \frac{1}{2}(1+1) = 1 = 1^2 = n^2 \quad \text{for } n=1 \quad \checkmark$

$$S_2 = \frac{2}{2}(1+3) = 4 = 2^2 = n^2 \quad \text{for } n=2 \quad \checkmark$$

$$S_3 = \frac{3}{2}(1+5) = 9 = 3^2 = n^2 \quad \text{for } n=3 \quad \checkmark$$

$$S_4 = \frac{4}{2}(1+7) = 16 = 4^2 = n^2 \quad \text{for } n=4 \quad \checkmark$$

- 10** $u_6 = 21$, $S_{17} = 0$; so we need to find u_1 and u_2

$$\begin{aligned}
 S_n &= \frac{n}{2}(2u_1 + (n-1)d) & \text{Also, } u_n &= u_1 + (n-1)d \\
 \therefore S_{17} &= \frac{17}{2}(2u_1 + 16d) = 0 & \therefore u_6 &= u_1 + 5d \\
 & \therefore u_1 + 8d = 0 & \therefore 21 &= -8d + 5d \quad \{\text{using (1)}\} \\
 & \therefore u_1 = -8d \quad \dots\dots (1) & \therefore 21 &= -3d \\
 & & \therefore d &= -7 \\
 & & \text{and } u_1 &= -8(-7) = 56 \\
 & & \text{so } u_2 &= 56 - 7 = 49
 \end{aligned}$$

i.e., the first two terms are 56 and 49.

- 11** Let the three consecutive terms be $x - d$, x and $x + d$.

$$\begin{aligned}
 \text{Now, sum of terms} &= 12 & \text{Also, product of terms} &= -80 \\
 \text{i.e., } (x - d) + x + (x + d) &= 12 & \text{i.e., } (4 - d)4(4 + d) &= -80 \\
 \therefore 3x &= 12 & \therefore 4(4^2 - d^2) &= -80 \\
 \therefore x &= 4 & \therefore 16 - d^2 &= -20 \\
 \text{So the terms are } 4 - d, 4, 4 + d & & \therefore d^2 &= 36 \\
 & & \therefore d &= \pm 6
 \end{aligned}$$

So, the three terms could be $4 - 6$, 4 , $4 + 6$, i.e., -2 , 4 , 10
 or $4 - (-6)$, 4 , $4 + (-6)$, i.e., 10 , 4 , -2

- 12** Let the five consecutive terms be $n - 2d$, $n - d$, n , $n + d$, $n + 2d$.

$$\begin{aligned}
 \text{Now, sum of terms} &= 40 \\
 \text{i.e., } (n - 2d) + (n - d) + n + (n + d) + (n + 2d) &= 40 \\
 \therefore 5n &= 40 \\
 \therefore n &= 8
 \end{aligned}$$

So the terms are $8 - 2d$, $8 - d$, 8 , $8 + d$, $8 + 2d$

$$\begin{aligned}
 \text{Also, product of middle and end terms} &= 8 \times (8 - 2d) \times (8 + 2d) = 224 \\
 \therefore 8(8^2 - 4d^2) &= 224 \\
 \therefore 64 - 4d^2 &= 28 \\
 \therefore 4d^2 &= 36 \\
 \therefore d^2 &= 9 \\
 \therefore d &= \pm 3
 \end{aligned}$$

So, the five terms could be $8 - 2(3)$, $8 - 3$, 8 , $8 + 3$, $8 + 2(3)$, i.e., 2 , 5 , 8 , 11 , 14
 or $8 - 2(-3)$, $8 - (-3)$, 8 , $8 + (-3)$, $8 + 2(-3)$, i.e., 14 , 11 , 8 , 5 , 2

EXERCISE 2E.3

- 1 a** The series is geometric with
 $u_1 = 12$, $r = \frac{1}{2}$, $n = 10$

$$\begin{aligned}
 \text{Now } S_n &= \frac{u_1(1 - r^n)}{1 - r} \\
 \text{i.e., } S_{10} &= \frac{12(1 - (\frac{1}{2})^{10})}{1 - \frac{1}{2}} \\
 &\doteq 23.9766 \\
 \text{So, } S_{10} &\doteq 24.0
 \end{aligned}$$

- b** The series is geometric with
 $u_1 = \sqrt{7}$, $r = \sqrt{7}$, $n = 12$

$$\begin{aligned}
 \text{Now } S_n &= \frac{u_1(r^n - 1)}{r - 1} \\
 \text{i.e., } S_{12} &= \frac{\sqrt{7}((\sqrt{7})^{12} - 1)}{\sqrt{7} - 1} \\
 &\doteq 189\,134
 \end{aligned}$$

c The series is geometric with

$$u_1 = 6, \quad r = -\frac{1}{2}, \quad n = 15$$

$$\text{Now } S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\begin{aligned} \text{i.e., } S_{15} &= \frac{6\left(1 - \left(-\frac{1}{2}\right)^{15}\right)}{1 - \left(-\frac{1}{2}\right)} \\ &\div 4.000 \end{aligned}$$

d The series is geometric with

$$u_1 = 1, \quad r = -\frac{1}{\sqrt{2}}, \quad n = 20$$

$$\text{Now } S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\begin{aligned} \text{i.e., } S_{20} &= \frac{1\left(1 - \left(-\frac{1}{\sqrt{2}}\right)^{20}\right)}{1 - \left(-\frac{1}{\sqrt{2}}\right)} \\ &\div 0.5852 \end{aligned}$$

2 a The series is geometric with

$$u_1 = \sqrt{3}, \quad r = \sqrt{3}$$

$$\begin{aligned} S_n &= \frac{u_1(r^n - 1)}{r - 1} \\ &= \frac{\sqrt{3}((\sqrt{3})^n - 1)}{\sqrt{3} - 1} \times \left(\frac{\sqrt{3} + 1}{\sqrt{3} + 1}\right) \\ &= \frac{(3 + \sqrt{3})((\sqrt{3})^n - 1)}{3 - 1} \\ &= \frac{3 + \sqrt{3}}{2} ((\sqrt{3})^n - 1) \end{aligned}$$

b The series is geometric with

$$u_1 = 12, \quad r = \frac{1}{2}$$

$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r} \\ &= \frac{12\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} \\ &= 24\left(1 - \left(\frac{1}{2}\right)^n\right) \end{aligned}$$

c The series is geometric with

$$u_1 = 0.9, \quad r = 0.1$$

$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r} \\ &= \frac{0.9(1 - (0.1)^n)}{1 - 0.1} \\ &= 1 - (0.1)^n \end{aligned}$$

d The series is geometric with

$$u_1 = 20, \quad r = -\frac{1}{2}$$

$$\begin{aligned} S_n &= \frac{u_1(1-r^n)}{1-r} \\ &= \frac{20\left(1 - \left(-\frac{1}{2}\right)^n\right)}{1 - \left(-\frac{1}{2}\right)} \\ &= \frac{20\left(1 - \left(-\frac{1}{2}\right)^n\right)}{\left(\frac{3}{2}\right)} \\ &= \frac{40}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right) \end{aligned}$$

3 a $A_2 = A_1 \times 1.06 + 2000$

$$= (A_0 \times 1.06 + 2000) \times 1.06 + 2000$$

$$= (2000 \times 1.06 + 2000) \times 1.06 + 2000$$

$$\text{i.e., } A_2 = 2000 + 2000 \times 1.06 + 2000 \times (1.06)^2 \quad \text{as required}$$

b $A_3 = A_2 \times 1.06 + 2000$

$$= [2000 + 2000 \times 1.06 + 2000 \times (1.06)^2] \times 1.06 + 2000 \quad \{\text{from a}\}$$

$$\text{i.e., } A_3 = 2000 [1 + 1.06 + (1.06)^2 + (1.06)^3] \quad \text{as required}$$

c $A_9 = 2000 [1 + 1.06 + (1.06)^2 + (1.06)^3 + (1.06)^4 + (1.06)^5 + (1.06)^6 + (1.06)^7 + (1.06)^8 + (1.06)^9]$

$$\text{i.e., } A_9 = 26\,361.59$$

$$\text{i.e., total bank balance after 10 years is } \$26\,361.59$$

$$4 \quad a \quad S_1 = \frac{1}{2}, \quad S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, \quad S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}, \quad S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}, \\ S_5 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{31}{32}$$

$$b \quad S_n = \frac{2^n - 1}{2^n} \qquad c \quad S_n = \frac{u_1(1 - r^n)}{1 - r}, \quad \text{where } u_1 = \frac{1}{2}, \quad r = \frac{1}{2} \\ = \frac{\frac{1}{2} \left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}}$$

$$\text{i.e., } S_n = 1 - \left(\frac{1}{2}\right)^n = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

d As n gets very large, i.e., as $n \rightarrow \infty$, $\left(\frac{1}{2}\right)^n \rightarrow 0$, so $S_n \rightarrow 1$ (from below)

e The diagram represents one whole unit divided into smaller and smaller fractions.
As $n \rightarrow \infty$, the area which the fraction represents becomes smaller and smaller, and the total area approaches one whole unit.

$$5 \quad a \quad \text{Total time of motion} = 1 + (90\% \times 1) + (90\% \times 1) + (90\% \times 90\% \times 1) \\ + (90\% \times 90\% \times 1) + (90\% \times 90\% \times 90\% \times 1) + \dots \\ = 1 + 0.9 + 0.9 + (0.9)^2 + (0.9)^2 + (0.9)^3 + \dots \\ = 1 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + \dots \quad \text{as required}$$

$$b \quad S_n = \frac{u_1(1 - r^n)}{1 - r}, \quad \text{where } u_1 = 2(0.9), \quad r = 0.9, \quad 'n' = n - 1 \\ \text{(since the term } u_1, \text{ used in calculating } S_n, \text{ is the second term of the series, not the first)}$$

$$\text{i.e., } S_n = \frac{2(0.9)(1 - (0.9)^{n-1})}{1 - 0.9} + 1 \\ = \frac{1.8(1 - (0.9)^{n-1})}{0.1} + 1$$

$$\text{i.e., } S_n = 1 + 18(1 - (0.9)^{n-1})$$

c For the ball to come to rest, n must approach infinity,
thus $(0.9)^{n-1} \rightarrow 0$ so $(1 - (0.9)^{n-1}) \rightarrow 1$ (from below)

$$\text{So, } S_n \rightarrow 1 + 18(1), \quad \text{i.e., } S_n \rightarrow 19$$

i.e., it takes 19 seconds for the ball to come to rest.

$$6 \quad a \quad i \quad u_1 = \frac{3}{10} \qquad ii \quad r = \frac{\left(\frac{3}{100}\right)}{\left(\frac{3}{10}\right)} = \frac{1}{10}$$

b We need to show that $0.\bar{3} = \frac{1}{3}$.

$$\text{Now } 0.\bar{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots$$

$$\text{So, let } S_n = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \dots$$

$$\text{Since } n \rightarrow \infty, \text{ then } S_\infty = \frac{u_1}{1 - r} = \frac{\frac{3}{10}}{1 - \left(\frac{1}{10}\right)} = \frac{1}{3}$$

$$\text{i.e., } 0.\bar{3} = \frac{1}{3} \quad \text{as required}$$

7 a $0.\overline{4} = 0.444444\dots$

$$= \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + \dots$$

which is a geometric series with
 $a = 0.4$, $r = 0.1$

$$\therefore S_{\infty} = \frac{a}{1-r} = \frac{0.4}{1-0.1} = \frac{0.4}{0.9}$$

$$= \frac{4}{9}$$

So, $0.\overline{4} = \frac{4}{9}$

c $0.\overline{312} = 0.312312312\dots$

$$= \frac{312}{10^3} + \frac{312}{10^6} + \frac{312}{10^9} + \dots$$

which is a geometric series with
 $a = 0.312$, $r = 0.001$

$$\therefore S_{\infty} = \frac{a}{1-r} = \frac{0.312}{0.999} = \frac{312}{999} = \frac{104}{333}$$

So, $0.\overline{312} = \frac{104}{333}$

8 Checking **4 d**:

$$S_{\infty} = \frac{u_1}{1-r}$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}}$$

i.e., $S_{\infty} = 1$ ✓

Checking **5 c**:

$$S_{\infty} = \frac{u_1}{1-r}$$

$$= \frac{2(0.9)}{1-0.9} + 1 \quad \{\text{since 1st term '1' is not part of the series}\}$$

i.e., $S_{\infty} = 19$ ✓

EXERCISE 2F

1 a $\sum_{r=1}^4 (3r-5) = -2 + 1 + 4 + 7 = 10$

b $\sum_{r=1}^5 (11-2r) = 9 + 7 + 5 + 3 + 1 = 25$

c $\sum_{r=1}^7 r(r+1) = 2 + 6 + 12 + 20 + 30 + 42 + 56 = 168$

d $\sum_{i=1}^5 10 \times 2^{i-1} = 10 + 20 + 40 + 80 + 160 = 310$

2 $u_n = 3n - 1$

$$u_1 + u_2 + u_3 + \dots + u_{20} = 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 + 35$$

$$+ 38 + 41 + 44 + 47 + 50 + 53 + 56 + 59$$

This series is arithmetic with $u_1 = 2$, $d = 3$ and $n = 20$

so its sum is $\frac{n}{2} [2u_1 + (n-1)d] = 10[4 + 19 \times 3] = 610$

3 a $\sum_{r=1}^{10} (2r+5) = 7 + 9 + 11 + \dots + 21 + 23 + 25$

This series is arithmetic with $u_1 = 7$, $d = 2$ and $n = 10$.

$$\therefore \text{sum} = \frac{n}{2} [2u_1 + (n-1)d] = \frac{10}{2} [14 + 9 \times 2] = 160$$

b $\sum_{r=1}^{15} (r-50) = (-49) + (-48) + (-47) + \dots + (-37) + (-36) + (-35)$

This series is arithmetic with $u_1 = -49$, $d = 1$ and $n = 15$.

$$\therefore \text{sum} = \frac{n}{2} [2u_1 + (n-1)d] = \frac{15}{2} [-98 + 14 \times 1] = -630$$

$$\text{c} \quad \sum_{r=1}^{20} \left(\frac{r+3}{2} \right) = 2 + \frac{5}{2} + 3 + \dots + \frac{21}{2} + 11 + \frac{23}{2}$$

This series is arithmetic with $u_1 = 2$, $r = \frac{1}{2}$ and $n = 20$.

$$\therefore \text{sum} = \frac{n}{2} [2u_1 + (n-1)d] = \frac{20}{2} [4 + 19 \times \frac{1}{2}] = 135$$

$$4 \quad \text{a} \quad \sum_{r=1}^{10} (3 \times 2^{r-1}) = 3 + 6 + 12 + \dots + 384 + 768 + 1536$$

This series is geometric with $u_1 = 3$, $r = 2$ and $n = 10$.

$$\therefore \text{sum} = \frac{u_1(r^n - 1)}{r - 1} = \frac{3(2^{10} - 1)}{1} = 3069$$

$$\text{b} \quad \sum_{r=1}^{12} \left(\frac{1}{2} \right)^{r-2} = 2 + 1 + \frac{1}{2} + \dots + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024}$$

This series is geometric with $u_1 = 2$, $r = \frac{1}{2}$ and $n = 12$.

$$\therefore \text{sum} = \frac{u_1(1 - r^n)}{1 - r} = \frac{2 \left(1 - \left(\frac{1}{2} \right)^{12} \right)}{\frac{1}{2}} = 4 \left(1 - \left(\frac{1}{2} \right)^{12} \right)$$

$$\therefore \text{sum} \div 3.999$$

$$\text{c} \quad \sum_{r=1}^{25} (6 \times (-2)^r) = -12 + 24 + (-48) + \dots + 100\,663\,296 + (-201\,326\,592)$$

This series is geometric with $u_1 = -12$, $r = -2$ and $n = 25$.

$$\therefore \text{sum} = \frac{u_1(1 - r^n)}{1 - r} = \frac{-12(1 - (-2)^{25})}{1 + 2} = -4(1 - (-2)^{25})$$

$$\therefore \text{sum} = -134\,217\,732$$

$$5 \quad \text{a} \quad \sum_{k=1}^5 k(k+1)(k+2) = 6 + 24 + 60 + 120 + 210 = 420$$

Note: This series is neither arithmetic nor geometric so the sum is found by adding the 5 terms.

$$\text{b} \quad \sum_{k=6}^{12} (100(1.2)^{k-3}) = 172.8 + 207.36 + 248.832 + 298.5984 + 358.31808 \\ + 429.981696 + 515.9780352$$

This series is geometric with $u_1 = 172.8$, $r = 1.2$ and $n = 7$

$$\therefore \text{sum} = \frac{u_1(r^n - 1)}{r - 1} = \frac{172.8((1.2)^7 - 1)}{0.2} \div 2231.87$$

$$6 \quad \text{a} \quad \text{The LHS is arithmetic with } u_1 = 5, d = 2, \text{ “}n\text{”} = n$$

$$\text{Now } \frac{n}{2} [2u_1 + (n-1)d] = 1517$$

$$\therefore \frac{n}{2} [10 + 2(n-1)] = 1517$$

$$\therefore \frac{n}{2} [2n + 8] = 1517$$

$$\therefore n(n+4) - 1517 = 0$$

$$\therefore n^2 + 4n - 1517 = 0 \quad \text{where } n > 1$$

$$\text{and } n = 37 \quad \{\text{using technology}\}$$

b The LHS is geometric with $u_1 = 2$, $r = 3$, “ n ” = n

$$\text{Now } u_1 \left(\frac{r^n - 1}{r - 1} \right) = 177\,146$$

$$\therefore 2 \left(\frac{3^n - 1}{3 - 1} \right) = 177\,146$$

$$\therefore 3^n - 1 = 177\,146$$

$$\therefore 3^n = 177\,147$$

$$\therefore n = 11 \quad \{\text{using technology}\}$$

EXERCISE 2G

1 11, 14, 17, 20, is arithmetic with $u_1 = 11$, $d = 3$

$$\begin{aligned} \text{and } S_n &= \frac{n}{2} [2u_1 + (n-1)d] \\ &= \frac{n}{2} [22 + 3(n-1)] \\ &= \frac{n}{2} [3n + 19] \end{aligned}$$

$$\begin{aligned} \text{Suppose } S_n &= 2000 \quad \therefore \frac{n}{2} [3n + 19] = 2000 \\ \therefore 3n^2 + 19n &= 4000 \end{aligned}$$

Using technology, listing terms of $3n^2 + 19n$, $n = 33$ given $S_n = 3894$
 $n = 34$ given $S_n = 4114$

$\therefore$ would sell the 2000th in week 34.

2 $u_1 = \$2795$, $r = 0.98$, $u_n = 500$

$$\begin{aligned} \text{Now } u_n &= u_1 \times r^{n-1} \\ \therefore 500 &= 2795 \times (0.98)^{n-1} \\ \therefore (0.98)^{n-1} &= 0.178\,89 \end{aligned}$$

Using technology, listing the terms of $(0.98)^{n-1}$, $n \div 86$

when $n = 86$, $u_{86} = 2795 \times (0.98)^{85} \div 501.88$ (after 85 months)

when $n = 87$, $u_{87} = 2795 \times (0.98)^{86} \div 491.84$ (after 86 months)

So, it is \$500 during the 85th month.

3 $u_2 = u_1 r = 6$ and so $u_1 = \frac{6}{r}$

$$S_3 = u_1 + u_1 r + u_1 r^2 = -14$$

$$\therefore \frac{6}{r} + 6 + 6r = -14$$

$$\therefore \frac{6}{r} + 20 + 6r = 0$$

$$\therefore 6 + 20r + 6r^2 = 0$$

$$\therefore 3r^2 + 10r + 3 = 0$$

$$\therefore (3r + 1)(r + 3) = 0$$

$$\therefore r = -\frac{1}{3} \text{ or } -3$$

$$\text{when } r = -\frac{1}{3}, \quad u_1 = -18$$

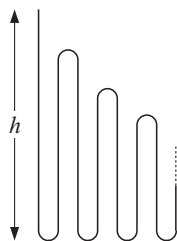
$$\text{when } r = -3, \quad u_1 = -2$$

and as $u_4 = u_1 r^3$ then

$$u_4 = -18 \times \left(-\frac{1}{3}\right)^3 \text{ or } -2 \times (-3)^3$$

$$\text{i.e., } u_4 = \frac{2}{3} \text{ or } 54$$

4



Total distance fallen

$$\begin{aligned}
 &= h + 2 \left(\frac{3}{4} \right) h + 2 \left(\frac{3}{4} \right) \left(\frac{3}{4} \right) h + \dots \\
 &= h + 2 \left(\frac{3}{4} \right) h \left[1 + \frac{3}{4} + \left(\frac{3}{4} \right)^2 + \left(\frac{3}{4} \right)^3 + \dots \right] \\
 &= h + \frac{3}{2} h \left(\frac{1}{1 - \frac{3}{4}} \right) \quad \left\{ \text{as } |r| = \left| \frac{3}{4} \right| < 1 \text{ and } S_{\infty} = \frac{u_1}{1 - r} \right\} \\
 &= h + \frac{3}{2} h(4) \\
 &= 7h
 \end{aligned}$$

$$\text{But } 7h = 490 \quad \therefore h = 70$$

 $\therefore$ the table top is 70 cm above the floor.

5

$$u_1 = 1$$

$$u_2 = u_1 + d \quad \{\text{for arithmetic}\}$$

$$\text{and } u_2 = u_1 r \quad \{\text{for geometric}\}$$

$$\therefore u_1 + d = u_1 r \quad \text{i.e., } 1 + d = r \quad \dots (1)$$

$$\text{Now } u_{14} = u_1 + 13d \quad \{\text{for arithmetic}\}$$

$$\text{and } u_3 = u_1 r^2 \quad \{\text{for geometric}\}$$

$$\therefore u_1 + 13d = 3u_1 r^2 \quad \text{i.e., } 1 + 13d = 3r^2 \quad \dots (2)$$

$$\text{From (1), } d = r - 1 \quad \therefore \text{ in (2) } \quad 1 + 13(r - 1) = 3r^2$$

$$\therefore 1 + 13r - 13 = 3r^2$$

$$\therefore 3r^2 - 13r + 12 = 0$$

$$\therefore (3r - 4)(r - 3) = 0$$

$$\therefore r = \frac{4}{3} \text{ or } 3$$

$$\text{When } r = \frac{4}{3}, \quad d = \frac{1}{3}: \quad u_{20} = u_1 + 19d = 1 + 19\left(\frac{1}{3}\right) = \frac{22}{3} = 7\frac{1}{3}$$

$$u_{20} = u_1 r^{19} = 1 \times \left(\frac{4}{3}\right)^{19} = \left(\frac{4}{3}\right)^{19}$$

$$\text{When } r = 3, \quad d = 2: \quad u_{20} = u_1 + 19d = 1 + 19(2) = 39$$

$$u_{20} = u_1 r^{19} = 1 \times 3^{19} = 3^{19}$$

$$\text{So the 20th terms are } 7\frac{1}{3} \quad \text{for arithmetic and } \left(\frac{4}{3}\right)^{19} \quad \text{for geometric or}$$

$$39 \quad \text{for arithmetic and } 3^{19} \quad \text{for geometric.}$$

6

$$\sum_{r=1}^{\infty} \left(\frac{3x}{2} \right)^{r-1} = \left(\frac{3x}{2} \right)^0 + \left(\frac{3x}{2} \right)^1 + \left(\frac{3x}{2} \right)^2 + \left(\frac{3x}{2} \right)^3 + \dots$$

$$= 1 + \frac{3x}{2} + \left(\frac{3x}{2} \right)^2 + \left(\frac{3x}{2} \right)^3 + \dots$$

$$= \frac{u_1}{1 - r} \quad \{\text{as it converges to 4 and is geometric}\}$$

$$= \frac{1}{1 - \frac{3x}{2}}$$

$$= \frac{2}{2 - 3x}$$

$$\therefore \frac{2}{2 - 3x} = 4 \quad \text{and so } 2 - 3x = \frac{1}{2}$$

$$\therefore 3x = 1\frac{1}{2}$$

$$\therefore x = \frac{1}{2}$$

$$\begin{aligned} \mathbf{7} \quad \mathbf{a} \quad S_n &= \frac{n(3n+11)}{2} \quad \therefore S_1 = u_1 = \frac{1(14)}{2} = 7 \\ &\text{and } S_2 = u_1 + u_2 = \frac{2(17)}{2} = 17 \quad \therefore u_1 = 7 \quad \text{and } u_2 = 10 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad u_1 &= 7 \quad \text{and } d = 3 \quad \therefore u_{20} = u_1 + 19d = 7 + 19 \times 3 = 64 \\ \therefore \text{the twentieth term is } 64. \end{aligned}$$

$$\begin{aligned} \mathbf{8} \quad \mathbf{a} \quad A_3 &= A_2 \times 1.03 - R \\ &= (\$8000 \times (1.03)^2 - 1.03R - R) \times 1.03 - R \\ &= \$8000 \times (1.03)^3 - (1.03)^2 R - (1.03)R - R \end{aligned}$$

$$\mathbf{b} \quad A_8 = \$8000 \times (1.03)^8 - (1.03)^7 R - (1.03)^6 R - (1.03)^5 R - \dots - 1.03R - R$$

We want A_8 to have a value of 0

$$\therefore R(1 + 1.03 + (1.03)^2 + (1.03)^3 + \dots + (1.03)^7) = \$8000 \times (1.03)^8$$

$$\therefore R \left(1 \left[\frac{(1.03)^8 - 1}{1.03 - 1} \right] \right) = \$8000 \times (1.03)^8$$

$$\therefore R = \frac{\$8000 \times (1.03)^8 \times 0.03}{(1.03)^8 - 1} \div \$1139.65$$

$$\mathbf{c} \quad \text{Notice in } \mathbf{b} \text{ that } \$8000 = P \quad \text{and } (1.03)^8 = \left(1 + \frac{3}{100}\right)^8 = \left(1 + \frac{r}{100}\right)^m$$

$$0.03 = \frac{3}{100} = \frac{r}{100}$$

$$\text{and } (1.03)^8 - 1 = \left(1 + \frac{r}{100}\right)^m - 1$$

$$\text{So, in the general case } R = \frac{P \times \left(1 + \frac{r}{100}\right)^m \times \frac{r}{100}}{\left(1 + \frac{r}{100}\right)^m - 1}$$

REVIEW SET 2A

$$\mathbf{1} \quad \mathbf{a} \quad u_n = 3^{n-2} \quad \therefore u_1 = 3^{-1} = \frac{1}{3}, \quad u_2 = 3^0 = 1, \quad u_3 = 3^1 = 3, \quad u_4 = 3^2 = 9$$

$$\mathbf{b} \quad u_n = \frac{3n+2}{n+3} \quad \therefore u_1 = \frac{5}{4}, \quad u_2 = \frac{8}{5}, \quad u_3 = \frac{11}{6}, \quad u_4 = \frac{14}{7} = 2$$

$$\begin{aligned} \mathbf{c} \quad u_n &= 2^n - (-3)^n \\ u_1 &= 2 - (-3) = 5, \quad u_2 = 4 - 9 = -5, \quad u_3 = 8 - (-27) = 35, \quad u_4 = 16 - 81 = -65 \end{aligned}$$

$$\mathbf{2} \quad u_n = 68 - 5n$$

$$\begin{aligned} \mathbf{a} \quad u_{n+1} - u_n &= 68 - 5(n+1) - [68 - 5n] & \mathbf{b} \quad u_1 &= 68 - 5(1) = 63, \quad d = -5 \\ &= 68 - 5n - 5 - 68 + 5n & \mathbf{c} \quad u_{37} &= 68 - 5(37) = -117 \\ &= -5 \quad \text{for all } n \end{aligned}$$

$\mathbf{d}$ Let $u_n = -200$, and we need to find n .

$$u_n = 68 - 5n = -200$$

$$\therefore 5n = 268$$

$$\therefore n = 53\frac{3}{5}$$

So, try the two values on either side of $n = 53\frac{3}{5}$, i.e., for $n = 53$ and $n = 54$:

$$\begin{aligned} u_{53} &= 68 - 5(53) & \text{and } u_{54} &= 68 - 5(54) \\ &= -197 & &= -202 \end{aligned}$$

i.e., the first term of the sequence less than -200 is $u_{54} = -202$.

3 a 3, 12, 48, 192,

$$\frac{12}{3} = 4 \quad \text{So, assuming the pattern continues, consecutive terms have a common ratio of 4.}$$

$$\frac{48}{12} = 4 \quad \therefore \text{the sequence is geometric with } u_1 = 3 \text{ and } r = 4.$$

$$\frac{192}{48} = 4 \quad \mathbf{b} \quad u_n = u_1 r^{n-1}$$

$$\therefore u_n = 3 \times 4^{n-1}$$

$$\text{i.e., } u_9 = 3 \times 4^8 = 196\,608$$

4 Since the terms are consecutive,

$$\text{then } (k-2) - 3k = k+7 - (k-2) \quad \{\text{equating common differences}\}$$

$$\therefore k-2-3k = k+7-k+2$$

$$\therefore -2-2k = 9$$

$$\therefore 2k = -11$$

$$\therefore k = -\frac{11}{2}$$

$$\mathbf{5} \quad u_7 = 31 \quad \therefore u_1 + 6d = 31 \quad \dots (1)$$

$$u_{15} = -17 \quad \therefore u_1 + 14d = -17 \quad \dots (2)$$

Solving simultaneously,

$$-u_1 - 6d = -31$$

$$u_1 + 14d = -17$$

$$\hline \therefore 8d = -48 \quad \{\text{adding the equations}\}$$

$$\therefore d = -6$$

$$\text{So in (1), } u_1 + 6(-6) = 31$$

$$\therefore u_1 - 36 = 31$$

$$\therefore u_1 = 67$$

$$\text{Now } u_n = u_1 + (n-1)d$$

$$\therefore u_n = 67 + (n-1)(-6)$$

$$\therefore u_n = 67 - 6n + 6$$

$$\therefore u_n = -6n + 73$$

$$\text{So } u_{34} = -6(34) + 73 = -131$$

$$\mathbf{6} \quad u_n = 6 \left(\frac{1}{2}\right)^{n-1}$$

$$\mathbf{a} \quad \frac{u_{n+1}}{u_n} = \frac{6 \left(\frac{1}{2}\right)^{n+1-1}}{6 \left(\frac{1}{2}\right)^{n-1}} = \frac{1}{2} \quad \text{for all } n$$

$$\mathbf{b} \quad u_1 = 6, \quad r = \frac{1}{2}$$

$$\mathbf{c} \quad u_{16} = 6 \left(\frac{1}{2}\right)^{15} = 0.000\,183$$

$$\therefore \{u_n\} \text{ is a geometric sequence.}$$

7 28, 23, 18, 13,

$$23 - 28 = -5 \quad \text{So, assuming that the pattern continues, consecutive terms differ by } -5.$$

$$18 - 23 = -5 \quad \therefore \text{the sequence is arithmetic with } u_1 = 28, \quad d = -5.$$

$$13 - 18 = -5$$

$$u_n = u_1 + (n-1)d$$

$$= 28 + (n-1)(-5)$$

$$= 28 - 5n + 5$$

$$= -5n + 33$$

$$S_n = \frac{n}{2} (2u_1 + (n-1)d)$$

$$= \frac{n}{2} (2 \times 28 + (n-1)(-5))$$

$$= \frac{n}{2} (56 - 5n + 5)$$

$$\text{i.e., } S_n = \frac{n}{2} (61 - 5n)$$

8 Since the terms are geometric, then $\frac{k}{4} = \frac{k^2 - 1}{k}$

$$\therefore k^2 = 4(k^2 - 1)$$

$$\therefore k^2 = 4k^2 - 4 \quad \therefore k = \pm \frac{2}{\sqrt{3}}$$

$$\therefore 3k^2 = 4 \quad \therefore k = \pm \frac{2\sqrt{3}}{3}$$

$$\therefore k^2 = \frac{4}{3}$$

9 $u_6 = \frac{16}{3} \quad \therefore u_1 \times r^5 = \frac{16}{3} \quad \dots (1)$ So $\frac{u_1 r^9}{u_1 r^5} = \frac{(\frac{256}{3})}{(\frac{16}{3})} \quad \{(2) \div (1)\}$

$$u_{10} = \frac{256}{3} \quad \therefore u_1 \times r^9 = \frac{256}{3} \quad \dots (2)$$

$$\therefore r^4 = 16$$

$$\therefore r = \pm 2$$

Substituting $r = 2$ into (1) gives

$$u_1 \times 2^5 = \frac{16}{3}$$

$$\therefore u_1 \times 32 = \frac{16}{3}$$

$$\therefore u_1 = \frac{1}{6}$$

Substituting $r = -2$ into (1) gives

$$u_1 \times (-2)^5 = \frac{16}{3}$$

$$\therefore u_1 \times (-32) = \frac{16}{3}$$

$$\therefore u_1 = -\frac{1}{6}$$

Now $u_n = u_1 r^{n-1} \quad \therefore u_n = \frac{1}{6} \times 2^{n-1} \quad \text{or} \quad -\frac{1}{6} \times 2^{n-1}$

REVIEW SET 2B

1 a 24, $23\frac{1}{4}$, $22\frac{1}{2}$,, -36 i.e., $u_1 = 24$, $u_n = -36$, and we need to find n .

The sequence is arithmetic with $d = -\frac{3}{4}$.

$$\text{Now } u_n = u_1 + (n-1)d$$

$$\therefore -36 = 24 + (n-1)(-\frac{3}{4})$$

$$\therefore -60 = -\frac{3}{4}n + \frac{3}{4}$$

$$\therefore \frac{3}{4}n = \frac{243}{4}$$

$$\therefore n = 81$$

i.e., there are 81 terms in the sequence.

b $u_{35} = 24 + (35-1)(-\frac{3}{4})$

$$= 24 - \frac{102}{4}$$

$$= -\frac{3}{2}$$

c $S_n = \frac{n}{2}(u_1 + u_n)$

i.e., $S_{81} = \frac{81}{2}(24 + (-36))$

$$= -486$$

2 Let the numbers be 23, $23 + d$, $23 + 2d$, $23 + 3d$, $23 + 4d$, $23 + 5d$, $23 + 6d$, 9

Then $23 + 7d = 9$

i.e., $7d = -14$

$\therefore d = -2$ i.e., the numbers are 23, 21, 19, 17, 15, 13, 11, 9.

3 a 86, 83, 80, 77,, i.e., the sequence is arithmetic with $u_1 = 86$, $d = -3$

$$u_n = u_1 + (n-1)d$$

i.e., $u_n = 86 + (n-1)(-3) = 86 - 3n + 3$

i.e., $u_n = 89 - 3n$

b $\frac{3}{4}, 1, \frac{7}{6}, \frac{9}{7}, \dots$ which can also be written as $\frac{3}{4}, \frac{5}{5}, \frac{7}{6}, \frac{9}{7}, \dots$

i.e., the numerator starts at 3 and increases by 2 each time,

whilst the denominator starts at 4 and increases by 1 each time.

$$\text{i.e., the } n\text{th term is } \frac{2n+1}{n+3}, \text{ i.e., } u_n = \frac{2n+1}{n+3}$$

- c** 100, 90, 81, 72.9, i.e., the sequence is geometric with $u_1 = 100$, $r = \frac{90}{100} = 0.9$

$$u_n = u_1 r^{n-1}$$

$$\text{i.e., } u_n = 100(0.9)^{n-1}$$

4 a $\sum_{r=1}^7 r^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2$
 $= 1 + 4 + 9 + 16 + 25 + 36 + 49$

b $\sum_{r=1}^8 \frac{r+3}{r+2} = \frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6} + \frac{8}{7} + \frac{9}{8} + \frac{10}{9} + \frac{11}{10}$

5 a $4 + 11 + 18 + 25 + \dots$

i.e., the series is arithmetic with $u_1 = 4$, $d = 7$, $u_r = u_1 + (r-1)d$
 $= 4 + 7(r-1)$

$$\text{i.e., } \sum_{r=1}^n (7r-3)$$

$$= 7r-3$$

b $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$ i.e., the series is geometric with $u_1 = \frac{1}{4}$, $R = \frac{1}{2}$,

$$u_r = u_1 R^{r-1} = \frac{1}{4} \times \left(\frac{1}{2}\right)^{r-1} = \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{r-1} = \left(\frac{1}{2}\right)^{r+1}$$

$$\text{i.e., } \sum_{r=1}^n \left(\frac{1}{2}\right)^{r+1}$$

6 a $3 + 9 + 15 + 21 + \dots$

i.e., the series is arithmetic with
 $u_1 = 3$, $d = 6$, $n = 23$

$$\text{Now } S_n = \frac{n}{2} (2u_1 + (n-1)d)$$

$$\text{i.e., } S_{23} = \frac{23}{2} (2 \times 3 + 6(23-1))$$

$$\therefore S_{23} = \frac{23}{2} (6 + 132)$$

$$= 1587$$

b $24 + 12 + 6 + 3 + \dots$

i.e., the series is geometric with
 $u_1 = 24$, $r = \frac{1}{2}$, $n = 12$

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

$$\text{i.e., } S_{12} = \frac{24(1-(\frac{1}{2})^{12})}{1-\frac{1}{2}}$$

$$= 48(1-(\frac{1}{2})^{12})$$

$$\text{i.e., } S_{12} \div 47.99$$

7 a $\sum_{r=1}^8 \left(\frac{31-3r}{2}\right) = 14 + 12\frac{1}{2} + 11 + 9\frac{1}{2} + 8 + 6\frac{1}{2} + 5 + 3\frac{1}{2}$

This series is arithmetic with $u_1 = 14$, $d = -1\frac{1}{2}$ and $n = 8$.

$$\therefore \text{ the sum is } \frac{8}{2} [28 + 7(-1\frac{1}{2})] = 70$$

b $\sum_{r=1}^{15} 50(0.8)^{r-1} \div 50 + 40 + 32 + \dots + 3.436 + 2.749 + 2.199$

This series is geometric with $u_1 = 50$, $r = 0.8$ and $n = 15$.

$$\therefore \text{ the sum is } \frac{50[1-(0.8)^{15}]}{1-0.8} \div 241.2$$

- 8** 5, 10, 20, 40, i.e., the sequence is geometric with $u_1 = 5$, $r = 2$

$$u_n = u_1 r^{n-1} = 5 \times 2^{n-1}$$

$$\text{Let } u_n = 10\,000 = 5 \times 2^{n-1}$$

$$\therefore 2000 = 2^{n-1}$$

$$\therefore n \div 11.97 \quad \{\text{using technology}\}$$

So, try the two values on either side of $n \div 11.97$, i.e., for $n = 11$ and $n = 12$:

$$u_{11} = 5 \times 2^{10} \quad \text{and} \quad u_{12} = 5 \times 2^{11}$$

$$= 5120 \quad \quad \quad = 10\,240$$

i.e., the first term to exceed 10 000 is $u_{12} = 10\,240$.

- 9 a** $u_6 = u_1 \times r^5$ is the amount after 5 years, where $r = 1.07$

$$= 6000 \times (1.07)^5$$

i.e., $u_6 = 8415.31$, i.e., the value of the investment will be 8415.31 Euro

- b** If interest is compounded quarterly, then $r = 1 + \frac{0.07}{4} = 1.0175$

$$\text{and } n = 5 \times 4 = 20$$

$$u_{21} = u_1 \times r^{20}$$

$$= 6000 \times (1.0175)^{20}$$

i.e., $u_{21} = 8488.67$, i.e., the value of the investment will be 8488.67 Euro

- c** If interest is compounded monthly, then $r = 1 + \frac{0.07}{12} = 1.0058\bar{3}$

$$\text{and } n = 5 \times 12 = 60$$

$$u_{61} = u_1 \times r^{60}$$

$$= 6000 \times (1.0058\bar{3})^{60}$$

i.e., $u_{61} = 8505.75$, i.e., the value of the investment will be 8505.75 Euro

REVIEW SET 2C

- 1** $u_6 = 24 \quad \therefore u_1 \times r^5 = 24 \quad \dots (1)$

$$u_{11} = 768 \quad \therefore u_1 \times r^{10} = 768 \quad \dots (2)$$

$$\text{So } \frac{u_1 r^{10}}{u_1 r^5} = \frac{768}{24} \quad \{(2) \div (1)\}$$

$$\therefore r^5 = 32$$

$$\therefore r = 2$$

$$\text{Substituting } r = 2 \text{ into (1) gives } u_1 \times 2^5 = 24 \quad \therefore u_1 = \frac{24}{32} = \frac{3}{4}$$

$$u_n = u_1 r^{n-1} = \left(\frac{3}{4}\right) 2^{n-1}$$

$$\begin{aligned} \mathbf{a} \quad u_{17} &= \left(\frac{3}{4}\right) 2^{17-1} \\ &= 49\,152 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad S_n &= \frac{u_1(r^n - 1)}{r - 1} \\ &= \frac{\frac{3}{4}(2^n - 1)}{2 - 1} \\ &= \frac{3}{4}(2^n - 1) \\ \therefore S_{15} &= \frac{3}{4}(2^{15} - 1) \\ &= 24\,575.25 \end{aligned}$$

- 2** $11 + 16 + 21 + 26 + \dots$ i.e., the series is arithmetic with $u_1 = 11$, $d = 5$

$$\begin{aligned} S_n &= \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(2 \times 11 + 5(n-1)) \\ &= \frac{n}{2}(22 + 5n - 5) \\ &= \frac{n}{2}(5n + 17) \end{aligned}$$

Let $S_n = 450$ and we need to find n ,

$$\text{i.e., } S_n = \frac{n}{2}(5n + 17) = 450$$

$$\therefore \frac{5}{2}n^2 + \frac{17}{2}n - 450 = 0$$

$$\therefore 5n^2 + 17n - 900 = 0$$

$$\therefore n \div -15.2, 11.8 \quad \{\text{using technology}\}$$

$$\text{but } n > 0, \therefore n \div 11.8$$

So, try the two values on either side of $n \div 11.8$, i.e., for $n = 11$ and $n = 12$:

$$\begin{aligned} S_{11} &= \frac{11}{2}(5(11) + 17) & \text{and} & & S_{12} &= \frac{12}{2}(5(12) + 17) \\ &= 396 & & & &= 462 \end{aligned}$$

i.e., 12 terms of the series are required to exceed a sum of 450.

- 3** $24, 8, \frac{8}{3}, \frac{8}{9}, \dots$ i.e., the sequence is geometric with $u_1 = 24$, $r = \frac{1}{3}$

$$\begin{aligned} u_n &= u_1 r^{n-1} \\ &= 24 \left(\frac{1}{3}\right)^{n-1} \end{aligned}$$

Let $u_n = 0.001$ and we need to find n .

$$\text{i.e., } u_n = 24 \left(\frac{1}{3}\right)^{n-1} = 0.001$$

$$\therefore \left(\frac{1}{3}\right)^{n-1} = \frac{0.001}{24}$$

$$\therefore n \div 10.18 \quad \{\text{using technology}\}$$

So, try the two values on either side of $n \div 10.18$, i.e., for $n = 10$ and $n = 11$:

$$\begin{aligned} u_{10} &= 24 \left(\frac{1}{3}\right)^9 & \text{and} & & u_{11} &= 24 \left(\frac{1}{3}\right)^{10} \\ &= 0.00122 & & & &= 0.000406 \end{aligned}$$

i.e., $u_{11} \div 0.000406$ is the first term of the sequence which is less than 0.001.

- 4 a** $128, 64, 32, 16, \dots, \frac{1}{512}$

i.e., the sequence is geometric with

$$u_1 = 128, \quad r = \frac{1}{2}, \quad u_n = \frac{1}{512}$$

$$\begin{aligned} u_n &= u_1 r^{n-1} \\ &= 128 \left(\frac{1}{2}\right)^{n-1} \end{aligned}$$

$$\text{i.e., } \frac{1}{512} = 128 \left(\frac{1}{2}\right)^{n-1}$$

$$\therefore \left(\frac{1}{2}\right)^{n-1} = \frac{1}{65536}$$

$$\therefore n = 17 \quad \{\text{using technology}\}$$

i.e., there are 17 terms in the sequence.

b $S_n = \frac{u_1(1-r^n)}{1-r}$

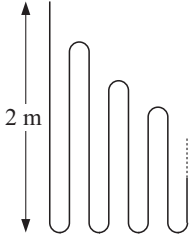
$$\text{i.e., } S_{17} = \frac{128 \left(1 - \left(\frac{1}{2}\right)^{17}\right)}{1 - \frac{1}{2}}$$

$$= 255.998$$

$$\text{i.e., } S_{17} \div 256$$

- 5 a** $u_{n+1} = u_1 \times r^n$ where $u_1 = 12\,500$, $r = 1 + \frac{0.0825}{2} = 1.041\,25$, $n = 5 \times 2 = 10$
i.e., $u_{n+1} = 12\,500 \times (1.041\,25)^{10}$
 $= 18\,726.65$, i.e., the value of the investment is \$18 726.65
- b** $u_{n+1} = u_1 \times r^n$ where $u_1 = 12\,500$, $r = 1 + \frac{0.0825}{12} = 1.006\,875$, $n = 5 \times 12 = 60$
i.e., $u_{n+1} = 12\,500 \times (1.006\,875)^{60}$
 $= 18\,855.74$, i.e., the value of the investment is \$18 855.74
- 6** $u_{n+1} = u_1 \times r^n$ where $u_{n+1} = 20\,000$, $r = 1 + \frac{0.09}{12} = 1.0075$, $n = 4 \times 12 = 48$
i.e., $20\,000 = u_1 \times (1.0075)^{48}$
i.e., $u_1 = 13\,972.28$, i.e., \$13 972.28 should be invested
- 7 a** $u_{n+1} = u_1 \times r^n$, where $u_1 = 3000$, $r = 1.05$, $n = 3$
i.e., $u_{n+1} = 3000 \times (1.05)^3$
 $= 3472.875$, i.e., approximately 3470 koalas
- b** $u_{n+1} = u_1 \times r^n$ where $u_1 = 3000$, $u_{n+1} = 5000$, $r = 1.05$
i.e., $5000 = 3000 \times (1.05)^n$
i.e., $n \div 10.47$
i.e., after 10.47 years the population will exceed 5000, i.e., in the year 2008.

8



Total distance travelled

$$\begin{aligned}
 &= 2 + 2 \times 0.8 \times 2 + 2 \times (0.8)^2 \times 2 + 2 \times (0.8)^3 \times 2 + \dots \\
 &= 2 + 4 \times 0.8 \times [1 + 0.8 + (0.8)^2 + (0.8)^3 + \dots] \\
 &= 2 + 3.2 \times \left[\frac{1}{1-0.8} \right] \quad \text{as } r = 0.8, |r| < 1 \therefore \text{converges to } \frac{u_1}{1-r} \\
 &= 2 + \frac{3.2}{0.2} \\
 &= 2 + 16 \\
 &= 18 \text{ metres}
 \end{aligned}$$

- 9 a** $\sum_{r=1}^{\infty} 50(2x-1)^{r-1}$ is a geometric series with $r = 2x-1$ and converges if $-1 < r < 1$
 $\therefore -1 < 2x-1 < 1$
 $\therefore 0 < 2x < 2$
 $\therefore 0 < x < 1$
- b** When $x = 0.3$, $2x-1 = 0.6-1 = -0.4$
and $\sum_{r=1}^{\infty} 50(2x-1)^{r-1} = 50(-0.4)^0 + 50(-0.4)^1 + 50(-0.4)^2 + \dots$ which is geometric
with $u_1 = 50$, $r = -0.4$
Now as $0 < 0.3 < 1$, the series converges and $S_{\infty} = \frac{u_1}{1-r} = \frac{50}{1+0.4} = \frac{50}{\frac{7}{5}} = 35\frac{5}{7}$

Chapter 3

EXPONENTS

EXERCISE 3A

- 1 a $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16,$
 $2^5 = 32, 2^6 = 64$
 c $5^1 = 5, 5^2 = 25, 5^3 = 125, 5^4 = 625$
 b $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81$
 d $7^1 = 7, 7^2 = 49, 7^3 = 343$

EXERCISE 3B

- 1 a $(-1)^3$
 $= (-1) \times (-1) \times (-1)$
 $= 1 \times (-1)$
 $= -1$
 b $(-1)^4$
 $= (-1)^3 \times (-1)$
 $= (-1) \times (-1)$
 $= 1$
 c $(-1)^{12}$
 $= 1$
 d $(-1)^{17}$
 $= -1$
 e $(-1)^6$
 $= 1$
 f -1^6
 $= -(1^6)$
 $= -1$
 g $-(-1)^6$
 $= -(1)$
 $= -1$
 h $(-2)^3$
 $= (-2) \times (-2) \times (-2)$
 $= 4 \times (-2)$
 $= -8$
 i -2^3
 $= -(2^3)$
 $= -8$
 j $-(-2)^3$
 $= -(-8)$
 $= 8$
 k $-(-5)^2$
 $= -(25)$
 $= -25$
 l $-(-5)^3$
 $= -(-125)$
 $= 125$
- 2 a 512 b -3125 c -243 d 16 807 e 512 f 6561 g -6561
 h 5.117 264 691 i -0.764 479 956 j -20.361 584 96
- 3 a $0.\overline{142857}$ b $0.\overline{142857}$ c $0.\overline{1}$ d $0.\overline{1}$ e 0.015 625 f 0.015 625
 g 1 h 1

We notice that $7^{-1} = \frac{1}{7^1}$, $3^{-2} = \frac{1}{3^2}$, $4^{-3} = \frac{1}{4^3}$ and $a^0 = 1$ for $a > 0$

- 4 $3^{33} = \underbrace{3^4 \times 3^4 \times 3^4 \times \dots \times 3^4}_{8 \text{ of these}} \times 3^1$ But $3^4 = 81$ i.e., ends in a 1
 $\therefore \underbrace{3^4 \times 3^4 \times 3^4 \times \dots \times 3^4}_{8 \text{ of these}}$ ends in a 1
 $\therefore 3^{33}$ ends in a 3

- 5 $7^1 = 7, 7^2 = 49, 7^3 = 343, 7^4 = 2401, 7^5 = 16\,807$

Now $7^{77} = \underbrace{7^4 \times 7^4 \times 7^4 \times \dots \times 7^4}_{19 \text{ of these}} \times 7^1$

So, ends in a 1.

$\therefore 7^{77}$ ends in $1 \times 7 = 7$.

EXERCISE 3C

$$\begin{aligned} 1 \quad a \quad & 7^3 \times 7^2 \\ & = 7^{3+2} \\ & = 7^5 \end{aligned}$$

$$\begin{aligned} b \quad & 5^4 \times 5^3 \\ & = 5^{4+3} \\ & = 5^7 \end{aligned}$$

$$\begin{aligned} c \quad & a^7 \times a^2 \\ & = a^{7+2} \\ & = a^9 \end{aligned}$$

$$\begin{aligned} d \quad & a^4 \times a \\ & = a^4 \times a^1 \\ & = a^{4+1} \\ & = a^5 \end{aligned}$$

$$\begin{aligned} e \quad & b^8 \times b^5 \\ & = b^{8+5} \\ & = b^{13} \end{aligned}$$

$$\begin{aligned} f \quad & a^3 \times a^n \\ & = a^{3+n} \end{aligned}$$

$$\begin{aligned} g \quad & b^7 \times b^m \\ & = b^{7+m} \end{aligned}$$

$$\begin{aligned} h \quad & m^4 \times m^2 \times m^3 \\ & = m^{4+2+3} \\ & = m^9 \end{aligned}$$

$$\begin{aligned} 2 \quad a \quad & \frac{5^9}{5^2} \\ & = 5^{9-2} \\ & = 5^7 \end{aligned}$$

$$\begin{aligned} b \quad & \frac{11^{13}}{11^9} \\ & = 11^{13-9} \\ & = 11^4 \end{aligned}$$

$$\begin{aligned} c \quad & 7^7 \div 7^4 \\ & = 7^{7-4} \\ & = 7^3 \end{aligned}$$

$$\begin{aligned} d \quad & \frac{a^6}{a^2} \\ & = a^{6-2} \\ & = a^4 \end{aligned}$$

$$\begin{aligned} e \quad & \frac{b^{10}}{b^7} \\ & = b^{10-7} \\ & = b^3 \end{aligned}$$

$$\begin{aligned} f \quad & \frac{p^5}{p^m} \\ & = p^{5-m} \end{aligned}$$

$$\begin{aligned} g \quad & \frac{y^a}{y^5} \\ & = y^{a-5} \end{aligned}$$

$$\begin{aligned} h \quad & b^{2x} \div b \\ & = b^{2x} \div b^1 \\ & = b^{2x-1} \end{aligned}$$

$$\begin{aligned} 3 \quad a \quad & (3^2)^4 \\ & = 3^{2 \times 4} \\ & = 3^8 \end{aligned}$$

$$\begin{aligned} b \quad & (5^3)^5 \\ & = 5^{3 \times 5} \\ & = 5^{15} \end{aligned}$$

$$\begin{aligned} c \quad & (2^4)^7 \\ & = 2^{4 \times 7} \\ & = 2^{28} \end{aligned}$$

$$\begin{aligned} d \quad & (a^5)^2 \\ & = a^{5 \times 2} \\ & = a^{10} \end{aligned}$$

$$\begin{aligned} e \quad & (p^4)^5 \\ & = p^{4 \times 5} \\ & = p^{20} \end{aligned}$$

$$\begin{aligned} f \quad & (b^5)^n \\ & = b^{5 \times n} \\ & = b^{5n} \end{aligned}$$

$$\begin{aligned} g \quad & (x^y)^3 \\ & = x^{y \times 3} \\ & = x^{3y} \end{aligned}$$

$$\begin{aligned} h \quad & (a^{2x})^5 \\ & = a^{2x \times 5} \\ & = a^{10x} \end{aligned}$$

$$\begin{aligned} 4 \quad a \quad & 8 \\ & = 2^3 \end{aligned}$$

$$\begin{aligned} b \quad & 25 \\ & = 5^2 \end{aligned}$$

$$\begin{aligned} c \quad & 27 \\ & = 3^3 \end{aligned}$$

$$\begin{aligned} d \quad & 4^3 \\ & = (2^2)^3 \\ & = 2^6 \end{aligned}$$

$$\begin{aligned} e \quad & 9^2 \\ & = (3^2)^2 \\ & = 3^4 \end{aligned}$$

$$\begin{aligned} f \quad & 3^a \times 9 \\ & = 3^a \times 3^2 \\ & = 3^{a+2} \end{aligned}$$

$$\begin{aligned} g \quad & 5^t \div 5 \\ & = 5^t \div 5^1 \\ & = 5^{t-1} \end{aligned}$$

$$\begin{aligned} h \quad & 3^n \times 9^n \\ & = 3^n \times (3^2)^n \\ & = 3^{n+2n} \\ & = 3^{3n} \end{aligned}$$

$$\begin{aligned} i \quad & \frac{16}{2^x} \\ & = \frac{2^4}{2^x} \\ & = 2^{4-x} \end{aligned}$$

$$\begin{aligned} j \quad & \frac{3^{x+1}}{3^{x-1}} \\ & = 3^{(x+1)-(x-1)} \\ & = 3^{x+1-x+1} \\ & = 3^2 \end{aligned}$$

$$\begin{aligned} k \quad & (5^4)^{x-1} \\ & = 5^{4(x-1)} \\ & = 5^{4x-4} \end{aligned}$$

$$\begin{aligned} l \quad & 2^x \times 2^{2-x} \\ & = 2^{x+2-x} \\ & = 2^2 \end{aligned}$$

$$\begin{aligned} m \quad & \frac{2^y}{4^x} \\ & = \frac{2^y}{(2^2)^x} \\ & = \frac{2^y}{2^{2x}} \\ & = 2^{y-2x} \end{aligned}$$

$$\begin{aligned} n \quad & \frac{4^y}{8^x} \\ & = \frac{(2^2)^y}{(2^3)^x} \\ & = \frac{2^{2y}}{2^{3x}} \\ & = 2^{2y-3x} \end{aligned}$$

$$\begin{aligned} o \quad & \frac{3^{x+1}}{3^{1-x}} \\ & = 3^{(x+1)-(1-x)} \\ & = 3^{x+1-1+x} \\ & = 3^{2x} \end{aligned}$$

$$\begin{aligned} p \quad & \frac{2^t \times 4^t}{8^{t-1}} \\ & = \frac{2^t \times (2^2)^t}{(2^3)^{t-1}} \\ & = \frac{2^t \times 2^{2t}}{2^{3t-3}} \\ & = 2^{3t-(3t-3)} \\ & = 2^3 \end{aligned}$$

5 a	$(ab)^3$ $= a^3b^3$	b	$(ac)^4$ $= a^4c^4$	c	$(bc)^5$ $= b^5c^5$	d	$(abc)^3$ $= a^3b^3c^3$
e	$(2a)^4$ $= 2^4a^4$ $= 16a^4$	f	$(5b)^2$ $= 5^2b^2$ $= 25b^2$	g	$(3n)^4$ $= 3^4n^4$ $= 81n^4$	h	$(2bc)^3$ $= 2^3b^3c^3$ $= 8b^3c^3$
i	$(4ab)^3$ $= 4^3a^3b^3$ $= 64a^3b^3$	j	$\left(\frac{a}{b}\right)^3$ $= \frac{a^3}{b^3}$	k	$\left(\frac{m}{n}\right)^4$ $= \frac{m^4}{n^4}$	l	$\left(\frac{2c}{d}\right)^5$ $= \frac{2^5c^5}{d^5}$ or $\frac{32c^5}{d^5}$
6 a	$(2b^4)^3$ $= 2^3b^{12}$ $= 8b^{12}$	b	$\left(\frac{3}{x^2y}\right)^2$ $= \frac{3^2}{x^4y^2}$ $= \frac{9}{x^4y^2}$	c	$(5a^4b)^2$ $= 5^2a^8b^2$ $= 25a^8b^2$	d	$\left(\frac{m^3}{2n^2}\right)^4$ $= \frac{m^{12}}{2^4n^8}$ $= \frac{m^{12}}{16n^8}$
e	$\left(\frac{3a^3}{b^5}\right)^3$ $= \frac{3^3a^9}{b^{15}}$ $= \frac{27a^9}{b^{15}}$	f	$(2m^3n^2)^5$ $= 2^5m^{15}n^{10}$ $= 32m^{15}n^{10}$	g	$\left(\frac{4a^4}{b^2}\right)^2$ $= \frac{4^2a^8}{b^4}$ $= \frac{16a^8}{b^4}$	h	$(5x^2y^3)^3$ $= 5^3x^6y^9$ $= 125x^6y^9$
i	$(-2a)^2$ $= (-2)^2a^2$ $= 4a^2$	j	$(-6b^2)^2$ $= (-6)^2b^4$ $= 36b^4$	k	$(-2a)^3$ $= (-2)^3a^3$ $= -8a^3$	l	$(-3m^2n^2)^3$ $= (-3)^3m^6n^6$ $= -27m^6n^6$
m	$(-2ab^4)^4$ $= (-2)^4a^4b^{16}$ $= 16a^4b^{16}$	n	$\left(\frac{-2a^2}{b^2}\right)^3$ $= \frac{(-2)^3a^6}{b^6}$ $= -\frac{8a^6}{b^6}$	o	$\left(\frac{-4a^3}{b}\right)^2$ $= \frac{(-4)^2a^6}{b^2}$ $= \frac{16a^6}{b^2}$	p	$\left(\frac{-3p^2}{q^3}\right)^2$ $= \frac{(-3)^2p^4}{q^6}$ $= \frac{9p^4}{q^6}$
7 a	$\frac{a^3}{a}$ $= \frac{a^3}{a^1}$ $= a^{3-1}$ or a^2	b	$4b^2 \times 2b^3$ $= 8b^{2+3}$ $= 8b^5$	c	$\frac{m^5n^4}{m^2n^3}$ $= m^{5-2}n^{4-3}$ $= m^3n^1$ $= m^3n$	d	$\frac{14a^7}{2a^2}$ $= \frac{14}{2}a^{7-2}$ $= 7a^5$
e	$\frac{12a^2b^3}{3ab}$ $= \frac{12}{3}a^{2-1}b^{3-1}$ $= 4ab^2$	f	$\frac{18m^7a^3}{4m^4a^3}$ $= \frac{18}{4}m^{7-4}a^{3-3}$ $= \frac{9}{2}m^3a^0$ $= \frac{9}{2}m^3$ or $\frac{9m^3}{2}$	g	$10hk^3 \times 4h^4$ $= 40 \times h^{1+4}k^3$ $= 40h^5k^3$	h	$\frac{m^{11}}{(m^2)^8}$ $= \frac{m^{11}}{m^{16}}$ $= m^{11-16}$ $= m^{-5}$ or $\frac{1}{m^5}$
i	$\frac{p^2 \times p^7}{(p^3)^2} = \frac{p^{2+7}}{p^6} = \frac{p^9}{p^6} = p^{9-6} = p^3$						

8 a $5^0 = 1$

e $2^2 = 4$

i $5^2 = 25$

b $3^{-1} = \frac{1}{3}$

f $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

j $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

c $6^{-1} = \frac{1}{6}$

g $2^3 = 8$

k $10^2 = 100$

d $8^0 = 1$

h $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

l $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

9 a $\left(\frac{2}{3}\right)^0 = 1$

b $\frac{4^3}{4^3} = 1$

c $3y^0$
 $= 3 \times 1$
 $= 3$

d $(3y)^0$
 $= 1$

e 2×3^0
 $= 2 \times 1$
 $= 2$

f $6^0 = 1$

g $\frac{5^2}{5^4}$
 $= 5^{2-4}$
 $= 5^{-2}$
 $= \frac{1}{5^2} \text{ or } \frac{1}{25}$

h $\frac{2^{10}}{2^{15}}$
 $= 2^{10-15}$
 $= 2^{-5}$
 $= \frac{1}{2^5} \text{ or } \frac{1}{32}$

i $\left(\frac{1}{3}\right)^{-1}$
 $= \left(\frac{3}{1}\right)^1$
 $= 3$

j $\left(\frac{2}{5}\right)^{-1}$
 $= \left(\frac{5}{2}\right)^1$
 $= 2\frac{1}{2}$

k $\left(\frac{4}{3}\right)^{-1}$
 $= \left(\frac{3}{4}\right)^1$
 $= \frac{3}{4}$

l $\left(\frac{1}{12}\right)^{-1}$
 $= \left(\frac{12}{1}\right)^1$
 $= 12$

m $\left(\frac{2}{3}\right)^{-2}$
 $= \left(\frac{3}{2}\right)^2$
 $= \frac{9}{4} \text{ or } 2\frac{1}{4}$

n $5^0 - 5^{-1}$
 $= 1 - \frac{1}{5}$
 $= \frac{4}{5}$

o $7^{-1} + 7^0$
 $= \frac{1}{7} + 1$
 $= 1\frac{1}{7} \text{ or } \frac{8}{7}$

p $2^0 + 2^1 + 2^{-1}$
 $= 1 + 2 + \frac{1}{2}$
 $= 3\frac{1}{2} \text{ or } \frac{7}{2}$

10 a $(2a)^{-1}$
 $= \frac{1}{2a}$

b $2a^{-1}$
 $= \frac{2}{1} \times \frac{1}{a}$
 $= \frac{2}{a}$

c $3b^{-1}$
 $= \frac{3}{1} \times \frac{1}{b}$
 $= \frac{3}{b}$

d $(3b)^{-1}$
 $= \frac{1}{3b}$

e $\left(\frac{2}{b}\right)^{-2}$
 $= \left(\frac{b}{2}\right)^2$
 $= \frac{b^2}{4}$

f $(2b)^{-2}$
 $= \frac{1}{(2b)^2}$
 $= \frac{1}{4b^2}$

g $(3n)^{-2}$
 $= \frac{1}{(3n)^2}$
 $= \frac{1}{9n^2}$

h $(3n^{-2})^{-1}$
 $= 3^{-1}n^2$
 $= \frac{1}{3}n^2$
 $= \frac{n^2}{3}$

i ab^{-1}
 $= \frac{a}{1} \times \frac{1}{b}$
 $= \frac{a}{b}$

j $(ab)^{-1}$
 $= \frac{1}{ab}$

k ab^{-2}
 $= a \times \frac{1}{b^2}$
 $= \frac{a}{b^2}$

l $(ab)^{-2}$
 $= \frac{1}{(ab)^2}$
 $= \frac{1}{a^2b^2}$

m $(2ab)^{-1}$
 $= \frac{1}{2ab}$

n $2(ab)^{-1}$
 $= \frac{2}{1} \times \frac{1}{ab}$
 $= \frac{2}{ab}$

o $2ab^{-1}$
 $= \frac{2a}{1} \times \frac{1}{b}$
 $= \frac{2a}{b}$

p $\frac{(ab)^2}{b^{-1}}$
 $= \frac{a^2b^2}{b^{-1}}$
 $= a^2b^{2-(-1)}$
 $= a^2b^3$

11 a $\frac{1}{3} = 3^{-1}$

b $\frac{1}{2} = 2^{-1}$

c $\frac{1}{5} = 5^{-1}$

d $\frac{1}{4}$
 $= \frac{1}{2^2}$
 $= 2^{-2}$

e $\frac{1}{27}$
 $= \frac{1}{3^3}$
 $= 3^{-3}$

f $\frac{1}{25}$
 $= \frac{1}{5^2}$
 $= 5^{-2}$

g $\frac{1}{8^x}$
 $= \frac{1}{(2^3)^x}$
 $= \frac{1}{2^{3x}}$
 $= 2^{-3x}$

h $\frac{1}{16^y}$
 $= \frac{1}{(2^4)^y}$
 $= \frac{1}{2^{4y}}$
 $= 2^{-4y}$

i $\frac{1}{81^a}$
 $= \frac{1}{(3^4)^a}$
 $= \frac{1}{3^{4a}}$
 $= 3^{-4a}$

j $\frac{9}{3^4}$
 $= \frac{3^2}{3^4}$
 $= 3^{2-4}$
 $= 3^{-2}$

k 25×5^{-4}
 $= 5^2 \times 5^{-4}$
 $= 5^{2+(-4)}$
 $= 5^{-2}$

l $\frac{5^{-1}}{5^2}$
 $= 5^{-1-2}$
 $= 5^{-3}$

m $2 \div 2^{-3}$
 $= 2^1 \div 2^{-3}$
 $= 2^{1-(-3)}$
 $= 2^4$

n 1
 $= 2^0$ or 3^0 or 5^0

o 6^{-3}
 $= (2 \times 3)^{-3}$
 $= 2^{-3} \times 3^{-3}$

p 4×10^2
 $= 2^2 \times (2 \times 5)^2$
 $= 2^2 \times 2^2 \times 5^2$
 $= 2^4 \times 5^2$

12 1 less day, i.e., 25 days

13 a $5^3 = 21 + 23 + 25 + 27 + 29$

b $7^3 = 43 + 45 + 47 + 49 + 51 + 53 + 55$

c $12^3 = 133 + 135 + 137 + 139 + 141 + 143 + 145 + 147 + 149 + 151 + 153 + 155$

14 We notice that 175 and 75 have a common factor of 25.

So 2^{175} and 5^{75}
 $= (2^7)^{25}$ $= (5^3)^{25}$
 $= (128)^{25}$ $= (125)^{25}$ $\therefore$ as $125 < 128$, 5^{75} is smaller.

EXERCISE 3D

1 a $\sqrt[5]{2}$
 $= 2^{\frac{1}{5}}$

b $\frac{1}{\sqrt[5]{2}}$
 $= \frac{1}{2^{\frac{1}{5}}}$
 $= 2^{-\frac{1}{5}}$

c $2\sqrt{2}$
 $= 2^1 \times 2^{\frac{1}{2}}$
 $= 2^{\frac{3}{2}}$

d $4\sqrt{2}$
 $= 2^2 \times 2^{\frac{1}{2}}$
 $= 2^{\frac{5}{2}}$

e $\frac{1}{\sqrt[3]{2}}$
 $= \frac{1}{2^{\frac{1}{3}}}$
 $= 2^{-\frac{1}{3}}$

f $2 \times \sqrt[3]{2}$
 $= 2^1 \times 2^{\frac{1}{3}}$
 $= 2^{\frac{4}{3}}$

g $\frac{4}{\sqrt{2}}$
 $= \frac{2^2}{2^{\frac{1}{2}}}$
 $= 2^{2-\frac{1}{2}}$
 $= 2^{\frac{3}{2}}$

h $(\sqrt{2})^3$
 $= \left(2^{\frac{1}{2}}\right)^3$
 $= 2^{\frac{1}{2} \times 3}$
 $= 2^{\frac{3}{2}}$

i $\frac{1}{\sqrt[3]{16}}$
 $= \frac{1}{(2^4)^{\frac{1}{3}}}$
 $= \frac{1}{2^{\frac{4}{3}}}$
 $= 2^{-\frac{4}{3}}$

j $\frac{1}{\sqrt{8}}$
 $= \frac{1}{(2^3)^{\frac{1}{2}}}$
 $= \frac{1}{2^{\frac{3}{2}}}$
 $= 2^{-\frac{3}{2}}$

2 a	$\sqrt[3]{3}$ $= 3^{\frac{1}{3}}$	b	$\frac{1}{\sqrt[3]{3}}$ $= \frac{1}{3^{\frac{1}{3}}}$ $= 3^{-\frac{1}{3}}$	c	$\sqrt[4]{3}$ $= 3^{\frac{1}{4}}$	d	$3\sqrt{3}$ $= 3^1 \times 3^{\frac{1}{2}}$ $= 3^{1\frac{1}{2}}$ $= 3^{\frac{3}{2}}$	e	$\frac{1}{9\sqrt{3}}$ $= \frac{1}{3^2 3^{\frac{1}{2}}}$ $= \frac{1}{3^{2\frac{1}{2}}}$ $= 3^{-\frac{5}{2}}$
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3 a	$\sqrt[3]{7}$ $= 7^{\frac{1}{3}}$	b	$\sqrt[4]{27}$ $= (3^3)^{\frac{1}{4}}$ $= 3^{\frac{3}{4}}$	c	$\sqrt[5]{16}$ $= (2^4)^{\frac{1}{5}}$ $= 2^{\frac{4}{5}}$	d	$\sqrt[3]{32}$ $= (2^5)^{\frac{1}{3}}$ $= 2^{\frac{5}{3}}$	e	$\sqrt[7]{49}$ $= (7^2)^{\frac{1}{7}}$ $= 7^{\frac{2}{7}}$
f	$\frac{1}{\sqrt[3]{7}}$ $= \frac{1}{7^{\frac{1}{3}}}$ $= 7^{-\frac{1}{3}}$	g	$\frac{1}{\sqrt[4]{27}}$ $= \frac{1}{(3^3)^{\frac{1}{4}}}$ $= \frac{1}{3^{\frac{3}{4}}}$ $= 3^{-\frac{3}{4}}$	h	$\frac{1}{\sqrt[5]{16}}$ $= \frac{1}{(2^4)^{\frac{1}{5}}}$ $= \frac{1}{2^{\frac{4}{5}}}$ $= 2^{-\frac{4}{5}}$	i	$\frac{1}{\sqrt[3]{32}}$ $= \frac{1}{(2^5)^{\frac{1}{3}}}$ $= \frac{1}{2^{\frac{5}{3}}}$ $= 2^{-\frac{5}{3}}$	j	$\frac{1}{\sqrt[7]{49}}$ $= \frac{1}{(7^2)^{\frac{1}{7}}}$ $= \frac{1}{7^{\frac{2}{7}}}$ $= 7^{-\frac{2}{7}}$

4 a 2.280 **b** 1.834 **c** 0.794 **d** 0.435

5 a 3 **b** 1.682 **c** 1.933 **d** 0.523

6 a	$4^{\frac{3}{2}}$ $= (2^2)^{\frac{3}{2}}$ $= 2^3$ $= 8$	b	$8^{\frac{5}{3}}$ $= (2^3)^{\frac{5}{3}}$ $= 2^5$ $= 32$	c	$16^{\frac{3}{4}}$ $= (2^4)^{\frac{3}{4}}$ $= 2^3$ $= 8$	d	$25^{\frac{3}{2}}$ $= (5^2)^{\frac{3}{2}}$ $= 5^3$ $= 125$	e	$32^{\frac{2}{5}}$ $= (2^5)^{\frac{2}{5}}$ $= 2^2$ $= 4$
f	$4^{-\frac{1}{2}}$ $= (2^2)^{-\frac{1}{2}}$ $= 2^{-1}$ $= \frac{1}{2}$	g	$9^{-\frac{3}{2}}$ $= (3^2)^{-\frac{3}{2}}$ $= 3^{-3}$ $= \frac{1}{3^3}$ $= \frac{1}{27}$	h	$8^{-\frac{4}{3}}$ $= (2^3)^{-\frac{4}{3}}$ $= 2^{-4}$ $= \frac{1}{2^4}$ $= \frac{1}{16}$	i	$27^{-\frac{4}{3}}$ $= (3^3)^{-\frac{4}{3}}$ $= 3^{-4}$ $= \frac{1}{3^4}$ $= \frac{1}{81}$	j	$125^{-\frac{2}{3}}$ $= (5^3)^{-\frac{2}{3}}$ $= 5^{-2}$ $= \frac{1}{5^2}$ $= \frac{1}{25}$

EXERCISE 3E

1 a	$x^2(x^3 + 2x^2 + 1)$ $= x^5 + 2x^4 + x^2$	b	$2^x(2^x + 1)$ $= 2^{2x} + 2^x$	c	$x^{\frac{1}{2}}(x^{\frac{1}{2}} + x^{-\frac{1}{2}})$ $= x^1 + x^0$ $= x + 1$
d	$e^x(e^x + 2)$ $= e^{2x} + 2e^x$	e	$3^x(2 - 3^{-x})$ $= 2 \times 3^x - 3^0$ $= 2 \times 3^x - 1$	f	$x^{\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}})$ $= x^2 + 2x^1 + 3x^0$ $= x^2 + 2x + 3$

$$\begin{aligned}\mathbf{g} \quad & 2^{-x}(2^x + 5) \\ &= 2^0 + 5 \times 2^{-x} \\ &= 1 + 5 \times 2^{-x}\end{aligned}$$

$$\begin{aligned}\mathbf{h} \quad & 5^{-x}(5^{2x} + 5^x) \\ &= 5^{-x+2x} + 5^{-x+x} \\ &= 5^x + 5^0 \\ &= 5^x + 1\end{aligned}$$

$$\begin{aligned}\mathbf{i} \quad & x^{-\frac{1}{2}}(x^2 + x^1 + x^{\frac{1}{2}}) \\ &= x^{1\frac{1}{2}} + x^{\frac{1}{2}} + x^0 \\ &= x^{\frac{3}{2}} + x^{\frac{1}{2}} + 1\end{aligned}$$

$$\begin{aligned}\mathbf{2} \quad \mathbf{a} \quad & (2^x + 1)(2^x + 3) \\ &= 2^{2x} + 3 \times 2^x + 2^x + 3 \\ &= 2^{2x} + 4 \times 2^x + 3 \\ &= 4^x + 2^2 \times 2^x + 3 \\ &= 4^x + 2^{2+x} + 3\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & (3^x + 2)(3^x + 5) \\ &= 3^{2x} + 5 \times 3^x + 2 \times 3^x + 10 \\ &= 9^x + 7 \times 3^x + 10\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad & (5^x - 2)(5^x - 4) \\ &= 5^{2x} - 4 \times 5^x - 2 \times 5^x + 8 \\ &= 25^x - 6 \times 5^x + 8\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad & (2^x + 3)^2 \\ &= (2^x)^2 + 2 \times 2^x \times 3 + 3^2 \\ &= 2^{2x} + 3 \times 2^{1+x} + 9 \\ &= 4^x + 3 \times 2^{x+1} + 9\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad & (3^x - 1)^2 \\ &= (3^x)^2 - 2 \times 3^x + 1 \\ &= 3^{2x} - 2 \times 3^x + 1 \\ &= 9^x - 2 \times 3^x + 1\end{aligned}$$

$$\begin{aligned}\mathbf{f} \quad & (4^x + 7)^2 \\ &= (4^x)^2 + 2 \times 4^x \times 7 + 7^2 \\ &= 4^{2x} + 7 \times 2 \times 2^{2x} + 49 \\ &= 2^{4x} + 7 \times 2^{2x+1} + 49\end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad & (x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} - 2) \\ &= (x^{\frac{1}{2}})^2 - 2^2 \\ &= x - 4\end{aligned}$$

$$\begin{aligned}\mathbf{h} \quad & (2^x + 3)(2^x - 3) \\ &= (2^x)^2 - 3^2 \\ &= 2^{2x} - 9\end{aligned}$$

$$\begin{aligned}\mathbf{i} \quad & (x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \\ &= x^1 - x^{-1} \\ &= x - \frac{1}{x}\end{aligned}$$

$$\begin{aligned}\mathbf{j} \quad & \left(x + \frac{2}{x}\right)^2 \\ &= x^2 + 2 \times x \times \frac{2}{x} + \frac{4}{x^2} \\ &= x^2 + 4 + \frac{4}{x^2}\end{aligned}$$

$$\begin{aligned}\mathbf{k} \quad & (e^x - e^{-x})^2 \\ &= (e^x)^2 - 2e^x e^{-x} + e^{-2x} \\ &= e^{2x} - 2 + e^{-2x}\end{aligned}$$

$$\begin{aligned}\mathbf{l} \quad & (5 - 2^{-x})^2 \\ &= 5^2 - 2 \times 5 \times 2^{-x} + (2^{-x})^2 \\ &= 25 - 5 \times 2^{1-x} + 2^{-2x}\end{aligned}$$

EXERCISE 3F

$$\begin{aligned}\mathbf{1} \quad \mathbf{a} \quad & 2^x = 2 \\ \therefore 2^x &= 2^1 \\ \therefore x &= 1\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & 2^x = 4 \\ \therefore 2^x &= 2^2 \\ \therefore x &= 2\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad & 3^x = 27 \\ \therefore 3^x &= 3^3 \\ \therefore x &= 3\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad & 2^x = 1 \\ \therefore 2^x &= 2^0 \\ \therefore x &= 0\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad & 2^x = \frac{1}{2} \\ \therefore 2^x &= 2^{-1} \\ \therefore x &= -1\end{aligned}$$

$$\begin{aligned}\mathbf{f} \quad & 3^x = \frac{1}{3} \\ \therefore 3^x &= 3^{-1} \\ \therefore x &= -1\end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad & 2^x = \frac{1}{8} \\ \therefore 2^x &= 2^{-3} \\ \therefore x &= -3\end{aligned}$$

$$\begin{aligned}\mathbf{h} \quad & 2^{x+1} = 8 \\ \therefore 2^{x+1} &= 2^3 \\ \therefore x+1 &= 3 \\ \therefore x &= 2\end{aligned}$$

$$\begin{aligned}\mathbf{i} \quad & 2^{x-2} = \frac{1}{4} \\ \therefore 2^{x-2} &= 2^{-2} \\ \therefore x-2 &= -2 \\ \therefore x &= 0\end{aligned}$$

$$\begin{aligned}\mathbf{j} \quad & 3^{x+1} = \frac{1}{27} \\ \therefore 3^{x+1} &= 3^{-3} \\ \therefore x+1 &= -3 \\ \therefore x &= -4\end{aligned}$$

$$\begin{aligned}\mathbf{k} \quad & 2^{x+1} = 64 \\ \therefore 2^{x+1} &= 2^6 \\ \therefore x+1 &= 6 \\ \therefore x &= 5\end{aligned}$$

$$\begin{aligned}\mathbf{l} \quad & 2^{1-2x} = \frac{1}{2} \\ \therefore 2^{1-2x} &= 2^{-1} \\ \therefore 1-2x &= -1 \\ \therefore -2x &= -2 \\ \therefore x &= 1\end{aligned}$$

2 a $4^x = 32$

$$\therefore 2^{2x} = 2^5$$

$$\therefore 2x = 5$$

$$\therefore x = \frac{5}{2}$$

d $49^x = \frac{1}{7}$

$$\therefore 7^{2x} = 7^{-1}$$

$$\therefore 2x = -1$$

$$\therefore x = -\frac{1}{2}$$

g $8^{x+2} = 32$

$$\therefore 2^{3(x+2)} = 2^5$$

$$\therefore 3x + 6 = 5$$

$$\therefore 3x = -1$$

$$\therefore x = -\frac{1}{3}$$

j $9^{x-3} = 3$

$$\therefore 3^{2(x-3)} = 3^1$$

$$\therefore 2x - 6 = 1$$

$$\therefore 2x = 7$$

$$\therefore x = \frac{7}{2}$$

m $4^x = 8^{-x}$

$$\therefore 2^{2x} = (2^3)^{-x}$$

$$\therefore 2x = -3x$$

$$\therefore 5x = 0$$

$$\therefore x = 0$$

p $\left(\frac{1}{2}\right)^{x+1} = 32$

$$\therefore (2^{-1})^{x+1} = 2^5$$

$$\therefore -x - 1 = 5$$

$$\therefore -x = 6$$

$$\therefore x = -6$$

3 a $4^{2x+1} = 8^{1-x}$

$$\therefore (2^2)^{2x+1} = (2^3)^{1-x}$$

$$\therefore 4x + 2 = 3 - 3x$$

$$\therefore 7x = 1$$

$$\therefore x = \frac{1}{7}$$

b $8^x = \frac{1}{4}$

$$\therefore 2^{3x} = 2^{-2}$$

$$\therefore 3x = -2$$

$$\therefore x = -\frac{2}{3}$$

e $4^x = \frac{1}{8}$

$$\therefore 2^{2x} = 2^{-3}$$

$$\therefore 2x = -3$$

$$\therefore x = -\frac{3}{2}$$

h $8^{1-x} = \frac{1}{4}$

$$\therefore 2^{3(1-x)} = 2^{-2}$$

$$\therefore 3 - 3x = -2$$

$$\therefore -3x = -5$$

$$\therefore x = \frac{5}{3}$$

k $\left(\frac{1}{2}\right)^{x+1} = 2$

$$\therefore (2^{-1})^{x+1} = 2^1$$

$$\therefore -x - 1 = 1$$

$$\therefore -x = 2$$

$$\therefore x = -2$$

n $\left(\frac{1}{4}\right)^{1-x} = 8$

$$\therefore (2^{-2})^{1-x} = 2^3$$

$$\therefore -2 + 2x = 3$$

$$\therefore 2x = 5$$

$$\therefore x = \frac{5}{2}$$

c $9^x = \frac{1}{3}$

$$\therefore 3^{2x} = 3^{-1}$$

$$\therefore 2x = -1$$

$$\therefore x = -\frac{1}{2}$$

f $25^x = \frac{1}{5}$

$$\therefore 5^{2x} = 5^{-1}$$

$$\therefore 2x = -1$$

$$\therefore x = -\frac{1}{2}$$

i $4^{2x-1} = \frac{1}{2}$

$$\therefore 2^{2(2x-1)} = 2^{-1}$$

$$\therefore 4x - 2 = -1$$

$$\therefore 4x = 1$$

$$\therefore x = \frac{1}{4}$$

l $\left(\frac{1}{3}\right)^{x+2} = 9$

$$\therefore (3^{-1})^{x+2} = 3^2$$

$$\therefore -x - 2 = 2$$

$$\therefore -x = 4$$

$$\therefore x = -4$$

o $\left(\frac{1}{7}\right)^x = 49$

$$\therefore (7^{-1})^x = 7^2$$

$$\therefore -x = 2$$

$$\therefore x = -2$$

So, no solutions exist.

4 a $3 \times 2^x = 24$

$$\therefore 2^x = 8$$

$$\therefore 2^x = 2^3$$

$$\therefore x = 3$$

b $7 \times 2^x = 56$

$$\therefore 2^x = 8$$

$$\therefore 2^x = 2^3$$

$$\therefore x = 3$$

c $3 \times 2^{x+1} = 24$

$$\therefore 2^{x+1} = 8$$

$$\therefore 2^{x+1} = 2^3$$

$$\therefore x + 1 = 3$$

$$\therefore x = 2$$

$$\mathbf{d} \quad 12 \times 3^{-x} = \frac{4}{3}$$

$$\therefore 3^{-x} = \frac{4}{3} \div 12$$

$$\therefore 3^{-x} = \frac{4}{3} \times \frac{1}{12}$$

$$\therefore 3^{-x} = \frac{1}{9}$$

$$\therefore 3^{-x} = 3^{-2}$$

$$\therefore x = 2$$

$$\mathbf{e} \quad 4 \times \left(\frac{1}{3}\right)^x = 36$$

$$\therefore \left(\frac{1}{3}\right)^x = 9$$

$$\therefore (3^{-1})^x = 3^2$$

$$\therefore 3^{-x} = 3^2$$

$$\therefore -x = 2$$

$$\therefore x = -2$$

$$\mathbf{f} \quad 5 \times \left(\frac{1}{2}\right)^x = 20$$

$$\therefore \left(\frac{1}{2}\right)^x = 4$$

$$\therefore (2^{-1})^x = 2^2$$

$$\therefore -x = 2$$

$$\therefore x = -2$$

EXERCISE 3G

- 1 a** When $x = \frac{1}{2}$, $y = 2^{\frac{1}{2}}$
from point A, $y \div 1.4$

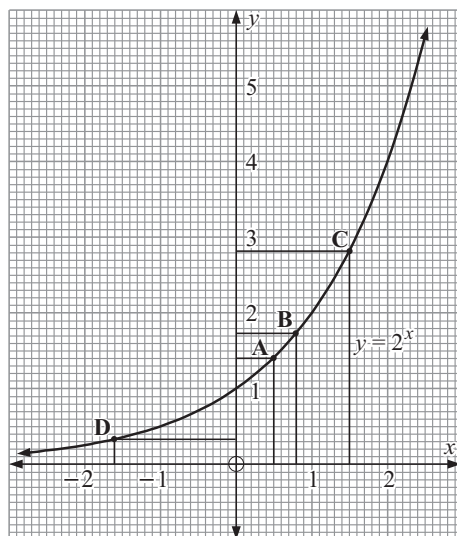
$$\therefore 2^{\frac{1}{2}} \div 1.4$$

- b** When $x = 0.8$, $y = 2^{0.8}$
from point B, $y \div 1.7$

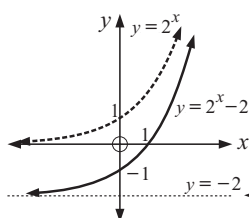
$$\therefore 2^{0.8} \div 1.7$$

- c** When $x = 1.5$, $y = 2^{1.5}$
from point C, $y \div 2.8$
 $\therefore 2^{1.5} \div 2.8$

- d** When $x = -1.6$, $y = 2^{-1.6}$
from point D, $y \div 0.3$
 $\therefore 2^{-1.6} \div 0.3$

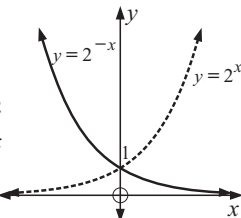


2 a



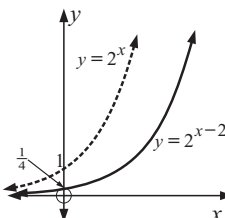
a vertical translation of 2 units downwards
 $y = -2$ is a HA

b



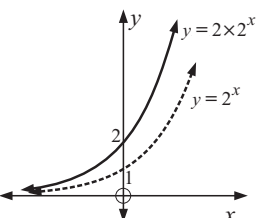
a reflection in the y -axis

c



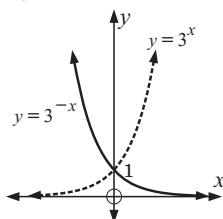
a vertical dilation of factor $\frac{1}{4}$

d



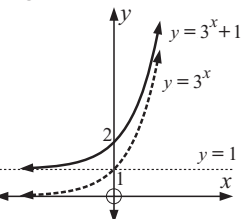
a vertical dilation of factor 2

4 a



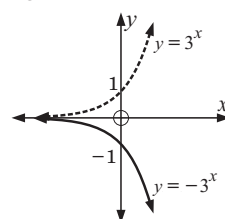
a reflection in the y -axis

b



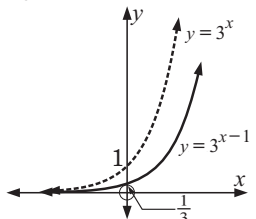
a vertical translation of 1 unit
 $y = 1$ is the HA

c

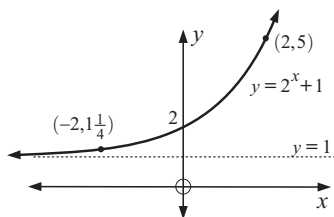


a reflection in the x -axis

d



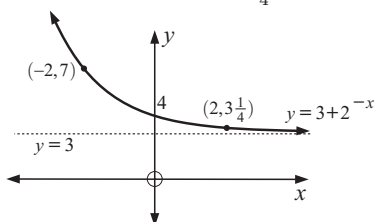
a vertical dilation of factor $\frac{1}{3}$

5 a

a vertical translation of 1 unit
Horizontal asymptote is $y = 1$.

$$\text{When } x = 2, \quad y = 4 + 1 = 5$$

$$\text{When } x = -2, \quad y = \frac{1}{4} + 1 = 1\frac{1}{4}$$

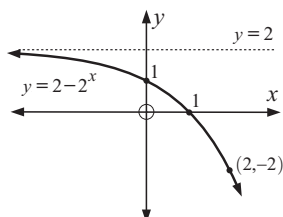
c

The HA is $y = 3$.

$$\text{When } x = 0, \quad y = 3 + 1 = 4$$

$$\text{When } x = 2, \quad y = 3 + \frac{1}{4} = 3\frac{1}{4}$$

$$\text{When } x = -2, \quad y = 3 + 2^2 = 7$$

b

The horizontal asymptote is $y = 2$.

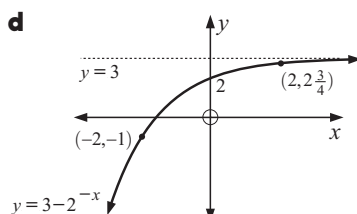
$$\text{When } x = 0, \quad y = 2 - 2^0 = 2 - 1 = 1$$

$\therefore$ y -intercept is 1

$$\text{When } x = 1, \quad y = 2 - 2 = 0$$

$$\text{When } x = 2, \quad y = 2 - 4 = -2$$

$$\text{When } x = -2, \quad y = 2 - \frac{1}{4} = 1\frac{3}{4}$$

d

The HA is $y = 3$.

$$\text{When } x = 0, \quad y = 3 - 1 = 2$$

$$\text{When } x = 2, \quad y = 3 - \frac{1}{4} = 2\frac{3}{4}$$

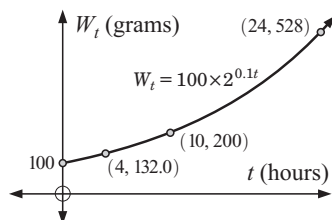
$$\text{When } x = -2, \quad y = 3 - 4 = -1$$

EXERCISE 3H**1 a** When $t = 0$, $W_0 = 100$ grams

$$\begin{aligned} \text{b i} \quad \text{When } t = 4, \\ W_4 &= 100 \times 2^{0.1 \times 4} \\ &= 100 \times 2^{0.4} \\ &\div 132 \text{ grams} \end{aligned}$$

$$\begin{aligned} \text{iii} \quad \text{When } t = 24, \\ W_{24} &= 100 \times 2^{0.1 \times 24} \\ &= 100 \times 2^{2.4} \\ &\div 528 \text{ grams} \end{aligned}$$

$$\begin{aligned} \text{ii} \quad \text{When } t = 10, \\ W_{10} &= 100 \times 2^1 \\ &= 200 \text{ grams} \end{aligned}$$

**2 a** $P_0 = 50$ (the initial population)

$$\begin{aligned} \text{b i} \quad \text{When } t = 2, \\ P_2 &= 50 \times 2^{0.3 \times 2} \\ &= 50 \times 2^{0.6} \\ &\div 75.785\dots \end{aligned}$$

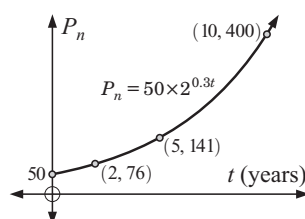
So, the expected population is 76 possums.

$$\begin{aligned} \text{iii} \quad \text{When } t = 10, \quad P_{10} &= 50 \times 2^{0.3 \times 10} \\ &= 50 \times 2^3 \\ &= 400 \end{aligned}$$

$$\begin{aligned} \text{ii} \quad \text{When } t = 5, \\ P_5 &= 50 \times 2^{0.3 \times 5} \\ &= 50 \times 2^{1.5} \\ &\div 141.421\dots \end{aligned}$$

So, the expected population is 141 possums.

So, the expected population is 400 possums.



- 3 a** When $t = 0$
 $V_0 = V_0 \times 2^0$
 $= V_0$
 So, the speed is V_0 .
- b** When $t = 20$
 $V_{20} = V_0 \times 2^{0.05 \times 20}$
 $= V_0 \times 2^1$
 $= 2V_0$
 So, the speed is $2V_0$.
- c** V_0 becomes $2V_0$
 a 100% increase.

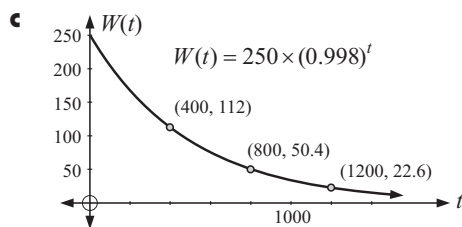
d $\left(\frac{V_{50} - V_{20}}{V_{20}} \right) \times 100\%$
 $= \left(\frac{V_0 \times 2^{2.5} - V_0 \times 2^1}{V_0 \times 2^1} \right) \times 100\%$
 $= \left(\frac{2^{2.5} - 2^1}{2^1} \right) \times 100\%$
 $\div 183\%$

This expression is the percentage increase in the speed from a speed at 20°C increased to the speed at 50°C .
 $V_{50} - V_{20}$ is the increase in speed.

- 4 a** $B_0 = 6$ pairs = 12 bears
- b** At year 2018, $t = 20$
 $\therefore B_{20} = 12 \times 2^{0.18 \times 20}$
 $= 12 \times 2^{3.6}$
 $\div 145.508\dots$
 $\div 146$ bears
- c** At year 2008, $t = 10$
 $\therefore \% \text{ increase} = \left(\frac{B_{20} - B_{10}}{B_{10}} \right) \times 100\%$
 $= \left(\frac{12 \times 2^{3.6} - 12 \times 2^{1.8}}{12 \times 2^{1.8}} \right) \times 100\%$
 $= \left(\frac{2^{3.6} - 2^{1.8}}{2^{1.8}} \right) \times 100\%$
 $\div 248\%$

EXERCISE 3I

- 1** $W_t = 250 \times (0.998)^t$ grams
- a** $W_0 = 250 \times (0.998)^0$
 $= 250 \times 1$
 $= 250$ grams
- b i** When $t = 400$
 W_{400}
 $= 250 \times (0.998)^{400}$
 $\div 112$ grams
- ii** When $t = 800$
 W_{800}
 $= 250 \times (0.998)^{800}$
 $\div 50.4$ grams
- iii** When $t = 1200$
 W_{1200}
 $= 250 \times (0.998)^{1200}$
 $\div 22.6$ grams



- d** When $W(t) = 125$
 $250 \times (0.998)^t = 125$
 $\therefore (0.998)^t = 0.5$
 $\therefore t \div 346.2$ {technology}
 i.e., it takes approximately 346 years

2 $R = 8000 \times (0.837)^t$

1 When $t = 0$, $R = 8000 \times (0.837)^0$
 $= 8000 \times 1$
 $= 8000$ rabbits

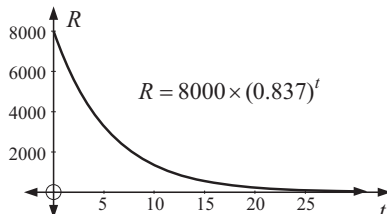
2 'to the power 3.5' means that
 $(0.837)^{3.5} = (0.837)^3 \times \sqrt{0.837}$

3 When $R = 80$,
 $8000 \times (0.837)^t = 80$
 $\therefore (0.837)^t = 0.01$
 and by trial-and-error methods, $t \doteq 26$
 i.e., it would take 26 weeks (approx)

4 Yes, after a very long time.

Notice that when $R = 1$
 $8000(0.837)^t = 1$
 $\therefore (0.837)^t = 0.000125$
 and by trial and error $t \doteq 51$
 i.e., after 51 weeks

5

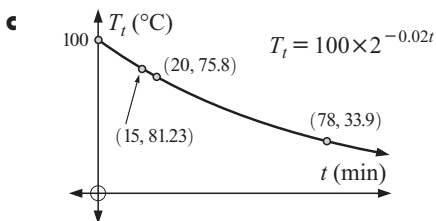


3 $T_t = 100 \times 2^{-0.02t}$

a $T_0 = 100 \times 2^0$
 $= 100 \times 1$
 $= 100^\circ\text{C}$

b i $T_{15} = 100 \times 2^{-0.02 \times 15}$
 $= 100 \times 2^{-0.3}$
 $\doteq 81.2^\circ\text{C}$

ii $T_{20} = 100 \times 2^{-0.02 \times 20}$
 $= 100 \times 2^{-0.4}$
 $\doteq 75.8^\circ\text{C}$

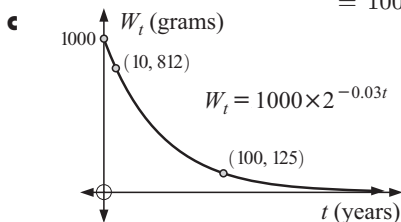


iii $T_{78} = 100 \times 2^{-0.02 \times 78}$
 $= 100 \times 2^{-1.56}$
 $\doteq 33.9^\circ\text{C}$

4 $W_t = 1000 \times 2^{-0.03t}$

a $W_0 = 1000 \times 2^0$
 $= 1000 \times 1$
 $= 1000$ grams

b i $W_{10} = 1000 \times 2^{-0.3}$
 $\doteq 812$ g



ii $W_{100} = 1000 \times 2^{-3}$
 $= 125$ g

iii $W_{1000} = 1000 \times 2^{-30}$
 $\doteq 9.31 \times 10^{-7}$ g

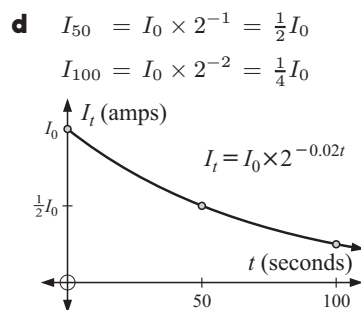
5 a When $t = 0$, $W_0 = W_0 2^0$
 $= W_0$
 $\therefore$ the original weight was W_0

b $\% \text{ loss} = \left(\frac{W_{1000} - W_0}{W_0} \right) \times 100\%$
 $= \left(\frac{W_0 \times 2^{-0.2} - W_0}{W_0} \right) \times 100\%$
 $= (2^{-0.2} - 1) \times 100\%$
 $\doteq -12.9\%$
 i.e., a 12.9% weight loss

6 a When $t = 0$, $I_0 = I_0 \times 2^0$
 $= I_0 \times 1$
 $= I_0$
 $\therefore$ the original current is I_0 amps

b When $t = 1$, $I_1 = I_0 \times 2^{-0.02}$
 $\doteq 0.9862 \times I_0$

$$\begin{aligned}
 \text{c } \% \text{ change} &= \left(\frac{I_1 - I_0}{I_0} \right) \times 100\% \\
 &\div \left(\frac{0.986I_0 - I_0}{I_0} \right) \times 100\% \\
 &\div (0.9862 - 1) \times 100\% \\
 &\div -0.0138 \times 100\% \\
 &\div 1.38\% \text{ loss}
 \end{aligned}$$



REVIEW SET 3A

1 a $-(-1)^{10}$
 $= -1 \quad \{(-1)^{10} = 1\}$

b $-(-3)^3$
 $= -[-27]$
 $= 27$

c $3^0 - 3^{-1}$
 $= 1 - \frac{1}{3}$
 $= \frac{2}{3}$

2 a $a^4b^5 \times a^2b^2$
 $= a^{4+2} \times b^{5+2}$
 $= a^6b^7$

b $6xy^5 \div 9x^2y^5$
 $= \frac{6}{9}x^{1-2}y^{5-5}$
 $= \frac{2}{3}x^{-1}y^0$
 $= \frac{2}{3x}$

c $\frac{5(x^2y)^2}{(5x^2)^2}$
 $= \frac{5 \times x^4y^2}{25x^4}$
 $= \frac{1}{5}x^0y^2 \text{ or } \frac{y^2}{5}$

3 a 2×2^{-4}
 $= 2^1 \times 2^{-4}$
 $= 2^{1+(-4)}$
 $= 2^{-3}$

b $16 \div 2^{-3}$
 $= 2^4 \div 2^{-3}$
 $= 2^{4-(-3)}$
 $= 2^7$

c 8^4
 $= (2^3)^4$
 $= 2^{12}$

4 a b^{-3}
 $= \frac{1}{b^3}$

b $(ab)^{-1}$
 $= \left(\frac{1}{ab} \right)$
 $= \frac{1}{ab}$

c ab^{-1}
 $= \frac{a}{1} \times \frac{1}{b}$
 $= \frac{a}{b}$

5 a $2^{x-3} = \frac{1}{32}$
 $\therefore 2^{x-3} = 2^{-5}$
 $\therefore x-3 = -5$
 $\therefore x = -2$

b $9^x = 27^{2-2x}$
 $\therefore (3^2)^x = (3^3)^{2-2x}$
 $\therefore 2x = 6 - 6x$
 $\therefore 8x = 6$
 $\therefore x = \frac{6}{8} = \frac{3}{4}$

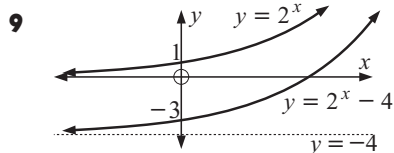
6 a $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = 4$

b $27^{-\frac{2}{3}} = (3^3)^{-\frac{2}{3}} = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

7 a 2.28 b 0.517 c 3.16

8 $f(x) = 3 \times 2^x$ a $f(0) = 3 \times 2^0$
 $= 3 \times 1$
 $= 3$

b $f(3) = 3 \times 2^3$ c $f(-2) = 3 \times 2^{-2}$
 $= 3 \times 8$
 $= 24$
 $= 3 \times \frac{1}{2^2}$
 $= \frac{3}{4}$



a $y = 2^x$ has y -intercept 1 and horizontal asymptote $y = 0$

b $y = 2^x - 4$ has y -intercept -3 and horizontal asymptote $y = -4$

10 $T = 80 \times (0.913)^t \text{ } ^\circ\text{C}$

a When $t = 0$, $T = 80 \times (0.913)^0$
 $= 80 \times 1$
 $= 80 \therefore$ initial temperature is 80°C

b i When $t = 12$,

$$T = 80 \times (0.913)^{12}$$

$$\div 26.8^\circ\text{C}$$

ii When $t = 24$,

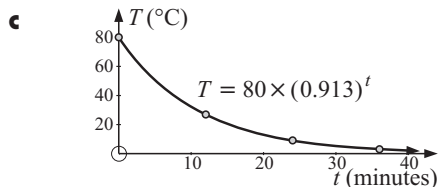
$$T = 80 \times (0.913)^{24}$$

$$\div 9.00^\circ\text{C}$$

iii When $t = 36$,

$$T = 80 \times (0.913)^{36}$$

$$\div 3.02^\circ\text{C}$$



d When $T = 25$

$$80 \times (0.913)^t = 25$$

$$\therefore 0.913^t = 0.3125$$

$$\therefore t \div 12.8 \text{ sec } \{\text{technology}\}$$

REVIEW SET 3B

1 a $-(-2)^3$
 $= -[-8]$
 $= 8$

b $5^{-1} - 5^0$
 $= \frac{1}{5} - 1$
 $= -\frac{4}{5}$

2 a $(a^7)^3$
 $= a^{7 \times 3}$
 $= a^{21}$

b $pq^2 \times p^3q^4$
 $= p^{1+3}q^{2+4}$
 $= p^4q^6$

c $\frac{8ab^5}{2a^4b^4}$
 $= \frac{8}{2}a^{1-4}b^{5-4}$
 $= 4a^{-3}b^1$
 $= \frac{4}{1} \times \frac{1}{a^3} \times b$
 $= \frac{4b}{a^3}$

3 a $\frac{1}{16}$
 $= \frac{1}{2^4}$
 $= 2^{-4}$

b $2^x \times 4$
 $= 2^x \times 2^2$
 $= 2^{x+2}$

c $4^x \div 8$
 $= (2^2)^x \div 2^3$
 $= 2^{2x} \div 2^3$
 $= 2^{2x-3}$

4 a $x^{-2} \times x^{-3}$
 $= x^{-2+(-3)}$
 $= x^{-5}$
 $= \frac{1}{x^5}$

b $2(ab)^{-2}$
 $= 2 \times \frac{1}{(ab)^2}$
 $= \frac{2}{a^2b^2}$

c $2ab^{-2}$
 $= 2a \times \left(\frac{1}{b^2}\right)$
 $= \frac{2a}{b^2}$

5 a $2^{x+1} = 32$
 $\therefore 2^{x+1} = 2^5$
 $\therefore x+1 = 5$
 $\therefore x = 4$

b $4^{x+1} = \left(\frac{1}{8}\right)^x$
 $\therefore (2^2)^{x+1} = (2^{-3})^x$
 $\therefore 2x+2 = -3x$
 $\therefore 5x = -2$
 $\therefore x = -\frac{2}{5}$

6 a $81 = 3^4$

b $1 = 3^0$

c $\frac{1}{27} = \frac{1}{3^3} = 3^{-3}$

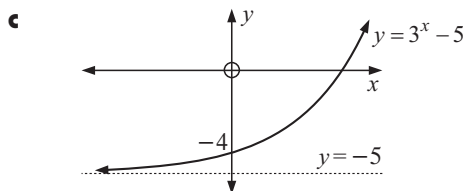
d $\frac{1}{243} = \frac{1}{3^5} = 3^{-5}$

$$\begin{aligned} 7 \quad a \quad \frac{27}{9^a} &= \frac{3^3}{(3^2)^a} \\ &= 3^{3-2a} \end{aligned}$$

$$\begin{aligned} b \quad (\sqrt{3})^{1-x} \times 9^{1-2x} &= (3^{\frac{1}{2}})^{1-x} \times (3^2)^{1-2x} \\ &= 3^{\frac{1}{2}-\frac{1}{2}x+2-4x} \\ &= 3^{\frac{5}{2}-\frac{9}{2}x} \end{aligned}$$

$$\begin{aligned} 8 \quad a \quad \text{When } x=0, \quad y &= 3^0 - 5 = 1 - 5 = -4 \\ \text{When } x=1, \quad y &= 3^1 - 5 = 3 - 5 = -2 \\ \text{When } x=2, \quad y &= 3^2 - 5 = 9 - 5 = 4 \\ \text{When } x=-1, \quad y &= 3^{-1} - 5 = \frac{1}{3} - 5 = -4\frac{2}{3} \\ \text{When } x=-2, \quad y &= 3^{-2} - 5 = \frac{1}{9} - 5 = -4\frac{8}{9} \end{aligned}$$

$$\begin{aligned} b \quad \text{as } x \rightarrow \infty, \quad 3^x &\rightarrow \infty, \quad \therefore y \rightarrow \infty \\ \text{as } x \rightarrow -\infty, \quad 3^x &\rightarrow 0, \quad \therefore y \rightarrow -5 \quad (\text{from above}) \end{aligned}$$



d $y = -5$ is the horizontal asymptote

$$\begin{aligned} 9 \quad a \quad 27^x &= 3 \\ \therefore (3^3)^x &= 3^1 \\ \therefore 3x &= 1 \\ \therefore x &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} b \quad 9^{1-x} &= 27^{x+2} \\ \therefore (3^2)^{1-x} &= (3^3)^{x+2} \\ \therefore 2-2x &= 3x+6 \\ \therefore -5x &= 4 \quad \text{and so } x = -\frac{4}{5} \end{aligned}$$

$$\begin{aligned} 10 \quad 4^x \times 2^y &= 16 \\ \therefore (2^2)^x \times 2^y &= 2^4 \\ \therefore 2x + y &= 4 \quad \dots (1) \end{aligned}$$

$$\begin{aligned} 8^x &= 2^{\frac{y}{2}} \\ \therefore (2^3)^x &= 2^{\frac{y}{2}} \\ \therefore 3x &= \frac{y}{2} \quad \dots (2) \end{aligned}$$

$$\begin{aligned} \text{From (2), } y &= 6x \\ \therefore \text{in (1), } 2x + 6x &= 4 \\ \therefore 8x &= 4 \\ \therefore x &= \frac{1}{2} \\ \text{and so } y &= 6 \times \frac{1}{2} = 3 \end{aligned}$$

REVIEW SET 3C

$$\begin{aligned} 1 \quad a \quad 4 \times 2^n &= 2^2 \times 2^n \\ &= 2^{2+n} \end{aligned}$$

$$\begin{aligned} b \quad 7^{-1} - 7^0 &= \frac{1}{7} - 1 \\ &= -\frac{6}{7} \end{aligned}$$

$$\begin{aligned} c \quad \left(\frac{2}{3}\right)^{-3} &= \left(\frac{3}{2}\right)^3 \\ &= \frac{27}{8} \\ &= 3\frac{3}{8} \end{aligned}$$

$$\begin{aligned} d \quad \left(\frac{2a^{-1}}{b^2}\right)^2 &= \frac{2^2 a^{-2}}{b^4} \\ &= \frac{4}{a^2 b^4} \end{aligned}$$

$$\begin{array}{r|l} 2 & 288 \\ 2 & 144 \\ 2 & 72 \\ 2 & 36 \\ 2 & 18 \\ 2 & 9 \\ 3 & 3 \end{array}$$

$$\therefore 288 = 2^5 \times 3^2$$

$$\begin{aligned} b \quad \frac{2^{x+1}}{2^{1-x}} &= 2^{x+1-(1-x)} \\ &= 2^{x+1-1+x} \\ &= 2^{2x} \end{aligned}$$

$$3 \quad a \quad 1 = 5^0$$

$$\begin{aligned} b \quad 5\sqrt{5} &= 5^1 \times 5^{\frac{1}{2}} \\ &= 5^{1\frac{1}{2}} \\ &= 5^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} c \quad \frac{1}{\sqrt[4]{5}} &= \frac{1}{5^{\frac{1}{4}}} \\ &= 5^{-\frac{1}{4}} \end{aligned}$$

$$\begin{aligned} d \quad 25^{a+3} &= (5^2)^{a+3} \\ &= 5^{2a+6} \end{aligned}$$

$$\begin{aligned}
 4 \quad a \quad & -(-2)^2 \\
 & = -[4] \\
 & = -4
 \end{aligned}$$

$$\begin{aligned}
 b \quad & \left(-\frac{1}{2}a^{-3}\right)^2 \\
 & = \left(-\frac{1}{2}\right)^2 a^{-6} \\
 & = \frac{1}{4}a^{-6} \\
 & = \frac{1}{4} \times \frac{1}{a^6} \\
 & = \frac{1}{4a^6}
 \end{aligned}$$

$$\begin{aligned}
 c \quad & (-3b^{-1})^{-3} \\
 & = (-3)^{-3}b^3 \\
 & = \frac{1}{(-3)^3}b^3 \\
 & = \frac{1}{-27}b^3 \\
 & = -\frac{b^3}{27}
 \end{aligned}$$

$$\begin{aligned}
 5 \quad a \quad & e^x(e^{-x} + e^x) \\
 & = e^0 + e^{2x} \\
 & = 1 + e^{2x}
 \end{aligned}$$

$$\begin{aligned}
 b \quad & (2^x + 5)^2 \\
 & = (2^x)^2 + 2 \times 2^x \times 5 + 5^2 \\
 & = 2^{2x} + 5 \times 2^{x+1} + 25 \\
 & = 4^x + 5 \times 2^{x+1} + 25 \\
 & \text{(or } 2^{2x} + 10 \times 2^x + 25)
 \end{aligned}$$

$$\begin{aligned}
 c \quad & (x^{\frac{1}{2}} - 7)(x^{\frac{1}{2}} + 7) \\
 & = (x^{\frac{1}{2}})^2 - 7^2 \\
 & = x^1 - 49 \\
 & = x - 49
 \end{aligned}$$

$$\begin{aligned}
 6 \quad a \quad & (3 - 2^a)^2 \\
 & = 3^2 - 2 \times 3 \times 2^a + (2^a)^2 \\
 & = 9 - 3 \times 2^{a+1} + 2^{2a} \\
 & \text{(or } 9 - 6 \times 2^a + 2^{2a})
 \end{aligned}$$

$$\begin{aligned}
 b \quad & (\sqrt{x} + 2)(\sqrt{x} - 2) \\
 & = (\sqrt{x})^2 - 2^2 \\
 & = x - 4
 \end{aligned}$$

$$\begin{aligned}
 c \quad & 2^{-x}(2^{2x} + 2^x) \\
 & = 2^{-x+2x} + 2^{-x+x} \\
 & = 2^x + 2^0 \\
 & = 2^x + 1
 \end{aligned}$$

$$\begin{aligned}
 7 \quad a \quad & 6 \times 2^x = 192 \\
 & \therefore 2^x = 32 \\
 & \therefore 2^x = 2^5 \\
 & \therefore x = 5
 \end{aligned}$$

$$\begin{aligned}
 b \quad & 4 \times \left(\frac{1}{3}\right)^x = 324 \\
 & \therefore \left(\frac{1}{3}\right)^x = 81 \\
 & \therefore (3^{-1})^x = 3^4 \\
 & \therefore 3^{-x} = 3^4 \\
 & \therefore x = -4
 \end{aligned}$$

$$8 \quad W = 1500 \times (0.993)^t \text{ grams}$$

a When $t = 0$,

$$\begin{aligned}
 W &= 1500 \times (0.993)^0 \\
 &= 1500 \times 1 \\
 &= 1500 \text{ grams}
 \end{aligned}$$

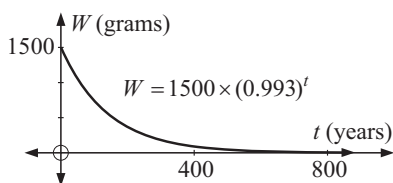
b i When $t = 400$,

$$\begin{aligned}
 W &= 1500 \times (0.993)^{400} \\
 &\doteq 90.3 \text{ grams}
 \end{aligned}$$

ii When $t = 800$,

$$\begin{aligned}
 W &= 1500 \times (0.993)^{800} \\
 &\doteq 5.4 \text{ grams}
 \end{aligned}$$

c



d When $W = 100$,

$$\begin{aligned}
 1500 \times (0.993)^t &= 100 \\
 \therefore (0.993)^t &\doteq 0.0667
 \end{aligned}$$

By trial and error

$$\begin{aligned}
 (0.993)^{300} &\doteq 0.1216 \\
 (0.993)^{350} &\doteq 0.08556 \\
 (0.993)^{380} &\doteq 0.06930 \\
 (0.993)^{385} &\doteq 0.06691 \\
 (0.993)^{386} &\doteq 0.06644
 \end{aligned}$$

i.e., about 386 years

Chapter 4

LOGARITHMS

EXERCISE 4A

- 1** **a** $10^4 = 10\,000$ **b** $10^{-1} = 0.1$ **c** $10^{\frac{1}{2}} = \sqrt{10}$ **d** $2^3 = 8$ **e** $2^{-2} = \frac{1}{4}$
f $3^{1.5} = \sqrt{27}$
- 2** **a** $\log_2 4 = 2$ **b** $\log_2(\frac{1}{8}) = -3$ **c** $\log_{10}(0.01) = -2$ **d** $\log_7 49 = 2$ **e** $\log_2 64 = 6$
f $\log_3(\frac{1}{27}) = -3$
- 3** **a** As $10^5 = 100\,000$
then $\log_{10}(100\,000) = 5$ **b** As $10^{-2} = 0.01$
then $\log_{10}(0.01) = -2$ **c** As $3^{\frac{1}{2}} = \sqrt{3}$
then $\log_3(\sqrt{3}) = \frac{1}{2}$
- d** As $2^3 = 8$
then $\log_2 8 = 3$ **e** As $2^6 = 64$
then $\log_2 64 = 6$ **f** As $2^7 = 128$
then $\log_2 128 = 7$
- g** As $5^2 = 25$
then $\log_5 25 = 2$ **h** As $5^3 = 125$
then $\log_5 125 = 3$ **i** As $2^{-3} = \frac{1}{8} = 0.125$
then $\log_2(0.125) = -3$
- j** As $9^{\frac{1}{2}} = 3$
then $\log_9 3 = \frac{1}{2}$ **k** As $4^2 = 16$
then $\log_4 16 = 2$ **l** As $36^{\frac{1}{2}} = \sqrt{36} = 6$
then $\log_{36} 6 = \frac{1}{2}$
- m** As $243 = 3^5$
then $\log_3 243 = 5$ **n** As $\sqrt[3]{2} = 2^{\frac{1}{3}}$
then $\log_2 \sqrt[3]{2} = \frac{1}{3}$ **o** As $a^n = a^n$
then $\log_n a^n = n$
- p** As $2 = 8^{\frac{1}{3}}$
then $\log_8 2 = \frac{1}{3}$ **q** As $\frac{1}{t} = t^{-1}$
then $\log_t \left(\frac{1}{t}\right) = -1$ **r** As $6\sqrt{6} = 6^1 \times 6^{\frac{1}{2}} = 6^{1\frac{1}{2}}$
then $\log_6(6\sqrt{6}) = 1\frac{1}{2}$
- s** As $1 = 4^0$
then $\log_4 1 = 0$ **t** As $9 = 9^1$
then $\log_9 9 = 1$
- 4** **a** $\div 2.18$ **b** $\div 1.40$ **c** $\div 1.87$ **d** $\div -0.0969$
- 5** **a** $\log_2 x = 3$
 $\therefore x = 2^3$
 $\therefore x = 8$ **b** $\log_4 x = \frac{1}{2}$
 $\therefore x = 4^{\frac{1}{2}}$
 $\therefore x = 2$ **c** $\log_x 81 = 4$
 $\therefore 81 = x^4$
 $\therefore x = \pm \sqrt[4]{81}$
 $\therefore x = \pm 3$
but $x > 0$
 $\therefore x = 3$ **d** $\log_2(x-6) = 3$
 $\therefore x-6 = 2^3$
 $\therefore x-6 = 8$
 $\therefore x = 14$
- 6** **a** $\log_2 4$
 $= \log_2 2^2$
 $= 2$ **b** $\log_3(\frac{1}{3})$
 $= \log_3 3^{-1}$
 $= -1$ **c** $\log_5(25\sqrt{5})$
 $= \log_5 \left(5^2 \times 5^{\frac{1}{2}}\right)$
 $= \log_5 5^{\frac{5}{2}}$
 $= \frac{5}{2}$ **d** $\log_3 \left(\frac{1}{\sqrt{3}}\right)$
 $= \log_3 \left(\frac{1}{3^{\frac{1}{2}}}\right)$
 $= \log_3 3^{-\frac{1}{2}}$
 $= -\frac{1}{2}$

EXERCISE 4B

- 1**
- a** $\log 10\,000$
 $= \log_{10} 10^4$
 $= 4$
- b** $\log 0.001$
 $= \log_{10} 10^{-3}$
 $= -3$
- c** $\log 10$
 $= \log_{10} 10^1$
 $= 1$
- d** $\log 1$
 $= \log_{10} 10^0$
 $= 0$
- e** $\log \sqrt{10}$
 $= \log_{10} 10^{\frac{1}{2}}$
 $= \frac{1}{2}$
- f** $\log \sqrt[3]{10}$
 $= \log_{10} 10^{\frac{1}{3}}$
 $= \frac{1}{3}$
- g** $\log \left(\frac{1}{\sqrt[4]{10}} \right)$
 $= \log_{10} 10^{-\frac{1}{4}}$
 $= -\frac{1}{4}$
- h** $\log 10\sqrt{10}$
 $= \log_{10} 10^{\frac{3}{2}}$
 $= \frac{3}{2}$
- i** $\log \sqrt[3]{100}$
 $= \log_{10} (10^2)^{\frac{1}{3}}$
 $= \log_{10} 10^{\frac{2}{3}}$
 $= \frac{2}{3}$
- j** $\log \left(\frac{100}{\sqrt{10}} \right)$
 $= \log_{10} \left(\frac{10^2}{10^{\frac{1}{2}}} \right)$
 $= \log_{10} 10^{\frac{3}{2}}$
 $= \frac{3}{2}$
- k** $\log (10 \times \sqrt[3]{10})$
 $= \log_{10} (10^1 \times 10^{\frac{1}{3}})$
 $= \log_{10} 10^{\frac{4}{3}}$
 $= \frac{4}{3}$
- l** $\log 1000\sqrt{10}$
 $= \log_{10} (10^3 \times 10^{\frac{1}{2}})$
 $= \log_{10} 10^{\frac{7}{2}}$
 $= \frac{7}{2}$
- m** $\log 10^n$
 $= \log_{10} 10^n$
 $= n$
- n** $\log (10^a \times 100)$
 $= \log_{10} (10^a \times 10^2)$
 $= \log_{10} (10^{a+2})$
 $= a + 2$
- o** $\log \left(\frac{10}{10^m} \right)$
 $= \log_{10} (10^{1-m})$
 $= 1 - m$
- p** $\log \left(\frac{10^a}{10^b} \right)$
 $= \log_{10} (10^{a-b})$
 $= a - b$

- 2** Use:
- a** $\boxed{\log} \ 10\,000 \ \boxed{\text{ENTER}}$
- b** $\boxed{\log} \ 0.001 \ \boxed{\text{ENTER}}$
- c** $\boxed{\log} \ \boxed{2\text{nd}} \ \boxed{\sqrt{}} \ 10 \ \boxed{)} \ \boxed{)} \ \boxed{\text{ENTER}}$
- d** $\boxed{\log} \ 10 \ \boxed{\wedge} \ \boxed{(} \ 1 \ \boxed{\div} \ 3 \ \boxed{)} \ \boxed{)} \ \boxed{\text{ENTER}}$
- e** $\boxed{\log} \ 100 \ \boxed{\wedge} \ \boxed{(} \ 1 \ \boxed{\div} \ 3 \ \boxed{)} \ \boxed{)} \ \boxed{\text{ENTER}}$
- f** $\boxed{\log} \ 10 \ \boxed{\times} \ \boxed{2\text{nd}} \ \boxed{\sqrt{}} \ 10 \ \boxed{)} \ \boxed{)} \ \boxed{\text{ENTER}}$
- g** $\boxed{\log} \ 1 \ \boxed{\div} \ \boxed{2\text{nd}} \ \boxed{\sqrt{}} \ 10 \ \boxed{)} \ \boxed{)} \ \boxed{\text{ENTER}}$
- h** $\boxed{\log} \ 1 \ \boxed{\div} \ 10 \ \boxed{\wedge} \ 0.25 \ \boxed{)} \ \boxed{\text{ENTER}}$

- 3**
- a** 6
 $= 10^{\log 6}$
 $\div 10^{0.7782}$
- b** 60
 $= 10^{\log 60}$
 $\div 10^{1.7782}$
- c** 6000
 $= 10^{\log 6000}$
 $\div 10^{3.7782}$
- d** 0.6
 $= 10^{\log(0.6)}$
 $= 10^{-0.2218}$
- e** 0.006
 $= 10^{\log(0.006)}$
 $= 10^{-2.2218}$
- f** 15
 $= 10^{\log 15}$
 $\div 10^{1.1761}$
- g** 1500
 $= 10^{\log 1500}$
 $= 10^{3.1761}$
- h** 1.5
 $= 10^{\log 1.5}$
 $= 10^{0.1761}$
- i** 0.15
 $= 10^{\log(0.15)}$
 $= 10^{-0.8239}$
- j** $0.000\,15$
 $= 10^{\log(0.000\,15)}$
 $= 10^{-3.8239}$

- 4**
- a i** $\log 3$
 $\div 0.477$
- ii** $\log 300$
 $\div 2.477$
- b** $300 = 3 \times 10^2$
 $= 10^{\log 3} \times 10^2$
 $= 10^{\log 3 + 2} \quad \therefore \log 300 = \log 3 + 2$

- 5**
- a i** $\log 5$
 $\div 0.699$
- ii** $\log 0.05$
 $= -1.301$
- b** $0.05 = 5 \times 10^{-2}$
 $= 10^{\log 5} \times 10^{-2}$
 $= 10^{\log 5 - 2}$
 $\therefore \log(0.05) = \log 5 - 2$

6 a $\log x = 2$ $\therefore x = 10^2$ $\therefore x = 100$	b $\log x = 1$ $\therefore x = 10^1$ $\therefore x = 10$	c $\log x = 0$ $\therefore x = 10^0$ $\therefore x = 1$	d $\log x = -1$ $\therefore x = 10^{-1}$ $\therefore x = \frac{1}{10}$
e $\log x = \frac{1}{2}$ $\therefore x = 10^{\frac{1}{2}}$ $\therefore x = \sqrt{10}$	f $\log x = -\frac{1}{2}$ $\therefore x = 10^{-\frac{1}{2}}$ $\therefore x = \frac{1}{10^{\frac{1}{2}}}$ $\therefore x = \frac{1}{\sqrt{10}}$	g $\log x \div 0.8351$ $\therefore x \div 10^{0.8351}$ $\therefore x \div 6.84$	h $\log x \div -3.1997$ $\therefore x \div 10^{-3.1997}$ $\therefore x \div 0.000\,631$

EXERCISE 4C

1 a $\log 8 + \log 2$ $= \log(8 \times 2)$ $= \log 16$	b $\log 8 - \log 2$ $= \log\left(\frac{8}{2}\right)$ $= \log 4$	c $\log 40 - \log 5$ $= \log\left(\frac{40}{5}\right)$ $= \log 8$
d $\log 4 + \log 5$ $= \log(4 \times 5)$ $= \log 20$	e $\log 5 + \log(0.4)$ $= \log(5 \times 0.4)$ $= \log 2$	f $\log 2 + \log 3 + \log 4$ $= \log(2 \times 3 \times 4)$ $= \log 24$
g $1 + \log 3$ $= \log 10^1 + \log 3$ $= \log(10 \times 3)$ $= \log 30$	h $\log 4 - 1$ $= \log 4 - \log 10^1$ $= \log\left(\frac{4}{10}\right)$ $= \log(0.4)$	i $\log 5 + \log 4 - \log 2$ $= \log\left(\frac{5 \times 4}{2}\right)$ $= \log 10$
j $2 + \log 2$ $= \log 10^2 + \log 2$ $= \log(100 \times 2)$ $= \log 200$	k $\log 40 - 2$ $= \log 40 - \log 10^2$ $= \log\left(\frac{40}{100}\right)$ $= \log(0.4)$	l $\log 6 - \log 2 - \log 3$ $= \log(6 \div 2 \div 3)$ $= \log 1$
m $\log 50 - 4$ $= \log 50 - \log 10^4$ $= \log\left(\frac{50}{10^4}\right)$ $= \log(0.005)$	n $3 - \log 50$ $= \log 10^3 - \log 50$ $= \log\left(\frac{1000}{50}\right)$ $= \log 20$	o $\log\left(\frac{4}{3}\right) + \log 3 + \log 7$ $= \log\left(\frac{4}{3} \times 3 \times 7\right)$ $= \log 28$
2 a $5 \log 2 + \log 3$ $= \log 2^5 + \log 3$ $= \log(2^5 \times 3)$ $= \log 96$	b $2 \log 3 + 3 \log 2$ $= \log 3^2 + \log 2^3$ $= \log(9 \times 8)$ $= \log 72$	c $3 \log 4 - \log 8$ $= \log 4^3 - \log 8$ $= \log\left(\frac{64}{8}\right)$ $= \log 8$
d $2 \log 5 - 3 \log 2$ $= \log 5^2 - \log 2^3$ $= \log\left(\frac{25}{8}\right)$	e $\frac{1}{2} \log 4 + \log 3$ $= \log 4^{\frac{1}{2}} + \log 3$ $= \log(2 \times 3)$ $= \log 6$	f $\frac{1}{3} \log\left(\frac{1}{8}\right)$ $= \log\left(\frac{1}{8}\right)^{\frac{1}{3}}$ $= \log(2^{-3})^{\frac{1}{3}}$ $= \log 2^{-1}$ $= \log\left(\frac{1}{2}\right) \text{ or } -\log 2$
g $3 - \log 2 - 2 \log 5$ $= \log 10^3 - \log 2 - \log 5^2$ $= \log(1000 \div 2 \div 25)$ $= \log 20$	h $1 - 3 \log 2 + \log 20$ $= \log 10^1 - \log 2^3 + \log 20$ $= \log(10 \div 8 \times 20)$ $= \log 25$	i $2 - \frac{1}{2} \log 4 - \log 5$ $= \log 10^2 - \log 4^{\frac{1}{2}} - \log 5$ $= \log(100 \div 2 \div 5)$ $= \log 10$ $= 1$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & \frac{\log 4}{\log 2} \\
 &= \frac{\log 2^2}{\log 2} \\
 &= \frac{2 \log 2}{\log 2} \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{\log 27}{\log 9} \\
 &= \frac{\log 3^3}{\log 3^2} \\
 &= \frac{3 \log 3}{2 \log 3} \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{\log 8}{\log 2} \\
 &= \frac{\log 2^3}{\log 2} \\
 &= \frac{3 \log 2}{\log 2} \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \frac{\log 3}{\log 9} \\
 &= \frac{\log 3}{\log 3^2} \\
 &= \frac{\log 3}{2 \log 3} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \frac{\log 25}{\log(0.2)} \\
 &= \frac{\log 5^2}{\log 5^{-1}} \\
 &= \frac{2 \log 5}{-1 \log 5} \\
 &= -2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \frac{\log 8}{\log(0.25)} \\
 &= \frac{\log 2^3}{\log 2^{-2}} \quad \{0.25 = \frac{1}{4} = \frac{1}{2^2}\} \\
 &= \frac{3 \log 2}{-2 \log 2} \\
 &= -\frac{3}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & \log 9 = \log 3^2 \\
 &= 2 \log 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \log \sqrt{2} = \log 2^{\frac{1}{2}} \\
 &= \frac{1}{2} \log 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \log \left(\frac{1}{8}\right) = \log \left(\frac{1}{2^3}\right) \\
 &= \log 2^{-3} \\
 &= -3 \log 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \log \left(\frac{1}{5}\right) = \log 5^{-1} \\
 &= -1 \log 5 \\
 &= -\log 5
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \log 5 = \log \left(\frac{10}{2}\right) \\
 &= \log 10^1 - \log 2 \\
 &= 1 - \log 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \log 5000 = \log \left(\frac{10\,000}{2}\right) \\
 &= \log 10^4 - \log 2 \\
 &= 4 - \log 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad & y = 2^x \\
 \therefore \log y &= \log 2^x \\
 \therefore \log y &= x \log 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & y = 20b^3 \\
 \therefore \log y &= \log(20b^3) \\
 \therefore \log y &= \log 20 + \log b^3 \\
 \therefore \log y &= \log 20 + 3 \log b
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & M = ad^4 \\
 \therefore \log M &= \log(ad^4) \\
 \therefore \log M &= \log a + \log d^4 \\
 \therefore \log M &= \log a + 4 \log d
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & T = 5\sqrt{d} = 5d^{\frac{1}{2}} \\
 \therefore \log T &= \log(5d^{\frac{1}{2}}) \\
 \therefore \log T &= \log 5 + \log d^{\frac{1}{2}} \\
 \therefore \log T &= \log 5 + \frac{1}{2} \log d
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & R = b\sqrt{l} = bl^{\frac{1}{2}} \\
 \therefore \log R &= \log(bl^{\frac{1}{2}}) \\
 \therefore \log R &= \log b + \log l^{\frac{1}{2}} \\
 \therefore \log R &= \log b + \frac{1}{2} \log l
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & Q = \frac{a}{b^n} \\
 \therefore \log Q &= \log \left(\frac{a}{b^n}\right) \\
 \therefore \log Q &= \log a - \log b^n \\
 \therefore \log Q &= \log a - n \log b
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & y = ab^x \\
 \therefore \log y &= \log(ab^x) \\
 \therefore \log y &= \log a + \log b^x \\
 \therefore \log y &= \log a + x \log b
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & F = \frac{20}{\sqrt{n}} = \frac{20}{n^{\frac{1}{2}}} \\
 \therefore \log F &= \log \left(\frac{20}{n^{\frac{1}{2}}}\right) \\
 \therefore \log F &= \log 20 - \log n^{\frac{1}{2}} \\
 \therefore \log F &= \log 20 - \frac{1}{2} \log n
 \end{aligned}$$

$$\mathbf{i} \quad L = \frac{ab}{c}$$

$$\therefore \log L = \log \left(\frac{ab}{c} \right)$$

$$\therefore \log L = \log ab - \log c$$

$$\therefore \log L = \log a + \log b - \log c$$

$$\mathbf{j} \quad N = \sqrt{\frac{a}{b}} = \left(\frac{a}{b} \right)^{\frac{1}{2}}$$

$$\therefore \log N = \log \left(\frac{a}{b} \right)^{\frac{1}{2}}$$

$$\therefore \log N = \frac{1}{2} \log \left(\frac{a}{b} \right)$$

$$\therefore \log N = \frac{1}{2} \{ \log a - \log b \}$$

$$\mathbf{k} \quad S = 200 \times 2^t$$

$$\therefore \log S = \log(200 \times 2^t)$$

$$\therefore \log S = \log 200 + \log 2^t$$

$$\therefore \log S = \log 200 + t \log 2$$

$$\mathbf{l} \quad y = \frac{a^m}{b^n}$$

$$\therefore \log y = \log \left(\frac{a^m}{b^n} \right)$$

$$\therefore \log y = \log a^m - \log b^n$$

$$\therefore \log y = m \log a - n \log b$$

$$\mathbf{6} \quad \mathbf{a} \quad \log D = \log e + \log 2$$

$$= \log(e \times 2)$$

$$\therefore D = 2e$$

$$\mathbf{b} \quad \log F = \log 5 - \log t$$

$$= \log \left(\frac{5}{t} \right)$$

$$\therefore F = \frac{5}{t}$$

$$\mathbf{c} \quad \log P = \frac{1}{2} \log x$$

$$= \log x^{\frac{1}{2}}$$

$$\therefore P = \sqrt{x}$$

$$\mathbf{d} \quad \log M = 2 \log b + \log c$$

$$= \log b^2 + \log c$$

$$= \log(b^2 c)$$

$$\therefore M = b^2 c$$

$$\mathbf{e} \quad \log B = 3 \log m - 2 \log n$$

$$= \log m^3 - \log n^2$$

$$= \log \left(\frac{m^3}{n^2} \right)$$

$$\therefore B = \frac{m^3}{n^2}$$

$$\mathbf{f} \quad \log N = -\frac{1}{3} \log p$$

$$= \log p^{-\frac{1}{3}}$$

$$= \log \left(\frac{1}{\sqrt[3]{p}} \right)$$

$$\therefore N = \frac{1}{\sqrt[3]{p}}$$

$$\mathbf{g} \quad \log P = 3 \log x + 1$$

$$= \log x^3 + \log 10^1$$

$$= \log(10x^3)$$

$$\therefore P = 10x^3$$

$$\mathbf{h} \quad \log Q = 2 - \log x$$

$$= \log 10^2 - \log x$$

$$= \log \left(\frac{100}{x} \right) \quad \therefore Q = \frac{100}{x}$$

$$\mathbf{7} \quad \mathbf{a} \quad \log_b 6$$

$$= \log_b(2 \times 3)$$

$$= \log_b 2 + \log_b 3$$

$$= p + q$$

$$\mathbf{b} \quad \log_b 108$$

$$= \log_b(2^2 3^3)$$

$$= 2 \log_b 2 + 3 \log_b 3$$

$$= 2p + 3q$$

$$\mathbf{c} \quad \log_b 45$$

$$= \log_b(3^2 5)$$

$$= 2 \log_b 3 + \log_b 5$$

$$= 2q + r$$

$$\mathbf{d} \quad \log_b \left(\frac{5\sqrt{3}}{2} \right)$$

$$= \log_b(5 \times 3^{\frac{1}{2}}) - \log_b 2$$

$$= \log_b 5 + \frac{1}{2} \log_b 3 - \log_b 2$$

$$= r + \frac{1}{2}q - p$$

$$\mathbf{e} \quad \log_b \left(\frac{5}{32} \right)$$

$$= \log_b 5 - \log_b 2^5$$

$$= \log_b 5 - 5 \log_b 2$$

$$= r - 5p$$

$$\mathbf{f} \quad \log_b(0.\bar{2})$$

$$= \log_b \left(\frac{2}{9} \right)$$

$$= \log_b 2 - \log_b 3^2$$

$$= p - 2q$$

$$\mathbf{8} \quad \mathbf{a} \quad \log_2(PR)$$

$$= \log_2 P + \log_2 R$$

$$= x + z$$

$$\mathbf{b} \quad \log_2(RQ^2)$$

$$= \log_2 R + \log_2 Q^2$$

$$= \log_2 R + 2 \log_2 Q$$

$$= z + 2y$$

$$\mathbf{c} \quad \log_2 \left(\frac{PR}{Q} \right)$$

$$= \log_2(PR) - \log_2 Q$$

$$= \log_2 P + \log_2 R - \log_2 Q$$

$$= x + z - y$$

$$\begin{aligned} \mathbf{d} \quad \log_2(P^2\sqrt{Q}) &= \log_2 P^2 + \log_2 Q^{\frac{1}{2}} \\ &= 2\log_2 P + \frac{1}{2}\log_2 Q \\ &= 2x + \frac{1}{2}y \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \log_2\left(\frac{Q^3}{\sqrt{R}}\right) &= \log_2 Q^3 - \log_2 R^{\frac{1}{2}} \\ &= 3\log_2 Q - \frac{1}{2}\log_2 R \\ &= 3y - \frac{1}{2}z \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \log_2\left(\frac{R^2\sqrt{Q}}{P^3}\right) &= \log_2 R^2 + \log_2 Q^{\frac{1}{2}} - \log_2 P^3 \\ &= 2\log_2 R + \frac{1}{2}\log_2 Q - 3\log_2 P \\ &= 2z + \frac{1}{2}y - 3x \end{aligned}$$

$$\begin{aligned} \mathbf{9} \quad \mathbf{a} \quad \log_t N^2 &= 1.72 \\ \therefore 2\log_t N &= 1.72 \\ \therefore \log_t N &= 1.72 \div 2 \\ &= 0.86 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \log_t(MN) &= \log_t M + \log_t N \\ &= 1.29 + 0.86 \\ &= 2.15 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \log_t\left(\frac{N^2}{\sqrt{M}}\right) &= \log_t N^2 - \log_t M^{\frac{1}{2}} \\ &= 2\log_t N - \frac{1}{2}\log_t M \\ &= 2(0.86) - \frac{1}{2}(1.29) \\ &= 1.075 \end{aligned}$$

$$\begin{aligned} \mathbf{10} \quad \mathbf{a} \quad \log_3 27 + \log_3\left(\frac{1}{3}\right) &= \log_3 x \\ \therefore \log_3(27 \times \frac{1}{3}) &= \log_3 x \\ \therefore \log_3 9 &= \log_3 x \\ \therefore x &= 9 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \log_5 x &= \log_5 8 - \log_5(6-x) \\ \therefore \log_5 x &= \log_5\left(\frac{8}{6-x}\right) \\ \therefore x &= \frac{8}{6-x} \quad \text{Note: } x > 0 \\ &\quad \text{and } 6-x > 0 \\ &\quad \text{i.e., } 0 < x < 6 \\ \therefore 6x - x^2 &= 8 \\ \therefore x^2 - 6x + 8 &= 0 \\ \therefore (x-2)(x-4) &= 0 \\ \therefore x &= 2 \text{ or } 4 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \log_5 125 - \log_5 \sqrt{5} &= \log_5 x \\ \therefore \log_5\left(\frac{125}{\sqrt{5}}\right) &= \log_5 x \\ \therefore x &= \frac{125}{\sqrt{5}} \text{ or } 25\sqrt{5} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \log_{20} x &= 1 + \log_{20} 10 \\ \therefore \log_{20} x &= \log_{20} 20^1 + \log_{20} 10 \\ &= \log_{20} 200 \\ \therefore x &= 200 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \log x + \log(x+1) &= \log 30 \\ \therefore \log[x(x+1)] &= \log 30 \\ \therefore x^2 + x &= 30 \\ \therefore x^2 + x - 30 &= 0 \\ \therefore (x+6)(x-5) &= 0 \\ \therefore x &= -6 \text{ or } 5 \\ \text{but } x > 0 \text{ for } \log x \text{ to exist} \\ \therefore x &= 5 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \log(x+2) - \log(x-2) &= \log 5 \\ \therefore \log\left(\frac{x+2}{x-2}\right) &= \log 5 \\ \therefore \frac{x+2}{x-2} &= 5 \\ \therefore x+2 &= 5x-10 \\ \therefore -4x &= -12 \\ \therefore x &= 3 \end{aligned}$$

Note: $x+2 > 0$ and $x-2 > 0 \quad \therefore x > 2 \quad \checkmark$

EXERCISE 4D

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad 2^x &= 10 \\ \therefore \log 2^x &= \log 10 \\ \therefore x \log 2 &= \log 10 \\ \therefore x &= \frac{\log 10}{\log 2} \div 3.32 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 3^x &= 20 \\ \therefore \log 3^x &= \log 20 \\ \therefore x \log 3 &= \log 20 \\ \therefore x &= \frac{\log 20}{\log 3} \div 2.73 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 4^x &= 100 \\ \therefore \log 4^x &= \log 100 \\ \therefore x \log 4 &= \log 100 \\ \therefore x &= \frac{\log 100}{\log 4} \div 3.32 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad (1.2)^x &= 1000 \\ \therefore x \log(1.2) &= \log 1000 \\ \therefore x &= \frac{\log 1000}{\log(1.2)} \\ \therefore x &\div 37.9 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad 2^x &= 0.08 \\ \therefore \log 2^x &= \log(0.08) \\ \therefore x \log 2 &= \log(0.08) \\ \therefore x &= \frac{\log(0.08)}{\log 2} \\ \therefore x &\div -3.64 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & 3^x = 0.000\,25 \\
 \therefore \log 3^x &= \log(0.000\,25) \\
 \therefore x \log 3 &= \log(0.000\,25) \\
 \therefore x &= \frac{\log(0.000\,25)}{\log 3} \doteq -7.55
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \left(\frac{3}{4}\right)^x = 10^{-4} \\
 \therefore \log(0.75)^x &= -4 \\
 \therefore x \log(0.75) &= -4 \\
 \therefore x &= \frac{-4}{\log(0.75)} \doteq 32.0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & 200 \times 2^{0.25t} = 600 \\
 \therefore 2^{0.25t} &= 3 \\
 \therefore \log(2^{0.25t}) &= \log 3 \\
 \therefore 0.25t \log 2 &= \log 3 \\
 \therefore t &= \frac{\log 3}{0.25 \times \log 2} \\
 \therefore t &\doteq 6.34
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 30 \times 3^{-0.25t} = 3 \\
 \therefore 3^{-0.25t} &= \frac{1}{10} \\
 \therefore \log 3^{-0.25t} &= \log(0.1) \\
 \therefore -0.25t \log 3 &= \log(0.1) \\
 \therefore t &= \frac{\log(0.1)}{-0.25 \times \log 3} \doteq 8.38
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & 50 \times 5^{-0.02t} = 1 \\
 \therefore 5^{-0.02t} &= \frac{1}{50} = 0.02 \\
 \therefore \log 5^{-0.02t} &= \log(0.02) \\
 \therefore -0.02t \log 5 &= \log(0.02) \\
 \therefore t &= \frac{\log(0.02)}{-0.02 \times \log 5} \\
 \therefore t &\doteq 122
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \left(\frac{1}{2}\right)^x = 0.005 \\
 \therefore \log(0.5)^x &= \log(0.005) \\
 \therefore x \log(0.5) &= \log(0.005) \\
 \therefore x &= \frac{\log(0.005)}{\log(0.5)} \doteq 7.64
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & (0.99)^x = 0.000\,01 \\
 \therefore \log(0.99)^x &= \log(0.000\,01) \\
 \therefore x \log(0.99) &= \log(0.000\,01) \\
 \therefore x &= \frac{\log(0.000\,01)}{\log(0.99)} \\
 \therefore x &\doteq 1146 \text{ or } 1150 \text{ (3 s.f.)}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 20 \times 2^{0.06t} = 450 \\
 \therefore 2^{0.06t} &= 22.5 \\
 \therefore \log(2^{0.06t}) &= \log(22.5) \\
 \therefore 0.06t \log 2 &= \log(22.5) \\
 \therefore t &= \frac{\log(22.5)}{0.06 \times \log 2} \\
 \therefore t &\doteq 74.9
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & 12 \times 2^{-0.05t} = 0.12 \\
 \therefore 2^{-0.05t} &= \frac{1}{100} \\
 \therefore \log 2^{-0.05t} &= \log(0.01) \\
 \therefore -0.05t \log 2 &= \log(0.01) \\
 \therefore t &= \frac{\log(0.01)}{-0.05 \times \log 2} \doteq 133
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & 300 \times 2^{0.005t} = 1000 \\
 \therefore 2^{0.005t} &= \frac{10}{3} \\
 \therefore \log 2^{0.005t} &= \log\left(\frac{10}{3}\right) \\
 \therefore 0.005t \log 2 &= \log\left(\frac{10}{3}\right) \\
 \therefore t &= \frac{\log\left(\frac{10}{3}\right)}{0.005 \times \log 2} \\
 \therefore t &\doteq 347
 \end{aligned}$$

EXERCISE 4E

$$\mathbf{1} \quad W_t = 20 \times 2^{0.15t} \text{ grams}$$

$$\begin{aligned}
 \mathbf{a} \quad & \text{When } W_t = 30, \\
 & 20 \times 2^{0.15t} = 30 \\
 \therefore 2^{0.15t} &= 1.5 \\
 \therefore \log 2^{0.15t} &= \log(1.5) \\
 \therefore 0.15t \log 2 &= \log(1.5) \\
 \therefore t &= \frac{\log(1.5)}{0.15 \times \log 2} \\
 \therefore t &\doteq 3.90 \text{ hours}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \text{When } W_t = 100, \\
 & 20 \times 2^{0.15t} = 100 \\
 \therefore 2^{0.15t} &= 5 \\
 \therefore \log 2^{0.15t} &= \log 5 \\
 \therefore 0.15t \log 2 &= \log 5 \\
 \therefore t &= \frac{\log 5}{0.15 \times \log 2} \\
 \therefore t &\doteq 15.5 \text{ hours}
 \end{aligned}$$

2 $T = 100 \times 2^{-0.03t} \text{ } ^\circ\text{C}$

a When $T = 25$,

$$100 \times 2^{-0.03t} = 25$$

$$\therefore 2^{-0.03t} = \frac{1}{4} = 2^{-2}$$

$$\therefore -0.03t = -2$$

$$\therefore t = \frac{2}{0.03}$$

$$\therefore t \doteq 66.7 \text{ min}$$

b When $T = 1$,

$$100 \times 2^{-0.03t} = 1$$

$$\therefore 2^{-0.03t} = 0.01$$

$$\therefore \log 2^{-0.03t} = \log(0.01)$$

$$\therefore -0.03t \log 2 = \log(0.01)$$

$$\therefore t = \frac{\log(0.01)}{-0.03 \times \log 2} \doteq 221 \text{ min}$$

3 $W_t = 1000 \times 2^{-0.04t}$ has $W_0 = 1000 \times 2^0 = 1000$ grams

a For the weight to halve,

$$W_t = 500$$

$$\therefore 1000 \times 2^{-0.04t} = 500$$

$$\therefore 2^{-0.04t} = \frac{1}{2} = 2^{-1}$$

$$\therefore -0.04t = -1$$

$$\therefore t = \frac{1}{0.04}$$

$$\therefore t = 25 \text{ years}$$

b For $W_t = 20$,

$$1000 \times 2^{-0.04t} = 20$$

$$\therefore 2^{-0.04t} = 0.02$$

$$\therefore \log 2^{-0.04t} = \log(0.02)$$

$$\therefore -0.04t \log 2 = \log(0.02)$$

$$\therefore t = \frac{\log(0.02)}{-0.04 \times \log 2}$$

$$\therefore t \doteq 141 \text{ years}$$

c When $W_t = 1\%$ of 1000 grams = 10 g,

$$1000 \times 2^{-0.04t} = 10$$

$$\therefore 2^{-0.04t} = 0.01$$

$$\therefore \log 2^{-0.04t} = \log(0.01)$$

$$\text{and so } -0.04 \log 2 = \log(0.01)$$

$$\therefore t = \frac{\log(0.01)}{-0.04 \times \log 2}$$

$$\therefore t \doteq 166 \text{ years}$$

4 $W = W_0 \times 2^{-0.0002t}$ grams

a When W is 25% of original,

$$W = \frac{1}{4} \text{ of } W_0$$

$$\therefore W_0 \times 2^{-0.0002t} = \frac{1}{4} \times W_0$$

$$\therefore 2^{-0.0002t} = 2^{-2}$$

$$\therefore 0.0002t = 2$$

$$\therefore t = \frac{2}{0.0002}$$

$$\therefore t = 10\,000$$

$$\therefore \text{it would take } 10\,000 \text{ years}$$

b When W is 0.1% of original,

$$W = \frac{0.1}{100} \text{ of } W_0$$

$$\therefore W_0 \times 2^{-0.0002t} = \frac{1}{1000} \times W_0$$

$$\therefore \log 2^{-0.0002t} = \log(0.001)$$

$$\therefore -0.0002t \log 2 = \log(0.001)$$

$$\therefore t = \frac{\log(0.001)}{-0.0002 \times \log 2}$$

$$\therefore t \doteq 49\,829$$

$$\text{i.e., it would take about } 49\,800 \text{ years}$$

5 $V = V_0 \times 2^{0.1t}$

$$\text{When } t = 0, V = V_0 \times 2^0 = V_0$$

So, we need to find t when

$$V = 3V_0$$

$$\text{i.e., } V_0 \times 2^{0.1t} = 3V_0$$

$$\therefore \log 2^{0.1t} = \log 3$$

$$\therefore 0.1t \log 2 = \log 3$$

$$\therefore t = \frac{\log 3}{0.1 \times \log 2} \doteq 15.8$$

$$\therefore \text{the temperature would be } 15.8^\circ\text{C}.$$

6 $I = I_0 \times 2^{-0.02t}$ amps

$$\text{When } t = 0, I = I_0 \times 2^0 = I_0 \text{ amps}$$

So, we need to find t when

$$I = 10\% \text{ of } I_0$$

$$\therefore I_0 \times 2^{-0.02t} = 0.1 \times I_0$$

$$\therefore \log 2^{-0.02t} = \log(0.1)$$

$$\therefore -0.02t \log 2 = \log(0.1)$$

$$\therefore t = \frac{\log(0.1)}{-0.02 \times \log 2} \doteq 166$$

$$\text{i.e., it would take } 166 \text{ seconds.}$$

$$7 \quad V = 50(1 - 2^{-0.2t})$$

$$\text{So when } V = 40, \quad 50(1 - 2^{-0.2t}) = 40$$

$$\therefore 1 - 2^{-0.2t} = 0.8$$

$$\therefore 2^{-0.2t} = 0.2$$

$$\therefore \log 2^{-0.2t} = \log(0.2)$$

$$\therefore -0.2t \log 2 = \log(0.2)$$

$$\therefore t = \frac{\log(0.2)}{-0.2 \times \log 2} \doteq 11.6 \quad \text{i.e., it would take 11.6 sec.}$$

EXERCISE 4F

$$1 \quad r = 107.5\%,$$

$$= 1.075$$

$$u_1 = 160\,000$$

$$u_{n+1} = 250\,000$$

$$u_{n+1} = u_1 \times r^n$$

$$\therefore 250\,000 = 160\,000 \times (1.075)^n$$

$$\therefore (1.075)^n = \frac{25}{16}$$

$$\therefore \log(1.075)^n = \log\left(\frac{25}{16}\right)$$

$$\therefore n \log(1.075) = \log\left(\frac{25}{16}\right)$$

$$\therefore n = \frac{\log\left(\frac{25}{16}\right)}{\log(1.075)} \doteq 6.1709 \dots$$

i.e., it would take 6.17 years

($\doteq$ 6 years, 62 days)

$$2 \quad u_1 = 10\,000$$

$$u_{n+1} = 15\,000$$

$$r = 104.8\%$$

$$= 1.048$$

$$u_{n+1} = u_1 \times r^n$$

$$\therefore 15\,000 = 10\,000 \times (1.048)^n$$

$$\therefore (1.048)^n = 1.5$$

$$\therefore \log(1.048)^n = \log(1.5)$$

$$\therefore n \log(1.048) = \log(1.5)$$

$$\therefore n = \frac{\log(1.5)}{\log(1.048)}$$

$$\therefore n \doteq 8.648 \dots$$

i.e., it would take 8.65 years

($\doteq$ 8 years, 237 days)

$$3 \quad \text{a} \quad 8.4\% \text{ p.a. compounded monthly}$$

$$\text{is } \frac{8.4\%}{12} = 0.7\% \text{ a month}$$

$$\text{So } T = 100\% + 0.7\%$$

$$= 100.7\%$$

$$= 1.007$$

$$\text{b} \quad u_1 = 15\,000 \quad \text{and} \quad u_{n+1} = 25\,000$$

$$u_{n+1} = u_1 \times r^n$$

$$\therefore 25\,000 = 15\,000 \times (1.007)^n$$

$$\therefore (1.007)^n = \frac{25}{15} = \frac{5}{3}$$

$$\therefore \log(1.007)^n = \log\left(\frac{5}{3}\right)$$

$$\therefore n \log(1.007) = \log\left(\frac{5}{3}\right)$$

$$\therefore n = \frac{\log\left(\frac{5}{3}\right)}{\log(1.007)} \doteq 73.23 \dots$$

i.e., it would take 74 months.

EXERCISE 4G

$$1 \quad \text{a} \quad \log_3 12$$

$$= \frac{\log 12}{\log 3}$$

$$\doteq 2.26$$

$$\text{b} \quad \log_{\frac{1}{2}} 1250$$

$$= \frac{\log 1250}{\log(0.5)}$$

$$\doteq -10.3$$

$$\text{c} \quad \log_3(0.067)$$

$$= \frac{\log(0.067)}{\log 3}$$

$$\doteq -2.46$$

$$\text{d} \quad \log_{0.4}(0.006\,984)$$

$$= \frac{\log(0.006\,984)}{\log(0.4)}$$

$$\doteq 5.42$$

$$2 \quad \text{a} \quad 2^x = 0.051$$

$$\therefore x = \log_2(0.051)$$

$$\therefore x = \frac{\log(0.051)}{\log 2}$$

$$\therefore x \doteq -4.29$$

$$\text{b} \quad 4^x = 213.8$$

$$\therefore x = \log_4 213.8$$

$$\therefore x = \frac{\log(213.8)}{\log 4}$$

$$\therefore x \doteq 3.87$$

$$\text{c} \quad 3^{2x+1} = 4.069$$

$$\therefore 2x+1 = \log_3(4.069)$$

$$\therefore 2x+1 = \frac{\log(4.069)}{\log 3}$$

$$\therefore 2x+1 \doteq 1.2774 \dots$$

$$\therefore 2x \doteq 0.2774 \dots$$

$$\therefore x \doteq 0.139$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & 25^x - 3 \times 5^x = 0 \\
 \therefore & 5^{2x} - 3 \times 5^x = 0 \\
 \therefore & 5^x(5^x - 3) = 0 \\
 \therefore & 5^x = 3 \\
 & \{\text{as } 5^x > 0 \text{ for all } x\} \\
 \therefore & x = \log_5 3 \\
 \therefore & x = \frac{\log 3}{\log 5} \\
 \therefore & x \div 0.683
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 8 \times 9^x - 3^x = 0 \\
 \therefore & 8 \times 3^{2x} - 3^x = 0 \\
 \therefore & 3^x(8 \times 3^x - 1) = 0 \\
 \therefore & 8 \times 3^x - 1 = 0 \\
 & \{\text{as } 3^x > 0 \text{ for all } x\} \\
 \therefore & 3^x = \frac{1}{8} \\
 \therefore & x = \log_3\left(\frac{1}{8}\right) \\
 \therefore & x = \frac{\log(\frac{1}{8})}{\log 3} \div -1.89
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & \log_4 x^3 + \log_2 \sqrt{x} = 8 \\
 \therefore & \frac{\log x^3}{\log 4} + \frac{\log x^{\frac{1}{2}}}{\log 2} = 8 \\
 \therefore & \frac{3 \log x}{2 \log 2} + \frac{\frac{1}{2} \log x}{\log 2} = 8 \\
 \therefore & \frac{3 \log x}{2 \log 2} + \frac{\log x}{2 \log 2} = 8 \\
 \therefore & \frac{4 \log x}{2 \log 2} = 8 \\
 \therefore & \log x = 4 \log 2 \\
 \therefore & \log x = \log 2^4 \\
 \therefore & x = 16
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \log_{16} x^5 = \log_{64} 125 - \log_4 \sqrt{x} \\
 \therefore & \frac{\log x^5}{\log 16} = \frac{\log 125}{\log 64} - \frac{\log x^{\frac{1}{2}}}{\log 4} \\
 \therefore & \frac{5 \log x}{4 \log 2} = \frac{\log 125}{6 \log 2} - \frac{\frac{1}{2} \log x}{2 \log 2} \\
 \therefore & \frac{15 \log x}{12 \log 2} = \frac{2 \log 125}{12 \log 2} - \frac{3 \log x}{12 \log 2} \\
 \therefore & 15 \log x = 2 \log 125 - 3 \log x \\
 \therefore & 18 \log x = 2 \log 125 \\
 \therefore & \log x = \frac{1}{9} \log 5^3 \\
 \therefore & \log x = \log(5^3)^{\frac{1}{9}} \\
 \therefore & x = 5^{\frac{1}{3}} \text{ or } x \div 1.71
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad & 4^x \times 5^{4x+3} = 10^{2x+3} \\
 \therefore & \log(4^x \times 5^{4x+3}) = \log 10^{2x+3} \\
 \therefore & x \log 4 + (4x+3) \log 5 = 2x+3 \\
 \therefore & x \log 4 + 4x \log 5 + 3 \log 5 = 2x+3 \\
 \therefore & x[\log 4 + 4 \log 5 - 2] = 3 - 3 \log 5
 \end{aligned}$$

$$\begin{aligned}
 \therefore & x = \frac{3 - 3 \log 5}{\log 4 + 4 \log 5 - 2} \\
 \therefore & x = \frac{\log 10^3 - \log 5^3}{\log 4 + \log 5^4 - \log 10^2} \\
 \therefore & x = \frac{\log(\frac{1000}{125})}{\log(\frac{4 \times 5^4}{10^2})} \\
 \therefore & x = \frac{\log 8}{\log 25} \text{ or } \log_{25} 8
 \end{aligned}$$

EXERCISE 4H

$$\mathbf{1} \quad \mathbf{a} \quad f(x) = \log_3(x+1)$$

i We require $x+1 > 0 \therefore x > -1$ So, the domain is $x \in]-1, \infty[$
the range is $y \in \mathcal{R}$

ii As $x \rightarrow -1$ (from the right), $y \rightarrow -\infty$, so $x = -1$ is a vertical asymptote.
As $x \rightarrow \infty$, $y \rightarrow \infty$.

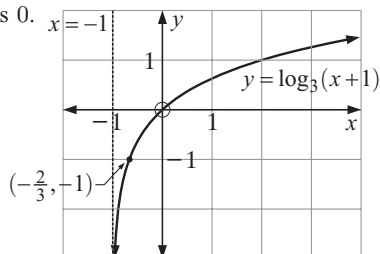
When $x = 0$, $y = \log_3 1 = 0 \therefore y$ -intercept is 0.

When $y = 0$, $\log_3(x+1) = 0$

$\therefore x+1 = 3^0$, so $x+1 = 1$, $x = 0$

$\therefore x$ -intercept is 0.

iii We graph, using $f(x) = \frac{\log(x+1)}{\log 3}$



$$\begin{array}{ll}
 \text{iv} & \text{If } f(x) = -1 \\
 & \text{then } \log_3(x+1) = -1 \\
 & \therefore x+1 = 3^{-1} \\
 & \therefore x = \frac{1}{3} - 1 \\
 & \text{i.e., } x = -\frac{2}{3} \\
 & \text{which checks with the graph}
 \end{array}
 \quad
 \begin{array}{ll}
 \text{v} & f \text{ is defined by } y = \log_3(x+1) \\
 & \therefore f^{-1} \text{ is defined by } x = \log_3(y+1) \\
 & \therefore y+1 = 3^x \\
 & \therefore y = 3^x - 1 \\
 & \therefore f^{-1}(x) = 3^x - 1
 \end{array}$$

and has HA $y = -1$

Its domain is $x \in \mathcal{R}$

range is $y \in]-1, \infty [$

b $f(x) = 1 - \log_3(x+1)$

i We require $x+1 > 0 \therefore x > -1$ So, domain is $x \in]-1, \infty [$

range is $y \in \mathcal{R}$

ii As $x \rightarrow -1$ (right), $y \rightarrow \infty$, so $x = -1$ is a vertical asymptote.

As $x \rightarrow \infty$, $y \rightarrow -\infty$.

When $x = 0$, $y = 1 - \log_3 1 = 1 - 0 = 1 \therefore y$ -intercept is 1.

When $y = 0$, $1 - \log_3(x+1) = 0 \therefore \log_3(x+1) = 1 \therefore x+1 = 3^1 = 3$
 $\therefore x = 2$

So, the x -intercept is 2.

iii We graph using $y = 1 - \frac{\log(x+1)}{\log 3}$

$$\begin{array}{ll}
 \text{iv} & \text{If } f(x) = -1, \quad 1 - \log_3(x+1) = -1 \\
 & \therefore \log_3(x+1) = 2 \\
 & \therefore x+1 = 3^2 \\
 & \therefore x = 8
 \end{array}$$

v f is defined by $y = 1 - \log_3(x+1)$

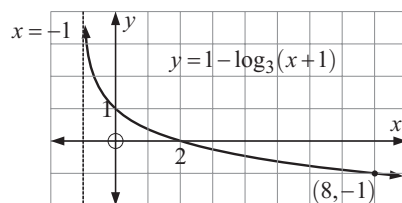
$\therefore f^{-1}$ is defined by $x = 1 - \log_3(y+1)$

$$\therefore \log_3(y+1) = 1 - x$$

$$\therefore y+1 = 3^{1-x}$$

$$\therefore y = 3^{1-x} - 1$$

$$\therefore f^{-1}(x) = 3^{1-x} - 1$$



It has HA, $y = -1$

domain is $x \in \mathcal{R}$

range is $y \in]-1, \infty [$

c $y = \log_5(x-2) - 2$

i We require $x-2 > 0 \therefore x > 2$. So, domain is $x \in]2, \infty [$

range is $y \in \mathcal{R}$

ii As $x \rightarrow 2$ (right), $y \rightarrow -\infty$. So, $x = 2$ is a vertical asymptote.

As $x \rightarrow \infty$, $y \rightarrow \infty$.

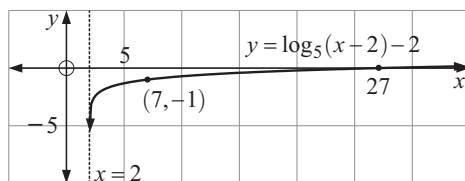
When $x = 0$, y is undefined $\therefore$ no y -intercept.

When $y = 0$, $\log_5(x-2) = 2 \therefore x-2 = 5^2 = 25$

$$\therefore x = 27$$

$\therefore x$ -intercept is 27

iii We graph using $y = \frac{\log(x-2)}{\log 5} - 2$



$$\begin{array}{ll}
 \text{iv} & \text{If } f(x) = -1 \\
 & \log_5(x-2) - 2 = -1 \\
 & \therefore \log_5(x-2) = 1 \\
 & \therefore x-2 = 5^1 \\
 & \therefore x = 5+2 \\
 & \therefore x = 7 \\
 \text{v} & f \text{ is defined by } y = \log_5(x-2) - 2 \\
 & \therefore f^{-1} \text{ is defined by } x = \log_5(y-2) - 2 \\
 & \therefore x+2 = \log_5(y-2) \\
 & \therefore y-2 = 5^{x+2} \\
 & \therefore y = 5^{x+2} + 2 \\
 & \text{i.e., } f^{-1}(x) = 5^{x+2} + 2
 \end{array}$$

It has a HA of $y = 2$

domain is $x \in \mathcal{R}$

range is $y \in] 2, \infty [$

d $y = 1 - \log_5(x-2)$

i We require $x-2 > 0 \quad \therefore x > 2$, So domain is $x \in] 2, \infty [$
range is $y \in \mathcal{R}$

ii As $x \rightarrow 2$ (right), $y \rightarrow \infty \quad \therefore x = 2$ is a VA.

As $x \rightarrow \infty$, $y \rightarrow -\infty$. When $x = 0$, y is undefined $\therefore$ no y -intercept.

When $y = 0$, $1 - \log_5(x-2) = 0$

$$\therefore \log_5(x-2) = 1$$

$$\therefore x-2 = 5^1$$

$$\therefore x = 7$$

So, x -intercept is 7.

iii We graph using $y = 1 - \frac{\log(x-2)}{\log 5}$

$$\begin{array}{ll}
 \text{iv} & \text{If } f(x) = -1, \text{ then } 1 - \log_5(x-2) = -1 \\
 & \therefore \log_5(x-2) = 2 \\
 & \therefore x-2 = 5^2 \\
 & \therefore x = 27
 \end{array}$$

v f is defined by $y = 1 - \log_5(x-2)$

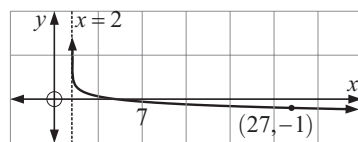
$$\therefore f^{-1} \text{ is defined by } x = 1 - \log_5(y-2)$$

$$\therefore \log_5(y-2) = 1-x$$

$$\therefore y-2 = 5^{1-x}$$

$$\therefore y = 5^{1-x} + 2$$

$$\text{i.e., } f^{-1}(x) = 5^{1-x} + 2$$



It has a HA of $y = 2$

domain is $x \in \mathcal{R}$

range is $y \in] 2, \infty [$

e $y = 1 - \log_2 x^2$

i We require $x^2 > 0$ which is true for all x except $x = 0$

$\therefore$ domain is $x \in \mathcal{R}$, but $x \neq 0$, range is $y \in \mathcal{R}$.

ii As $x \rightarrow 0$ (right or left), $y \rightarrow \infty \quad \therefore x = 0$ is a VA.

As $x \rightarrow \infty$, $y \rightarrow -\infty$,

as $x \rightarrow -\infty$, $y \rightarrow -\infty$.

When $x = 0$, $y = 1 - \log_2 0$

which is undefined $\therefore$ no y -intercept.

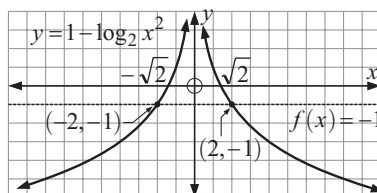
When $y = 0$, $\log_2 x^2 = 1$

$$\therefore x^2 = 2^1 = 2$$

$$\therefore x = \pm\sqrt{2}$$

$\therefore$ x -intercepts are $\sqrt{2}$ and $-\sqrt{2}$

iii We graph using $y = 1 - \frac{\log x^2}{\log 2}$



$$\begin{aligned} \text{iv} \quad & \text{When } f(x) = -1 \quad \therefore x^2 = 2^2 \\ & 1 - \log_2 x^2 = -1 \quad \therefore x^2 = 4 \\ & \therefore \log_2 x^2 = 2 \quad \therefore x = \pm 2 \end{aligned}$$

$$\begin{aligned} \text{v} \quad & \text{If } f(x) = 1 - \log_2 x^2, \quad x > 0 \\ & \text{then } f^{-1}(x) \text{ exists and is defined} \\ & \quad \text{by } x = 1 - \log_2 y^2, \quad y > 0 \\ & \therefore \log_2 y^2 = 1 - x \\ & \therefore y^2 = 2^{1-x} \\ & \therefore y = 2^{\frac{1-x}{2}} \quad \text{as } y > 0 \\ & \text{i.e., } f^{-1}(x) = 2^{\frac{1-x}{2}} \end{aligned}$$

$$\begin{aligned} & \text{If } f(x) = 1 - \log_2 x^2, \quad x < 0 \\ & \text{then } f^{-1}(x) \text{ also exists and is defined} \\ & \quad \text{by } x = 1 - \log_2 y^2, \quad y < 0 \\ & \therefore \log_2 y^2 = 1 - x \\ & \therefore y^2 = 2^{1-x} \\ & \therefore y = -2^{\frac{1-x}{2}} \quad \text{as } y < 0 \\ & \therefore f^{-1}(x) = -2^{\frac{1-x}{2}} \end{aligned}$$

$$\text{f} \quad f(x) = \log_2(x^2 - 3x - 4)$$

$$\begin{aligned} \text{i} \quad & \text{We require } x^2 - 3x - 4 > 0 \quad \therefore \text{domain is } x \in] -\infty, -1 [\text{ or }] 4, \infty [\\ & \text{i.e., } (x-4)(x+1) > 0 \quad \text{range is } y \in \mathcal{R} \end{aligned}$$

$$\begin{array}{c} + \quad | \quad - \quad | \quad + \\ \leftarrow -1 \quad \quad \quad 4 \rightarrow \end{array}$$

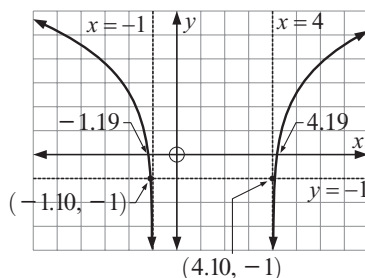
$$\begin{aligned} \text{ii} \quad & \text{As } x \rightarrow -1 \text{ (left), } y \rightarrow -\infty \\ & \text{As } x \rightarrow 4 \text{ (right), } y \rightarrow -\infty \quad \therefore x = -1, x = 4 \text{ are VAs.} \\ & \text{As } x \rightarrow \infty, y \rightarrow \infty \text{ and as } x \rightarrow -\infty, y \rightarrow \infty. \\ & \text{When } x = 0, y = \log_2(-4) \text{ which is undefined } \therefore \text{no } y\text{-intercept.} \end{aligned}$$

$$\begin{aligned} & \text{When } y = 0, \quad x^2 - 3x - 4 = 2^0 = 1 \\ & \therefore x^2 - 3x - 5 = 0 \\ & \therefore x \doteq -1.19 \text{ or } 4.19 \quad \{\text{from technology}\} \end{aligned}$$

$$\text{i.e., } x\text{-intercepts are } \doteq -1.19, 4.19$$

$$\text{iii} \quad \text{We graph using } y = \frac{\log(x^2 - 3x - 4)}{\log 2}$$

$$\begin{aligned} \text{iv} \quad & \text{If } f(x) = -1 \text{ then} \\ & \log_2(x^2 - 3x - 4) = -1 \\ & \therefore x^2 - 3x - 4 = 2^{-1} \\ & \text{i.e., } x^2 - 3x - 4.5 = 0 \\ & \therefore x \doteq -1.10 \text{ or } 4.10 \\ & \quad \{\text{using technology}\} \end{aligned}$$



$$\begin{aligned} \text{v} \quad & \text{If } f(x) = \log_2(x^2 - 3x - 4), \quad x > 4 \\ & \text{then } f^{-1} \text{ is defined by} \end{aligned}$$

$$\begin{aligned} & x = \log_2(y^2 - 3y - 4), \quad y > 4 \\ & \therefore y^2 - 3y - 4 = 2^x, \quad y > 0 \\ & \therefore y^2 - 3y - [4 + 2^x] = 0, \quad y > 0 \end{aligned}$$

$$\therefore y = \frac{3 \pm \sqrt{9 + 4[4 + 2^x]}}{2}, \quad y > 0$$

$$\therefore y = \frac{3 + \sqrt{25 + 2^{x+2}}}{2} \quad \text{as } y > 0$$

$$\therefore f^{-1}(x) = \frac{3 + \sqrt{25 + 2^{x+2}}}{2}$$

$$\begin{aligned} & \text{If } f(x) = \log_2(x^2 - 3x - 4), \quad x < -1 \\ & \text{by the same working} \end{aligned}$$

$$y = \frac{3 - \sqrt{25 + 2^{x+2}}}{2} \quad \text{as } y < 0$$

$$\text{i.e., } f^{-1}(x) = \frac{3 - \sqrt{25 + 2^{x+2}}}{2}$$

REVIEW SET 4A

$$\begin{aligned} 1 \quad a \quad & \log_4 64 \\ &= \log_4 4^3 \\ &= 3 \end{aligned}$$

$$\begin{aligned} b \quad & \log_2 256 \\ &= \log_2 2^8 \\ &= 8 \end{aligned}$$

$$\begin{aligned} c \quad & \log_2(0.25) \\ &= \log_2\left(\frac{1}{4}\right) \\ &= \log_2 2^{-2} \\ &= -2 \end{aligned}$$

$$\begin{aligned} d \quad & \log_{25} 5 \\ &= \log_{25} 25^{\frac{1}{2}} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} e \quad & \log_8 1 \\ &= \log_8 8^0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} f \quad & \log_6 6 \\ &= \log_6 6^1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} g \quad & \log_{81} 3 \\ &= \log_{81} 81^{\frac{1}{4}} \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} h \quad & \log_9(0.\bar{1}) \\ &= \log_9\left(\frac{1}{9}\right) \\ &= \log_9 9^{-1} \\ &= -1 \end{aligned}$$

$$\begin{aligned} i \quad & \log_{27} 3 \\ &= \log_{27} 27^{\frac{1}{3}} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} j \quad & \log_k \sqrt{k} \\ &= \log_k k^{\frac{1}{2}} \\ &= \frac{1}{2} \\ &(k > 0, k \neq 1) \end{aligned}$$

$$\begin{aligned} 2 \quad a \quad & \log \sqrt{10} \\ &= \log 10^{\frac{1}{2}} \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} b \quad & \log\left(\frac{1}{\sqrt[3]{10}}\right) \\ &= \log 10^{-\frac{1}{3}} \\ &= -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} c \quad & \log(10^a \times 10^{b+1}) \\ &= \log 10^{a+b+1} \\ &= a + b + 1 \end{aligned}$$

$$\begin{aligned} 3 \quad a \quad & \log_2 x = -3 \\ &\therefore x = 2^{-3} \\ &\therefore x = \frac{1}{8} \end{aligned}$$

$$\begin{aligned} b \quad & \log_5 x \div 2.743 \\ &\therefore x \div 5^{2.743} \\ &\therefore x \div 82.7 \end{aligned}$$

$$\begin{aligned} c \quad & \log_3 x \div -3.145 \\ &\therefore x \div 3^{-3.145} \\ &\therefore x \div 0.0316 \end{aligned}$$

$$\begin{aligned} 4 \quad a \quad & P = 3 \times b^x \\ &\therefore \log P = \log(3 \times b^x) \\ &\therefore \log P = \log 3 + \log b^x \\ &\therefore \log P = \log 3 + x \log b \end{aligned}$$

$$\begin{aligned} b \quad & m = \frac{n^3}{p^2} \quad \therefore \log m = \log\left(\frac{n^3}{p^2}\right) \\ &\therefore \log m = \log n^3 - \log p^2 \\ &\therefore \log m = 3 \log n - 2 \log p \end{aligned}$$

$$\begin{aligned} 5 \quad a \quad & \log_2 k \div 1.699 + x \\ &\therefore k \div 2^{1.699+x} \\ &\therefore k \div 2^{1.699} \times 2^x \\ &\therefore k \div 3.25 \times 2^x \end{aligned}$$

$$\begin{aligned} b \quad & \log_a Q = 3 \log_a P + \log_a R \\ &= \log_a (P^3 \times R) \\ &\therefore Q = P^3 R \end{aligned}$$

$$\begin{aligned} c \quad & \log A \div 5 \log B - 2.602 \\ &\therefore \log A - \log B^5 \div -2.602 \\ &\therefore \log\left(\frac{A}{B^5}\right) \div -2.602 \\ &\therefore \frac{A}{B^5} \div 10^{-2.602} \div 0.0025 \\ &\therefore A \div \frac{B^5}{400} \end{aligned}$$

$$\begin{aligned} 6 \quad a \quad & 5^x = 7 \\ &\therefore \log 5^x = \log 7 \\ &\therefore x \log 5 = \log 7 \\ &\therefore x = \frac{\log 7}{\log 5} \\ &\therefore x \div 1.21 \end{aligned}$$

$$\begin{aligned} b \quad & 20 \times 2^{2x+1} = 500 \\ &\therefore 2^{2x+1} = 25 \\ &\therefore \log 2^{2x+1} = \log 25 \\ &\therefore (2x+1) \log 2 = \log 25 \\ &\therefore 2x+1 = \frac{\log 25}{\log 2} \div 4.6438 \dots \\ &\therefore 2x \div 3.6438 \dots \\ &\therefore x \div 1.82 \end{aligned}$$

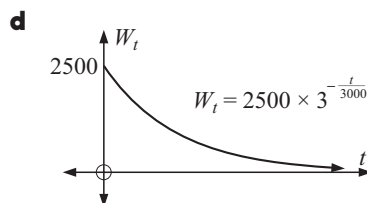
$$7 \quad W_t = 2500 \times 3^{-\frac{t}{3000}} \text{ grams}$$

$$\begin{aligned} \mathbf{a} \quad W_0 &= 2500 \times 3^0 \\ &= 2500 \times 1 \\ &= 2500 \text{ grams} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \% \text{ loss} &= \left(\frac{W_{1500} - W_0}{W_0} \right) \times 100\% \\ &= \left(\frac{2500 \times 3^{-\frac{1}{2}} - 2500}{2500} \right) \times 100\% \\ &= (3^{-\frac{1}{2}} - 1) \times 100\% \\ &\div -42.3\% \\ \text{i.e., a loss of } 42.3\% \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{We need } t \text{ when } W_t &= 30\% \text{ of } 2500 \text{ g} \\ \text{i.e., } 2500 \times 3^{-\frac{t}{3000}} &= 0.3 \times 2500 \\ \therefore \log 3^{-\frac{t}{3000}} &= \log(0.3) \\ \therefore -\frac{t}{3000} \times \log 3 &= \log(0.3) \\ \therefore t &= \frac{-\log(0.3) \times 3000}{\log 3} \\ \therefore t &\div 3287.7 \dots \end{aligned}$$

i.e., about 3290 years



$$\begin{aligned} \mathbf{8} \quad 16^x - 5 \times 8^x &= 0 \\ \therefore 2^x \times 8^x - 5 \times 8^x &= 0 \\ \therefore 8^x(2^x - 5) &= 0 \\ \therefore 2^x &= 5 \quad \text{as } 8^x > 0 \text{ for all } x \\ \therefore x &= \log_2 5 = \frac{\log 5}{\log 2} \div 2.32 \end{aligned}$$

REVIEW SET 4B

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad \log \sqrt{1000} &= \log (10^3)^{\frac{1}{2}} \\ &= \log 10^{\frac{3}{2}} \\ &= \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \log \left(\frac{10}{\sqrt[3]{10}} \right) &= \log \left(\frac{10^1}{10^{\frac{1}{3}}} \right) \\ &= \log 10^{\frac{2}{3}} = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \log \left(\frac{10^a}{10^{-b}} \right) &= \log (10^{a-(-b)}) \\ &= \log 10^{a+b} \\ &= a + b \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad \log x &= 3 \\ \therefore x &= 10^3 \\ \therefore x &= 1000 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \log_3(x+2) &= 1.732 \\ \therefore x+2 &= 3^{1.732} \\ \therefore x+2 &\div 6.7046 \\ \therefore x &\div 4.7046 \\ \therefore x &\div 4.70 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \log_2 \left(\frac{x}{10} \right) &= -0.671 \\ \therefore \frac{x}{10} &= 2^{-0.671} \\ \therefore \frac{x}{10} &\div 0.62807 \\ \therefore x &\div 6.28 \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad \log 16 + 2 \log 3 &= \log 16 + \log 3^2 \\ &= \log(16 \times 9) \\ &= \log 144 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \log_2 16 - 2 \log_2 3 &= \log_2 16 - \log_2 3^2 \\ &= \log_2 \left(\frac{16}{9} \right) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 2 + \log_4 5 &= \log 4^2 + \log 5 \\ &= \log_4(16 \times 5) \\ &= \log_4 80 \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \log T &= 2 \log x - \log y \\ \therefore \log T &= \log x^2 - \log y \\ \therefore \log T &= \log \left(\frac{x^2}{y} \right) \\ \therefore T &= \frac{x^2}{y} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \log_2 K &= \log_2 n + \frac{1}{2} \log_2 t \\ \therefore \log_2 K &= \log_2 n + \log_2 t^{\frac{1}{2}} \\ \therefore \log_2 K &= \log_2(n \times \sqrt{t}) \\ \therefore K &= n\sqrt{t} \end{aligned}$$

$$\begin{array}{lll}
 \mathbf{5} \quad \mathbf{a} & 3^x = 300 & \mathbf{b} \quad 30 \times 5^{1-x} = 0.15 \\
 & \therefore \log 3^x = \log 300 & \therefore 5^{1-x} = 0.005 \\
 & \therefore x \log 3 = \log 300 & \therefore \log 5^{1-x} = \log(0.005) \\
 & \therefore x = \frac{\log 300}{\log 3} & \therefore (1-x) \log 5 = \log(0.005) \\
 & \therefore x \div 5.19 & \therefore 1-x = \frac{\log(0.005)}{\log 5} \\
 & & \therefore 1-x \div -3.292 \\
 & & \therefore x \div 4.29 \\
 \mathbf{c} & 3^{x+2} = 2^{1-x} & \\
 & \therefore \log 3^{x+2} = \log 2^{1-x} & \\
 & \therefore (x+2) \log 3 = (1-x) \log 2 & \\
 & \therefore x \log 3 + 2 \log 3 = \log 2 - x \log 2 & \\
 & \therefore x(\log 3 + \log 2) = \log 2 - 2 \log 3 & \\
 & \therefore x \log 6 = \log\left(\frac{2}{9}\right) & \\
 & \therefore x = \frac{\log\left(\frac{2}{9}\right)}{\log 6} \div -0.839 &
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{6} \quad \mathbf{a} & \log_2 36 & \mathbf{b} \quad \log_2 54 \\
 & = \log_2(2^2 \times 3^2) & = \log_2(2 \times 3^3) \\
 & = \log_2 2^2 + \log_2 3^2 & = \log_2 2 + \log_2 3^3 \\
 & = 2 \log_2 2 + 2 \log_2 3 & = \log_2 2 + 3 \log_2 3 \\
 & = 2A + 2B & = A + 3B \\
 \mathbf{c} & \log_2(8\sqrt{3}) & \\
 & = \log_2(2^3 \times 3^{\frac{1}{2}}) & \\
 & = \log_2 2^3 + \log_2 3^{\frac{1}{2}} & \\
 & = 3 \log_2 2 + \frac{1}{2} \log_2 3 & \\
 & = 3A + \frac{1}{2}B &
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{d} & \log_2(20.25) \\
 & = \log_2(20\frac{1}{4}) \\
 & = \log_2\left(\frac{81}{4}\right) \\
 & = \log_2\left(\frac{3^4}{2^2}\right) \\
 & = \log_2 3^4 - \log_2 2^2 \\
 & = 4 \log_2 3 - 2 \log_2 2 \\
 & = 4B - 2A \\
 \mathbf{e} & \log_2(0.\bar{8}) \\
 & = \log_2\left(\frac{8}{9}\right) \\
 & = \log_2\left(\frac{2^3}{3^2}\right) \\
 & = \log_2 2^3 - \log_2 3^2 \\
 & = 3 \log_2 2 - 2 \log_2 3 \\
 & = 3A - 2B
 \end{array}$$

$$\mathbf{7} \quad g(x) = \log_3(x+2) - 2$$

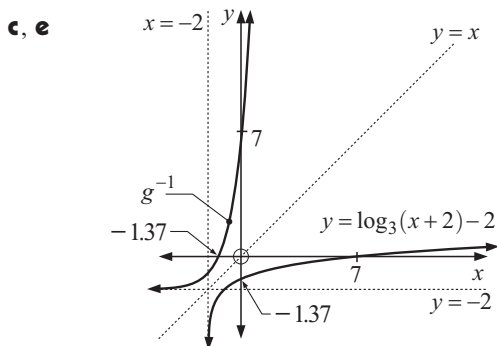
a We require $x+2 > 0$, i.e., $x > -2$ $\therefore$ domain is $x \in]-2, \infty[$ range is $y \in \mathbb{R}$

b If $x \rightarrow -2$ (right), $y \rightarrow -\infty$ $\therefore$ VA is $x = -2$.

As $x \rightarrow \infty$, $y \rightarrow \infty$.

When $x = 0$, $g(0) = \log_3 2 - 2 \div -1.37$ $\therefore$ y -intercept $\div -1.37$

When $y = 0$, $\log_3(x+2) = 2$ $\therefore$ $x+2 = 3^2$
 $\therefore$ $x = 7$ So, the x -intercept is 7.



$$\begin{array}{ll}
 \mathbf{d} & g^{-1} \text{ is defined by } x = \log_3(y+2) - 2 \\
 & \therefore \log_3(y+2) = x+2 \\
 & \therefore y+2 = 3^{x+2} \\
 & \therefore y = 3^{x+2} - 2 \\
 & \therefore g^{-1}(x) = 3^{x+2} - 2
 \end{array}$$

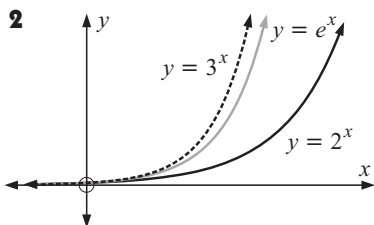
$$\begin{array}{ll}
 \mathbf{8} & \log_4 a^5 + \log_2 a^{\frac{3}{2}} = \log_8 625 \\
 & \therefore \frac{\log a^5}{\log 4} + \frac{\log a^{\frac{3}{2}}}{\log 2} = \frac{\log 5^4}{\log 8} \\
 & \therefore \frac{5 \log a}{2 \log 2} + \frac{\frac{3}{2} \log a}{\log 2} = \frac{4 \log 5}{3 \log 2} \\
 & \therefore \frac{5}{2} \log a + \frac{3}{2} \log a = \frac{4}{3} \log 5 \\
 & \therefore 4 \log a = \frac{4}{3} \log 5 \\
 & \therefore \log a = \frac{1}{3} \log 5 \\
 & \therefore \log a = \log \sqrt[3]{5} \\
 & \text{and so } a = \sqrt[3]{5}
 \end{array}$$

Chapter 5

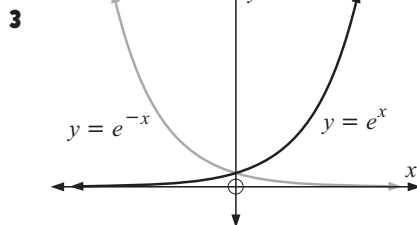
NATURAL LOGARITHMS

EXERCISE 5A

1 $e^1 \div 2.718\ 281\ 828\ \dots$



The graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$.



One is the other reflected in the y -axis.

4 When $x = 0$, $y = ae^0 = a \times 1 = a \quad \therefore$ the y -intercept is a .

5 a The graph of $y = e^x$ is entirely above the x -axis.

So $y > 0$ for all x

i.e., $e^x > 0$ for all x

$\therefore 2e^x > 0$ for all x

$\therefore y$ cannot be negative if $y = 2e^x$

b i When $x = -20$, $y = 2e^{-20} \div 4.1 \times 10^{-9}$

ii When $x = 20$, $y = 2e^{20} \div 9.7 \times 10^8$

6 a $\div 7.39$ b $\div 20.1$ c $\div 2.01$ d $\div 1.65$ e $\div 0.368$

7 a $\sqrt{e} = e^{\frac{1}{2}}$

b $e\sqrt{e}$
 $= e^1 e^{\frac{1}{2}}$
 $= e^{\frac{3}{2}}$

c $\frac{1}{\sqrt{e}}$
 $= \frac{1}{e^{\frac{1}{2}}}$
 $= e^{-\frac{1}{2}}$

d $\frac{1}{e^2} = e^{-2}$

8 a $(e^{0.36})^{\frac{t}{2}}$
 $= e^{0.36 \times \frac{t}{2}}$
 $= e^{0.18t}$

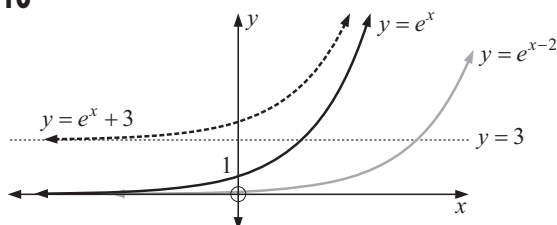
b $(e^{0.064})^{\frac{t}{16}}$
 $= e^{0.064 \times \frac{t}{16}}$
 $= e^{0.004t}$

c $(e^{-0.04})^{\frac{t}{8}}$
 $= e^{-0.04 \times \frac{t}{8}}$
 $= e^{-0.005t}$

d $(e^{-0.836})^{\frac{t}{5}}$
 $= e^{-0.836 \times \frac{t}{5}}$
 $= e^{-0.167t}$

9 a 10.074 b 0.099 261 c 125.09 d 0.007 994 5 e 41.914 f 42.429 g 3540.3
 h 0.006 342 4

10



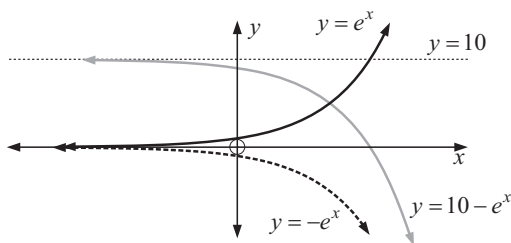
Domain of f , g and h is $\{x : x \in \mathbb{R}\}$

Range of f is $\{y : y > 0\}$

Range of g is $\{y : y > 0\}$

Range of h is $\{y : y > 3\}$

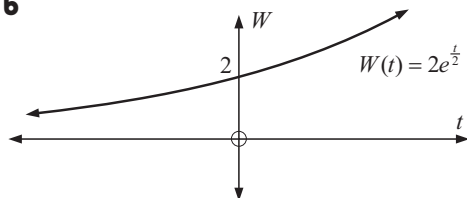
11

Domain of f , g and h is $\{x : x \in \mathcal{R}\}$ Range of f is $\{y : y > 0\}$ Range of g is $\{y : y < 0\}$ Range of h is $\{y : y < 10\}$

12 $W(t) = 2e^{\frac{t}{2}}$ grams

$$\begin{array}{llll} \text{a i} & W(0) = 2e^0 & \text{ii} & W\left(\frac{1}{2}\right) = 2e^{\frac{1}{4}} \\ & = 2 \times 1 & & \div 2.57 \text{ g} \\ & = 2 \text{ grams} & & \end{array} \quad \begin{array}{ll} \text{iii} & W\left(1\frac{1}{2}\right) = 2e^{\frac{3}{4}} \\ & \div 4.23 \text{ g} \end{array} \quad \begin{array}{ll} \text{iv} & W(6) = 2e^3 \\ & \div 40.2 \text{ g} \end{array}$$

b



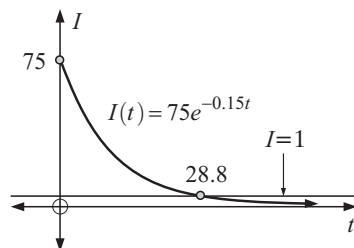
13 $I(t) = 75e^{-0.15t}$

$$\text{a i} \quad I(1) = 75e^{-0.15} = 64.6 \text{ amps}$$

$$\text{ii} \quad I(10) = 75e^{-1.5} \div 16.7 \text{ amps}$$

$$\text{c} \quad \text{We need to solve } 75e^{-0.15t} = 1. \\ \text{Using technology, } t \div 28.8 \text{ sec.}$$

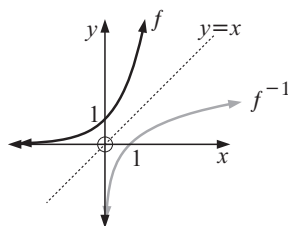
b



14 a $f(x) = e^x$

$$\begin{array}{l} \text{i.e., } y = e^x \text{ and has inverse } x = e^y \\ \therefore y = \log_e x \\ \text{i.e., } f^{-1}(x) = \log_e x \end{array}$$

b



EXERCISE 5B

1 a $\ln e^3$
 $= 3 \quad \{\ln e^a = a\}$

b $\ln 1$
 $= \ln e^0$
 $= 0$

c $\ln \sqrt[3]{e}$
 $= \ln e^{\frac{1}{3}}$
 $= \frac{1}{3}$

d $\ln \left(\frac{1}{e^2}\right)$
 $= \ln e^{-2}$
 $= -2$

3 $\ln x$ exists only when $x > 0$. $\{\text{See the graph of } y = \ln x.\}$

So, $\ln(-2)$ and $\ln(0)$ do not exist.Note: If $\ln(-2) = a$

then $-2 = e^a$

and $e^a = -2$ has no solutions as $e^a > 0$ for all a .

$$\begin{array}{llll}
 \mathbf{4} & \mathbf{a} & \ln e^a & \mathbf{b} & \ln(e \times e^a) & \mathbf{c} & \ln(e^a \times e^b) & \mathbf{d} & \ln(e^a)^b & \mathbf{f} & \ln\left(\frac{e^a}{e^b}\right) \\
 & & = a & & = \ln e^{1+a} & & = \ln(e^{a+b}) & & = \ln e^{ab} & & = \ln(e^{a-b}) \\
 & & & & = 1 + a & & = a + b & & = ab & & = a - b
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{5} & \mathbf{a} & e^{1.7918} & \mathbf{b} & e^{4.0943} & \mathbf{c} & e^{8.6995} & \mathbf{d} & e^{-0.5108} & \mathbf{e} & e^{-5.1160} \\
 & \mathbf{f} & e^{2.7081} & \mathbf{g} & e^{7.3132} & \mathbf{h} & e^{0.4055} & \mathbf{i} & e^{-1.8971} & \mathbf{j} & e^{-8.8049}
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{6} & \mathbf{a} & \ln x = 3 & \mathbf{b} & \ln x = 1 & \mathbf{c} & \ln x = 0 & \mathbf{d} & \ln x = -1 \\
 & & \therefore x = e^3 & & \therefore x = e^1 & & \therefore x = e^0 & & \therefore x = e^{-1} \\
 & & \therefore x \div 20.1 & & \therefore x = e \div 2.72 & & \therefore x = 1 & & \therefore x \div 0.368 \\
 & \mathbf{e} & \ln x = -5 & \mathbf{f} & \ln x \div 0.835 & \mathbf{g} & \ln x \div 2.145 & \mathbf{h} & \ln x \div -3.2971 \\
 & & \therefore x = e^{-5} & & \therefore x \div e^{0.835} & & \therefore x \div e^{2.145} & & \therefore x \div e^{-3.2971} \\
 & & \therefore x \div 0.00674 & & \therefore x \div 2.30 & & \therefore x \div 8.54 & & \therefore x \div 0.0370
 \end{array}$$

EXERCISE 5C

$$\begin{array}{llll}
 \mathbf{1} & \mathbf{a} & \ln 8 + \ln 2 & \mathbf{b} & \ln 8 - \ln 2 & \mathbf{c} & \ln 40 - \ln 5 & \mathbf{d} & \ln 4 + \ln 5 \\
 & & = \ln(8 \times 2) & & = \ln\left(\frac{8}{2}\right) & & = \ln\left(\frac{40}{5}\right) & & = \ln(4 \times 5) \\
 & & = \ln 16 & & = \ln 4 & & = \ln 8 & & = \ln 20 \\
 & \mathbf{e} & \ln 5 + \ln(0.4) & \mathbf{f} & \ln 2 + \ln 3 + \ln 4 & \mathbf{g} & 1 + \ln 3 & \mathbf{h} & \ln 4 - 1 \\
 & & = \ln(5 \times 0.4) & & = \ln(2 \times 3 \times 4) & & = \ln e^1 + \ln 3 & & = \ln 4 - \ln e^1 \\
 & & = \ln 2 & & = \ln 24 & & = \ln(e \times 3) & & = \ln\left(\frac{4}{e}\right) \\
 & \mathbf{i} & \ln 5 + \ln 4 - \ln 2 & \mathbf{j} & 2 + \ln 2 & \mathbf{k} & \ln 40 - 2 & \mathbf{l} & \ln 6 - \ln 2 - \ln 3 \\
 & & = \ln(5 \times 4) - \ln 2 & & = \ln e^2 + \ln 2 & & = \ln 40 - \ln e^2 & & = \ln\left(\frac{6}{2}\right) - \ln 3 \\
 & & = \ln\left(\frac{20}{2}\right) & & = \ln(e^2 \times 2) & & = \ln\left(\frac{40}{e^2}\right) & & = \ln 3 - \ln 3 \\
 & & = \ln 10 & & = \ln(2e^2) & & & & = \ln\left(\frac{3}{3}\right) = \ln 1
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{2} & \mathbf{a} & 5 \ln 2 + \ln 3 & \mathbf{b} & 2 \ln 3 + 3 \ln 2 & \mathbf{c} & 3 \ln 4 - \ln 8 \\
 & & = \ln 2^5 + \ln 3 & & = \ln 3^2 + \ln 2^3 & & = \ln 4^3 - \ln 8 \\
 & & = \ln(2^5 \times 3) & & = \ln(9 \times 8) & & = \ln\left(\frac{64}{8}\right) \\
 & & = \ln 96 & & = \ln 72 & & = \ln 8 \\
 & \mathbf{d} & 2 \ln 5 - 3 \ln 2 & \mathbf{e} & \frac{1}{2} \ln 4 + \ln 3 & \mathbf{f} & \frac{1}{3} \ln\left(\frac{1}{8}\right) \\
 & & = \ln 5^2 - \ln 2^3 & & = \ln 4^{\frac{1}{2}} + \ln 3 & & = \frac{1}{3} \ln(2^{-3}) \\
 & & = \ln\left(\frac{25}{8}\right) & & = \ln 2 + \ln 3 & & = \frac{1}{3} \times -3 \ln 2 \\
 & & & & = \ln(2 \times 3) & & = -\ln 2 \quad (\text{or } \ln\left(\frac{1}{2}\right)) \\
 & & & & = \ln 6 & & \\
 & \mathbf{g} & -\ln 2 & \mathbf{h} & -\ln\left(\frac{1}{3}\right) & \mathbf{i} & -2 \ln\left(\frac{1}{4}\right) \\
 & & = -1 \ln 2 & & = -1 \ln(3^{-1}) & & = \ln(2^{-2})^{-2} \\
 & & = \ln 2^{-1} & & = \ln(3^{-1})^{-1} & & = \ln 2^4 \\
 & & = \ln\left(\frac{1}{2}\right) & & = \ln 3 & & = \ln 16 \text{ or } 4 \ln 2
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{3} & \mathbf{a} & \ln 9 & \mathbf{b} & \ln \sqrt{2} & \mathbf{c} & \ln\left(\frac{1}{8}\right) & \mathbf{d} & \ln\left(\frac{1}{5}\right) & \mathbf{e} & \ln\left(\frac{1}{\sqrt{2}}\right) \\
 & & = \ln 3^2 & & = \ln 2^{\frac{1}{2}} & & = \ln\left(\frac{1}{2^3}\right) & & = \ln 5^{-1} & & = \ln 2^{-\frac{1}{2}} \\
 & & = 2 \ln 3 & & = \frac{1}{2} \ln 2 & & = \ln(2^{-3}) & & = -1 \ln 5 & & = -\frac{1}{2} \ln 2 \\
 & & & & & & = -3 \ln 2 & & = -\ln 5 & &
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{f} & \ln\left(\frac{e}{5}\right) & \mathbf{g} & \ln \sqrt[3]{5} \\
 & = \ln e^1 - \ln 5 & & = \ln 5^{\frac{1}{3}} \\
 & = 1 - \ln 5 & & = \frac{1}{3} \ln 5 \\
 \mathbf{h} & \ln\left(\frac{1}{32}\right) & \mathbf{i} & \ln\left(\frac{1}{\sqrt[5]{2}}\right) = \ln\left(\frac{1}{2^{\frac{1}{5}}}\right) \\
 & = \ln 2^{-5} & & = \ln 2^{-\frac{1}{5}} = -\frac{1}{5} \ln 2 \\
 & = -5 \ln 2 & &
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{4} \quad \mathbf{a} & \ln D = \ln x + 1 & \mathbf{b} \quad \ln F = -\ln p + 2 \\
 & \therefore \ln D - \ln x = 1 & \therefore \ln F + \ln p = 2 \\
 & \therefore \ln\left(\frac{D}{x}\right) = 1 & \therefore \ln(Fp) = 2 \\
 & \therefore \frac{D}{x} = e^1 & \therefore Fp = e^2 \\
 & \therefore D = ex & \therefore F = \frac{e^2}{p}
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{d} & \ln M = 2 \ln y + 3 & \mathbf{e} \quad \ln B = 3 \ln t - 1 \\
 & \therefore \ln M - 2 \ln y = 3 & \therefore \ln B - \ln t^3 = -1 \\
 & \therefore \ln\left(\frac{M}{y^2}\right) = 3 & \therefore \ln\left(\frac{B}{t^3}\right) = -1 \\
 & \therefore \frac{M}{y^2} = e^3 & \therefore \frac{B}{t^3} = e^{-1} \\
 & \therefore M = e^3 y^2 & \therefore B = \frac{t^3}{e}
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{g} & \ln Q \div 3 \ln x + 2.159 & \mathbf{h} \quad \ln D \div 0.4 \ln n - 0.6582 \\
 & \therefore \ln Q - 3 \ln x \div 2.159 & \therefore \ln D - \ln n^{0.4} \div -0.6582 \\
 & \therefore \ln\left(\frac{Q}{x^3}\right) \div 2.159 & \therefore \ln\left(\frac{D}{n^{0.4}}\right) \div -0.6582 \\
 & \therefore \frac{Q}{x^3} \div e^{2.159} & \therefore \frac{D}{n^{0.4}} \div e^{-0.6582} \\
 & \therefore \frac{Q}{x^3} \div 8.66 & \therefore \frac{D}{n^{0.4}} \div 0.518 \\
 & \therefore Q \div 8.66x^3 & \therefore D \div 0.518n^{0.4}
 \end{array}$$

EXERCISE 5D

$$\begin{array}{lll}
 \mathbf{1} \quad \mathbf{a} & e^x = 10 & \mathbf{b} \quad e^x = 1000 \\
 & \therefore x = \ln 10 & \therefore x = \ln 1000 \\
 & \therefore x \div 2.303 & \therefore x \div 6.908 \\
 & & \mathbf{c} \quad e^x = 0.00862 \\
 & & \therefore x = \ln(0.00862) \\
 & & \therefore x \div -4.754 \\
 \mathbf{d} & e^{\frac{x}{2}} = 5 & \mathbf{e} \quad e^{\frac{x}{3}} = 157.8 \\
 & \therefore \frac{x}{2} = \ln 5 & \therefore \frac{x}{3} = \ln(157.8) \\
 & \therefore x = 2 \ln 5 & \therefore x = 3 \ln(157.8) \\
 & \therefore x \div 3.219 & \therefore x \div 15.18 \\
 \mathbf{g} & 20 \times e^{0.06x} = 8.312 & \mathbf{h} \quad 50e^{-0.03x} = 0.816 \\
 & \therefore e^{0.06x} = 0.4156 & \therefore e^{-0.03x} = 0.01632 \\
 & \therefore 0.06x = \ln(0.4156) & \therefore -0.03x = \ln(0.01632) \\
 & \therefore x = \frac{\ln(0.4156)}{0.06} & \therefore x = \frac{\ln(0.01632)}{-0.03} \\
 & \therefore x \div -14.63 & \therefore x \div 137.2 \\
 & & \mathbf{i} \quad 41.83e^{0.652x} = 1000 \\
 & & \therefore e^{0.652x} \div 23.91 \\
 & & \therefore 0.652x \div \ln(23.91) \\
 & & \therefore x \div \frac{\ln(23.91)}{0.652} \\
 & & \therefore x \div 4.868
 \end{array}$$

EXERCISE 5E

1 $M_t = 20e^{0.15t}$

a When $M_t = 25$,

$$20e^{0.15t} = 25$$

$$\therefore e^{0.15t} = \frac{25}{20}$$

$$\therefore 0.15t = \ln\left(\frac{25}{20}\right)$$

$$\therefore t = \frac{\ln(1.25)}{0.15}$$

$$\therefore t \doteq 1.488$$

i.e., after 1 hour 29 min

b When $M_t = 100$,

$$20e^{0.15t} = 100$$

$$\therefore e^{0.15t} = 5$$

$$\therefore 0.15t = \ln 5$$

$$\therefore t = \frac{\ln 5}{0.15}$$

$$\therefore t \doteq 10.73$$

i.e., after 10 hours 44 min

2 $M_t = 1000e^{-0.04t} \quad \therefore M_0 = 1000e^0 = 1000 \text{ g}$

a For M_t to halve,

$$M_t = 500$$

$$\therefore 1000e^{-0.04t} = 500$$

$$\therefore e^{-0.04t} = 0.5$$

$$\therefore -0.04t = \ln(0.5)$$

$$\therefore t = \frac{\ln(0.5)}{-0.04}$$

$$\therefore t \doteq 17.3 \text{ years}$$

b For $M_t = 25 \text{ g}$,

$$\therefore 1000e^{-0.04t} = 25$$

$$\therefore e^{-0.04t} = 0.025$$

$$\therefore -0.04t = \ln(0.025)$$

$$\therefore t = \frac{\ln(0.025)}{-0.04}$$

$$\therefore t \doteq 92.2 \text{ years}$$

c For $M_t = 1\% \text{ of } M_0$

$$\therefore 1000e^{-0.04t} = 0.01 \times 1000$$

$$\therefore e^{-0.04t} = 0.01$$

$$\therefore -0.04t = \ln(0.01)$$

$$\therefore t = \frac{\ln(0.01)}{-0.04}$$

$$\therefore t \doteq 115 \text{ years}$$

3 $V = 50(1 - e^{-0.2t}) \text{ m/s}$

So, when $V = 40$,

$$50(1 - e^{-0.2t}) = 40$$

$$\therefore 1 - e^{-0.2t} = 0.8$$

$$\therefore e^{-0.2t} = 0.2$$

$$\therefore -0.2t = \ln(0.2)$$

$$\therefore t = \frac{\ln(0.2)}{-0.2}$$

$$\therefore t \doteq 8.05 \text{ sec}$$

i.e., it would take 8.05 sec

4 $T_m = (225 \times e^{-0.17m} - 6) \text{ } ^\circ\text{C}$

So, when $T_m = 0$,

$$225e^{-0.17m} - 6 = 0$$

$$\therefore 225e^{-0.17m} = 6$$

$$\therefore e^{-0.17m} = \frac{6}{225}$$

$$\therefore -0.17m = \ln\left(\frac{6}{225}\right)$$

$$\therefore m = \frac{\ln\left(\frac{6}{225}\right)}{-0.17}$$

$$\therefore m \doteq 21.3$$

i.e., it would take 21.3 minutes

EXERCISE 5F**1 a i**

$$f(x) = e^x + 5$$

$$\text{i.e., } y = e^x + 5$$

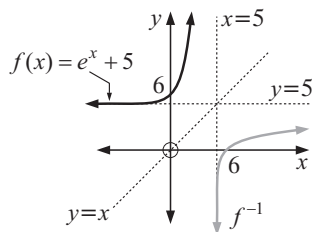
has inverse function

$$x = e^y + 5$$

$$\therefore x - 5 = e^y$$

$$\therefore y = \ln(x - 5)$$

$$\therefore f^{-1}(x) = \ln(x - 5)$$

ii**iii** domain of f is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > 5\}$ domain of f^{-1} is $\{x : x > 5\}$, range is $\{y : y \in \mathcal{R}\}$ **b i**

$$f(x) = e^{x+1} - 3$$

$$\text{i.e., } y = e^{x+1} - 3$$

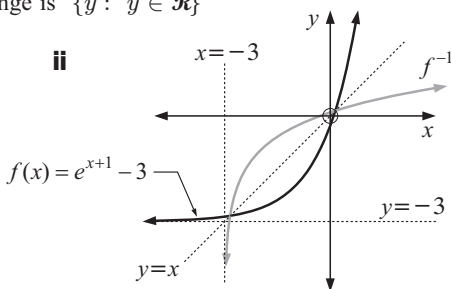
has inverse function

$$x = e^{y+1} - 3$$

$$\text{i.e., } x + 3 = e^{y+1}$$

$$\text{i.e., } y + 1 = \ln(x + 3)$$

$$\therefore f^{-1}(x) = \ln(x + 3) - 1$$

ii**iii** domain of f is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > -3\}$ domain of f^{-1} is $\{x : x > -3\}$, range is $\{y : y \in \mathcal{R}\}$ **c i**

$$f(x) = \ln x - 4$$

$$\text{i.e., } y = \ln x - 4 \text{ and}$$

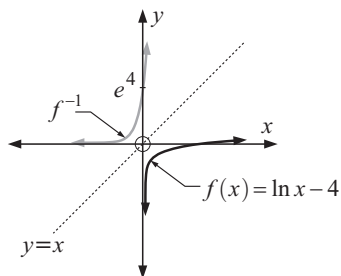
has inverse function

$$x = \ln y - 4$$

$$\text{i.e., } x + 4 = \ln y$$

$$\therefore y = e^{x+4}$$

$$\text{i.e., } f^{-1}(x) = e^{x+4}$$

ii**iii** domain of f is $\{x : x > 0\}$, range is $\{y : y \in \mathcal{R}\}$ domain of f^{-1} is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > 0\}$ **d i**

$$f(x) = \ln(x - 1) + 2, \quad x > 1$$

$$\text{i.e., } y = \ln(x - 1) + 2 \text{ and}$$

has inverse function

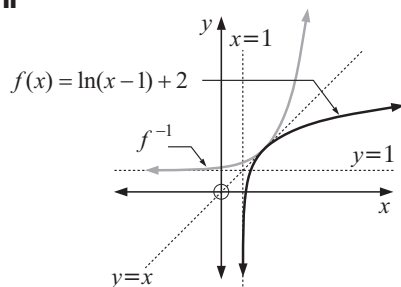
$$x = \ln(y - 1) + 2$$

$$\text{i.e., } \ln(y - 1) = x - 2$$

$$\therefore y - 1 = e^{x-2}$$

$$\therefore y = e^{x-2} + 1$$

$$\text{or } f^{-1}(x) = e^{x-2} + 1$$

ii**iii** domain of f is $\{x : x > 1\}$, range is $\{y : y \in \mathcal{R}\}$ domain of f^{-1} is $\{x : x \in \mathcal{R}\}$, range is $\{y : y > 1\}$

2 $f(x) = e^{2x}$ and $g(x) = 2x - 1$

a Now for $f(x) = e^{2x}$, i.e., $y = e^{2x}$
the inverse function is

$$\begin{aligned} x &= e^{2y} \\ \therefore 2y &= \ln x \\ \therefore y &= \frac{1}{2} \ln x \\ \text{i.e., } f^{-1}(x) &= \frac{1}{2} \ln x \\ \therefore (f^{-1} \circ g)(x) &= f^{-1}(g(x)) \\ &= \frac{1}{2} \ln(g(x)) \\ &= \frac{1}{2} \ln(2x - 1) \end{aligned}$$

b $(g \circ f)(x)$
 $= g(f(x))$
 $= 2f(x) - 1$
 $= 2e^{2x} - 1$ i.e., $y = 2e^{2x} - 1$

which has inverse function

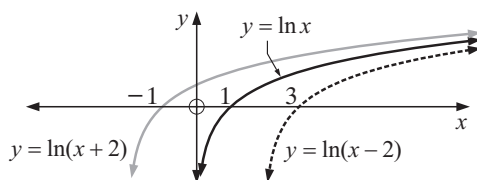
$$\begin{aligned} x &= 2e^{2y} - 1 \\ \text{i.e., } x + 1 &= 2e^{2y} \\ \therefore \frac{x+1}{2} &= e^{2y} \\ \therefore 2y &= \ln\left(\frac{x+1}{2}\right) \\ \therefore y &= \frac{1}{2} \ln\left(\frac{x+1}{2}\right) \\ \therefore (g \circ f)^{-1}(x) &= \frac{1}{2} \ln\left(\frac{x+1}{2}\right) \end{aligned}$$

3 a $y = \ln x$ cuts the x -axis when $y = 0$
 $\therefore \ln x = 0$
 $\therefore x = e^0 = 1$

So, graph A is that of $y = \ln x$.

Note: x -intercept of $y = \ln(x - 2)$
is when $x - 2 = e^0 = 1$
i.e., $x = 3$

b The x -intercept of $y = \ln(x + 2)$
occurs when $x + 2 = e^0 = 1$
 $\therefore x = -1$



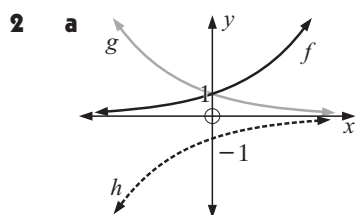
c $y = \ln x$ has a VA of $x = 0$
 $y = \ln(x + 2)$ has a VA of $x = -2$

$y = \ln(x - 2)$ has a VA of $x = 2$

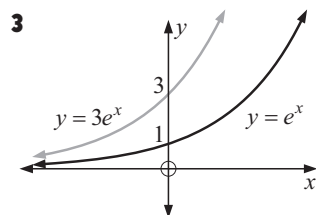
4 Since $y = \ln x^2$ then $y = 2 \ln x$ {log law}
i.e., new y -values are $2 \times$ old y -values. So, she is correct.

REVIEW SET 5A

1 a $\div 54.6$ **b** $\div 22.2$ **c** $\div 0.0613$ **d** $\div 6.07$

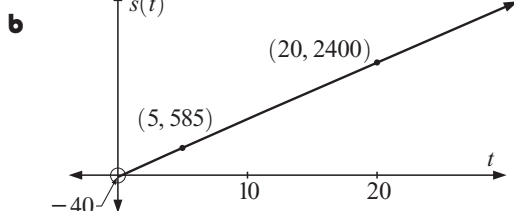


b i g is the reflection of f in the y -axis
ii h is the reflection of g in the x -axis



4 $s(t) = 120t - 40e^{-\frac{t}{5}}$ metres

a i $s(0) = 0 - 40e^0$
 $= -40$ m
ii $s(5) = 600 - 40e^{-1}$
 $\div 585$ m
iii $s(20) = 2400 - 40e^{-4}$
 $\div 2399.3$ m

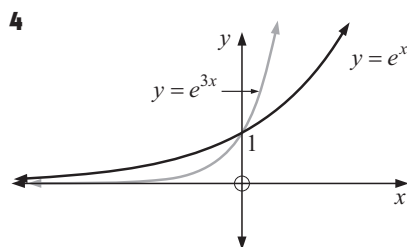
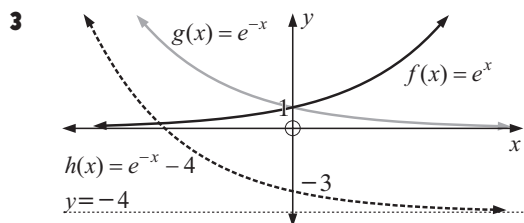


- 5** a $\ln(e^5) = 5$
 {as $\ln e^a = a$ } b $\ln(\sqrt{e}) = \ln e^{\frac{1}{2}}$
 $= \frac{1}{2}$ c $\ln\left(\frac{1}{e}\right) = \ln e^{-1}$
 $= -1$
- 6** a $\ln(e^{2x}) = 2x$ b $\ln(e^2 e^x) = \ln(e^{2+x})$
 $= 2 + x$ c $\ln\left(\frac{e}{e^x}\right) = \ln(e^{1-x})$
 $= 1 - x$
- 7** a $\ln x = 5$
 $\therefore x = e^5$
 $\therefore x \div 148$ b $3 \ln x + 2 = 0$
 $\therefore 3 \ln x = -2$
 $\therefore \ln x = -\frac{2}{3}$
 $\therefore x = e^{-\frac{2}{3}}$
 $\therefore x \div 0.513$
- 8** a $\ln 6 + \ln 4$
 $= \ln(6 \times 4)$
 $= \ln 24$ b $\ln 60 - \ln 20$
 $= \ln\left(\frac{60}{20}\right)$
 $= \ln 3$ c $\ln 4 + \ln 1$
 $= \ln 4 + 0$
 $= \ln 4$ d $\ln 200 - \ln 8 + \ln 5$
 $= \ln\left(\frac{200}{8}\right) + \ln 5$
 $= \ln\left(\frac{200}{8} \times 5\right)$
 $= \ln 125$
- 9** a $\ln 32 = \ln 2^5$
 $= 5 \ln 2$ b $\ln 125 = \ln 5^3$
 $= 3 \ln 5$ c $\ln 729 = \ln 3^6$
 $= 6 \ln 3$
- 10** a $e^x = 400$
 $\therefore x = \ln 400$
 $\therefore x \div 5.99$ b $e^{2x+1} = 11$
 $\therefore 2x + 1 = \ln 11$
 $\therefore 2x = \ln 11 - 1$
 $\therefore x = \frac{\ln 11 - 1}{2}$
 $\therefore x \div 0.699$ c $25e^{\frac{x}{2}} = 750$
 $\therefore e^{\frac{x}{2}} = 30$
 $\therefore \frac{x}{2} = \ln 30$
 $\therefore x = 2 \ln 30$
 $\therefore x \div 6.80$
- d** $e^{2x} = 7e^x - 12$ $\therefore e^{2x} - 7e^x + 12 = 0$ $\therefore e^x = 3$ or $e^x = 4$
 $\therefore (e^x - 3)(e^x - 4) = 0$ $\therefore x = \ln 3$ or $\ln 4$
 $\therefore x \div 1.10$ or 1.39

REVIEW SET 5B

- 1** a $\frac{1}{e^3}$
 $= e^{-3}$ b $\frac{\sqrt{e}}{e^2}$
 $= \frac{e^{\frac{1}{2}}}{e^2}$
 $= e^{\frac{1}{2}-2}$
 $= e^{-\frac{3}{2}}$ c $e^3 \sqrt{e^3}$
 $= e^3 \times (e^3)^{\frac{1}{2}}$
 $= e^{3+\frac{3}{2}}$
 $= e^{\frac{9}{2}}$ d $\sqrt{10e}$
 $= (e^{\ln 10} e)^{\frac{1}{2}}$
 $= e^{\frac{1}{2}(\ln 10 + 1)}$
 $(\div e^{1.65})$

- 2** a $\div 26.9401$ b $\div 0.109447$



Each function has domain $\{x : x \in \mathcal{R}\}$

Range of f is $\{y : y > 0\}$

Range of g is $\{y : y > 0\}$

Range of h is $\{y : y > -4\}$

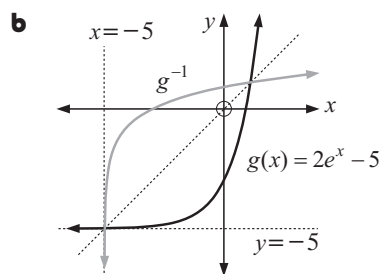
$$\begin{array}{ll}
 \mathbf{5} \quad \mathbf{a} & \ln(e\sqrt{e}) \\
 & = \ln(e^1 e^{\frac{1}{2}}) \\
 & = \ln e^{1\frac{1}{2}} \\
 & = 1\frac{1}{2} \\
 \mathbf{b} & \ln\left(\frac{1}{e^3}\right) \\
 & = \ln e^{-3} \\
 & = -3 \\
 \mathbf{c} & \ln\left(\frac{e}{\sqrt{e^5}}\right) = \ln\left(\frac{e^1}{e^{\frac{5}{2}}}\right) = \ln\left(e^{1-\frac{5}{2}}\right) = \ln e^{-\frac{3}{2}} = -\frac{3}{2}
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{6} \quad \mathbf{a} & 20 = e^{\ln 20} \\
 & \div e^{3.00} \\
 \mathbf{b} & 3000 = e^{\ln 3000} \\
 & \div e^{8.01} \\
 \mathbf{c} & 0.075 = e^{\ln(0.075)} \\
 & \div e^{-2.59}
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{7} \quad \mathbf{a} & 4 \ln 2 + 2 \ln 3 & \mathbf{b} & \frac{1}{2} \ln 9 - \ln 2 \\
 & = \ln 2^4 + \ln 3^2 & & = \ln 9^{\frac{1}{2}} - \ln 2 \\
 & = \ln(16 \times 9) & & = \ln 3 - \ln 2 \\
 & = \ln 144 & & = \ln\left(\frac{3}{2}\right) \\
 & & \mathbf{c} & 2 \ln 5 - 1 \\
 & & & = \ln 5^2 - \ln e^1 \\
 & & & = \ln\left(\frac{25}{e}\right) \\
 & & \mathbf{d} & \frac{1}{4} \ln 81 \\
 & & & = \ln(3^4)^{\frac{1}{4}} \\
 & & & = \ln 3^1 \\
 & & & = \ln 3
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{8} \quad \mathbf{a} & \ln P = 1.5 \ln Q + \ln T \\
 & \therefore \ln P = \ln Q^{\frac{3}{2}} + \ln T \\
 & = \ln\left(Q^{\frac{3}{2}} T\right) \\
 & \therefore P = Q^{\frac{3}{2}} T \\
 \mathbf{b} & \ln M = 1.2 - 0.5 \ln N \\
 & \therefore \ln M + \ln N^{\frac{1}{2}} = 1.2 \\
 & \therefore \ln(M\sqrt{N}) = 1.2 \\
 & \therefore M\sqrt{N} = e^{1.2} \\
 & \therefore M \div \frac{3.32}{\sqrt{N}}
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{9} \quad \mathbf{a} & g(x) = 2e^x - 5 \text{ has inverse} \\
 & \text{function } x = 2e^y - 5 \\
 & \therefore 2e^y = x + 5 \\
 & \therefore e^y = \frac{x+5}{2} \\
 & \text{i.e., } y = \ln\left(\frac{x+5}{2}\right) \\
 & \therefore g^{-1}(x) = \ln\left(\frac{x+5}{2}\right)
 \end{array}$$



$$\begin{array}{ll}
 \mathbf{c} & \text{domain of } g \text{ is } \{x : x \in \mathcal{R}\}, \text{ range is } \{y : y > -5\} \\
 & \text{domain of } g^{-1} \text{ is } \{x : x > -5\}, \text{ range is } \{y : y \in \mathcal{R}\}
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{10} & W_t = 8000 \times e^{-\frac{t}{20}} \text{ grams} \\
 & W_0 = 8000e^0 \\
 & = 8000 \times 1 \\
 & = 8000 \text{ grams}
 \end{array}$$

$$\mathbf{b} \quad \text{When } W_t = 1000,$$

$$8000e^{-\frac{t}{20}} = 1000$$

$$\therefore e^{-\frac{t}{20}} = \frac{1}{8}$$

$$\therefore -\frac{t}{20} = \ln\left(\frac{1}{8}\right)$$

$$\therefore t = -20 \ln\left(\frac{1}{8}\right)$$

$$\therefore t \div 41.6 \text{ weeks}$$

$$\mathbf{a} \quad \text{When } W_t = \frac{1}{2} \times 8000 \text{ grams,}$$

$$8000e^{-\frac{t}{20}} = 4000$$

$$\therefore e^{-\frac{t}{20}} = 0.5$$

$$\therefore -\frac{t}{20} = \ln(0.5)$$

$$\therefore t = -20 \ln(0.5) \div 13.9 \text{ weeks}$$

$$\mathbf{c} \quad \text{When } W_t = 0.1\% \text{ of } W_0$$

$$= \frac{1}{1000} \text{ of } 8000 \text{ g}$$

$$= 8 \text{ g,}$$

$$8000e^{-\frac{t}{20}} = 8$$

$$\therefore e^{-\frac{t}{20}} = 0.001$$

$$\therefore -\frac{t}{20} = \ln(0.001)$$

$$\therefore t = -20 \ln(0.001) \div 138 \text{ weeks}$$

Chapter 6

GRAPHING AND TRANSFORMING FUNCTIONS

EXERCISE 6A

1 $f(x) = x$

a $f(2x) = 2x$

b $f(x) + 2$
 $= x + 2$

c $\frac{1}{2}f(x) = \frac{1}{2}x$

d $2f(x) + 3$
 $= 2x + 3$

2 $f(x) = x^3$

a $f(4x)$
 $= (4x)^3$
 $= 64x^3$

b $\frac{1}{2}f(2x)$
 $= \frac{1}{2}(2x)^3$
 $= \frac{1}{2} \times 8x^3$
 $= 4x^3$

c $f(x+1)$
 $= (x+1)^3$
 $= x^3 + 3x^2 + 3x + 1$

d $2f(x+1) - 3$
 $= 2(x+1)^3 - 3$
 $= 2(x^3 + 3x^2 + 3x + 1) - 3$
 $= 2x^3 + 6x^2 + 6x - 1$

3 $f(x) = 2^x$

a $f(2x) = 2^{2x}$
 $= 4^x$

b $f(-x) + 1$
 $= 2^{-x} + 1$

c $f(x-2) + 3$
 $= 2^{x-2} + 3$

d $2f(x) + 3$
 $= 2 \times 2^x + 3$
 $= 2^{1+x} + 3$

4 $f(x) = \frac{1}{x}$

a $f(-x)$
 $= \frac{1}{(-x)}$
 $= -\frac{1}{x}$

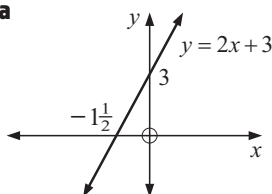
b $f(\frac{1}{2}x)$
 $= \frac{1}{\frac{1}{2}x}$
 $= \frac{2}{x}$

c $2f(x) + 3$
 $= 2\left(\frac{1}{x}\right) + 3$
 $= \frac{2}{x} + 3$

d $3f(x-1) + 2$
 $= 3\left(\frac{1}{x-1}\right) + 2$
 $= \frac{3}{x-1} + 2$

EXERCISE 6B

1 a

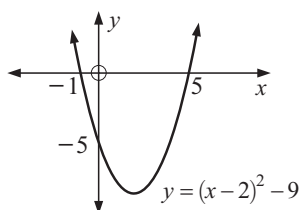


b i When $y = 0$, $2x + 3 = 0 \therefore x = -\frac{3}{2}$
 $\therefore$ x-intercept is $-1\frac{1}{2}$

ii When $x = 0$, $y = 0 + 3 = 3$
 $\therefore$ y-intercept is 3

iii As $y = 2x + 3$, the slope is 2 {the coefficient of x }

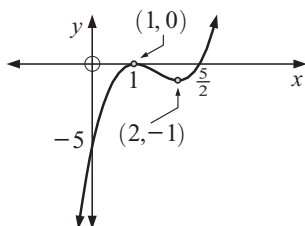
2 a



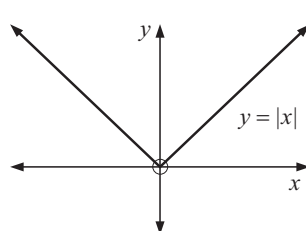
b When $y = 0$,
 $(x-2)^2 - 9 = 0$
 $\therefore (x-2)^2 = 9$
 $\therefore x-2 = \pm 3$
 $\therefore x = 2+3$ or $2-3$
 $\therefore x = 5$ or -1
 $\therefore$ x-intercepts are 5 and -1

When $x = 0$,
 $y = (-2)^2 - 9$
 $= 4 - 9$
 $= -5$
 $\therefore$ y-intercept is -5

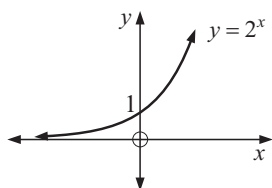
3 a, b



4

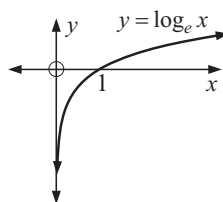


5



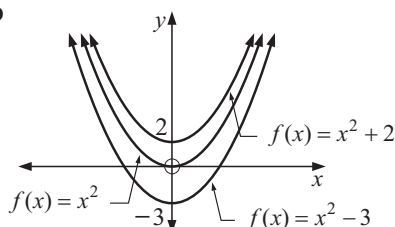
When $x = 0$,
 $y = 2^0 = 1$ ✓
 $2^x > 0$ for all
 x as the graph
 is always above
 the y -axis. ✓

6



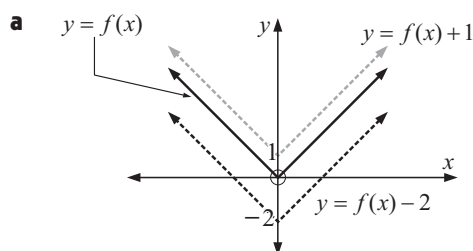
EXERCISE 6C.1

1 a, b

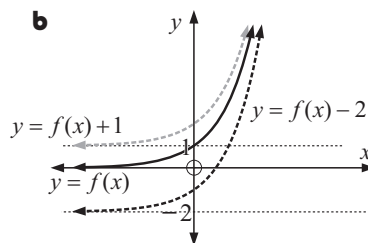


- c i** If $b > 0$, the function is translated vertically upwards through b units.
- ii** If $b < 0$, the function is translated vertically downwards $|b|$ units.

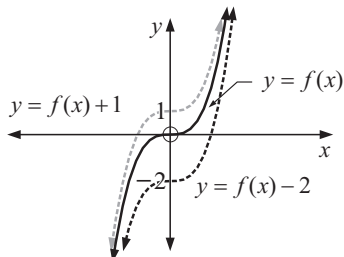
2



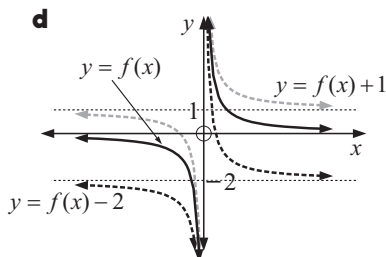
b



c



d

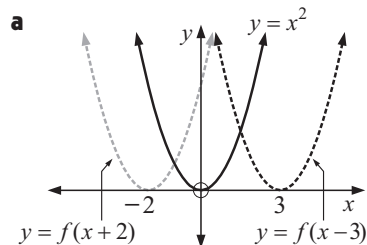


Summary: For $y = f(x) + b$, $y = f(x)$ is translated upwards b units.

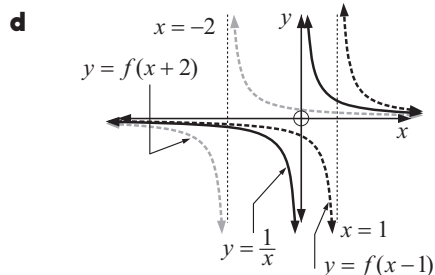
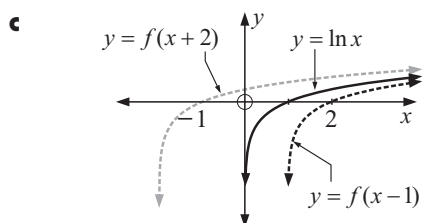
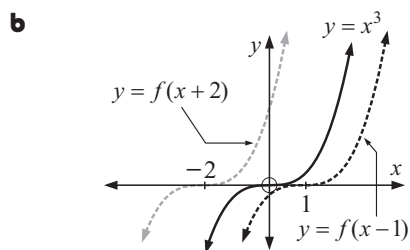
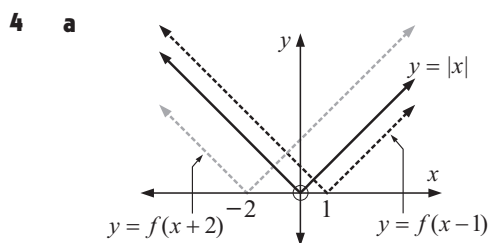
If $b > 0$ movement is vertically upwards.

If $b < 0$ movement is vertically downwards.

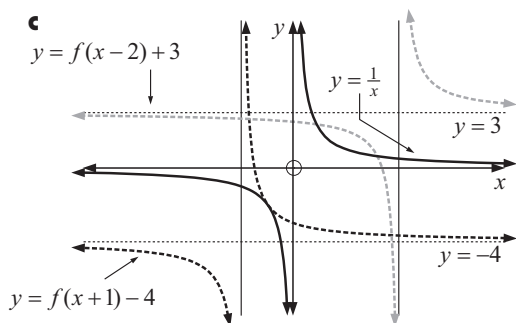
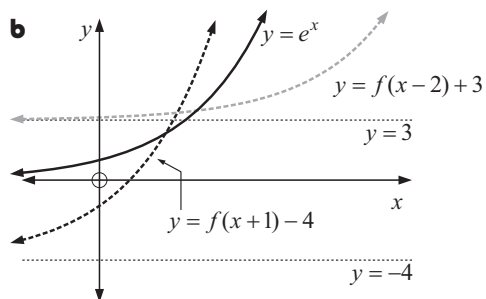
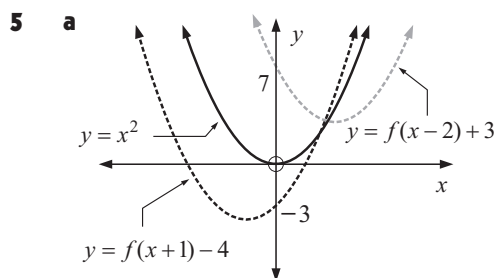
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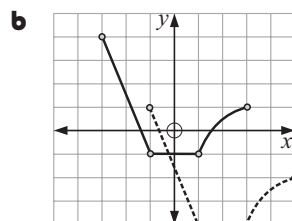
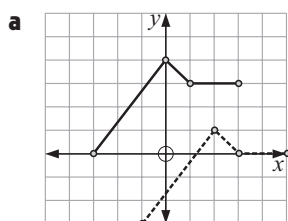
- b i** If $a > 0$, the graph is translated a units right.
- ii** If $a < 0$, the graph is translated $|a|$ units left.

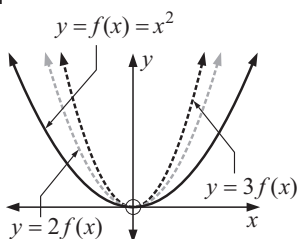
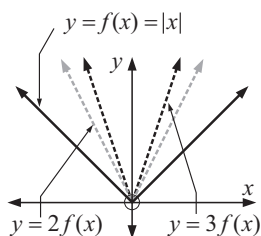
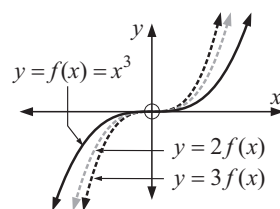
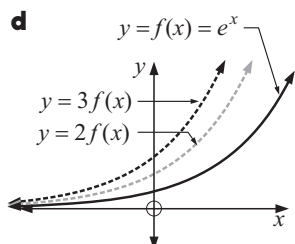
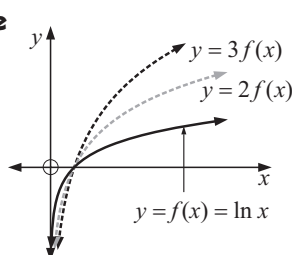
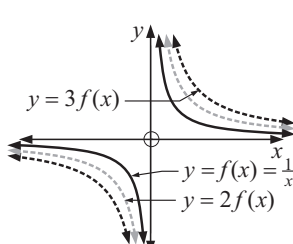
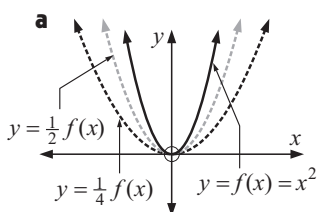
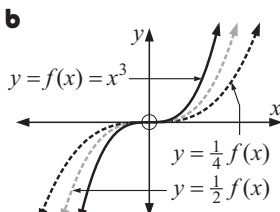
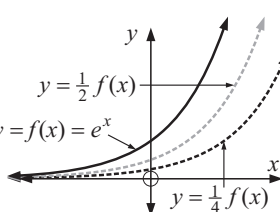


$y = f(x - a)$ is a horizontal translation of $y = f(x)$ through $\begin{bmatrix} a \\ 0 \end{bmatrix}$.

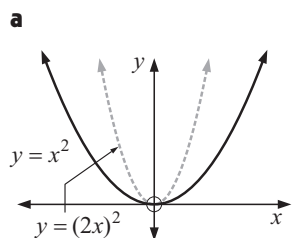
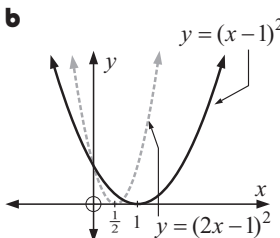
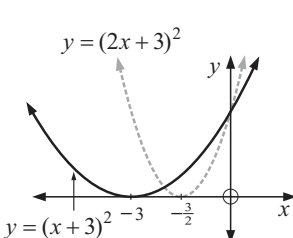
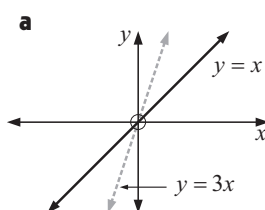
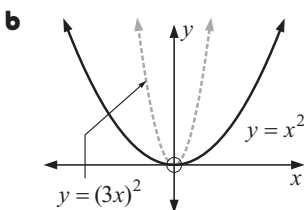
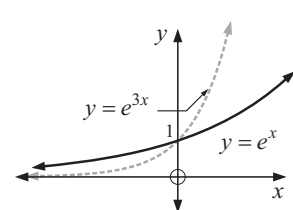


6 A translation of $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

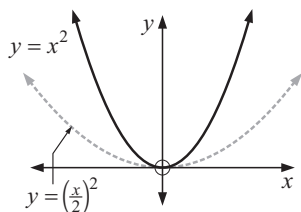


EXERCISE 6C.2
1 a

b

c

d

e

f

2 a

b

c

3

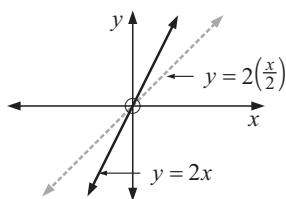
p affects the vertical stretching or compression of the graph of $y = f(x)$ by a factor of p . If $p > 1$, stretching occurs. If $0 < p < 1$, compression occurs.

4 a

b

c

5 a

b

c


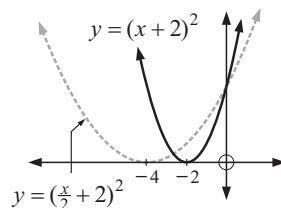
6 a



b



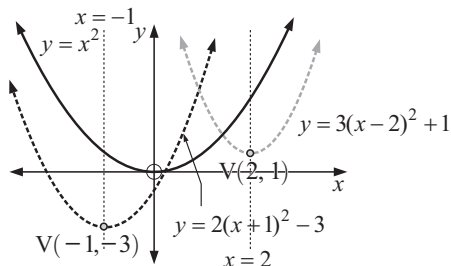
c



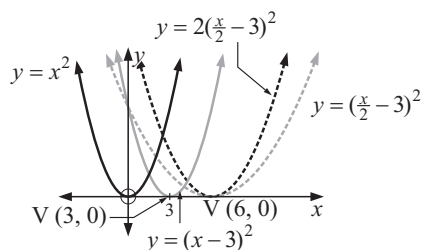
7 k affects the horizontal stretching or compression of $y = f(x)$ by a factor of $\frac{1}{k}$.

If $k > 1$, it moves closer to the y -axis. If $0 < k < 1$, it moves further from the y -axis.

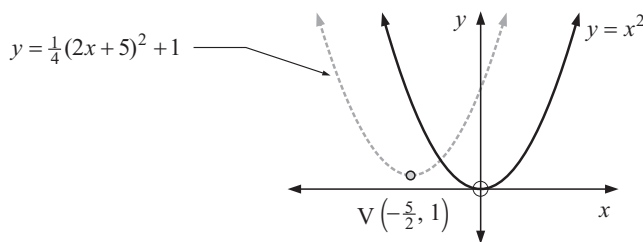
8 a



b

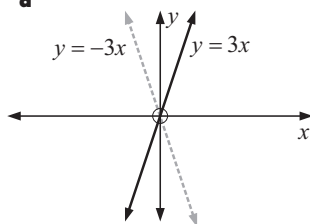


c

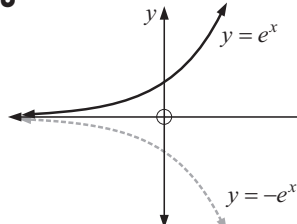


EXERCISE 6C.3

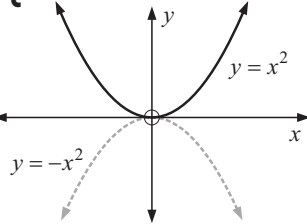
1 a



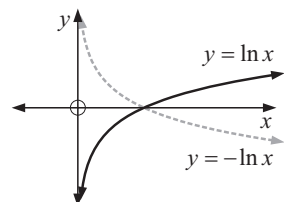
b



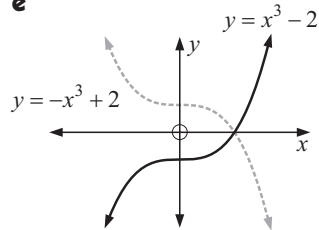
c



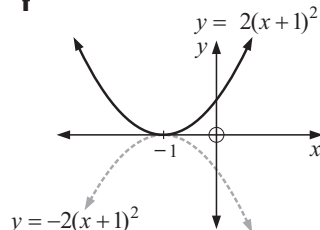
d



e



f



2 $y = -f(x)$ is the reflection of $y = f(x)$ in the x -axis.

3 a i $f(x) = 2x + 1$

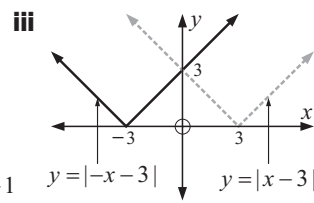
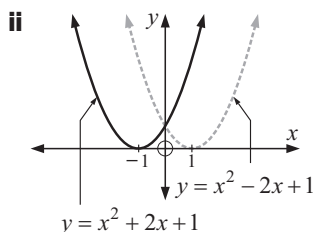
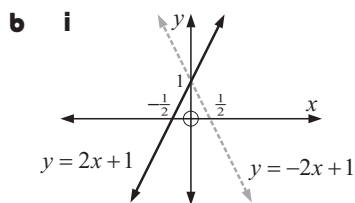
$$\begin{aligned}\therefore f(-x) &= 2(-x) + 1 \\ &= -2x + 1\end{aligned}$$

ii $f(x) = x^2 + 2x + 1$

$$\begin{aligned}\therefore f(-x) &= (-x)^2 + 2(-x) + 1 \\ &= x^2 - 2x + 1\end{aligned}$$

iii $f(x) = |x - 3|$

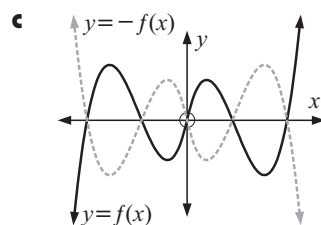
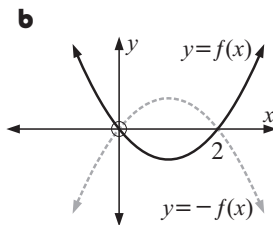
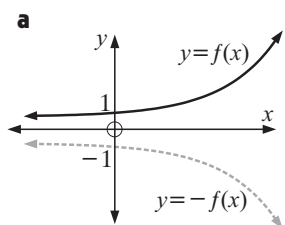
$$\begin{aligned}\therefore f(-x) &= |-x - 3| \\ &= |-(x + 3)| \\ &= |x + 3|\end{aligned}$$



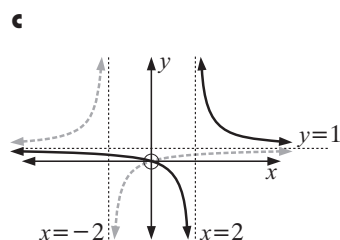
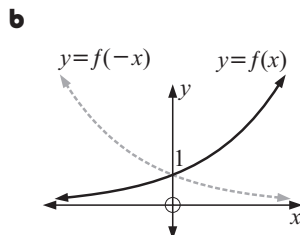
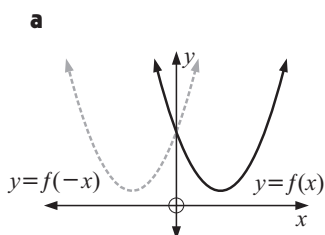
4 $y = f(-x)$ is the reflection of $y = f(x)$ in the y -axis.

EXERCISE 6D

1 $y = -f(x)$ is obtained from $y = f(x)$ by reflecting it in the x -axis.



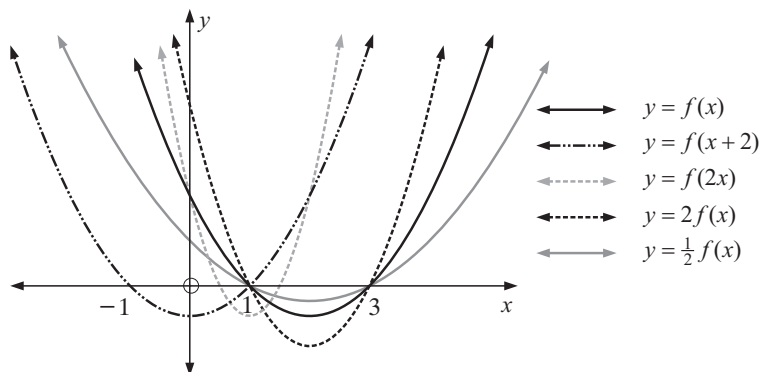
2 $y = f(-x)$ is obtained from $y = f(x)$ by reflecting it in the y -axis.



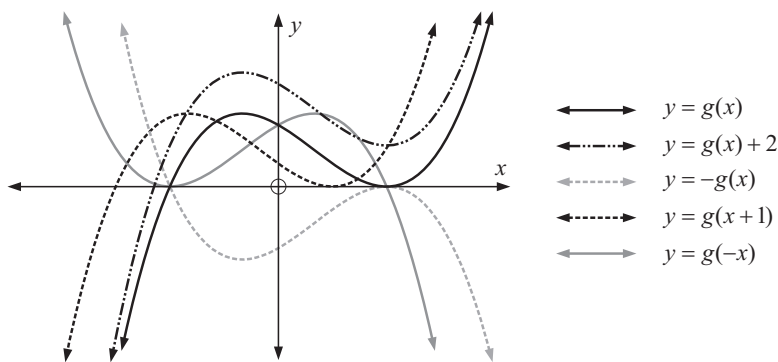
3 $y = 2x^4$ and $y = 6x^4$ are 'thinner' than $y = x^4$ and $y = \frac{1}{2}x^4$ is 'fatter'

$\therefore$ **a** is **A**, **b** is **B**, **c** is **D** and **d** is **C**

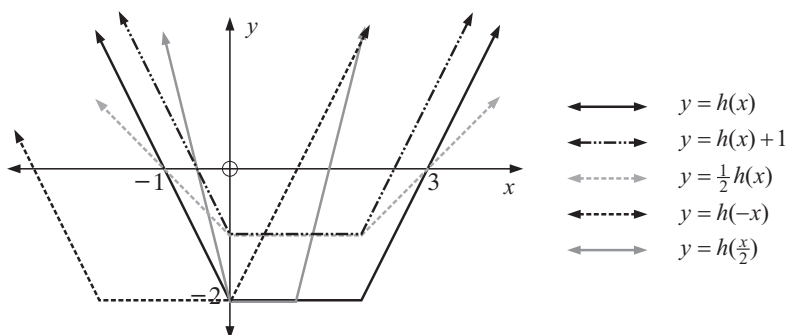
4



5



6

**EXERCISE 6E**

1 a i Under a vertical stretch of factor $\frac{1}{2}$, $y = \frac{1}{x}$ becomes $y = \frac{1}{2} \left(\frac{1}{x} \right)$ i.e., $y = \frac{1}{2x}$.

ii Under a horizontal stretch of factor 3, $y = \frac{1}{x}$ becomes $y = \frac{1}{\left(\frac{x}{3}\right)}$, i.e., $y = \frac{3}{x}$.

iii Under a horizontal translation of -3 , $y = \frac{1}{x}$ becomes $y = \frac{1}{x+3}$.

iv Under a vertical translation of 4, $y = \frac{1}{x}$ becomes $y - 4 = \frac{1}{x}$ i.e., $y = \frac{1}{x} + 4$
i.e., $y = \frac{4x+1}{x}$

v Under all four transformations

$$\frac{1}{x} \text{ becomes } \frac{1}{2x} \text{ becomes } \frac{1}{2\left(\frac{x}{3}\right)} \text{ or } \frac{3}{2x} \text{ becomes } \frac{3}{2(x+3)}$$

$$\text{So, finally } y - 4 = \frac{3}{2(x+3)} \text{ i.e., } y = \frac{3}{2(x+3)} + 4$$

$$\text{or } y = \frac{3+8(x+3)}{2x+6}$$

$$\text{i.e., } y = \frac{8x+27}{2x+6}$$

b From $y = \frac{3}{2(x+3)} + 4$ we see that y is undefined when $x+3=0$ i.e., $x=-3$

$\therefore$ VA is $x=-3$. Domain is $\{x : x \in R, x \neq -3\}$

Also, if $y=4$, $\frac{3}{2(x+3)}=0$ which is not possible as $3 \neq 0$

$\therefore y=4$ is the HA. Range is $\{y : y \in R, y \neq 4\}$

2 a

$$y = \frac{2x+4}{x-1}$$

$$= \frac{2(x-1)+6}{x-1}$$

$$= 2 + \frac{6}{x-1}$$

$$\therefore y-2 = \frac{6}{x-1}$$

$$\therefore (x-1)(y-2) = 6$$

$$\therefore x-1 \neq 0, \quad y-2 \neq 0$$

$$\{\text{otherwise } 0=6\}$$

$$\therefore \text{VA is } x=1, \quad \text{HA is } y=2$$

b To get $y = \frac{2x+4}{x-1} = 2 + \frac{6}{x-1}$ from

$$y = \frac{1}{x} \quad \text{we}$$

- vertically stretch by factor 6

$$\frac{1}{x} \text{ becomes } 6\left(\frac{1}{x}\right) = \frac{6}{x} \text{ then}$$

- translate $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ so $y = \frac{6}{x}$ becomes

$$y-2 = \frac{6}{x-1} \quad \text{i.e., } y = 2 + \frac{6}{x-1}$$

3

a i $y = \frac{2x+3}{x+1}$

$$\therefore y = \frac{2(x+1)+1}{x+1}$$

$$\therefore y = 2 + \frac{1}{x+1}$$

$$\therefore (y-2)(x+1) = 1$$

$$\therefore y \neq 2, \quad x \neq -1$$

$$\text{VA is } x = -1$$

$$\text{HA is } y = 2$$

ii

 Cuts x -axis when

$$y = 0 \quad \therefore 2x+3 = 0$$

$$\therefore x = -\frac{3}{2}$$

$$\therefore x\text{-intercept is } -\frac{3}{2}$$

 Cuts y -axis when $x = 0$

$$\therefore y = \frac{3}{1} = 3$$

$$\therefore y\text{-intercept is } 3$$

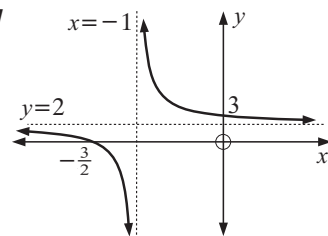
iii

 As $x \rightarrow -1$ (from left), $y \rightarrow -\infty$

 As $x \rightarrow -1$ (from right), $y \rightarrow \infty$

 As $x \rightarrow -\infty$, $y \rightarrow 2$ (from below)

 As $x \rightarrow \infty$, $y \rightarrow 2$ (from above)

iv

v Translate it $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$
b i

$$y = \frac{3}{x-2}$$

$$\therefore (x-2)y = 3$$

$$\therefore x-2 \neq 0, \quad y \neq 0$$

$$\therefore \text{VA is } x = 2$$

$$\text{HA is } y = 0$$

ii

 Cuts x -axis when

$$y = 0 \quad \therefore \frac{3}{x-2} = 0$$

which is not possible.

 $\therefore$ no x -intercept

 Cuts y -axis when $x = 0$

$$\therefore y = \frac{3}{-2} = -1\frac{1}{2}$$

$$\therefore y\text{-intercept is } -1\frac{1}{2}$$

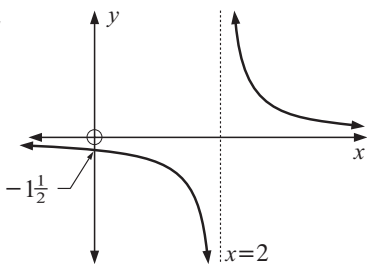
iii

 As $x \rightarrow 2$ (from left), $y \rightarrow -\infty$

 As $x \rightarrow 2$ (from right), $y \rightarrow \infty$

 As $x \rightarrow \infty$, $y \rightarrow 0$ (from above)

 As $x \rightarrow -\infty$, $y \rightarrow 0$ (from below)

iv

v Vertically stretch with factor 3, then translate $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

c i

$$y = \frac{2x-1}{3-x}$$

$$= \frac{-2x+1}{x-3}$$

$$= \frac{-2(x-3)-5}{x-3}$$

$$= -2 - \frac{5}{x-3}$$

$$\therefore (y+2)(x-3) = -5$$

$$\therefore x \neq 3, \quad y \neq -2$$

$$\therefore \text{VA is } x = 3$$

$$\text{HA is } y = -2$$

ii

 Cuts x -axis when

$$y = 0 \quad \therefore 2x-1 = 0$$

$$\therefore x = \frac{1}{2}$$

$$\therefore x\text{-intercept is } \frac{1}{2}$$

 Cuts y -axis when $x = 0$

$$\therefore y = \frac{-1}{3} = -\frac{1}{3}$$

$$\therefore y\text{-intercept is } -\frac{1}{3}$$

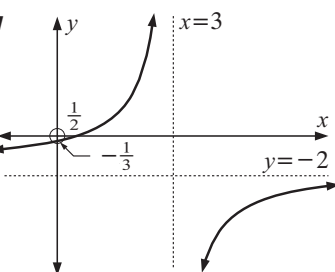
iii

 As $x \rightarrow 3$ (from left), $y \rightarrow \infty$

 As $x \rightarrow 3$ (from right), $y \rightarrow -\infty$

 As $x \rightarrow -\infty$, $y \rightarrow -2$ (from above)

 As $x \rightarrow \infty$, $y \rightarrow -2$ (from below)

iv

v Vertically stretch with factor -5 , then translate $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

$$\begin{aligned} \text{d i } y &= \frac{5x-1}{2x+1} \\ &= \frac{\frac{5}{2}x - \frac{1}{2}}{x + \frac{1}{2}} \\ &= \frac{\frac{5}{2}(x + \frac{1}{2}) - \frac{7}{4}}{x + \frac{1}{2}} \\ &= \frac{5}{2} - \frac{\frac{7}{4}}{x + \frac{1}{2}} \end{aligned}$$

$$\therefore (y - \frac{5}{2})(x + \frac{1}{2}) = -\frac{7}{4}$$

$$\therefore y \neq \frac{5}{2}, x \neq -\frac{1}{2}$$

$$\therefore \text{VA is } x = -\frac{1}{2}, \text{HA is } y = \frac{5}{2}$$

$$\begin{aligned} \text{ii } &\text{Cuts } x\text{-axis when} \\ &y = 0 \therefore 5x - 1 = 0 \\ &\therefore x = \frac{1}{5} \\ &\therefore x\text{-intercept is } \frac{1}{5} \end{aligned}$$

$$\text{Cuts } y\text{-axis when } x = 0$$

$$\therefore y = \frac{-1}{1} = -1$$

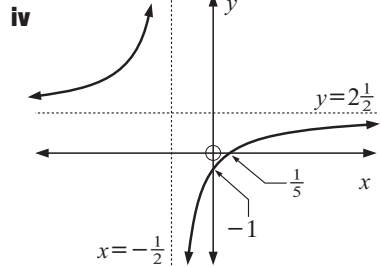
$$\therefore y\text{-intercept is } -1$$

$$\text{iii } \text{As } x \rightarrow -\frac{1}{2} \text{ (from left), } y \rightarrow \infty$$

$$\text{As } x \rightarrow -\frac{1}{2} \text{ (from right), } y \rightarrow -\infty$$

$$\text{As } x \rightarrow \infty, y \rightarrow 2\frac{1}{2} \text{ (from below)}$$

$$\text{As } x \rightarrow -\infty, y \rightarrow 2\frac{1}{2} \text{ (from above)}$$



v Vertically stretch with factor $-\frac{7}{4}$, then translate $\begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{bmatrix}$.

$$4 \quad N = 20 + \frac{100}{t+2} \quad \text{weeds/ha}$$

$$\text{a When } t = 0$$

$$N = 20 + \frac{100}{2}$$

$$= 20 + 50$$

$$= 70 \text{ weeds/ha}$$

$$\text{b When } t = 8,$$

$$N = 20 + \frac{100}{10}$$

$$= 20 + 10$$

$$= 30 \text{ weeds/ha}$$

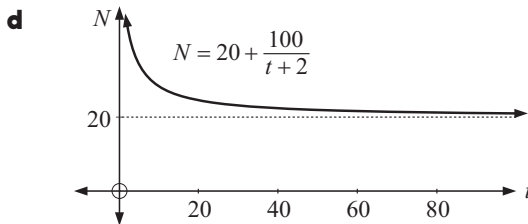
$$\text{c When } N = 40,$$

$$20 + \frac{100}{t+2} = 40$$

$$\therefore \frac{100}{t+2} = 20$$

$$\therefore t+2 = 5$$

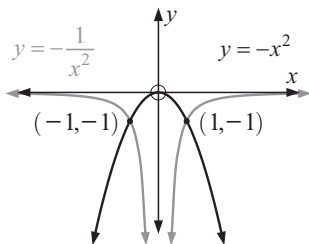
$$\therefore t = 3 \text{ days}$$



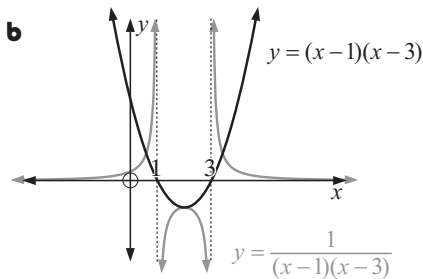
e No, the number of weeds/ha will approach 20 (from above).

EXERCISE 6F

1 a



b



$$2 \quad \text{Notice that } f(x) = \frac{1}{f(x)} \quad \text{i.e., } y = \frac{1}{y} \quad \therefore y^2 = 1 \quad \therefore y = \pm 1.$$

$$\text{For 1a When } y = 1, -x^2 = 1 \text{ which has no real solutions}$$

$$\text{When } y = -1, -x^2 = -1 \therefore x^2 = 1 \therefore x = \pm 1$$

$$\text{So invariant points are } (1, -1) \text{ and } (-1, -1).$$

$$\text{For 1b When } y = 1, (x-1)(x-3) = 1 \quad \text{when } y = -1, (x-1)(x-3) = -1$$

$$\therefore x^2 - 4x + 3 = 1$$

$$\therefore x^2 - 4x + 2 = 0$$

$$\therefore x = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

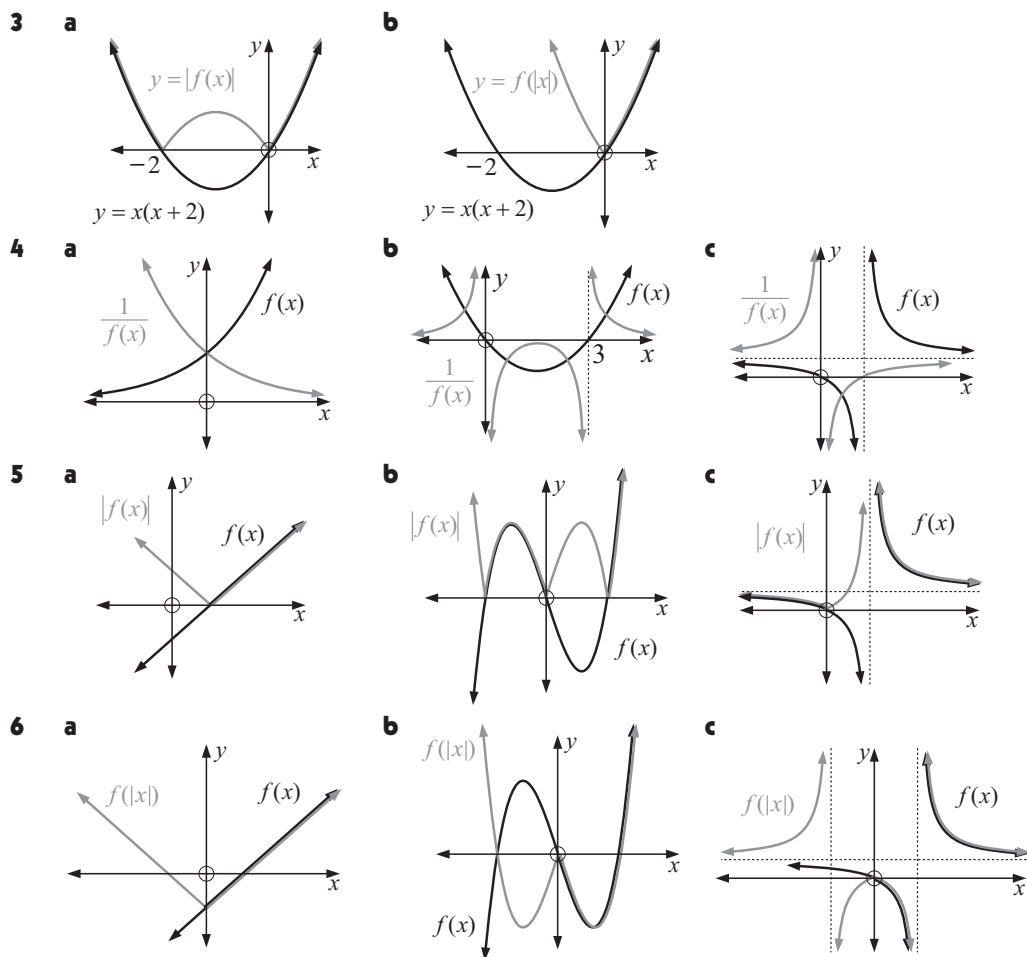
$$\therefore x^2 - 4x + 3 = -1$$

$$\therefore x^2 - 4x + 4 = 0$$

$$\therefore (x-2)^2 = 0$$

$$\therefore x = 2$$

$$\therefore \text{invariant points are } (2 - \sqrt{2}, 1), (2, -1), (2 + \sqrt{2}, 1)$$



REVIEW SET 6A

1 $f(x) = x^2 - 2x$

a

$$\begin{aligned} f(3) &= 3^2 - 2(3) \\ &= 9 - 6 \\ &= 3 \end{aligned}$$

d

$$\begin{aligned} f(-x) &= (-x)^2 - 2(-x) \\ &= x^2 + 2x \end{aligned}$$

b

$$\begin{aligned} f(-2) &= (-2)^2 - 2(-2) \\ &= 4 + 4 \\ &= 8 \end{aligned}$$

e

$$\begin{aligned} 3f(x) - 2 &= 3(x^2 - 2x) - 2 \\ &= 3x^2 - 6x - 2 \end{aligned}$$

c

$$\begin{aligned} f(2x) &= (2x)^2 - 2(2x) \\ &= 4x^2 - 4x \end{aligned}$$

2 $f(x) = 5 - x - x^2$

a

$$\begin{aligned} f(4) &= 5 - 4 - 4^2 \\ &= 1 - 16 \\ &= -15 \end{aligned}$$

b

$$\begin{aligned} f(-1) &= 5 - (-1) - (-1)^2 \\ &= 5 + 1 - 1 \\ &= 5 \end{aligned}$$

c

$$\begin{aligned} f(x-1) &= 5 - (x-1) - (x-1)^2 \\ &= 5 - x + 1 - [x^2 - 2x + 1] \\ &= 6 - x - x^2 + 2x - 1 \\ &= -x^2 + x + 5 \end{aligned}$$

d $f\left(\frac{x}{2}\right)$

$$= 5 - \left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)^2$$

$$= 5 - \frac{x}{2} - \frac{x^2}{4}$$

e $2f(x) - f(-x)$

$$= 2(5 - x - x^2) - [5 - (-x) - (-x)^2]$$

$$= 10 - 2x - 2x^2 - [5 + x - x^2]$$

$$= 10 - 2x - 2x^2 - 5 - x + x^2$$

$$= -x^2 - 3x + 5$$

3 $f(x) = \frac{4}{x}$

a $f(-4)$

$$= \frac{4}{-4}$$

$$= -1$$

b $f(2x)$

$$= \frac{4}{2x}$$

$$= \frac{2}{x}$$

c $f\left(\frac{x}{2}\right)$

$$= \frac{4}{\frac{x}{2}}$$

$$= 4 \times \frac{2}{x}$$

$$= \frac{8}{x}$$

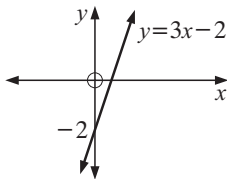
d $4f(x+2) - 3$

$$= 4\left(\frac{4}{x+2}\right) - 3$$

$$= \frac{16}{x+2} - 3$$

$$\left(\text{or } \frac{16 - 3(x+2)}{x+2} = \frac{10 - 3x}{x+2}\right)$$

4 a



b i When $y = 0$,
 $3x - 2 = 0$

$$\therefore x = \frac{2}{3}$$

$$\therefore x\text{-intercept is } \frac{2}{3}$$

ii When $x = 0$,
 $y = 0 - 2 = -2$

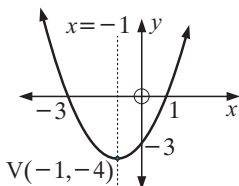
$$\therefore y\text{-intercept is } -2$$

iii As $y = 3x - 2$, the slope is 3 {coefficient of x }

c i When $x = 0.3$,
 $y = 3(0.3) - 2$
 $= 0.9 - 2$
 $= -1.1$

ii When $y = 0.7$,
 $3x - 2 = 0.7$
 $\therefore 3x = 2.7$
 $\therefore x = 0.9$

5 a



b i When $y = 0$,
 $(x+1)^2 - 4 = 0$

$$\therefore (x+1)^2 = 4$$

$$\therefore x+1 = \pm 2$$

$$\therefore x = 2 - 1 \text{ or } -2 - 1$$

$$\text{i.e., } x = 1 \text{ or } -3$$

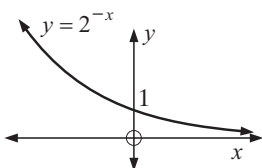
$$\therefore x\text{-intercepts are } 1, -3$$

ii When $x = 0$,
 $y = 1^2 - 4$
 $= -3$

$$\therefore y\text{-intercept is } -3$$

c Since $y = (x+1)^2 - 4$ then $y+4 = (x+1)^2$ which is obtained from $y = x^2$ under a translation of $\begin{bmatrix} -1 \\ -4 \end{bmatrix}$. So, the vertex must be $(-1, -4)$.

6 a



b i $x \rightarrow \infty$ means x is very large and positive.

We see the graph approaching the x -axis

$$\text{i.e., } y \rightarrow 0 \therefore \text{true.}$$

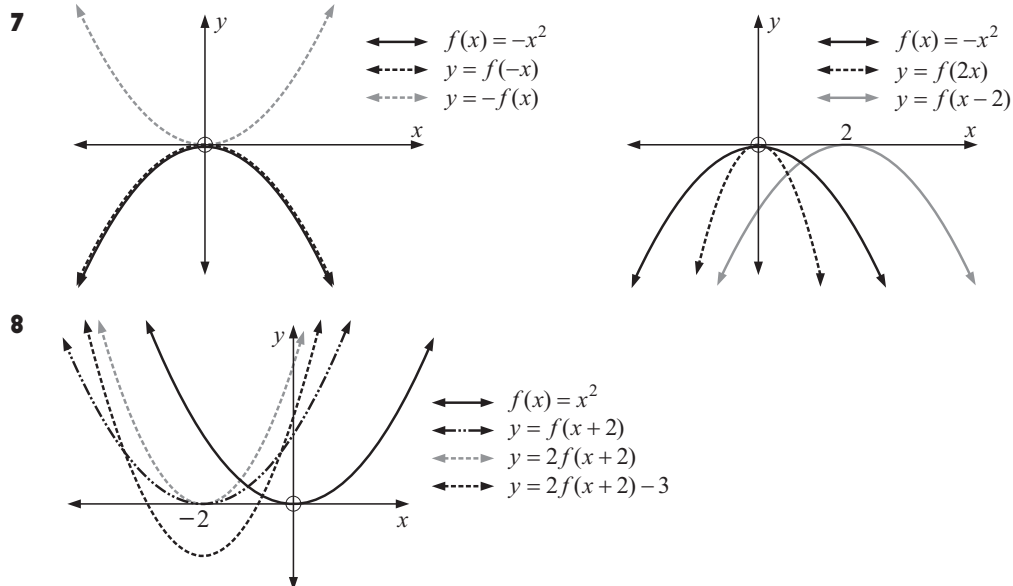
ii $x \rightarrow -\infty$ means x is very large and negative.

We see the graph heading for ∞ and positive

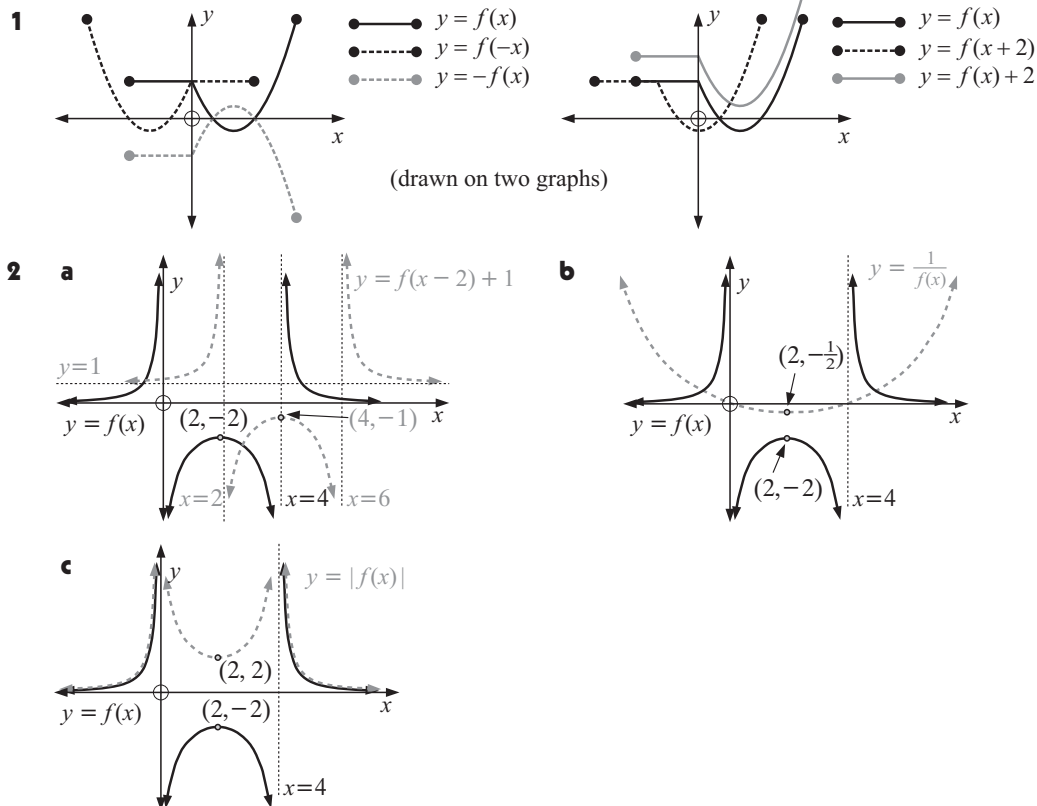
$$\therefore \text{statement is false.}$$

- iii The y -intercept is 1, when $x = 0$, $y = 2^0 = 1 \quad \therefore$ **false**.
 iv The graph is above the x -axis for all $x \quad \therefore 2^{-x} > 0$ for all $x \quad \therefore$ **true**.

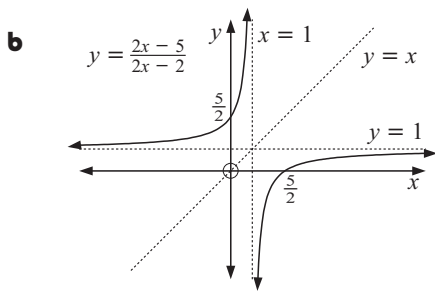
So that you can see the answers more easily in question 7, they have been drawn on two graphs.



REVIEW SET 6B



- 3 a** $y = \frac{1}{x}$ under a reflection in the y -axis becomes $y = \frac{1}{(-x)} = -\frac{1}{x}$
- $y = -\frac{1}{x}$ under a vertical stretch of factor 3 becomes $y = 3\left(-\frac{1}{x}\right) = -\frac{3}{x}$
- $y = -\frac{3}{x}$ under a horizontal stretch of factor $\frac{1}{2}$ becomes $y = \frac{-3}{(2x)} = \frac{-3}{2x}$
- $y = -\frac{3}{2x}$ under a translation of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ becomes $y - 1 = \frac{-3}{2(x - 1)}$



i.e., $y = 1 - \frac{3}{2x - 2}$

i.e., $y = \frac{2x - 2}{2x - 2} - \frac{3}{2x - 2}$

i.e., $y = \frac{2x - 5}{2x - 2}$

domain of $f(x)$ is $\{x: x \neq 1\}$

range of $f(x)$ is $\{y: y \neq 1\}$

- c** Yes, since it is a one-to-one function (passes both the vertical and horizontal line tests).

d $f(x) = y = \frac{2x - 5}{2x - 2}$

$\therefore$ inverse function is $x = \frac{2y - 5}{2y - 2}$

$\therefore (2y - 2)x = 2y - 5$

$\therefore 2xy - 2x = 2y - 5$

$\therefore y(2x - 2) = 2x - 5$

$\therefore y = \frac{2x - 5}{2x - 2}$

$\therefore f^{-1}(x) = f(x) = \frac{2x - 5}{2x - 2}$

i.e., it is a self-inverse function.

Also, the graph of $f(x)$ is symmetrical about the line $y = x$.

4 $y = \frac{2x - 3}{3x + 5}$

$= \frac{\frac{2}{3}x - 1}{x + \frac{5}{3}}$

$= \frac{\frac{2}{3}\left(x + \frac{5}{3}\right) - 1 - \frac{10}{9}}{x + \frac{5}{3}}$

$= \frac{\frac{2}{3}\left(x + \frac{5}{3}\right) - \frac{19}{9}}{x + \frac{5}{3}}$

$= \frac{2}{3} - \frac{\frac{19}{9}}{\left(x + \frac{5}{3}\right)}$

$\therefore (y - \frac{2}{3})(x + \frac{5}{3}) = -\frac{19}{9}$

$\therefore x \neq -\frac{5}{3}, y \neq \frac{2}{3}$

$\therefore$ VA is $x = -\frac{5}{3}$, HA is $y = \frac{2}{3}$

When $x = 0$, $y = -\frac{3}{5}$

$\therefore$ y -intercept is $-\frac{3}{5}$

When $y = 0$, $2x - 3 = 0$

$\therefore x = \frac{3}{2}$

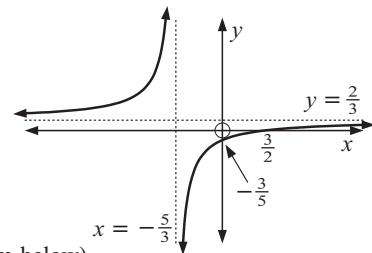
$\therefore$ x -intercept is $\frac{3}{2}$

As $x \rightarrow +\infty$, $y \rightarrow \frac{2}{3}$ (from below)

As $x \rightarrow -\infty$, $y \rightarrow \frac{2}{3}$ (from above)

As $x \rightarrow -\frac{5}{3}$ (from the left), $y \rightarrow +\infty$

As $x \rightarrow -\frac{5}{3}$ (from the right), $y \rightarrow -\infty$



5 VA is $x = 3$, HA is $y = -2$

$\therefore$ function has form $(x - 3)(y + 2) = k$ where k is a constant.

But $(0, 0)$ lies on the curve $\therefore (0 - 3)(0 + 2) = k$

$$\therefore k = -6$$

$\therefore$ curve is $(x - 3)(y + 2) = -6$

$$\text{i.e., } y + 2 = \frac{-6}{x - 3}$$

$$\therefore y = \frac{-6}{x - 3} - \frac{2}{1}$$

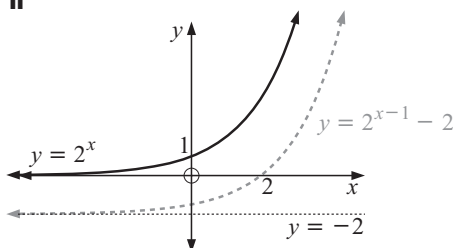
$$\therefore y = \frac{-6 - 2x + 6}{x - 3}$$

$$\text{i.e., } y = \frac{-2x}{x - 3}$$

6 a i Under translation $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$,

$$\begin{aligned} y &= 2^x \\ \text{becomes } y + 2 &= 2^{x-1} \\ \text{i.e., } y &= 2^{x-1} - 2 \end{aligned}$$

ii



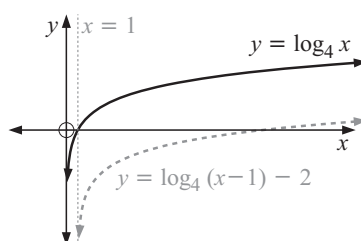
iii For $y = 2^x$, HA is $y = 0$,
no VA
For $y = 2^{x-1} - 2$, HA is
 $y = -2$, no VA

iv For $y = 2^x$,
domain is $\{x: x \in \mathcal{R}\}$,
range is $\{y: y > 0\}$
For $y = 2^{x-1} - 2$,
domain is $\{x: x \in \mathcal{R}\}$,
range is $\{y: y > -2\}$

b i Under translation $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$,

$$\begin{aligned} y &= \log_4 x \\ \text{becomes } y + 2 &= \log_4(x - 1) \\ \text{i.e., } y &= \log_4(x - 1) - 2 \end{aligned}$$

ii



iii For $y = \log_4 x$, VA is $x = 0$,
no HA
For $y = \log_4(x - 1) - 2$,
VA is $x = 1$, no HA

iv For $y = \log_4 x$,
domain is $\{x: x > 0\}$,
range is $\{y: y \in \mathcal{R}\}$
For $y = \log_4(x - 1) - 2$,
domain is $\{x: x > 1\}$,
range is $\{y \in \mathcal{R}\}$

Chapter 7

QUADRATIC EQUATIONS AND FUNCTIONS

EXERCISE 7A

- 1 a** $y = 3x^2 - 4x + 1$ is of the form $y = ax^2 + bx + c$, where $a = 3$, $b = -4$ and $c = 1$.
Hence it is a quadratic function.
- b** $y = 5x - 7$ is of the form $y = ax^2 + bx + c$, but requires $a = 0$.
Hence it is not a quadratic function.
- c** $y = -x^2$ is of the form $y = ax^2 + bx + c$, where $a = -1$, $b = 0$ and $c = 0$.
Hence it is a quadratic function.
- d** $y = \frac{2}{3}x^2 + 4$ is of the form $y = ax^2 + bx + c$, where $a = \frac{2}{3}$, $b = 0$ and $c = 4$.
Hence it is a quadratic function.
- e** If $2y + 3x^2 - 5 = 0$, then $2y = -3x^2 + 5 \quad \therefore y = -\frac{3}{2}x^2 + \frac{5}{2}$
This is of the form $f(x) = ax^2 + bx + c$, where $a = -\frac{3}{2}$, $b = 0$ and $c = \frac{5}{2}$.
Hence it is a quadratic function.
- f** $y = 5x^3 + x - 6$ contains the term x^3 , so is not of the form $y = ax^2 + bx + c$.
Hence it is not a quadratic function.

- 2 a** When $x = 3$, $y = 3^2 + 5 \times 3 - 4$
 $= 9 + 15 - 4$
 $= 20$
- b** When $x = -3$, $y = 2 \times (-3)^2 + 9$
 $= 2 \times 9 + 9$
 $= 27$
- c** When $x = 1$, $y = -2 \times 1^2 + 3 \times 1 - 5$
 $= -2 + 3 - 5$
 $= -4$
- d** When $x = 4$, $y = 4 \times 4^2 - 7 \times 4 + 1$
 $= 64 - 28 + 1$
 $= 37$

- 3 a** $f(2)$
 $= 2^2 - 2 \times 2 + 3$
 $= 4 - 4 + 3$
 $= 3$
- b** $f(-3)$
 $= 4 - (-3)^2$
 $= 4 - 9$
 $= -5$
- c** $f(0)$
 $= -\frac{1}{4} \times 0^2 + 3 \times 0 - 4$
 $= -4$
- d** $f(2)$
 $= \frac{1}{2} \times 2^2 + 3 \times 2$
 $= 2 + 6$
 $= 8$

- 4 a** $f(x) = 5x^2 - 10$
 $\therefore f(0) = 5 \times 0 - 10$
 $= -10$
 $\therefore f(0) \neq 5$
 $\therefore$ the function is not satisfied by the ordered pair.
- b** $y = 2x^2 + 5x - 3$
when $x = 4$,
 $y = 2 \times 4^2 + 5 \times 4 - 3$
 $= 32 + 20 - 3$
 $= 49$
 $\therefore$ the function is not satisfied by the ordered pair.
- c** $y = -2x^2 + 3x$
when $x = -\frac{1}{2}$,
 $y = -2 \times (-\frac{1}{2})^2 + 3(-\frac{1}{2})$
 $= -2 \times \frac{1}{4} - \frac{3}{2}$
 $= -\frac{1}{2} - \frac{3}{2}$
 $= -2$
 $\therefore$ the function is not satisfied by the ordered pair.
- d** $y = -7x^2 + 8x + 15$
when $x = -1$,
 $y = -7 \times (-1)^2 + 8 \times (-1) + 15$
 $= -7 - 8 + 15$
 $= 0$
 $\therefore$ the function is not satisfied by the ordered pair.

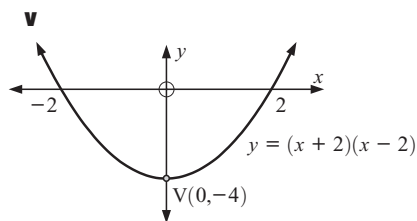
EXERCISE 7B.1

- 1 a i** $y = (x + 2)(x - 2)$ has x -intercepts -2 and 2 .

ii The axis of symmetry is midway between the x -intercepts, i.e., $x = 0$.

iii When $x = 0$, $y = 2 \times (-2) = -4$
 $\therefore$ the vertex is at $(0, -4)$

iv From **iii**, when $x = 0$, $y = -4$
 $\therefore$ the y -intercept is -4 .

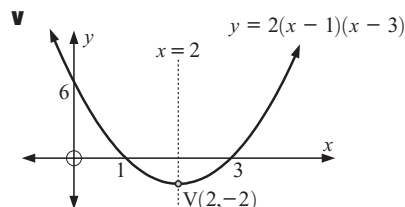


- b i** $y = 2(x - 1)(x - 3)$ has x -intercepts 1 and 3 .

ii The axis of symmetry is midway between the x -intercepts, i.e., $x = 2$.

iii When $x = 2$, $y = 2 \times 1 \times (-1) = -2$
 $\therefore$ the vertex is at $(2, -2)$

iv When $x = 0$, $y = 2(-1)(-3) = 6$
 $\therefore$ the y -intercept is 6 .

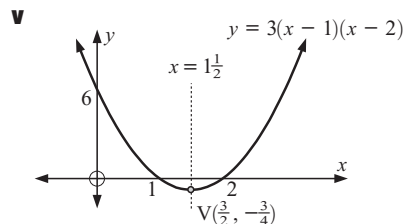


- c i** $y = 3(x - 1)(x - 2)$ has x -intercepts 1 and 2 .

ii The axis of symmetry is midway between the x -intercepts, i.e., $x = \frac{3}{2}$.

iii When $x = \frac{3}{2}$, $y = 3 \times \frac{1}{2} \times (-\frac{1}{2}) = -\frac{3}{4}$
 $\therefore$ the vertex is at $(\frac{3}{2}, -\frac{3}{4})$

iv When $x = 0$, $y = 3(-1)(-2) = 6$
 $\therefore$ the y -intercept is 6 .

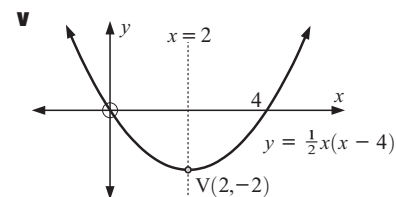


- d i** $y = \frac{1}{2}x(x - 4)$ has x -intercepts 0 and 4 .

ii The axis of symmetry is midway between the x -intercepts, i.e., $x = 2$.

iii When $x = 2$, $y = \frac{1}{2} \times 2 \times (-2) = -2$
 $\therefore$ the vertex is at $(2, -2)$

iv When $x = 0$, $y = 0$ also.
 $\therefore$ the y -intercept is 0 .

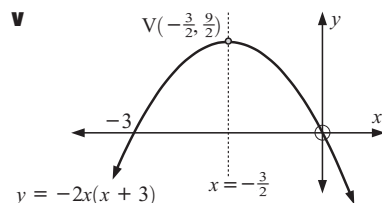


- e i** $y = -2x(x + 3)$ has x -intercepts 0 and -3 .

ii The axis of symmetry is midway between the x -intercepts, i.e., $x = -\frac{3}{2}$.

iii When $x = -\frac{3}{2}$, $y = -2 \times (-\frac{3}{2}) \times (\frac{3}{2}) = \frac{9}{2}$
 $\therefore$ the vertex is at $(-\frac{3}{2}, \frac{9}{2})$

iv When $x = 0$, $y = 0$ also.
 $\therefore$ the y -intercept is 0 .

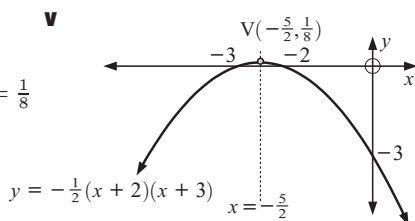


- f i** $y = -\frac{1}{2}(x + 2)(x + 3)$ has x -intercepts -2 and -3 .

ii The axis of symmetry is midway between the x -intercepts, i.e., $x = -\frac{5}{2}$.

iii When $x = -\frac{5}{2}$, $y = -\frac{1}{2} \times (-\frac{1}{2}) \times (\frac{1}{2}) = \frac{1}{8}$
 $\therefore$ the vertex is at $(-\frac{5}{2}, \frac{1}{8})$

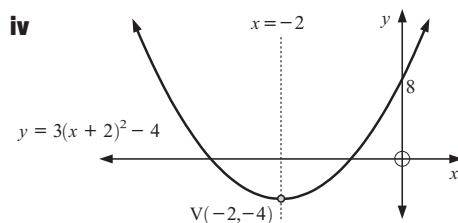
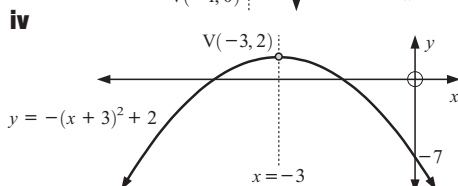
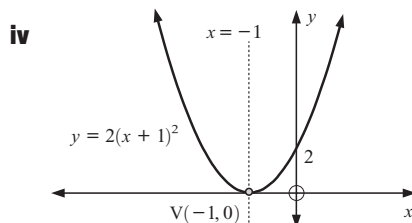
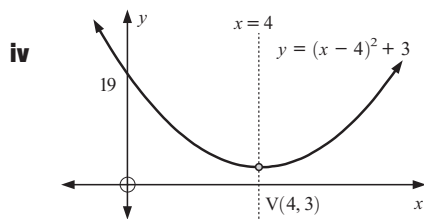
iv When $x = 0$, $y = -\frac{1}{2} \times 2 \times 3 = -3$
 $\therefore$ the y -intercept is -3 .



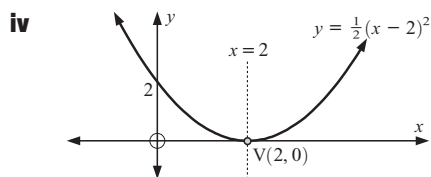
- 2 a** $y = x(x - 2)$ has x -intercepts 0 and 2.
Its axis of symmetry is $x = 1$, so its vertex has $y = 1 \times (-1) = -1$.
Hence the graph is **B**.
- b** $y = 3x(x - 2)$ also has x -intercepts 0 and 2.
Its axis of symmetry is $x = 1$, so its vertex has $y = 3 \times 1 \times (-1) = -3$.
Hence the graph is **A**.
- c** $y = -x(x - 2)$ also has x -intercepts 0 and 2.
 $\therefore$ its graph must be the remaining one like this, i.e., **F**.
Check: axis of symmetry is $x = 1$,
so its vertex has $y = -1 \times (-1) = 1$, which is true.
- d** $y = (x + 2)(x - 1)$ has x -intercepts -2 and 1 .
Its axis of symmetry is $x = -\frac{1}{2}$, so its vertex has $y = (\frac{3}{2}) \times (-\frac{3}{2}) = -\frac{9}{4}$.
Hence the graph is **D**.
- e** $y = 2(x + 2)(x - 1)$ also has x -intercepts -2 and 1 .
Its axis of symmetry is $x = -\frac{1}{2}$, so its vertex has $y = 2 \times (\frac{3}{2}) \times (-\frac{3}{2}) = -\frac{9}{2}$.
Hence the graph is **E**.
- f** $y = -2(x + 2)(x - 1)$ also has x -intercepts -2 and 1 .
Its graph must be the only one remaining, i.e., **C**.
Check: axis of symmetry is $x = -\frac{1}{2}$,
so its vertex has $y = -2 \times (\frac{3}{2}) \times (-\frac{3}{2}) = \frac{9}{2}$, which is true.

EXERCISE 7B.2

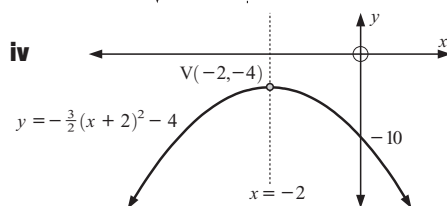
- 1 a i** $y = (x - 4)^2 + 3$ has axis of symmetry $x = 4$.
ii Its vertex is $(4, 3)$.
iii When $x = 0$, $y = (-4)^2 + 3 = 19$
 $\therefore$ the y -intercept is 19.
- b i** $y = 2(x + 1)^2$ has axis of symmetry $x = -1$.
ii Its vertex is $(-1, 0)$.
iii When $x = 0$, $y = 2 \times 1^2 = 2$
 $\therefore$ the y -intercept is 2.
- c i** $y = -(x + 3)^2 + 2$ has axis of symmetry $x = -3$.
ii Its vertex is $(-3, 2)$.
iii When $x = 0$, $y = -3^2 + 2 = -7$
 $\therefore$ the y -intercept is -7 .
- d i** $y = 3(x + 2)^2 - 4$ has axis of symmetry $x = -2$.
ii Its vertex is $(-2, -4)$.
iii When $x = 0$, $y = 3 \times 2^2 - 4 = 8$
 $\therefore$ the y -intercept is 8.



- e i** $y = \frac{1}{2}(x-2)^2$ has axis of symmetry $x = 2$.
- ii** Its vertex is $(2, 0)$.
- iii** When $x = 0$, $y = \frac{1}{2} \times (-2)^2 = 2$
 $\therefore$ the y -intercept is 2.



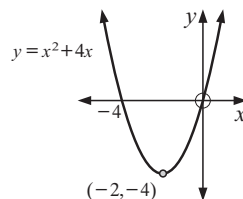
- f i** $y = -\frac{3}{2}(x+2)^2 - 4$ has axis of symmetry $x = -2$.
- ii** Its vertex is $(-2, -4)$.
- iii** When $x = 0$, $y = -\frac{3}{2} \times 2^2 - 4 = -10$
 $\therefore$ the y -intercept is -10 .



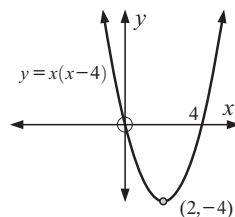
- 2 a** $y = -(x+1)^2 + 3$ has vertex $(-1, 3)$, so its graph must be **G**.
- b** $y = -2(x-3)^2 + 2$ has vertex $(3, 2)$, so its graph must be **A**.
- c** $y = x^2 + 2$ has vertex $(0, 2)$, so its graph must be **E**.
- d** $y = -(x-1)^2 + 1$ has vertex $(1, 1)$, so its graph must be **B**.
- e** $y = (x-2)^2 - 2$ has vertex $(2, -2)$, so its graph must be **I**.
- f** $y = \frac{1}{3}(x+3)^2 - 3$ has vertex $(-3, -3)$, so its graph must be **C**.
- g** $y = -x^2$ has vertex $(0, 0)$, so its graph must be **D**.
- h** $y = -\frac{1}{2}(x-1)^2 + 1$ has vertex $(1, 1)$, so its graph must be **F**.
- i** $y = 2(x+2)^2 - 1$ has vertex $(-2, -1)$, so its graph must be **H**.

- 3 a** The graph has x -intercepts 0 and 4, and its axis of symmetry is midway between them, i.e., $x = 2$.
- b** The graph has x -intercepts -5 and 0, and its axis of symmetry is midway between them, i.e., $x = -\frac{5}{2}$.
- c** The graph has x -intercepts -1 and 3, and its axis of symmetry is midway between them, i.e., $x = 1$.
- d** The graph touches the x -axis at its vertex $(3, 0)$
 $\therefore$ its axis of symmetry is $x = 3$.
- e** The graph touches the x -axis at its vertex $(-4, 0)$
 $\therefore$ its axis of symmetry is $x = -4$.
- f** Both $(-8, -5)$ and $(0, -5)$ lie on the graph. Since they have the same y -coordinate, the axis of symmetry must lie midway between them, i.e., $x = -4$.

- 4 a i** $y = x^2 + 4x$ has x -intercepts -4 and 0, and y -intercept 0.
- ii** The axis of symmetry is midway between the x -intercepts, i.e., $x = -2$.
- iii** When $x = -2$, $y = (-2)^2 + 4(-2) = -4$
 $\therefore$ the vertex is $(-2, -4)$.



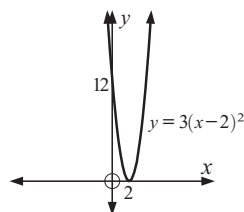
- b i** $y = x(x-4)$ has x -intercepts 0 and 4, and y -intercept 0.
- ii** The axis of symmetry is midway between the x -intercepts, i.e., $x = 2$.
- iii** When $x = 2$, $y = 2(-2) = -4$
 $\therefore$ the vertex is $(2, -4)$.



- c i** $y = 3(x - 2)^2$ has x -intercept 2
and y -intercept $= 3 \times (-2)^2 = 12$.

- ii** Since the vertex is when $x = 2$,
 $x = 2$ is the axis of symmetry.

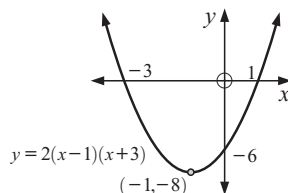
- iii** The vertex is $(2, 0)$.



- d i** $y = 2(x - 1)(x + 3)$ has x -intercepts -3 and 1 ,
and y -intercept -6 .

- ii** The axis of symmetry is midway
between the x -intercepts, i.e., $x = -1$.

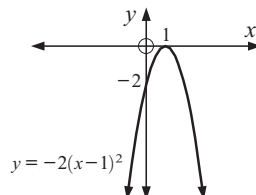
- iii** When $x = -1$, $y = 2(-2)(2) = -8$
 $\therefore$ the vertex is $(-1, -8)$.



- e i** $y = -2(x - 1)^2$ has x -intercept 1
and y -intercept $= -2 \times (-1)^2 = -2$.

- ii** Since the vertex is when $x = 1$,
 $x = 1$ is the axis of symmetry.

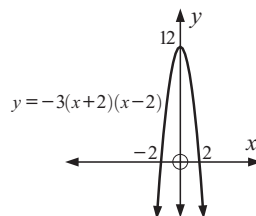
- iii** The vertex is $(1, 0)$.



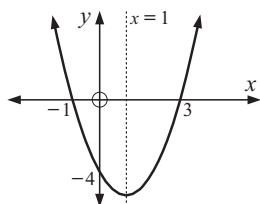
- f i** $y = -3(x + 2)(x - 2)$ has x -intercepts -2 and 2 ,
and y -intercept $= -3(2)(-2) = 12$.

- ii** The axis of symmetry is midway
between the x -intercepts, i.e., $x = 0$.

- iii** Since the y -intercept is 12 ,
the vertex is $(0, 12)$.

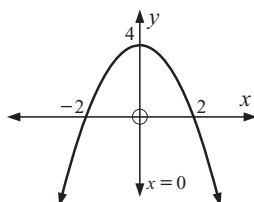


5 a i



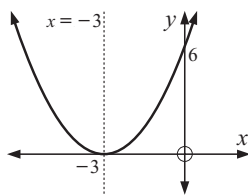
- ii** The axis of symmetry lies halfway between
the x -intercepts,
i.e., the axis of symmetry is $x = 1$.

b i

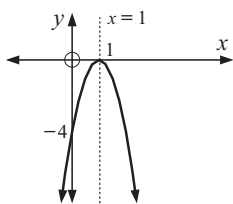


- ii** The axis of symmetry lies halfway between
the x -intercepts,
i.e., the axis of symmetry is $x = 0$.

c i



- ii** Since the vertex is at $x = -3$,
the axis of symmetry is $x = -3$.

d i**ii**

Since the vertex is at $x = 1$,
the axis of symmetry is $x = 1$.

- 6 a** The axis of symmetry, $x = -3$, lies midway between the x -intercepts.
 $\therefore$ since one x -intercept is -1 , the other is -5
 $\therefore$ the x -intercepts are -1 and -5 .
- b** Since the graph touches the x -axis at 3 and it is a quadratic function,
the only x -intercept is 3 (touching).

EXERCISE 7C**1 a**

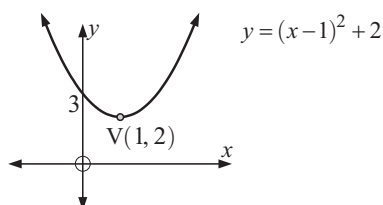
$$y = x^2 - 2x + 3$$

$$\therefore y = x^2 - 2x + 1^2 + 3 - 1^2$$

$$\therefore y = (x - 1)^2 + 2$$

$$\therefore \text{vertex is } (1, 2),$$

$$y\text{-intercept is } 3$$

**b**

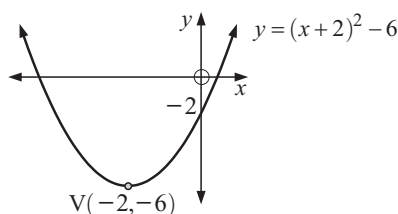
$$y = x^2 + 4x - 2$$

$$\therefore y = x^2 + 4x + 2^2 - 2 - 2^2$$

$$\therefore y = (x + 2)^2 - 6$$

$$\therefore \text{vertex is } (-2, -6),$$

$$y\text{-intercept is } -2$$

**c**

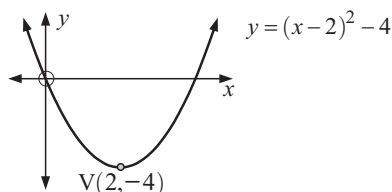
$$y = x^2 - 4x$$

$$\therefore y = x^2 - 4x + 2^2 - 2^2$$

$$\therefore y = (x - 2)^2 - 4$$

$$\therefore \text{vertex is } (2, -4),$$

$$y\text{-intercept is } 0$$

**d**

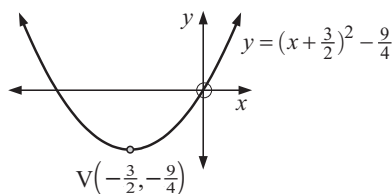
$$y = x^2 + 3x$$

$$\therefore y = x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2$$

$$\therefore y = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$$

$$\therefore \text{vertex is } \left(-\frac{3}{2}, -\frac{9}{4}\right),$$

$$y\text{-intercept is } 0$$

**e**

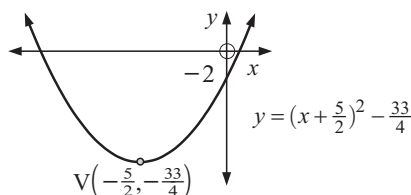
$$y = x^2 + 5x - 2$$

$$\therefore y = x^2 + 5x + \left(\frac{5}{2}\right)^2 - 2 - \left(\frac{5}{2}\right)^2$$

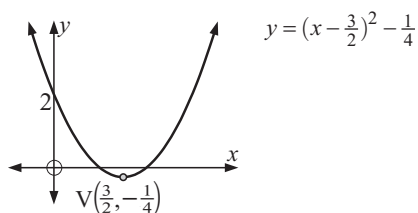
$$\therefore y = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$$

$$\therefore \text{vertex is } \left(-\frac{5}{2}, -\frac{33}{4}\right),$$

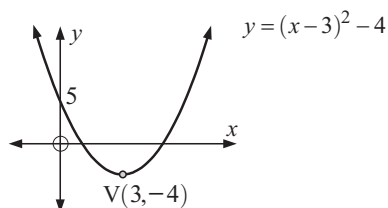
$$y\text{-intercept is } -2$$



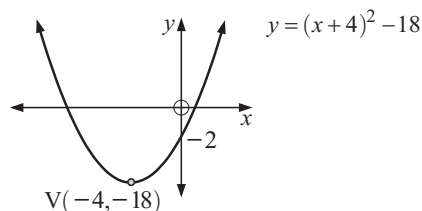
f $y = x^2 - 3x + 2$
 $\therefore y = x^2 - 3x + \left(\frac{3}{2}\right)^2 + 2 - \left(\frac{3}{2}\right)^2$
 $\therefore y = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$
 $\therefore$ vertex is $\left(\frac{3}{2}, -\frac{1}{4}\right)$,
 y -intercept is 2



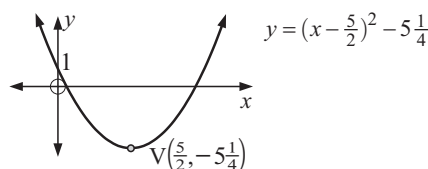
g $y = x^2 - 6x + 5$
 $\therefore y = x^2 - 6x + 3^2 + 5 - 3^2$
 $\therefore y = (x - 3)^2 - 4$
 $\therefore$ vertex is $(3, -4)$,
 y -intercept is 5



h $y = x^2 + 8x - 2$
 $\therefore y = x^2 + 8x + 4^2 - 2 - 4^2$
 $\therefore y = (x + 4)^2 - 18$
 $\therefore$ vertex is $(-4, -18)$,
 y -intercept is -2



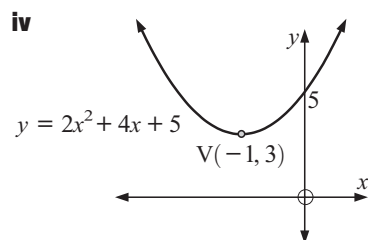
i $y = x^2 - 5x + 1$
 $\therefore y = x^2 - 5x + \left(\frac{5}{2}\right)^2 + 1 - \left(\frac{5}{2}\right)^2$
 $\therefore y = \left(x - \frac{5}{2}\right)^2 - 5\frac{1}{4}$
 $\therefore$ vertex is $\left(\frac{5}{2}, -5\frac{1}{4}\right)$,
 y -intercept is 1



2 a i $y = 2x^2 + 4x + 5$
 $= 2\left[x^2 + 2x + \frac{5}{2}\right]$
 $= 2\left[x^2 + 2x + 1^2 - 1^2 + \frac{5}{2}\right]$
 $= 2\left[(x + 1)^2 + \frac{3}{2}\right]$
 $= 2(x + 1)^2 + 3$

ii The vertex is $(-1, 3)$.

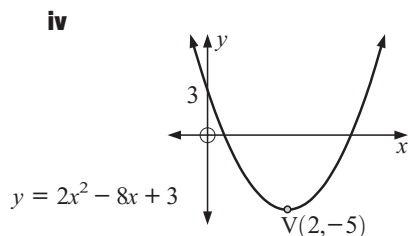
iii When $x = 0$, $y = 5$
 $\therefore$ the y -intercept is 5



b i $y = 2x^2 - 8x + 3$
 $= 2\left[x^2 - 4x + \frac{3}{2}\right]$
 $= 2\left[x^2 - 4x + 2^2 - 2^2 + \frac{3}{2}\right]$
 $= 2\left[(x - 2)^2 - \frac{5}{2}\right]$
 $= 2(x - 2)^2 - 5$

ii The vertex is $(2, -5)$.

iii When $x = 0$, $y = 3$
 $\therefore$ the y -intercept is 3



c i $y = 2x^2 - 6x + 1$
 $= 2 \left[x^2 - 3x + \frac{1}{2} \right]$
 $= 2 \left[x^2 - 3x + \left(\frac{3}{2} \right)^2 - \left(\frac{3}{2} \right)^2 + \frac{1}{2} \right]$
 $= 2 \left[\left(x - \frac{3}{2} \right)^2 - \frac{7}{4} \right]$
 $= 2 \left(x - \frac{3}{2} \right)^2 - \frac{7}{2}$

ii The vertex is $\left(\frac{3}{2}, -\frac{7}{2} \right)$.

d i $y = 3x^2 - 6x + 5$
 $= 3 \left[x^2 - 2x + \frac{5}{3} \right]$
 $= 3 \left[x^2 - 2x + 1^2 - 1^2 + \frac{5}{3} \right]$
 $= 3 \left[(x - 1)^2 + \frac{2}{3} \right]$
 $= 3(x - 1)^2 + 2$

ii The vertex is $(1, 2)$.

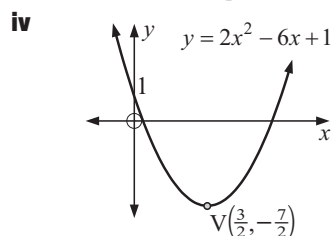
e i $y = -x^2 + 4x + 2$
 $= -[x^2 - 4x - 2]$
 $= -[x^2 - 4x + 2^2 - 2^2 - 2]$
 $= -[(x - 2)^2 - 6]$
 $= -(x - 2)^2 + 6$

ii The vertex is $(2, 6)$.

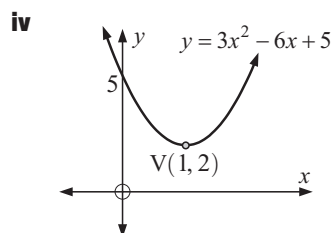
f i $y = -2x^2 - 5x + 3$
 $= -2 \left[x^2 + \frac{5}{2}x - \frac{3}{2} \right]$
 $= -2 \left[x^2 + \frac{5}{2}x + \left(\frac{5}{4} \right)^2 - \left(\frac{5}{4} \right)^2 - \frac{3}{2} \right]$
 $= -2 \left[\left(x + \frac{5}{4} \right)^2 - \frac{25}{16} - \frac{24}{16} \right]$
 $= -2 \left[\left(x + \frac{5}{4} \right)^2 - \frac{49}{16} \right]$
 $= -2 \left(x + \frac{5}{4} \right)^2 + \frac{49}{8}$

ii The vertex is $\left(-\frac{5}{4}, \frac{49}{8} \right)$.

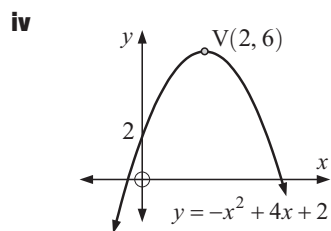
iii When $x = 0$, $y = 1$
 $\therefore$ the y -intercept is 1



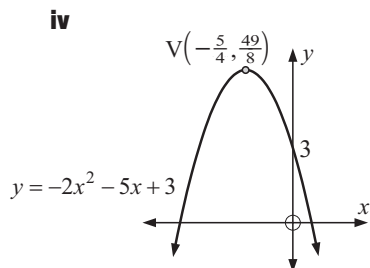
iii When $x = 0$, $y = 5$
 $\therefore$ the y -intercept is 5



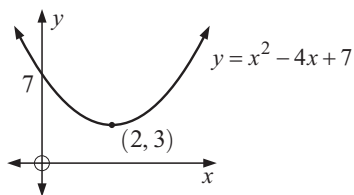
iii When $x = 0$, $y = 2$
 $\therefore$ the y -intercept is 2



iii When $x = 0$, $y = 3$
 $\therefore$ the y -intercept is 3



3 a Using technology, the graph is



Since the vertex is at $(2, 3)$,
the function must be of the form

$$y = a(x - 2)^2 + 3 \quad \text{for some } a.$$

$\therefore$ when $x = 0$,

$$y = a(-2)^2 + 3 = 4a + 3$$

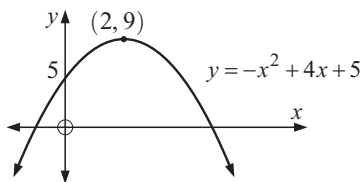
but the y -intercept is 7

$$\therefore 4a + 3 = 7$$

$$\therefore a = 1$$

$\therefore$ the equation is $y = (x - 2)^2 + 3$

c Using technology, the graph is



Since the vertex is at $(2, 9)$,
the function must be of the form

$$y = a(x - 2)^2 + 9 \quad \text{for some } a.$$

$\therefore$ when $x = 0$,

$$y = a(-2)^2 + 9 = 4a + 9$$

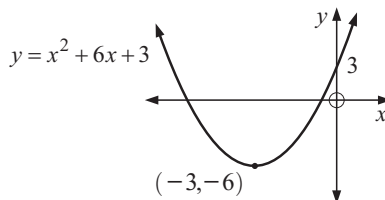
but the y -intercept is 5

$$\therefore 4a + 9 = 5$$

$$\therefore a = -1$$

$\therefore$ the equation is $y = -(x - 2)^2 + 9$

b Using technology, the graph is



Since the vertex is at $(-3, -6)$,
the function must be of the form

$$y = a(x + 3)^2 - 6 \quad \text{for some } a.$$

$\therefore$ when $x = 0$,

$$y = a \times 3^2 - 6 = 9a - 6$$

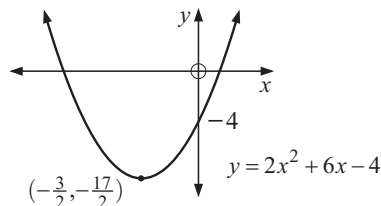
but the y -intercept is 3

$$\therefore 9a - 6 = 3$$

$$\therefore a = 1$$

$\therefore$ the equation is $y = (x + 3)^2 - 6$

d Using technology, the graph is



Since the vertex is at $(-\frac{3}{2}, -\frac{17}{2})$,
the function must be of the form

$$y = a(x + \frac{3}{2})^2 - \frac{17}{2} \quad \text{for some } a.$$

$\therefore$ when $x = 0$,

$$y = a(\frac{3}{2})^2 - \frac{17}{2} = \frac{9}{4}a - \frac{17}{2}$$

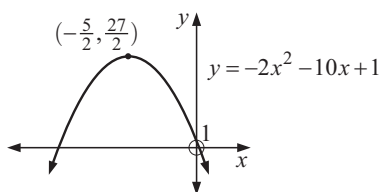
but the y -intercept is -4

$$\therefore \frac{9}{4}a - \frac{17}{2} = -4$$

$$\therefore \frac{9}{4}a = \frac{9}{2}$$

$$\therefore a = \frac{4}{9} \times \frac{9}{2} = 2$$

$\therefore$ the equation is $y = 2(x + \frac{3}{2})^2 - \frac{17}{2}$

e Using technology, the graph is

Since the vertex is at $(-\frac{5}{2}, \frac{27}{2})$,

the function must be of the form

$$y = a(x + \frac{5}{2})^2 + \frac{27}{2} \quad \text{for some } a.$$

$\therefore$ when $x = 0$,

$$y = a(\frac{5}{2})^2 + \frac{27}{2} = \frac{25}{4}a + \frac{27}{2}$$

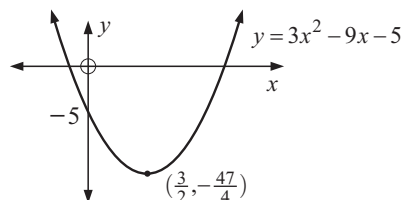
but the y -intercept is 1

$$\therefore \frac{25}{4}a + \frac{27}{2} = 1$$

$$\therefore \frac{25}{4}a = -\frac{25}{2}$$

$$\therefore a = -\frac{25}{2} \times \frac{4}{25} = -2$$

$\therefore$ the equation is $y = -2(x + \frac{5}{2})^2 + \frac{27}{2}$

f Using technology, the graph is

Since the vertex is at $(\frac{3}{2}, -\frac{47}{4})$,

the function must be of the form

$$y = a(x - \frac{3}{2})^2 - \frac{47}{4} \quad \text{for some } a.$$

$\therefore$ when $x = 0$,

$$y = a(-\frac{3}{2})^2 - \frac{47}{4} = \frac{9}{4}a - \frac{47}{4}$$

but the y -intercept is -5

$$\therefore \frac{9}{4}a - \frac{47}{4} = -5$$

$$\therefore \frac{9}{4}a = \frac{27}{4}$$

$$\therefore a = 3$$

$\therefore$ the equation is $y = 3(x - \frac{3}{2})^2 - \frac{47}{4}$

EXERCISE 7D.1

1 a $4x^2 + 7x = 0$

$$\therefore x(4x + 7) = 0$$

$$\therefore x = 0 \text{ or } 4x + 7 = 0$$

{Null Factor law}

$$\therefore x = 0 \text{ or } -\frac{7}{4}$$

d $2x^2 - 11x = 0$

$$\therefore x(2x - 11) = 0$$

$$\therefore x = 0 \text{ or } 2x - 11 = 0$$

{Null Factor law}

$$\therefore x = 0 \text{ or } \frac{11}{2}$$

g $x^2 - 5x + 6 = 0$

$$\therefore (x - 2)(x - 3) = 0$$

$$\therefore x - 2 = 0 \text{ or } x - 3 = 0$$

{Null Factor law}

$$\therefore x = 2 \text{ or } 3$$

b $6x^2 + 2x = 0$

$$\therefore 2x(3x + 1) = 0$$

$$\therefore x = 0 \text{ or } 3x + 1 = 0$$

{Null Factor law}

$$\therefore x = 0 \text{ or } -\frac{1}{3}$$

e $3x^2 = 8x$

$$\therefore 3x^2 - 8x = 0$$

$$\therefore x(3x - 8) = 0$$

$$\therefore x = 0 \text{ or } 3x - 8 = 0$$

{Null Factor law}

$$\therefore x = 0 \text{ or } \frac{8}{3}$$

c $3x^2 - 7x = 0$

$$\therefore x(3x - 7) = 0$$

$$\therefore x = 0 \text{ or } 3x - 7 = 0$$

{Null Factor law}

$$\therefore x = 0 \text{ or } \frac{7}{3}$$

f $9x = 6x^2$

$$\therefore 6x^2 - 9x = 0$$

$$\therefore 3x(2x - 3) = 0$$

$$\therefore x = 0 \text{ or } 2x - 3 = 0$$

{Null Factor law}

$$\therefore x = 0 \text{ or } \frac{3}{2}$$

h $x^2 = 2x + 8$

$$\therefore x^2 - 2x - 8 = 0$$

$$\therefore (x - 4)(x + 2) = 0$$

$$\therefore x - 4 = 0 \text{ or } x + 2 = 0$$

{Null Factor law}

$$\therefore x = -2 \text{ or } 4$$

i $x^2 + 21 = 10x$

$$\therefore x^2 - 10x + 21 = 0$$

$$\therefore (x - 3)(x - 7) = 0$$

$$\therefore x - 3 = 0 \text{ or } x - 7 = 0$$

{Null Factor law}

$$\therefore x = 3 \text{ or } 7$$

j $9 + x^2 = 6x$

$$\therefore x^2 - 6x + 9 = 0$$

$$\therefore (x - 3)^2 = 0$$

$$\therefore x - 3 = 0$$

$$\therefore x = 3$$

k $x^2 + x = 12$

$$\therefore x^2 + x - 12 = 0$$

$$\therefore (x + 4)(x - 3) = 0$$

$$\therefore x + 4 = 0 \text{ or } x - 3 = 0$$

{Null Factor law}

$$\therefore x = -4 \text{ or } 3$$

l $x^2 + 8x = 33$

$$\therefore x^2 + 8x - 33 = 0$$

$$\therefore (x + 11)(x - 3) = 0$$

$$\therefore x + 11 = 0 \text{ or } x - 3 = 0$$

{Null Factor law}

$$\therefore x = -11 \text{ or } 3$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad 9x^2 - 12x + 4 &= 0 \\ \therefore (3x - 2)^2 &= 0 \\ \therefore x &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad 2x^2 - 13x - 7 &= 0 \\ \therefore (2x + 1)(x - 7) &= 0 \\ \therefore x &= -\frac{1}{2} \text{ or } 7 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 3x^2 &= 16x + 12 \\ \therefore 3x^2 - 16x - 12 &= 0 \\ \therefore (3x + 2)(x - 6) &= 0 \\ \therefore x &= -\frac{2}{3} \text{ or } 6 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad 3x^2 + 5x &= 2 \\ \therefore 3x^2 + 5x - 2 &= 0 \\ \therefore (3x - 1)(x + 2) &= 0 \\ \therefore x &= \frac{1}{3} \text{ or } -2 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad 2x^2 + 3 &= 5x \\ \therefore 2x^2 - 5x + 3 &= 0 \\ \therefore (2x - 3)(x - 1) &= 0 \\ \therefore x &= \frac{3}{2} \text{ or } 1 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad 3x^2 &= 4x + 4 \\ \therefore 3x^2 - 4x - 4 &= 0 \\ \therefore (3x + 2)(x - 2) &= 0 \\ \therefore x &= -\frac{2}{3} \text{ or } 2 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad 3x^2 &= 10x + 8 \\ \therefore 3x^2 - 10x - 8 &= 0 \\ \therefore (3x + 2)(x - 4) &= 0 \\ \therefore x &= -\frac{2}{3} \text{ or } 4 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad 4x^2 + 4x &= 3 \\ \therefore 4x^2 + 4x - 3 &= 0 \\ \therefore (2x + 3)(2x - 1) &= 0 \\ \therefore x &= -\frac{3}{2} \text{ or } \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad 4x^2 &= 11x + 3 \\ \therefore 4x^2 - 11x - 3 &= 0 \\ \therefore (4x + 1)(x - 3) &= 0 \\ \therefore x &= -\frac{1}{4} \text{ or } 3 \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad 12x^2 &= 11x + 15 \\ \therefore 12x^2 - 11x - 15 &= 0 \\ \therefore (4x + 3)(3x - 5) &= 0 \\ \therefore x &= -\frac{3}{4} \text{ or } \frac{5}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad 7x^2 + 6x &= 1 \\ \therefore 7x^2 + 6x - 1 &= 0 \\ \therefore (7x - 1)(x + 1) &= 0 \\ \therefore x &= \frac{1}{7} \text{ or } -1 \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad 15x^2 + 2x &= 56 \\ \therefore 15x^2 + 2x - 56 &= 0 \\ \therefore (15x - 28)(x + 2) &= 0 \\ \therefore x &= \frac{28}{15} \text{ or } -2 \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad (x + 1)^2 &= 2x^2 - 5x + 11 \\ \therefore x^2 + 2x + 1 &= 2x^2 - 5x + 11 \\ \therefore x^2 - 7x + 10 &= 0 \\ \therefore (x - 2)(x - 5) &= 0 \\ \therefore x &= 2 \text{ or } 5 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (x + 2)(1 - x) &= -4 \\ \therefore x - x^2 + 2 - 2x &= -4 \\ \therefore x^2 + x - 6 &= 0 \\ \therefore (x + 3)(x - 2) &= 0 \\ \therefore x &= -3 \text{ or } 2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad 5 - 4x^2 &= 3(2x + 1) + 2 \\ \therefore 5 - 4x^2 &= 6x + 3 + 2 \\ \therefore 4x^2 + 6x &= 0 \\ \therefore 2x(2x + 3) &= 0 \\ \therefore x &= 0 \text{ or } -\frac{3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad x + \frac{2}{x} &= 3 \\ \therefore x^2 + 2 &= 3x \\ \therefore x^2 - 3x + 2 &= 0 \\ \therefore (x - 1)(x - 2) &= 0 \\ \therefore x &= 1 \text{ or } 2 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad 2x - \frac{1}{x} &= -1 \\ \therefore 2x^2 - 1 &= -x \\ \therefore 2x^2 + x - 1 &= 0 \\ \therefore (2x - 1)(x + 1) &= 0 \\ \therefore x &= \frac{1}{2} \text{ or } -1 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \frac{x + 3}{1 - x} &= -\frac{9}{x} \\ \therefore x(x + 3) &= -9(1 - x) \\ \therefore x^2 + 3x &= -9 + 9x \\ \therefore x^2 - 6x + 9 &= 0 \\ \therefore (x - 3)^2 &= 0 \\ \therefore x &= 3 \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \text{If } y = 1, \text{ then } x^2 + 6x + 10 &= 1 \\ \therefore x^2 + 6x + 9 &= 0 \\ \therefore (x + 3)^2 &= 0 \\ \therefore x &= -3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{If } y = 2, \text{ then } x^2 + 5x + 8 &= 2 \\ \therefore x^2 + 5x + 6 &= 0 \\ \therefore (x + 3)(x + 2) &= 0 \\ \therefore x &= -3 \text{ or } -2 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \text{If } y = -3, \text{ then } x^2 - 5x + 1 &= -3 \\ \therefore x^2 - 5x + 4 &= 0 \\ \therefore (x - 4)(x - 1) &= 0 \\ \therefore x &= 4 \text{ or } 1 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \text{If } y = -3, \text{ then } 3x^2 &= -3 \\ \therefore x^2 &= -1 \\ \therefore \text{there is no solution} \end{aligned}$$

5 a $f(x) = 5$

$$\therefore 3x^2 - 2x + 5 = 5$$

$$\therefore 3x^2 - 2x = 0$$

$$\therefore x(3x - 2) = 0$$

$$\therefore x = 0 \text{ or } \frac{2}{3}$$

c $f(x) = -4$

$$\therefore -2x^2 - 13x + 3 = -4$$

$$\therefore 2x^2 + 13x - 7 = 0$$

$$\therefore (2x - 1)(x + 7) = 0$$

$$\therefore x = \frac{1}{2} \text{ or } -7$$

b $f(x) = 1$

$$\therefore x^2 - x - 5 = 1$$

$$\therefore x^2 - x - 6 = 0$$

$$\therefore (x + 2)(x - 3) = 0$$

$$\therefore x = -2 \text{ or } 3$$

d $f(x) = -17$

$$\therefore 2x^2 - 12x + 1 = -17$$

$$\therefore 2x^2 - 12x + 18 = 0$$

$$\therefore 2(x^2 - 6x + 9) = 0$$

$$\therefore 2(x - 3)^2 = 0$$

$$\therefore x = 3$$

6 a i $h(1) = 30 - 5 = 25 \text{ m}$ **ii** $h(5) = 30 \times 5 - 5 \times 5^2 = 150 - 125 = 25 \text{ m}$ **iii** $h(3) = 30 \times 3 - 5 \times 3^2 = 90 - 45 = 45 \text{ m}$

b i When $h = 40 \text{ m}$,

$$30t - 5t^2 = 40$$

$$\therefore 5t^2 - 30t + 40 = 0$$

$$\therefore 5(t^2 - 6t + 8) = 0$$

$$\therefore 5(t - 4)(t - 2) = 0$$

$$\therefore t = 4 \text{ or } 2 \text{ seconds}$$

ii When $h = 0 \text{ m}$,

$$30t - 5t^2 = 0$$

$$\therefore 5t(6 - t) = 0$$

$$\therefore t = 0 \text{ or } 6 \text{ seconds}$$

c We get two answers in each case because the object is projected up, and gravity brings it back down.

7 a i $P(0) = -\frac{1}{4} \times 0 + 16 \times 0 - 30 = -\30 **b** If $P(x) = 57$, then $-\frac{1}{4}x^2 + 16x - 30 = 57$

$$\therefore \frac{1}{4}x^2 - 16x + 87 = 0$$

ii $P(10) = -\frac{1}{4} \times 10^2 + 16 \times 10 - 30$

$$= -25 + 160 - 30$$

$$= \$105$$

$$\therefore x^2 - 64x + 348 = 0$$

$$\therefore (x - 6)(x - 58) = 0$$

$$\therefore x = 6 \text{ or } 58$$

i.e., either 6 or 58 cakes are made.

EXERCISE 7D.2

1 a $(x + 5)^2 = 2$

$$\therefore x + 5 = \pm\sqrt{2}$$

$$\therefore x = -5 \pm \sqrt{2}$$

b $(x + 6)^2 = 11$

$$\therefore x + 6 = \pm\sqrt{11}$$

$$\therefore x = -6 \pm \sqrt{11}$$

c $(x - 4)^2 = 8$

$$\therefore x - 4 = \pm\sqrt{8}$$

$$\therefore x = 4 \pm 2\sqrt{2}$$

d $(x - 8)^2 = 7$

$$\therefore x - 8 = \pm\sqrt{7}$$

$$\therefore x = 8 \pm \sqrt{7}$$

e $2(x + 3)^2 = 10$

$$\therefore (x + 3)^2 = 5$$

$$\therefore x + 3 = \pm\sqrt{5}$$

$$\therefore x = -3 \pm \sqrt{5}$$

f $3(x - 2)^2 = 18$

$$\therefore (x - 2)^2 = 6$$

$$\therefore x - 2 = \pm\sqrt{6}$$

$$\therefore x = 2 \pm \sqrt{6}$$

g $(x + 1)^2 + 1 = 11$

$$\therefore (x + 1)^2 = 10$$

$$\therefore x + 1 = \pm\sqrt{10}$$

$$\therefore x = -1 \pm \sqrt{10}$$

h $(2x + 1)^2 = 3$

$$\therefore 2x + 1 = \pm\sqrt{3}$$

$$\therefore 2x = -1 \pm \sqrt{3}$$

$$\therefore x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & x^2 - 4x + 1 = 0 \\
 & \therefore x^2 - 4x = -1 \\
 \therefore & x^2 - 4x + (-2)^2 = -1 + (-2)^2 \\
 & \therefore (x-2)^2 = 3 \\
 & \therefore x-2 = \pm\sqrt{3} \\
 & \therefore x = 2 \pm \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & x^2 - 14x + 46 = 0 \\
 & \therefore x^2 - 14x = -46 \\
 \therefore & x^2 - 14x + (-7)^2 = -46 + (-7)^2 \\
 & \therefore (x-7)^2 = 3 \\
 & \therefore x-7 = \pm\sqrt{3} \\
 & \therefore x = 7 \pm \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & x^2 + 6x + 7 = 0 \\
 & \therefore x^2 + 6x = -7 \\
 \therefore & x^2 + 6x + 3^2 = -7 + 3^2 \\
 & \therefore (x+3)^2 = 2 \\
 & \therefore x+3 = \pm\sqrt{2} \\
 & \therefore x = -3 \pm \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & x^2 + 6x = 2 \\
 \therefore & x^2 + 6x + 3^2 = 2 + 3^2 \\
 & \therefore (x+3)^2 = 11 \\
 & \therefore x+3 = \pm\sqrt{11} \\
 & \therefore x = -3 \pm \sqrt{11}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & x^2 + 6x = -11 \\
 \therefore & x^2 + 6x + 3^2 = -11 + 3^2 \\
 & \therefore (x+3)^2 = -2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & 2x^2 + 4x + 1 = 0 \\
 \therefore & x^2 + 2x + \frac{1}{2} = 0 \\
 & \therefore x^2 + 2x = -\frac{1}{2} \\
 \therefore & x^2 + 2x + 1^2 = -\frac{1}{2} + 1^2 \\
 & \therefore (x+1)^2 = \frac{1}{2} \\
 & \therefore x+1 = \pm\frac{1}{\sqrt{2}} \\
 & \therefore x = -1 \pm \frac{1}{\sqrt{2}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 3x^2 + 12x + 5 = 0 \\
 \therefore & x^2 + 4x + \frac{5}{3} = 0 \\
 & \therefore x^2 + 4x = -\frac{5}{3} \\
 \therefore & x^2 + 4x + 2^2 = -\frac{5}{3} + 2^2 \\
 & \therefore (x+2)^2 = \frac{7}{3} \\
 & \therefore x+2 = \pm\sqrt{\frac{7}{3}} \\
 & \therefore x = -2 \pm \sqrt{\frac{7}{3}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & x^2 + 6x + 2 = 0 \\
 & \therefore x^2 + 6x = -2 \\
 \therefore & x^2 + 6x + 3^2 = -2 + 3^2 \\
 & \therefore (x+3)^2 = 7 \\
 & \therefore x+3 = \pm\sqrt{7} \\
 & \therefore x = -3 \pm \sqrt{7}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & x^2 = 4x + 3 \\
 & \therefore x^2 - 4x = 3 \\
 \therefore & x^2 - 4x + (-2)^2 = 3 + (-2)^2 \\
 & \therefore (x-2)^2 = 7 \\
 & \therefore x-2 = \pm\sqrt{7} \\
 & \therefore x = 2 \pm \sqrt{7}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & x^2 = 2x + 6 \\
 & \therefore x^2 - 2x = 6 \\
 \therefore & x^2 - 2x + (-1)^2 = 6 + (-1)^2 \\
 & \therefore (x-1)^2 = 7 \\
 & \therefore x-1 = \pm\sqrt{7} \\
 & \therefore x = 1 \pm \sqrt{7}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & x^2 + 10 = 8x \\
 & \therefore x^2 - 8x = -10 \\
 \therefore & x^2 - 8x + (-4)^2 = -10 + (-4)^2 \\
 & \therefore (x-4)^2 = 6 \\
 & \therefore x-4 = \pm\sqrt{6} \\
 & \therefore x = 4 \pm \sqrt{6}
 \end{aligned}$$

$\therefore x$ has no real solutions, since the perfect square cannot be negative.

$$\begin{aligned}
 \mathbf{b} \quad & 2x^2 - 10x + 3 = 0 \\
 \therefore & x^2 - 5x + \frac{3}{2} = 0 \\
 & \therefore x^2 - 5x = -\frac{3}{2} \\
 \therefore & x^2 - 5x + \left(-\frac{5}{2}\right)^2 = -\frac{3}{2} + \left(-\frac{5}{2}\right)^2 \\
 & \therefore \left(x - \frac{5}{2}\right)^2 = -\frac{3}{2} + \frac{25}{4} \\
 & \therefore \left(x - \frac{5}{2}\right)^2 = \frac{19}{4} \\
 & \therefore x - \frac{5}{2} = \pm\frac{\sqrt{19}}{2} \\
 & \therefore x = \frac{5}{2} \pm \frac{\sqrt{19}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & 3x^2 = 6x + 4 \\
 & \therefore x^2 = 2x + \frac{4}{3} \\
 & \therefore x^2 - 2x = \frac{4}{3} \\
 \therefore & x^2 - 2x + (-1)^2 = \frac{4}{3} + (-1)^2 \\
 & \therefore (x-1)^2 = \frac{7}{3} \\
 & \therefore x-1 = \pm\sqrt{\frac{7}{3}} \\
 & \therefore x = 1 \pm \sqrt{\frac{7}{3}}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & 5x^2 - 15x + 2 = 0 \\
 \therefore & x^2 - 3x + \frac{2}{5} = 0 \\
 \therefore & x^2 - 3x = -\frac{2}{5} \\
 \therefore & x^2 - 3x + \left(-\frac{3}{2}\right)^2 = -\frac{2}{5} + \left(-\frac{3}{2}\right)^2 \\
 \therefore & \left(x - \frac{3}{2}\right)^2 = -\frac{2}{5} + \frac{9}{4} = \frac{37}{20} \\
 \therefore & x - \frac{3}{2} = \pm \sqrt{\frac{37}{20}} \\
 \therefore & x = \frac{3}{2} \pm \sqrt{\frac{37}{20}}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 4x^2 + 4x = 5 \\
 \therefore & x^2 + x = \frac{5}{4} \\
 \therefore & x^2 + x + \left(\frac{1}{2}\right)^2 = \frac{5}{4} + \left(\frac{1}{2}\right)^2 \\
 \therefore & \left(x + \frac{1}{2}\right)^2 = \frac{6}{4} \\
 \therefore & x + \frac{1}{2} = \pm \frac{\sqrt{6}}{2} \\
 \therefore & x = -\frac{1}{2} \pm \frac{\sqrt{6}}{2}
 \end{aligned}$$

EXERCISE 7E

$$\begin{aligned}
 1 \quad \text{a} \quad & x^2 - 4x - 3 = 0 \\
 \text{has } & a = 1, \quad b = -4, \quad c = -3
 \end{aligned}$$

$$\begin{aligned}
 \therefore x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)} \\
 &= \frac{4 \pm \sqrt{28}}{2} \\
 &= \frac{4 \pm 2\sqrt{7}}{2} \\
 &= 2 \pm \sqrt{7}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & x^2 + 1 = 4x \\
 \therefore & x^2 - 4x + 1 = 0 \\
 \text{which has } & a = 1, \quad b = -4, \quad c = 1
 \end{aligned}$$

$$\begin{aligned}
 \therefore x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} \\
 &= \frac{4 \pm \sqrt{12}}{2} \\
 &= \frac{4 \pm 2\sqrt{3}}{2} \\
 &= 2 \pm \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & x^2 - 4x + 2 = 0 \\
 \text{has } & a = 1, \quad b = -4, \quad c = 2
 \end{aligned}$$

$$\begin{aligned}
 \therefore x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} \\
 &= \frac{4 \pm \sqrt{8}}{2} \\
 &= \frac{4 \pm 2\sqrt{2}}{2} \\
 &= 2 \pm \sqrt{2}
 \end{aligned}$$

$$\text{g} \quad x^2 - 2\sqrt{2}x + 2 = 0 \quad \text{has } a = 1, \quad b = -2\sqrt{2}, \quad c = 2$$

$$\begin{aligned}
 \therefore x &= \frac{-(-2\sqrt{2}) \pm \sqrt{(-2\sqrt{2})^2 - 4(1)(2)}}{2(1)} \\
 &= \frac{2\sqrt{2} \pm \sqrt{8 - 8}}{2} = \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & x^2 + 6x + 7 = 0 \\
 \text{has } & a = 1, \quad b = 6, \quad c = 7
 \end{aligned}$$

$$\begin{aligned}
 \therefore x &= \frac{-6 \pm \sqrt{6^2 - 4(1)(7)}}{2(1)} \\
 &= \frac{-6 \pm \sqrt{8}}{2} \\
 &= \frac{-6 \pm 2\sqrt{2}}{2} \\
 &= -3 \pm \sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & x^2 + 4x = 1 \\
 \therefore & x^2 + 4x - 1 = 0 \\
 \text{which has } & a = 1, \quad b = 4, \quad c = -1
 \end{aligned}$$

$$\begin{aligned}
 \therefore x &= \frac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)} \\
 &= \frac{-4 \pm \sqrt{20}}{2} \\
 &= \frac{-4 \pm 2\sqrt{5}}{2} \\
 &= -2 \pm \sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & 2x^2 - 2x - 3 = 0 \\
 \text{has } & a = 2, \quad b = -2, \quad c = -3
 \end{aligned}$$

$$\begin{aligned}
 \therefore x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-3)}}{2(2)} \\
 &= \frac{2 \pm \sqrt{28}}{4} \\
 &= \frac{2 \pm 2\sqrt{7}}{4} \\
 &= \frac{1}{2} \pm \frac{\sqrt{7}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & (3x+1)^2 = -2x \\
 \therefore & 9x^2 + 6x + 1 = -2x \\
 \therefore & 9x^2 + 8x + 1 = 0 \\
 \text{which has } & a = 9, \quad b = 8, \quad c = 1 \\
 \therefore x = & \frac{-8 \pm \sqrt{8^2 - 4(9)(1)}}{2(9)} \\
 = & \frac{-8 \pm \sqrt{28}}{18} \\
 = & \frac{-8 \pm 2\sqrt{7}}{18} \quad \text{or} \quad -\frac{4}{9} \pm \frac{\sqrt{7}}{9}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & (x+2)(x-1) = 2-3x \\
 \therefore & x^2 - x + 2x - 2 = 2-3x \\
 \therefore & x^2 + 4x - 4 = 0 \\
 \text{which has } & a = 1, \quad b = 4, \quad c = -4 \\
 \therefore x = & \frac{-4 \pm \sqrt{4^2 - 4(1)(-4)}}{2(1)} \\
 = & \frac{-4 \pm \sqrt{32}}{2} \\
 = & \frac{-4 \pm 4\sqrt{2}}{2} \\
 = & -2 \pm 2\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & (x-2)^2 = 1+x \\
 \therefore & x^2 - 4x + 4 = 1+x \\
 \therefore & x^2 - 5x + 3 = 0 \\
 \text{which has } & a = 1, \quad b = -5, \quad c = 3 \\
 \therefore x = & \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)} \\
 = & \frac{5 \pm \sqrt{25-12}}{2} \\
 = & \frac{5}{2} \pm \frac{\sqrt{13}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & x - \frac{1}{x} = 1 \\
 \therefore & x^2 - 1 = x \\
 \therefore & x^2 - x - 1 = 0 \\
 \text{which has } & a = 1, \quad b = -1, \quad c = -1 \\
 \therefore x = & \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \\
 = & \frac{1 \pm \sqrt{1+4}}{2} \\
 = & \frac{1}{2} \pm \frac{\sqrt{5}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & (x+3)(2x+1) = 9 \\
 \therefore & 2x^2 + x + 6x + 3 = 9 \\
 \therefore & 2x^2 + 7x - 6 = 0 \\
 \text{which has } & a = 2, \quad b = 7, \quad c = -6 \\
 \therefore x = & \frac{-7 \pm \sqrt{7^2 - 4(2)(-6)}}{2(2)} \\
 = & \frac{-7 \pm \sqrt{49+48}}{4} \\
 = & -\frac{7}{4} \pm \frac{\sqrt{97}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & (2x+1)^2 = 3-x \\
 \therefore & 4x^2 + 4x + 1 = 3-x \\
 \therefore & 4x^2 + 5x - 2 = 0 \\
 \text{which has } & a = 4, \quad b = 5, \quad c = -2 \\
 \therefore x = & \frac{-5 \pm \sqrt{5^2 - 4(4)(-2)}}{2(4)} \\
 = & \frac{-5 \pm \sqrt{25+32}}{8} \\
 = & -\frac{5}{8} \pm \frac{\sqrt{57}}{8}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \frac{x-1}{2-x} = 2x+1 \\
 \therefore & x-1 = (2x+1)(2-x) \\
 \therefore & x-1 = 4x-2x^2+2-x \\
 \therefore & 2x^2-2x-3 = 0 \\
 \text{which has } & a = 2, \quad b = -2, \quad c = -3 \\
 \therefore x = & \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-3)}}{2(2)} \\
 = & \frac{2 \pm \sqrt{28}}{4} \\
 = & \frac{2 \pm 2\sqrt{7}}{4} \quad \text{or} \quad \frac{1}{2} \pm \frac{\sqrt{7}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & 2x - \frac{1}{x} = 3 \\
 \therefore & 2x^2 - 1 = 3x \\
 \therefore & 2x^2 - 3x - 1 = 0 \\
 \text{which has } & a = 2, \quad b = -3, \quad c = -1 \\
 \therefore x = & \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)} \\
 = & \frac{3 \pm \sqrt{9+8}}{4} \\
 = & \frac{3}{4} \pm \frac{\sqrt{17}}{4}
 \end{aligned}$$

EXERCISE 7F

- 1** **a** $x = -3.414$ or -0.586 **b** $x = 0.317$ or -6.317 **c** $x = 2.77$ or -1.27
 d $x = -1.08$ or 3.41 **e** $x = -0.892$ or 3.64 **f** $x = 1.34$ or -2.54
- 2** **a** $x = -4.83$ or 0.828 **b** $x = -1.57$ or 0.319 **c** $x = 0.697$ or 4.30
 d $x = -0.823$ or 1.82 **e** $x = -0.618$ or 1.62 **f** $x = -0.281$ or 1.78

EXERCISE 7G

- 1** Let the smaller of the integers be x .
 Since they differ by 12,
 the other integer is $(x + 12)$.
 $\therefore$ the sum of their squares is

$$x^2 + (x + 12)^2 = 74$$

$$\therefore x^2 + x^2 + 24x + 144 = 74$$

$$\therefore 2x^2 + 24x + 70 = 0$$

$$\therefore x^2 + 12x + 35 = 0$$

$$\therefore (x + 7)(x + 5) = 0$$

$$\therefore x = -7 \text{ or } -5$$
 $\therefore$ the larger integer is 5 or 7
 i.e., the integers are -7 and 5 , or -5 and 7
- 2** Let the number be x , so its reciprocal is $\frac{1}{x}$.
 They have sum $x + \frac{1}{x} = 5\frac{1}{5}$

$$\therefore x^2 + 1 = \frac{26}{5}x$$

$$\therefore x^2 - \frac{26}{5}x + 1 = 0$$

$$\therefore 5x^2 - 26x + 5 = 0$$

$$\therefore (5x - 1)(x - 5) = 0$$

$$\therefore x = \frac{1}{5} \text{ or } 5$$
 i.e., the number is either $\frac{1}{5}$ or 5
- 3** Let the number be x so its square is x^2 .
 $\therefore$ the sum is $x + x^2 = 210$

$$\therefore x^2 + x - 210 = 0$$

$$\therefore (x + 15)(x - 14) = 0$$

$$\therefore x = -15 \text{ or } 14$$
 But x is a natural number, so $x > 0$,
 $\therefore$ the number is 14.
- 4** Suppose the numbers are x and $(x + 2)$.
 Then $x(x + 2) = 360$

$$\therefore x^2 + 2x - 360 = 0$$

$$\therefore (x + 20)(x - 18) = 0$$

$$\therefore x = -20 \text{ or } 18$$
 $\therefore$ the numbers are -20 and -18 ,
 or 18 and 20 .
- 5** Suppose the numbers are x and $(x + 2)$.
 Then $x(x + 2) = 255$

$$\therefore x^2 + 2x - 255 = 0$$

$$\therefore (x + 17)(x - 15) = 0$$

$$\therefore x = -17 \text{ or } 15$$
 $\therefore$ the numbers are -17 and -15 ,
 or 15 and 17 .
- 6** If the polygon has n sides, then

$$\frac{n}{2}(n - 3) = 90$$

$$\therefore \frac{1}{2}n^2 - \frac{3}{2}n = 90$$

$$\therefore n^2 - 3n - 180 = 0$$

$$\therefore (n - 15)(n + 12) = 0$$

$$\therefore n = -12 \text{ or } 15$$
 $\therefore$ the polygon has 15 sides. $\{\text{as } n > 0\}$
- 7** If the width of the rectangle is w cm,
 then its length is $(w + 4)$ cm.
 $\therefore$ the area is $w(w + 4) = 26$

$$\therefore w^2 + 4w - 26 = 0$$
 which has $a = 1$, $b = 4$, $c = -26$

$$\therefore w = \frac{-4 \pm \sqrt{4^2 - 4(1)(-26)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{120}}{2} = -2 \pm \sqrt{30}$$
- But $w > 0$, so $w = -2 + \sqrt{30}$

$$\div 3.477 \text{ cm}$$
 i.e., the width is approximately 3.48 cm.

- 8 a** The base has sides of length x cm, so the areas of the top and bottom surfaces are both x^2 cm².

The box has height $(x + 1)$ cm, so the area of each of the side faces is $x(x + 1)$ cm.

$\therefore$ the total surface area is

$$\begin{aligned} A &= 2x^2 + 4x(x + 1) \\ &= 2x^2 + 4x^2 + 4x \\ &= 6x^2 + 4x \text{ cm}^2 \end{aligned}$$

b $6x^2 + 4x = 240$

$$\therefore 3x^2 + 2x - 120 = 0$$

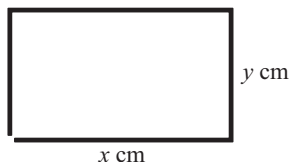
$$\therefore (3x + 20)(x - 6) = 0$$

$$\therefore x = -\frac{20}{3} \text{ or } 6$$

but $x > 0$, so $x = 6$ cm

$\therefore$ the box is 6 cm $\times$ 6 cm $\times$ 7 cm

10



Suppose one side of the rectangle has length x cm and the other has length y cm.

The perimeter is $(2x + 2y)$ cm,

so $2x + 2y = 20$

$$\therefore 2y = 20 - 2x$$

$$\therefore y = 10 - x$$

The area of the rectangle is therefore

$$x(10 - x) \text{ cm}^2.$$

$\therefore$ if the area is 30 cm², then

$$x(10 - x) = 30$$

$$\therefore 10x - x^2 = 30$$

$$\therefore x^2 - 10x + 30 = 0$$

which has $a = 1$, $b = -10$, $c = 30$

$$\therefore x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(30)}}{2(1)}$$

$$= \frac{10 \pm \sqrt{100 - 120}}{2}$$

$$= \frac{10 \pm \sqrt{-20}}{2}$$

$\therefore x$ has no real solutions, so it is not possible.

- 9** Suppose the tin plate was x cm $\times$ x cm.

When 3 cm $\times$ 3 cm squares are cut from the corners, the base of the open box formed is $(x - 6)$ cm $\times$ $(x - 6)$ cm.

The open box has height 3 cm, so its volume

$$\text{is } 3 \times (x - 6) \times (x - 6) = 80$$

$$\therefore 3(x^2 - 12x + 36) = 80$$

$$\therefore 3x^2 - 36x + 108 = 80$$

$$\therefore 3x^2 - 36x + 28 = 0$$

which has $a = 3$, $b = -36$, $c = 28$

$$\therefore x = \frac{-(-36) \pm \sqrt{(-36)^2 - 4(3)(28)}}{2(3)}$$

$$= \frac{36 \pm \sqrt{960}}{6} \text{ and since } x > 0,$$

$$x = 6 + \frac{\sqrt{960}}{6} \div 11.16 \text{ cm}$$

$\therefore$ the original piece of tinplate was about 11.2 cm square.

- 11** The smaller rectangle is similar to the original rectangle.

$$\therefore \frac{AB}{AD} = \frac{BC}{BY}$$

$$\text{But } AD = BC \text{ and } BY = AB - AY \\ = AB - AD$$

$$\therefore \frac{AB}{AD} = \frac{AD}{AB - AD}$$

Suppose $AB = x$ units, and $AD = BC = 1$ unit

$$\therefore \frac{x}{1} = \frac{1}{x - 1}$$

$$\therefore x(x - 1) = 1$$

$$\therefore x^2 - x - 1 = 0$$

which has $a = 1$, $b = -1$, $c = -1$

$$\therefore x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

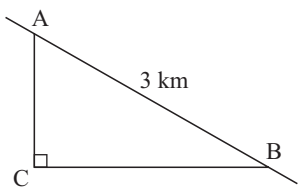
$$= \frac{1 \pm \sqrt{1 + 4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore x = \frac{1 + \sqrt{5}}{2}, \text{ since } x > 0$$

But $\frac{AB}{AD} = x$, which is the golden ratio

$\therefore$ the golden ratio is $\frac{1 + \sqrt{5}}{2}$

12

Suppose AC is x hundred metres,
so BC is $(x + 4)$ hundred metres.

Now $AC^2 + BC^2 = AB^2$ {Pythagoras}

$$\therefore x^2 + (x + 4)^2 = 30^2$$

$$\therefore x^2 + x^2 + 8x + 16 = 900$$

$$\therefore 2x^2 + 8x - 884 = 0$$

$$\therefore x^2 + 4x - 442 = 0$$

which has $a = 1$, $b = 4$, $c = -442$

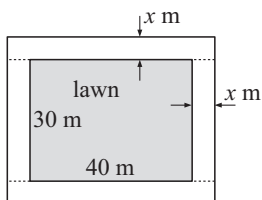
$$\therefore x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-442)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{1784}}{2}$$

$$\therefore \text{since } x > 0, x = \frac{-4 + \sqrt{1784}}{2} \doteq 19.12$$

$\therefore AC \doteq 19.12$ hundred metres and $BC \doteq 23.12$ hundred metres

$\therefore$ since the paddock is triangular, its area is $\frac{1}{2} \times 19.12 \times 23.12 \doteq 221$ hectares.

13

Suppose the concrete has width x m around the lawn. We divide the concrete up into four regions as shown.

The smaller regions have area $30x$ m², whilst the larger regions have area $x(40 + 2x)$ m².

Now the total area of concrete is one quarter the area of the lawn.

$$\therefore 2 \times 30x + 2 \times x(40 + 2x) = \frac{1}{4} \times 30 \times 40$$

$$\therefore 60x + 80x + 4x^2 = 300$$

$$\therefore 4x^2 + 140x = 300$$

$$\therefore x^2 + 35x - 75 = 0$$

which has $a = 1$, $b = 35$, $c = -75$

$$\therefore x = \frac{-35 \pm \sqrt{35^2 - 4(1)(-75)}}{2(1)}$$

$$= \frac{-35 \pm \sqrt{1525}}{2}$$

But $x > 0$,

$$\text{so } x = \frac{-35 + \sqrt{1525}}{2} \doteq 2.026 \text{ m}$$

$\therefore$ the path is about 2.03 m wide.

14 Suppose Hassan's speed is h kmph.

We know that $\text{speed} = \frac{\text{distance}}{\text{time}}$,

so $\text{time} = \frac{\text{distance}}{\text{speed}}$

$$\therefore \text{if it takes Hassan } t \text{ hours, } t = \frac{40}{h} \dots (1)$$

Now Chuong says he will drive home at speed $(h + 40)$ kmph and arrive in time $(t - \frac{1}{3})$ hrs.

$$\therefore t - \frac{1}{3} = \frac{40}{h + 40}$$

$$\text{i.e., } t = \frac{40}{h + 40} + \frac{1}{3} \dots (2)$$

$$\therefore \text{using (1) and (2), } \frac{40}{h} = \frac{40}{h + 40} + \frac{1}{3}$$

$$\therefore 40(h + 40) = 40h + \frac{1}{3}h(h + 40)$$

$$\therefore 40h + 1600 = 40h + \frac{1}{3}h^2 + \frac{40}{3}h$$

$$\therefore h^2 + 40h - 4800 = 0$$

which has $a = 1$, $b = 40$, $c = -4800$

$$\therefore h = \frac{-40 \pm \sqrt{40^2 - 4(1)(-4800)}}{2(1)}$$

$$= \frac{-40 \pm \sqrt{1600 + 19200}}{2}$$

$$= \frac{-40 \pm \sqrt{20800}}{2}$$

$$\text{But } h > 0, \text{ so } h = \frac{-40 + \sqrt{20800}}{2} \doteq 52.1 \text{ kmph}$$

i.e., Hassan's speed is approximately 52.1 kmph.

- 15** Suppose the speed of the plane is x kmph.

We know $\text{speed} = \frac{\text{distance}}{\text{time}}$, so $\text{time} = \frac{\text{distance}}{\text{speed}}$

Using the information given, $\frac{1000}{x} = \frac{1000}{x-120} - \frac{1}{2}$

$$\therefore 1000(x-120) = 1000x - \frac{1}{2}x(x-120)$$

$$\therefore 1000x - 120\,000 = 1000x - \frac{1}{2}x^2 + 60x$$

$$\therefore x^2 - 120x - 240\,000 = 0$$

which has $a = 1$, $b = -120$, $c = -240\,000$

$$\therefore x = \frac{-(-120) \pm \sqrt{(-120)^2 - 4(1)(-240\,000)}}{2(1)}$$

$$= \frac{120 \pm \sqrt{974\,400}}{2}$$

But $x > 0$, so $x = \frac{120 + \sqrt{974\,400}}{2} \div 553.6$ kmph, $\therefore$ the plane has speed approximately 554 kmph.

- 16** Suppose the express train travels at x kmph (on average).

We know $\text{speed} = \frac{\text{distance}}{\text{time}}$, so $\text{time} = \frac{\text{distance}}{\text{speed}}$

$\therefore$ it takes the express train $\frac{105}{x}$ hours and the normal train $\frac{105}{x-10}$ hours.

$$\therefore \frac{105}{x} + \frac{1}{2} = \frac{105}{x-10}$$

$$\therefore 105(x-10) + \frac{1}{2}x(x-10) = 105x$$

$$\therefore 105x - 1050 + \frac{1}{2}x^2 - 5x = 105x$$

$$\therefore x^2 - 10x - 2100 = 0$$

which has $a = 1$, $b = -10$, $c = -2100$

$$\therefore x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(-2100)}}{2(1)} = \frac{10 \pm \sqrt{8500}}{2}$$

But $x > 0$, so $x = \frac{10 + \sqrt{8500}}{2} \div 51.1$ kmph

$\therefore$ the express train travels on average at about 51.1 kmph.

- 17** Suppose n elderly citizens ended up going on the trip, so the cost per person was $\$ \frac{160}{n}$.

If the original number of elderly citizens had gone, there would have been $(n+8)$,

and the cost per person would have been $\$ \frac{160}{n+8}$.

$$\text{Hence } \frac{160}{n} = \frac{160}{n+8} + 1$$

$$\therefore 160(n+8) = 160n + n(n+8)$$

$$\therefore 160n + 1280 = 160n + n^2 + 8n$$

$$\therefore n^2 + 8n - 1280 = 0$$

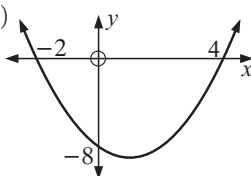
$$\therefore (n-32)(n+40) = 0$$

$\therefore$ since $n > 0$, $n = 32$, i.e., 32 elderly citizens went on the trip.

EXERCISE 7H

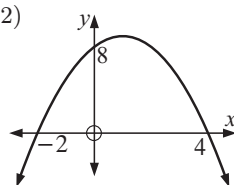
1 a $y = (x - 4)(x + 2)$

 has x -intercepts
 -2 and 4

 and y -intercept
 -8


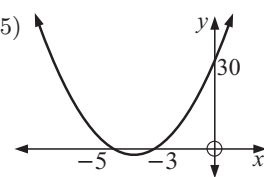
b $y = -(x - 4)(x + 2)$

 has x -intercepts
 -2 and 4

 and y -intercept 8


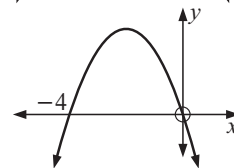
c $y = 2(x + 3)(x + 5)$

 has x -intercepts
 -5 and -3

 and y -intercept
 30


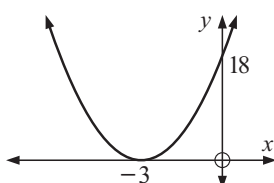
d $y = -3x(x + 4)$

 has x -intercepts
 0 and -4

 and y -intercept 0


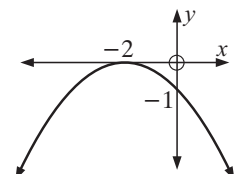
e $y = 2(x + 3)^2$

 has x -intercept
 -3

 and y -intercept
 18


f $y = -\frac{1}{4}(x + 2)^2$

 has x -intercept
 -2

 and y -intercept
 -1

2 a The average of the x -intercepts is 1 $\therefore$ the axis of symmetry is $x = 1$.

b The average of the x -intercepts is 1 $\therefore$ the axis of symmetry is $x = 1$.

c The average of the x -intercepts is -4 $\therefore$ the axis of symmetry is $x = -4$.

d The average of the x -intercepts is -2 $\therefore$ the axis of symmetry is $x = -2$.

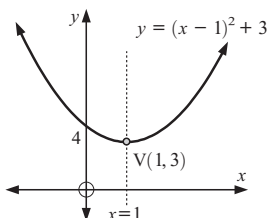
e The only x -intercept is -3 , so the axis of symmetry is $x = -3$.

f The only x -intercept is -2 , so the axis of symmetry is $x = -2$.

3 a The vertex is

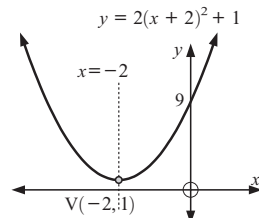
 $(1, 3)$.

 The axis of
 symmetry is
 $x = 1$.

 The y -intercept
 is 4 .

b The vertex is

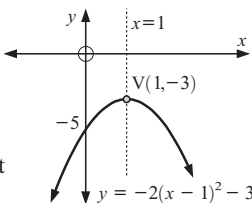
 $(-2, 1)$.

 The axis of
 symmetry is
 $x = -2$.

 The y -intercept
 is 9 .

c The vertex is

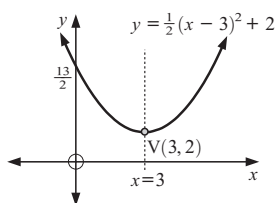
 $(1, -3)$.

 The axis of
 symmetry is
 $x = 1$.

 The y -intercept
 is -5 .

d The vertex is

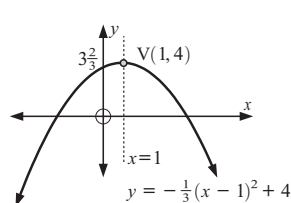
 $(3, 2)$.

 The axis of
 symmetry is
 $x = 3$.

 The y -intercept
 is $\frac{13}{2}$.

e The vertex is

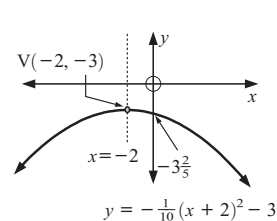
 $(1, 4)$.

 The axis of
 symmetry is
 $x = 1$.

 The y -intercept
 is $3\frac{2}{3}$.

f The vertex is

 $(-2, -3)$.

 The axis of
 symmetry is
 $x = -2$.

 The y -intercept
 is $-3\frac{2}{5}$.


4 a $y = x^2 - 4x + 2$
 has $a = 1$, $b = -4$, $c = 2$
 $\therefore -\frac{b}{2a} = -\frac{(-4)}{2(1)} = 2$
 $\therefore$ the axis of symmetry is $x = 2$
 When $x = 2$,
 $y = 2^2 - 4 \times 2 + 2 = -2$
 $\therefore$ the vertex is at $(2, -2)$.

c $y = 2x^2 + 4$
 has $a = 2$, $b = 0$, $c = 4$
 $\therefore -\frac{b}{2a} = -\frac{0}{2(2)} = 0$
 $\therefore$ the axis of symmetry is $x = 0$
 When $x = 0$, $y = 4$
 $\therefore$ the vertex is at $(0, 4)$.

e $y = 2x^2 + 8x - 7$
 has $a = 2$, $b = 8$, $c = -7$
 $\therefore -\frac{b}{2a} = -\frac{8}{2(2)} = -2$
 $\therefore$ the axis of symmetry is $x = -2$
 When $x = -2$,
 $y = 2(-2)^2 + 8(-2) - 7 = -15$
 $\therefore$ the vertex is at $(-2, -15)$.

g $y = 2x^2 + 6x - 1$
 has $a = 2$, $b = 6$, $c = -1$
 $\therefore -\frac{b}{2a} = -\frac{6}{2(2)} = -\frac{3}{2}$
 $\therefore$ the axis of symmetry is $x = -\frac{3}{2}$
 When $x = -\frac{3}{2}$,
 $y = 2(-\frac{3}{2})^2 + 6(-\frac{3}{2}) - 1$
 $= \frac{9}{2} - 9 - 1$
 $= -\frac{11}{2}$
 $\therefore$ the vertex is at $(-\frac{3}{2}, -\frac{11}{2})$.

i $y = -\frac{1}{2}x^2 + x - 5$
 has $a = -\frac{1}{2}$, $b = 1$, $c = -5$
 $\therefore -\frac{b}{2a} = -\frac{1}{2(-\frac{1}{2})} = 1$
 $\therefore$ the axis of symmetry is $x = 1$
 When $x = 1$,
 $y = -\frac{1}{2}(1)^2 + 1 - 5 = -\frac{9}{2}$
 $\therefore$ the vertex is at $(1, -\frac{9}{2})$.

b $y = x^2 + 2x - 3$
 has $a = 1$, $b = 2$, $c = -3$
 $\therefore -\frac{b}{2a} = -\frac{2}{2(1)} = -1$
 $\therefore$ the axis of symmetry is $x = -1$
 When $x = -1$,
 $y = (-1)^2 + 2(-1) - 3 = -4$
 $\therefore$ the vertex is at $(-1, -4)$.

d $y = -3x^2 + 1$
 has $a = -3$, $b = 0$, $c = 1$
 $\therefore -\frac{b}{2a} = -\frac{0}{2(-3)} = 0$
 $\therefore$ the axis of symmetry is $x = 0$
 When $x = 0$, $y = 1$
 $\therefore$ the vertex is at $(0, 1)$.

f $y = -x^2 - 4x - 9$
 has $a = -1$, $b = -4$, $c = -9$
 $\therefore -\frac{b}{2a} = -\frac{(-4)}{2(-1)} = -2$
 $\therefore$ the axis of symmetry is $x = -2$
 When $x = -2$, $y = -(-2)^2 - 4(-2) - 9$
 $= -4 + 8 - 9$
 $= -5$
 $\therefore$ the vertex is at $(-2, -5)$.

h $y = 2x^2 - 10x + 3$
 has $a = 2$, $b = -10$, $c = 3$
 $\therefore -\frac{b}{2a} = -\frac{(-10)}{2(2)} = \frac{5}{2}$
 $\therefore$ the axis of symmetry is $x = \frac{5}{2}$
 When $x = \frac{5}{2}$, $y = 2(\frac{5}{2})^2 - 10(\frac{5}{2}) + 3$
 $= \frac{25}{2} - \frac{50}{2} + 3$
 $= -\frac{19}{2}$
 $\therefore$ the vertex is at $(\frac{5}{2}, -\frac{19}{2})$.

j $y = -2x^2 + 8x - 2$
 has $a = -2$, $b = 8$, $c = -2$
 $\therefore -\frac{b}{2a} = -\frac{8}{2(-2)} = 2$
 $\therefore$ the axis of symmetry is $x = 2$
 When $x = 2$, $y = -2(2)^2 + 8(2) - 2$
 $= -8 + 16 - 2$
 $= 6$
 $\therefore$ the vertex is at $(2, 6)$.

5 a When $y = 0$, $x^2 - 9 = 0$
 $\therefore (x+3)(x-3) = 0$
 $\therefore x = \pm 3$
 $\therefore$ the x -intercepts are ± 3

c When $y = 0$, $x^2 + 7x + 10 = 0$
 $\therefore (x+5)(x+2) = 0$
 $\therefore x = -5$ or -2
 $\therefore$ the x -intercepts are -5 and -2

e When $y = 0$, $4x - x^2 = 0$
 $\therefore x(4-x) = 0$
 $\therefore x = 0$ or 4
 $\therefore$ the x -intercepts are 0 and 4

g When $y = 0$, $-2x^2 - 4x - 2 = 0$
 $\therefore x^2 + 2x + 1 = 0$
 $\therefore (x+1)^2 = 0$
 $\therefore x = -1$
 $\therefore$ the x -intercept is -1 (touching)

i When $y = 0$, $x^2 - 4x + 1 = 0$
 $a = 1$, $b = -4$ and $c = 1$
 $\therefore x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$
 $= \frac{4 \pm \sqrt{12}}{2}$
 $= \frac{4 \pm 2\sqrt{3}}{2}$
 $= 2 \pm \sqrt{3}$

$\therefore$ the x -intercepts are $2 \pm \sqrt{3}$

k When $y = 0$, $x^2 - 6x - 2 = 0$
 $a = 1$, $b = -6$ and $c = -2$
 $\therefore x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-2)}}{2(1)}$
 $= \frac{6 \pm \sqrt{44}}{2}$
 $= \frac{6 \pm 2\sqrt{11}}{2}$
 $= 3 \pm \sqrt{11}$
 $\therefore$ the x -intercepts are $3 \pm \sqrt{11}$

b When $y = 0$, $2x^2 - 6 = 0$
 $\therefore x^2 - 3 = 0$
 $\therefore (x+\sqrt{3})(x-\sqrt{3}) = 0$
 $\therefore x = \pm\sqrt{3}$
 $\therefore$ the x -intercepts are $\pm\sqrt{3}$

d When $y = 0$, $x^2 + x - 12 = 0$
 $\therefore (x+4)(x-3) = 0$
 $\therefore x = -4$ or 3
 $\therefore$ the x -intercepts are -4 and 3

f When $y = 0$, $-x^2 - 6x - 8 = 0$
 $\therefore x^2 + 6x + 8 = 0$
 $\therefore (x+4)(x+2) = 0$
 $\therefore x = -4$ or -2
 $\therefore$ the x -intercepts are -4 and -2

h When $y = 0$, $4x^2 - 24x + 36 = 0$
 $\therefore x^2 - 6x + 9 = 0$
 $\therefore (x-3)^2 = 0$
 $\therefore x = 3$
 $\therefore$ the x -intercept is 3 (touching)

j When $y = 0$, $x^2 + 4x - 3 = 0$
 $a = 1$, $b = 4$ and $c = -3$
 $\therefore x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-3)}}{2(1)}$
 $= \frac{-4 \pm \sqrt{28}}{2}$
 $= \frac{-4 \pm 2\sqrt{7}}{2}$
 $= -2 \pm \sqrt{7}$

$\therefore$ the x -intercepts are $-2 \pm \sqrt{7}$

l When $y = 0$, $x^2 + 8x + 11 = 0$
 $a = 1$, $b = 8$ and $c = 11$
 $\therefore x = \frac{-8 \pm \sqrt{8^2 - 4(1)(11)}}{2(1)}$
 $= \frac{-8 \pm \sqrt{20}}{2}$
 $= \frac{-8 \pm 2\sqrt{5}}{2}$
 $= -4 \pm \sqrt{5}$
 $\therefore$ the x -intercepts are $-4 \pm \sqrt{5}$

- 6 a i** $y = x^2 - 2x + 5$
 has $a = 1$, $b = -2$, $c = 5$
 $\therefore -\frac{b}{2a} = -\frac{(-2)}{2(1)} = 1$
 $\therefore$ the axis of symmetry is $x = 1$

iii When $x = 0$, $y = 5$,
 so the y -intercept is 5
 When $y = 0$, $x^2 - 2x + 5 = 0$
 $\therefore x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)}$
 $= \frac{2 \pm \sqrt{4 - 20}}{2}$

This has no real solutions,
 so there are no x -intercepts.

- b i** $y = x^2 + 4x - 1$
 has $a = 1$, $b = 4$, $c = -1$
 $\therefore -\frac{b}{2a} = -\frac{4}{2(1)} = -2$
 $\therefore$ the axis of symmetry is $x = -2$

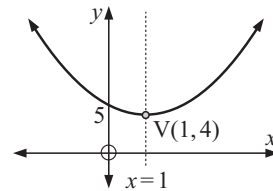
iii When $x = 0$, $y = -1$,
 so the y -intercept is -1 .
 When $y = 0$, $x^2 + 4x - 1 = 0$
 $\therefore x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-1)}}{2(1)}$
 $= \frac{-4 \pm \sqrt{20}}{2}$
 $= \frac{-4 \pm 2\sqrt{5}}{2}$
 $= -2 \pm \sqrt{5}$
 $\therefore$ the x -intercepts are $-2 \pm \sqrt{5}$

- c i** $y = 2x^2 - 5x + 2$
 has $a = 2$, $b = -5$, $c = 2$
 $\therefore -\frac{b}{2a} = -\frac{(-5)}{2(2)} = \frac{5}{4}$
 $\therefore$ the axis of symmetry is $x = \frac{5}{4}$

iii When $x = 0$, $y = 2$,
 so the y -intercept is 2.
 When $y = 0$, $2x^2 - 5x + 2 = 0$
 $\therefore (2x - 1)(x - 2) = 0$
 $\therefore x = \frac{1}{2}$ or 2
 $\therefore$ the x -intercepts are $\frac{1}{2}$ and 2

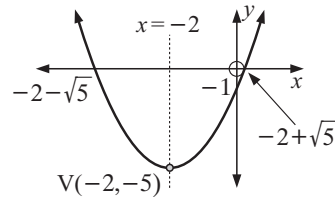
- ii** When $x = 1$,
 $y = 1^2 - 2(1) + 5$
 $= 1 - 2 + 5$
 $= 4$
 $\therefore$ the vertex is at $(1, 4)$

iv



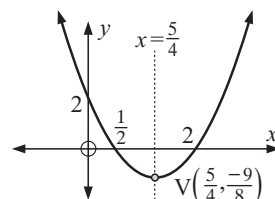
- ii** When $x = -2$,
 $y = (-2)^2 + 4(-2) - 1$
 $= 4 - 8 - 1$
 $= -5$
 $\therefore$ the vertex is at $(-2, -5)$

iv



- ii** When $x = \frac{5}{4}$,
 $y = 2(\frac{5}{4})^2 - 5(\frac{5}{4}) + 2$
 $= \frac{50}{16} - \frac{25}{4} + 2$
 $= -\frac{9}{8}$
 $\therefore$ the vertex is at $(\frac{5}{4}, -\frac{9}{8})$

iv



d i $y = -x^2 + 3x - 2$
 has $a = -1$, $b = 3$, $c = -2$
 $\therefore -\frac{b}{2a} = -\frac{3}{2(-1)} = \frac{3}{2}$
 $\therefore$ the axis of symmetry is $x = \frac{3}{2}$

iii When $x = 0$, $y = -2$,
 so the y -intercept is -2 .
 When $y = 0$, $-x^2 + 3x - 2 = 0$
 $\therefore x^2 - 3x + 2 = 0$
 $\therefore (x-1)(x-2) = 0$
 $\therefore x = 1$ or 2
 $\therefore$ the x -intercepts are 1 and 2

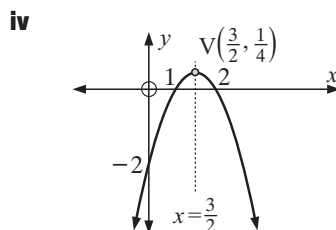
e i $y = -3x^2 + 4x - 1$
 has $a = -3$, $b = 4$, $c = -1$
 $\therefore -\frac{b}{2a} = -\frac{4}{2(-3)} = \frac{2}{3}$
 $\therefore$ the axis of symmetry is $x = \frac{2}{3}$

iii When $x = 0$, $y = -1$,
 so the y -intercept is -1 .
 When $y = 0$, $-3x^2 + 4x - 1 = 0$
 $\therefore 3x^2 - 4x + 1 = 0$
 $\therefore (3x-1)(x-1) = 0$
 $\therefore x = \frac{1}{3}$ or 1
 $\therefore$ the x -intercepts are $\frac{1}{3}$ and 1

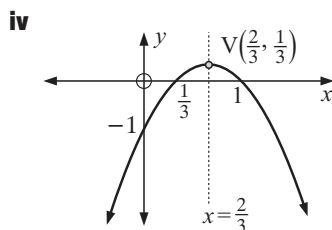
f i $y = -2x^2 + x + 1$
 has $a = -2$, $b = 1$, $c = 1$
 $\therefore -\frac{b}{2a} = -\frac{1}{2(-2)} = \frac{1}{4}$
 $\therefore$ the axis of symmetry is $x = \frac{1}{4}$

iii When $x = 0$, $y = 1$,
 so the y -intercept is 1 .
 When $y = 0$, $-2x^2 + x + 1 = 0$
 $\therefore 2x^2 - x - 1 = 0$
 $\therefore (2x+1)(x-1) = 0$
 $\therefore x = -\frac{1}{2}$ or 1
 $\therefore$ the x -intercepts are $-\frac{1}{2}$ and 1

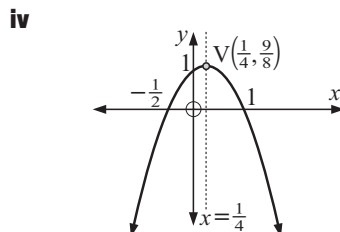
ii When $x = \frac{3}{2}$,
 $y = -(\frac{3}{2})^2 + 3(\frac{3}{2}) - 2$
 $= -\frac{9}{4} + \frac{9}{2} - 2$
 $= \frac{1}{4}$
 $\therefore$ the vertex is at $(\frac{3}{2}, \frac{1}{4})$



ii When $x = \frac{2}{3}$,
 $y = -3(\frac{2}{3})^2 + 4(\frac{2}{3}) - 1$
 $= -\frac{4}{3} + \frac{8}{3} - 1$
 $= \frac{1}{3}$
 $\therefore$ the vertex is at $(\frac{2}{3}, \frac{1}{3})$



ii When $x = \frac{1}{4}$,
 $y = -2(\frac{1}{4})^2 + \frac{1}{4} + 1$
 $= -\frac{1}{8} + \frac{1}{4} + 1$
 $= \frac{9}{8}$
 $\therefore$ the vertex is at $(\frac{1}{4}, \frac{9}{8})$



g i $y = 6x - x^2$
 has $a = -1$, $b = 6$, $c = 0$
 $\therefore -\frac{b}{2a} = -\frac{6}{2(-1)} = 3$

$\therefore$ the axis of symmetry is $x = 3$

iii When $x = 0$, $y = 0$,
 so the y -intercept is 0.
 When $y = 0$, $6x - x^2 = 0$
 $\therefore x(6 - x) = 0$
 $\therefore x = 0$ or 6

$\therefore$ the x -intercepts are 0 and 6

h i $y = -x^2 - 6x - 8$
 has $a = -1$, $b = -6$, $c = -8$
 $\therefore -\frac{b}{2a} = -\frac{(-6)}{2(-1)} = -3$

$\therefore$ the axis of symmetry is $x = -3$

iii When $x = 0$, $y = -8$,
 so the y -intercept is -8 .
 When $y = 0$, $-x^2 - 6x - 8 = 0$
 $\therefore x^2 + 6x + 8 = 0$
 $\therefore (x + 4)(x + 2) = 0$
 $\therefore x = -4$ or -2

$\therefore$ the x -intercepts are -4 and -2

i i $y = -\frac{1}{4}x^2 + 2x + 1$
 has $a = -\frac{1}{4}$, $b = 2$, $c = 1$
 $\therefore -\frac{b}{2a} = -\frac{2}{2(-\frac{1}{4})} = 4$

$\therefore$ the axis of symmetry is $x = 4$

iii When $x = 0$, $y = 1$,
 so the y -intercept is 1.
 When $y = 0$, $-\frac{1}{4}x^2 + 2x + 1 = 0$
 $\therefore x^2 - 8x - 4 = 0$
 which has $a = 1$, $b = -8$, $c = -4$

$$\therefore x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-4)}}{2(1)}$$

$$= \frac{8 \pm \sqrt{80}}{2}$$

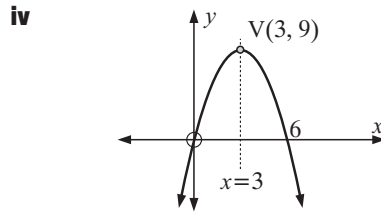
$$= \frac{8 \pm 4\sqrt{5}}{2}$$

$$= 4 \pm 2\sqrt{5}$$

$\therefore$ the x -intercepts are $4 \pm 2\sqrt{5}$

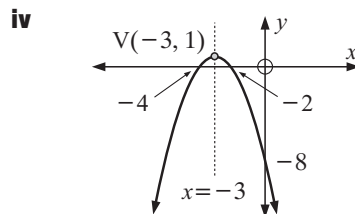
ii When $x = 3$,
 $y = 6 \times 3 - 3^2$
 $= 9$

$\therefore$ the vertex is at $(3, 9)$



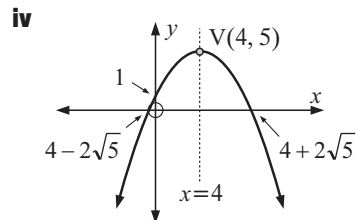
ii When $x = -3$,
 $y = -(-3)^2 - 6(-3) - 8$
 $= -9 + 18 - 8$
 $= 1$

$\therefore$ the vertex is at $(-3, 1)$



ii When $x = 4$,
 $y = -\frac{1}{4}(4)^2 + 2(4) + 1$
 $= -4 + 8 + 1$
 $= 5$

$\therefore$ the vertex is at $(4, 5)$



EXERCISE 71.1

1 a $x^2 + 7x - 2 = 0$

has $a = 1$, $b = 7$, $c = -2$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 7^2 - 4(1)(-2) \\ &= 57\end{aligned}$$

Since $\Delta > 0$, there are 2 distinct real solutions.

c $2x^2 + 3x - 1 = 0$

has $a = 2$, $b = 3$, $c = -1$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 3^2 - 4(2)(-1) \\ &= 17\end{aligned}$$

Since $\Delta > 0$, there are 2 distinct real solutions.

e $x^2 + x + 6 = 0$

has $a = 1$, $b = 1$, $c = 6$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 1^2 - 4(1)(6) \\ &= -23\end{aligned}$$

Since $\Delta < 0$, there are no real roots.

2 a $2x^2 + 7x - 4 = 0$

has $a = 2$, $b = 7$, $c = -4$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 7^2 - 4(2)(-4) \\ &= 81\end{aligned}$$

$\therefore \sqrt{\Delta} = 9$, so the equation has rational roots.

c $2x^2 + 6x + 1 = 0$

has $a = 2$, $b = 6$, $c = 1$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 6^2 - 4(2)(1) \\ &= 28\end{aligned}$$

$\therefore \sqrt{\Delta} = \sqrt{28} = 2\sqrt{7}$, so the equation does not have rational roots.

e $4x^2 - 3x + 3 = 0$

has $a = 4$, $b = -3$, $c = 3$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-3)^2 - 4(4)(3) \\ &= -39\end{aligned}$$

Since $\Delta < 0$, the equation does not have rational roots.

b $x^2 + 4\sqrt{2}x + 8 = 0$

has $a = 1$, $b = 4\sqrt{2}$, $c = 8$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (4\sqrt{2})^2 - 4(1)(8) \\ &= 32 - 32 \\ &= 0\end{aligned}$$

$\therefore$ there is one repeated real root.

d $6x^2 + 5x - 4 = 0$

has $a = 6$, $b = 5$, $c = -4$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 5^2 - 4(6)(-4) \\ &= 121\end{aligned}$$

Since $\Delta > 0$, there are 2 distinct real solutions.

f $9x^2 + 6x + 1 = 0$

has $a = 9$, $b = 6$, $c = 1$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 6^2 - 4(9)(1) \\ &= 0\end{aligned}$$

$\therefore$ there is one repeated real root.

b $3x^2 - 7x - 6 = 0$

has $a = 3$, $b = -7$, $c = -6$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-7)^2 - 4(3)(-6) \\ &= 121\end{aligned}$$

$\therefore \sqrt{\Delta} = 11$, so the equation has rational roots.

d $6x^2 + 19x + 10 = 0$

has $a = 6$, $b = 19$, $c = 10$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 19^2 - 4(6)(10) \\ &= 121\end{aligned}$$

$\therefore \sqrt{\Delta} = 11$, so the equation has rational roots.

f $8x^2 - 10x - 3 = 0$

has $a = 8$, $b = -10$, $c = -3$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-10)^2 - 4(8)(-3) \\ &= 196\end{aligned}$$

$\therefore \sqrt{\Delta} = 14$, so the equation has rational roots.

3 a $x^2 + 3x + m = 0$ has $a = 1$, $b = 3$, $c = m$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac = 3^2 - 4(1)m \\ &= 9 - 4m\end{aligned}$$

i For a repeated root,

$$\Delta = 0$$

$$\therefore 9 - 4m = 0$$

$$\therefore 4m = 9$$

$$\therefore m = \frac{9}{4}$$

ii For two distinct real roots,

$$\Delta > 0$$

$$\therefore 9 - 4m > 0$$

$$\therefore 4m < 9$$

$$\therefore m < \frac{9}{4}$$

iii For no real roots,

$$\Delta < 0$$

$$\therefore 9 - 4m < 0$$

$$\therefore 4m > 9$$

$$\therefore m > \frac{9}{4}$$

b $x^2 - 5x + m = 0$ has $a = 1$, $b = -5$, $c = m$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac = (-5)^2 - 4(1)m \\ &= 25 - 4m\end{aligned}$$

i For a repeated root,

$$\Delta = 0$$

$$\therefore 25 - 4m = 0$$

$$\therefore 4m = 25$$

$$\therefore m = \frac{25}{4}$$

ii For two distinct real roots,

$$\Delta > 0$$

$$\therefore 25 - 4m > 0$$

$$\therefore 4m < 25$$

$$\therefore m < \frac{25}{4}$$

iii For no real roots,

$$\Delta < 0$$

$$\therefore 25 - 4m < 0$$

$$\therefore 4m > 25$$

$$\therefore m > \frac{25}{4}$$

c $mx^2 - x + 1 = 0$ has $a = m$, $b = -1$, $c = 1$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac = (-1)^2 - 4m(1) \\ &= 1 - 4m\end{aligned}$$

i For a repeated root,

$$\Delta = 0$$

$$\therefore 1 - 4m = 0$$

$$\therefore 4m = 1$$

$$\therefore m = \frac{1}{4}$$

ii For two distinct real roots,

$$\Delta > 0$$

$$\therefore 1 - 4m > 0$$

$$\therefore 4m < 1$$

$$\therefore m < \frac{1}{4}$$

iii For no real roots,

$$\Delta < 0$$

$$\therefore 1 - 4m < 0$$

$$\therefore 4m > 1$$

$$\therefore m > \frac{1}{4}$$

d $mx^2 + 2x + 3 = 0$ has $a = m$, $b = 2$, $c = 3$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac = 2^2 - 4m(3) \\ &= 4 - 12m\end{aligned}$$

i For a repeated root,

$$\Delta = 0$$

$$\therefore 4 - 12m = 0$$

$$\therefore 12m = 4$$

$$\therefore m = \frac{1}{3}$$

ii For two distinct real roots,

$$\Delta > 0$$

$$\therefore 4 - 12m > 0$$

$$\therefore 12m < 4$$

$$\therefore m < \frac{1}{3}$$

iii For no real roots,

$$\Delta < 0$$

$$\therefore 4 - 12m < 0$$

$$\therefore 12m > 4$$

$$\therefore m > \frac{1}{3}$$

e $2x^2 + 7x + m = 0$ has $a = 2$, $b = 7$, $c = m$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac = 7^2 - 4(2)m \\ &= 49 - 8m\end{aligned}$$

i For a repeated root,

$$\Delta = 0$$

$$\therefore 49 - 8m = 0$$

$$\therefore 8m = 49$$

$$\therefore m = \frac{49}{8}$$

ii For two distinct real roots,

$$\Delta > 0$$

$$\therefore 49 - 8m > 0$$

$$\therefore 8m < 49$$

$$\therefore m < \frac{49}{8}$$

iii For no real roots,

$$\Delta < 0$$

$$\therefore 49 - 8m < 0$$

$$\therefore 8m > 49$$

$$\therefore m > \frac{49}{8}$$

f $mx^2 - 5x + 4 = 0$ has $a = m$, $b = -5$, $c = 4$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac = (-5)^2 - 4m(4) \\ &= 25 - 16m\end{aligned}$$

i For a repeated root,

$$\Delta = 0$$

$$\therefore 25 - 16m = 0$$

$$\therefore 16m = 25$$

$$\therefore m = \frac{25}{16}$$

ii For two distinct real roots,

$$\Delta > 0$$

$$\therefore 25 - 16m > 0$$

$$\therefore 16m < 25$$

$$\therefore m < \frac{25}{16}$$

iii For no real roots,

$$\Delta < 0$$

$$\therefore 25 - 16m < 0$$

$$\therefore 16m > 25$$

$$\therefore m > \frac{25}{16}$$

EXERCISE 71.2

1 a $y = x^2 + 7x - 2$

has $a = 1$, $b = 7$, $c = -2$

$$\therefore \Delta = b^2 - 4ac$$

$$= 7^2 - 4(1)(-2)$$

$$= 57$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

c $y = -2x^2 + 3x + 1$

has $a = -2$, $b = 3$, $c = 1$

$$\therefore \Delta = b^2 - 4ac$$

$$= 3^2 - 4(-2)(1)$$

$$= 17$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

e $y = -x^2 + x + 6$

has $a = -1$, $b = 1$, $c = 6$

$$\therefore \Delta = b^2 - 4ac$$

$$= 1^2 - 4(-1)(6)$$

$$= 25$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

2 a $x^2 - 3x + 6$

has $a = 1$, $b = -3$, $c = 6$

$$\therefore \Delta = b^2 - 4ac$$

$$= (-3)^2 - 4(1)(6)$$

$$= -15$$

$$\therefore \text{since } a > 0 \text{ and } \Delta < 0,$$

$$x^2 - 3x + 6 > 0 \text{ for all } x.$$

c $2x^2 - 4x + 7$

has $a = 2$, $b = -4$, $c = 7$

$$\therefore \Delta = b^2 - 4ac$$

$$= (-4)^2 - 4(2)(7)$$

$$= -40$$

$$\therefore \text{since } a > 0 \text{ and } \Delta < 0,$$

$$2x^2 - 4x + 7 > 0 \text{ for all } x.$$

$$\therefore \text{it is positive definite.}$$

b $y = x^2 + 4\sqrt{2}x + 8$

has $a = 1$, $b = 4\sqrt{2}$, $c = 8$

$$\therefore \Delta = b^2 - 4ac$$

$$= (4\sqrt{2})^2 - 4(1)(8)$$

$$= 0$$

$$\therefore \text{the graph touches the } x\text{-axis.}$$

d $y = 6x^2 + 5x - 4$

has $a = 6$, $b = 5$, $c = -4$

$$\therefore \Delta = b^2 - 4ac$$

$$= 5^2 - 4(6)(-4)$$

$$= 121$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

f $y = 9x^2 + 6x + 1$

has $a = 9$, $b = 6$, $c = 1$

$$\therefore \Delta = b^2 - 4ac$$

$$= 6^2 - 4(9)(1)$$

$$= 0$$

$$\therefore \text{the graph touches the } x\text{-axis.}$$

b $4x - x^2 - 6$

has $a = -1$, $b = 4$, $c = -6$

$$\therefore \Delta = b^2 - 4ac$$

$$= 4^2 - 4(-1)(-6)$$

$$= -8$$

$$\therefore \text{since } a < 0 \text{ and } \Delta < 0,$$

$$4x - x^2 - 6 < 0 \text{ for all } x.$$

d $-2x^2 + 3x - 4$

has $a = -2$, $b = 3$, $c = -4$

$$\therefore \Delta = b^2 - 4ac$$

$$= 3^2 - 4(-2)(-4)$$

$$= -23$$

$$\therefore \text{since } a < 0 \text{ and } \Delta < 0,$$

$$-2x^2 + 3x - 4 < 0 \text{ for all } x.$$

$$\therefore \text{it is negative definite.}$$

3 $3x^2 + kx - 1$

has $a = 3$, $b = k$, $c = -1$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= k^2 - 4(3)(-1) \\ &= k^2 + 12\end{aligned}$$

Now $k^2 \geq 0$ for all k

$\therefore k^2 + 12 > 0$ for all k

$\therefore \Delta > 0$ for all k

$\therefore 3x^2 + kx - 1$ has two real distinct roots for all k .

$\therefore$ it can never be positive definite.

4 $2x^2 + kx + 2$

has $a = 2$, $b = k$, $c = 2$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= k^2 - 4(2)(2) \\ &= k^2 - 16\end{aligned}$$

Now $2x^2 + kx + 2$ has $a > 0$.

$\therefore$ it is positive definite provided $k^2 - 16 < 0$

$\therefore k^2 < 16$

$\therefore -4 < k < 4$

EXERCISE 7J**1 a** The x -intercepts are 1 and 2.

$\therefore y = a(x-1)(x-2)$

for some $a \neq 0$.But the y -intercept is 4.

$\therefore a(-1)(-2) = 4$

$\therefore 2a = 4$

$\therefore a = 2$

$\therefore y = 2(x-1)(x-2)$

c The x -intercepts are 1 and 3.

$\therefore y = a(x-1)(x-3)$

for some $a \neq 0$.But the y -intercept is 3.

$\therefore a(-1)(-3) = 3$

$\therefore 3a = 3$

$\therefore a = 1$

$\therefore y = (x-1)(x-3)$

e The graph touches the x -axis when $x = 1$.

$\therefore y = a(x-1)^2$ for some $a \neq 0$.

But the y -intercept is -3 .

$\therefore a(-1)^2 = -3$

$\therefore a = -3$

$\therefore y = -3(x-1)^2$

b The graph touches the x -axis when $x = 2$.

$\therefore y = a(x-2)^2$ for some $a \neq 0$.

But the y -intercept is 8.

$\therefore a(-2)^2 = 8$

$\therefore 4a = 8$

$\therefore a = 2$

$\therefore y = 2(x-2)^2$

d The x -intercepts are -1 and 3 .

$\therefore y = a(x+1)(x-3)$

for some $a \neq 0$.But the y -intercept is 3.

$\therefore a(1)(-3) = 3$

$\therefore a = -1$

$\therefore y = -(x+1)(x-3)$

f The x -intercepts are -2 and 3 .

$\therefore y = a(x+2)(x-3)$

for some $a \neq 0$.But the y -intercept is 12.

$\therefore a(2)(-3) = 12$

$\therefore -6a = 12$

$\therefore a = -2$

$\therefore y = -2(x+2)(x-3)$

2 a $y = 2(x-1)(x-4)$

has x -intercepts 1 and 4,
and y -intercept 8 $\therefore$ its graph is **C**

d $y = (x+1)(x-4)$

has x -intercepts -1 and 4 ,
and y -intercept -4 $\therefore$ its graph is **F**

b $y = -(x+1)(x-4)$

has x -intercepts -1 and 4 ,
and y -intercept 4 $\therefore$ its graph is **E**

e $y = 2(x+4)(x-1)$

has x -intercepts -4 and 1 ,
and y -intercept -8 $\therefore$ its graph is **G**

c $y = (x-1)(x-4)$

has x -intercepts 1 and 4,
and y -intercept 4 $\therefore$ its graph is **B**

f $y = -3(x+4)(x-1)$

has x -intercepts -4 and 1 ,
and y -intercept 12 $\therefore$ its graph is **H**

g $y = -(x-1)(x-4)$

has x -intercepts 1 and 4,

and y -intercept -4

$\therefore$ its graph is **A**

h $y = -3(x-1)(x-4)$

has x -intercepts 1 and 4,

and y -intercept -12

$\therefore$ its graph is **D**

- 3 a** As the axis of symmetry is $x = 3$,
the other x -intercept is 4.

$$\therefore y = a(x-2)(x-4)$$

for some $a \neq 0$.

But the y -intercept $= 12$

$$\therefore a(-2)(-4) = 12$$

$$\therefore 8a = 12$$

$$\therefore a = \frac{12}{8} = \frac{3}{2}$$

$$\therefore y = \frac{3}{2}(x-2)(x-4)$$

- b** As the axis of symmetry is $x = -1$,
the other x -intercept is 2.

$$\therefore y = a(x+4)(x-2)$$

for some $a \neq 0$.

But the y -intercept $= 4$

$$\therefore a(4)(-2) = 4$$

$$\therefore -8a = 4$$

$$\therefore a = -\frac{1}{2}$$

$$\therefore y = -\frac{1}{2}(x+4)(x-2)$$

- c** The graph touches the x -axis at $x = -3$,

$$\therefore y = a(x+3)^2 \text{ for some } a \neq 0.$$

But the y -intercept $= -12$

$$\therefore a(3)^2 = -12$$

$$\therefore 9a = -12$$

$$\therefore a = -\frac{12}{9} = -\frac{4}{3} \quad \therefore y = -\frac{4}{3}(x+3)^2$$

- 4 a** Since the x -intercepts are 5 and 1, the
equation is $y = a(x-5)(x-1)$

for some $a \neq 0$.

But when $x = 2$, $y = -9$

$$\therefore -9 = a(2-5)(2-1)$$

$$\therefore -9 = a(-3)(1)$$

$$\therefore -3a = -9$$

$$\therefore a = 3$$

$\therefore$ the equation is

$$y = 3(x-5)(x-1)$$

$$\text{i.e., } y = 3(x^2 - 6x + 5)$$

$$\text{i.e., } y = 3x^2 - 18x + 15$$

- b** Since the x -intercepts are 2 and $-\frac{1}{2}$, the
equation is $y = a(x-2)(x+\frac{1}{2})$

for some $a \neq 0$.

But when $x = 3$, $y = -14$

$$\therefore -14 = a(3-2)(3+\frac{1}{2})$$

$$\therefore -14 = a(1)(\frac{7}{2})$$

$$\therefore \frac{7}{2}a = -14$$

$$\therefore a = -4$$

$\therefore$ the equation is

$$y = -4(x-2)(x+\frac{1}{2})$$

$$\text{i.e., } y = -4(x^2 - \frac{3}{2}x - 1)$$

$$\text{i.e., } y = -4x^2 + 6x + 4$$

- c** Since the graph touches the x -axis at 3,
its equation is $y = a(x-3)^2$,

for some $a \neq 0$.

But when $x = -2$, $y = -25$

$$\therefore -25 = a(-2-3)^2$$

$$\therefore -25 = 25a$$

$$\therefore a = -1$$

$\therefore$ the equation is

$$y = -(x-3)^2$$

$$\text{i.e., } y = -(x^2 - 6x + 9)$$

$$\text{i.e., } y = -x^2 + 6x - 9$$

- d** Since the graph touches the x -axis at -2 ,
its equation is $y = a(x+2)^2$,

for some $a \neq 0$.

But when $x = -1$, $y = 4$

$$\therefore 4 = a(-1+2)^2$$

$$\therefore 4 = a$$

$\therefore$ the equation is

$$y = 4(x+2)^2$$

$$\text{i.e., } y = 4(x^2 + 4x + 4)$$

$$\text{i.e., } y = 4x^2 + 16x + 16$$

- e** Since the graph cuts the x -axis at 3 and has axis of symmetry $x = 2$, it must also cut the x -axis at 1.
 $\therefore$ the x -intercepts are 3 and 1, and the equation is $y = a(x - 3)(x - 1)$ for some $a \neq 0$.

But when $x = 5$, $y = 12$

$$\therefore 12 = a(5 - 3)(5 - 1)$$

$$\therefore 12 = a(2)(4)$$

$$\therefore 8a = 12$$

$$\therefore a = \frac{3}{2}$$

$\therefore$ the equation is

$$y = \frac{3}{2}(x - 3)(x - 1)$$

$$\text{i.e., } y = \frac{3}{2}(x^2 - 4x + 3)$$

$$\text{i.e., } y = \frac{3}{2}x^2 - 6x + \frac{9}{2}$$

- 5 a** The vertex is $(2, 4)$,
 so the quadratic has equation
 $y = a(x - 2)^2 + 4$ for some $a \neq 0$.
 But the graph passes through the origin
 $\therefore 0 = a(0 - 2)^2 + 4$
 $\therefore 4a + 4 = 0$
 $\therefore a = -1$
 $\therefore$ the equation is $y = -(x - 2)^2 + 4$

- c** The vertex is $(3, 8)$,
 so the quadratic has equation
 $y = a(x - 3)^2 + 8$ for some $a \neq 0$.
 But the graph passes through $(1, 0)$
 $\therefore 0 = a(1 - 3)^2 + 8$
 $\therefore 0 = 4a + 8$
 $\therefore a = -2$
 $\therefore$ the equation is $y = -2(x - 3)^2 + 8$

- e** The vertex is $(2, 3)$,
 so the quadratic has equation
 $y = a(x - 2)^2 + 3$ for some $a \neq 0$.
 But the graph passes through $(3, 1)$
 $\therefore 1 = a(3 - 2)^2 + 3$
 $\therefore 1 = a + 3$
 $\therefore a = -2$
 $\therefore$ the equation is $y = -2(x - 2)^2 + 3$

- f** Since the graph cuts the x -axis at 5 and has axis of symmetry $x = 1$, it must also cut the x -axis at -3 .
 $\therefore$ the x -intercepts are 5 and -3 , and the equation is $y = a(x - 5)(x + 3)$ for some $a \neq 0$.

But when $x = 2$, $y = 5$

$$\therefore 5 = a(2 - 5)(2 + 3)$$

$$\therefore 5 = a(-3)(5)$$

$$\therefore -3a = 1$$

$$\therefore a = -\frac{1}{3}$$

$\therefore$ the equation is

$$y = -\frac{1}{3}(x - 5)(x + 3)$$

$$\text{i.e., } y = -\frac{1}{3}(x^2 - 2x - 15)$$

$$\text{i.e., } y = -\frac{1}{3}x^2 + \frac{2}{3}x + 5$$

- b** The vertex is $(2, -1)$,
 so the quadratic has equation
 $y = a(x - 2)^2 - 1$ for some $a \neq 0$.
 But the graph passes through $(0, 7)$
 $\therefore 7 = a(0 - 2)^2 - 1$
 $\therefore 7 = 4a - 1$
 $\therefore 4a = 8$
 $\therefore a = 2$
 $\therefore$ the equation is $y = 2(x - 2)^2 - 1$

- d** The vertex is $(4, -6)$,
 so the quadratic has equation
 $y = a(x - 4)^2 - 6$ for some $a \neq 0$.
 But the graph passes through $(7, 0)$
 $\therefore 0 = a(7 - 4)^2 - 6$
 $\therefore 9a - 6 = 0$
 $\therefore a = \frac{2}{3}$
 $\therefore$ the equation is $y = \frac{2}{3}(x - 4)^2 - 6$

- f** The vertex is $(\frac{1}{2}, -\frac{3}{2})$,
 so the quadratic has equation
 $y = a(x - \frac{1}{2})^2 - \frac{3}{2}$ for some $a \neq 0$.
 But the graph passes through $(\frac{3}{2}, \frac{1}{2})$
 $\therefore \frac{1}{2} = a(\frac{3}{2} - \frac{1}{2})^2 - \frac{3}{2}$
 $\therefore \frac{1}{2} = a - \frac{3}{2}$
 $\therefore a = 2$
 $\therefore$ the equation is $y = 2(x - \frac{1}{2})^2 - \frac{3}{2}$

EXERCISE 7K

1 a $y = x^2 - 2x + 8$ meets $y = x + 6$

when $x^2 - 2x + 8 = x + 6$

$$\therefore x^2 - 3x + 2 = 0$$

$$\therefore (x-1)(x-2) = 0$$

$$\therefore x = 1 \text{ or } 2$$

Substituting into $y = x + 6$,

when $x = 1$, $y = 7$

and when $x = 2$, $y = 8$

$$\therefore \text{the graphs meet at } (1, 7) \text{ and } (2, 8)$$

c $y = x^2 - 4x + 3$ meets $y = 2x - 6$

when $x^2 - 4x + 3 = 2x - 6$

$$\therefore x^2 - 6x + 9 = 0$$

$$\therefore (x-3)^2 = 0$$

$$\therefore x = 3$$

Substituting into $y = 2x - 6$,

when $x = 3$, $y = 0$

$$\therefore \text{the graphs touch at } (3, 0)$$

b $y = -x^2 + 3x + 9$ meets $y = 2x - 3$

when $-x^2 + 3x + 9 = 2x - 3$

$$\therefore x^2 - x - 12 = 0$$

$$\therefore (x-4)(x+3) = 0$$

$$\therefore x = 4 \text{ or } -3$$

Substituting into $y = 2x - 3$,

when $x = -3$, $y = 2(-3) - 3 = -9$

and when $x = 4$, $y = 2(4) - 3 = 5$

$$\therefore \text{the graphs meet at } (-3, -9) \text{ and } (4, 5)$$

d $y = -x^2 + 4x - 7$ meets $y = 5x - 4$

when $-x^2 + 4x - 7 = 5x - 4$

$$\therefore x^2 + x + 3 = 0$$

which has $a = 1$, $b = 1$, $c = 3$

$$\therefore x = \frac{-1 \pm \sqrt{1^2 - 4(1)(3)}}{2}$$

$$= \frac{-1 \pm \sqrt{-11}}{2}$$

$$\therefore \text{there are no real solutions}$$

$$\therefore \text{the graphs do not meet.}$$

2 a (0.59, 5.59) and (3.41, 8.41) **b** (3, -4) touching **c** graphs do not meet

d (-2.56, -18.81) and (1.56, 1.81)

3 a $y = x^2$ meets $y = x + 2$

when $x^2 = x + 2$

$$\therefore x^2 - x - 2 = 0$$

$$\therefore (x+1)(x-2) = 0$$

$$\therefore x = -1 \text{ or } 2$$

Substituting into $y = x + 2$,

when $x = -1$, $y = 1$

and when $x = 2$, $y = 4$

$$\therefore \text{the graphs meet at } (-1, 1) \text{ and } (2, 4).$$

b $y = x^2 + 2x - 3$ meets $y = x - 1$

when $x^2 + 2x - 3 = x - 1$

$$\therefore x^2 + x - 2 = 0$$

$$\therefore (x-1)(x+2) = 0$$

$$\therefore x = 1 \text{ or } -2$$

Substituting into $y = x - 1$,

when $x = 1$, $y = 0$

and when $x = -2$, $y = -3$

$$\therefore \text{the graphs meet at } (1, 0) \text{ and } (-2, -3).$$

c $y = 2x^2 - x + 3$ meets $y = 2 + x + x^2$

when $2x^2 - x + 3 = 2 + x + x^2$

$$\therefore x^2 - 2x + 1 = 0$$

$$\therefore (x-1)^2 = 0$$

$$\therefore x = 1$$

Substituting into $y = 2 + x + x^2$,

when $x = 1$, $y = 2 + 1 + 1 = 4$

$$\therefore \text{the graphs meet at } (1, 4)$$

d Now if $xy = 4$, then $y = \frac{4}{x}$

$xy = 4$ meets $y = x + 3$

when $\frac{4}{x} = x + 3$

$$\therefore 4 = x^2 + 3x$$

$$\therefore x^2 + 3x - 4 = 0$$

$$\therefore (x+4)(x-1) = 0$$

$$\therefore x = -4 \text{ or } 1$$

Substituting into $y = x + 3$,

when $x = -4$, $y = -1$

and when $x = 1$, $y = 4$

$$\therefore \text{the graphs meet at } (-4, -1) \text{ and } (1, 4)$$

EXERCISE 7L

1 a $H(t) = 36t - 2t^2$

has $a = -2$ and $b = 36$

Since $a < 0$,

the graph is



The maximum height reached occurs

$$\text{when } t = -\frac{b}{2a} = -\frac{36}{2(-2)} = 9$$

i.e., the maximum height is reached after 9 seconds.

2 a $C(x) = x^2 - 24x + 244$

has $a = 1$, $b = -24$ and $c = 244$

Since $a > 0$,

the graph is



The minimum cost occurs

$$\text{when } x = -\frac{b}{2a} = -\frac{(-24)}{2(1)} = 12$$

i.e., the minimum cost is when twelve skateboards are produced.

b $H(9) = 36 \times 9 - 2 \times 9^2$
 $= 324 - 162$
 $= 162$

$\therefore$ the maximum height reached is 162 m.

c The ball hits the ground when $H(t) = 0$

$$\therefore 36t - 2t^2 = 0$$

$$\therefore 2t(18 - t) = 0$$

$$\therefore t = 0 \text{ or } 18$$

$\therefore$ the ball hits the ground after 18 seconds.

b $C(12) = 12^2 - 24(12) + 244$
 $= 144 - 288 + 244$
 $= 100$

$\therefore$ the minimum cost is \$100.

c $C(0) = 0 - 24(0) + 244$
 $= 244$

$\therefore$ if no skateboards are made, there is a fixed cost of \$244.

3 $v(t) = -4t^2 + 12t + 80$ m/s

a $v(0) = 80$ m/s was the speed when the brakes were applied.

b When $v(t) = 88$, $-4t^2 + 12t + 80 = 88$

$$\therefore -4t^2 + 12t - 8 = 0$$

$$\therefore t^2 - 3t + 2 = 0$$

$$\therefore (t - 1)(t - 2) = 0$$

$$\therefore t = 1 \text{ or } 2$$

Since the car was travelling downhill, it was accelerating. Therefore, when the brake was applied, it took a while before the deceleration due to the brake cancelled the acceleration due to the hill. $\therefore$ the speed of the vehicle increased for a short time after the brake was applied. It passed 88 m/s after one second, then again after two seconds as the car slowed down.

c Axis of symmetry is $t = \frac{-b}{2a} = \frac{-12}{-8} = 1\frac{1}{2}$, $\therefore$ maximum velocity occurred at $1\frac{1}{2}$ sec.

d $\therefore$ maximum velocity was $-4(\frac{3}{2})^2 + 12(\frac{3}{2}) + 80$ m/s
 $= -9 + 18 + 80$ m/s
 $= 89$ m/s

Note: Actually the units for this question should have been kmph to be physically realistic.

4 a $P(n) = 84n - 45 - 2n^2$ has $a = -2$, $b = 84$ and $c = -45$

Since $a < 0$, the graph is



$$\therefore \text{the maximum value of } P \text{ is when } n = -\frac{b}{2a} = -\frac{84}{2(-2)} = 21$$

$\therefore$ the maximum profit occurs for a fleet of 21 taxis.

b $P(21) = 84 \times 21 - 45 - 2 \times 21^2$
 $= 837$

$\therefore$ the maximum profit is \$837 per hour.

c $P(0) = -45$, so if no taxis are on the road, \$45 is lost per hour.

- 5 a** Dusk corresponds to $t = 0$.

$$\text{Now } T(t) = \frac{1}{4}t^2 - 5t + 30$$

$$\therefore T(0) = 30^\circ\text{C}$$

$\therefore$ the temperature at dusk was 30°C .

$$\begin{aligned}\text{c } T(10) &= \frac{1}{4} \times 10^2 - 5 \times 10 + 30 \\ &= 5^\circ\text{C}\end{aligned}$$

$\therefore$ the minimum temperature was 5°C .

$$\text{b } T(t) = \frac{1}{4}t^2 - 5t + 30$$

$$\text{has } a = \frac{1}{4}, \quad b = -5 \quad \text{and} \quad c = 30$$

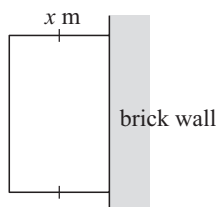
Since $a > 0$,  the graph is

$\therefore$ the temperature was a minimum

$$\text{when } t = -\frac{b}{2a} = -\frac{(-5)}{2(\frac{1}{4})} = 10$$

Now 10 hours after 7 pm is 5 am, so the temperature was a minimum at 5 am.

- 6 a**



If the sides shown are x m long, and the total length of fencing is 40 m, then the other side must have length $(40 - 2x)$ m.

$$\begin{aligned}\therefore \text{ the area is } A &= x(40 - 2x) \text{ m}^2 \\ \text{i.e., } A &= -2x^2 + 40x \text{ m}^2\end{aligned}$$

$$\text{b } A = -2x^2 + 40x$$

$$\text{has } a = -2 \quad \text{and} \quad b = 40$$

Since $a < 0$,

the graph of A has shape 

$\therefore$ the area A is maximised

$$\text{when } x = -\frac{b}{2a} = -\frac{40}{2(-2)} = 10$$

$\therefore$ the area is maximised when $x = 10$.

$$\begin{aligned}\text{c } A(10) &= -2 \times 10^2 + 40 \times 10 \\ &= -200 + 400 \\ &= 200 \text{ m}^2\end{aligned}$$

$\therefore$ the maximum area is 200 m^2 .

- 7 a** With the axes as described, the parabola has vertex $(0, 70)$.

$\therefore$ its equation is $y = ax^2 + 70$ for some $a \neq 0$.

The end of the bridge is 80 m from A,

so the arch meets the vertical end supports at the point $(80, 6)$.

Since $(80, 6)$ must lie on the curve, $6 = a(80)^2 + 70$

$$\therefore 6400a = -64 \quad \text{and so} \quad a = -\frac{1}{100}$$

$$\therefore \text{ the arch has equation } y = -\frac{1}{100}x^2 + 70$$

- b** The supports occur every 10 m.

$$\text{When } x = 10, \quad y = -\frac{1}{100} \times 10^2 + 70 = 69 \text{ m}$$

$$\text{When } x = 20, \quad y = -\frac{1}{100} \times 20^2 + 70 = 66 \text{ m}$$

$$\text{When } x = 30, \quad y = -\frac{1}{100} \times 30^2 + 70 = 61 \text{ m}$$

$$\text{When } x = 40, \quad y = -\frac{1}{100} \times 40^2 + 70 = 54 \text{ m}$$

$$\text{When } x = 50, \quad y = -\frac{1}{100} \times 50^2 + 70 = 45 \text{ m}$$

$$\text{When } x = 60, \quad y = -\frac{1}{100} \times 60^2 + 70 = 34 \text{ m}$$

$$\text{When } x = 70, \quad y = -\frac{1}{100} \times 70^2 + 70 = 21 \text{ m}$$

$\therefore$ the other supports have lengths 21 m, 34 m, 45 m, 54 m, 61 m, 66 m and 69 m.

REVIEW SET 7A

1 a The x -intercepts are -2 and 1 .

b The axis of symmetry lies midway between the x -intercepts, so its equation is $x = -\frac{1}{2}$.

c When $x = -\frac{1}{2}$, $y = -2(-\frac{1}{2} + 2)(-\frac{1}{2} - 1)$ **e**

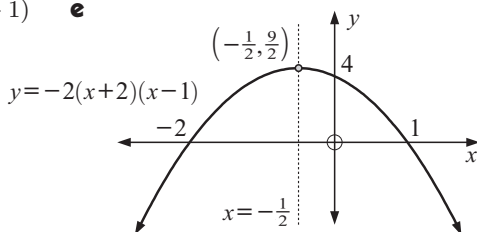
$$= -2(\frac{3}{2})(-\frac{3}{2})$$

$$= \frac{9}{2}$$

$\therefore$ the vertex is $(-\frac{1}{2}, \frac{9}{2})$

d When $x = 0$, $y = -2(2)(-1) = 4$

$\therefore$ the y -intercept is 4



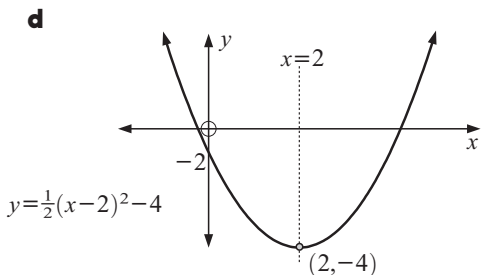
2 a The axis of symmetry is $x = 2$.

b When $x = 2$, $y = \frac{1}{2}(2 - 2)^2 - 4$
 $= -4$

$\therefore$ the vertex is $(2, -4)$

c When $x = 0$, $y = \frac{1}{2}(-2)^2 - 4$
 $= -2$

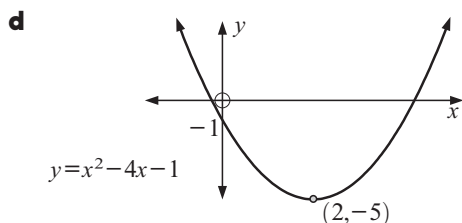
$\therefore$ the y -intercept is -2



3 a $y = x^2 - 4x - 1$
 $= x^2 - 4x + 4 - 4 - 1$
 $= (x - 2)^2 - 5$

b The vertex is $(2, -5)$

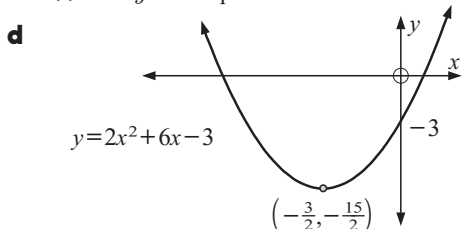
c When $x = 0$, $y = (-2)^2 - 5 = -1$
 $\therefore$ the y -intercept is -1



4 a $y = 2x^2 + 6x - 3$
 $= 2[x^2 + 3x - \frac{3}{2}]$
 $= 2[x^2 + 3x + (\frac{3}{2})^2 - (\frac{3}{2})^2 - \frac{3}{2}]$
 $= 2[x^2 + 3x + \frac{9}{4} - \frac{9}{4} - \frac{3}{2}]$
 $= 2(x^2 + 3x + \frac{9}{4}) - \frac{9}{2} - 3$
 $= 2(x + \frac{3}{2})^2 - \frac{15}{2}$

b The vertex is $(-\frac{3}{2}, -\frac{15}{2})$.

c When $x = 0$, $y = -3$
 $\therefore$ the y -intercept is -3 .



5 a $x^2 - 11x = 60$
 $\therefore x^2 - 11x - 60 = 0$
 $\therefore (x + 4)(x - 15) = 0$
 $\therefore x = -4$ or 15

b $3x^2 - x - 10 = 0$
 $\therefore (3x + 5)(x - 2) = 0$
 $\therefore x = -\frac{5}{3}$ or 2

c $3x^2 - 12x = 0$
 $\therefore 3x(x - 4) = 0$
 $\therefore x = 0$ or 4

6 a $x^2 + 10 = 7x$
 $\therefore x^2 - 7x + 10 = 0$
 $\therefore (x - 2)(x - 5) = 0$
 $\therefore x = 2$ or 5

b $x + \frac{12}{x} = 7$
 $\therefore x^2 + 12 = 7x$
 $\therefore x^2 - 7x + 12 = 0$
 $\therefore (x - 3)(x - 4) = 0$
 $\therefore x = 3$ or 4

c $2x^2 - 7x + 3 = 0$
 $\therefore (2x - 1)(x - 3) = 0$
 $\therefore x = \frac{1}{2}$ or 3

$$7 \quad x^2 + 7x - 4 = 0$$

$$\therefore x^2 + 7x + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 - 4 = 0$$

$$\therefore \left(x + \frac{7}{2}\right)^2 - \frac{49}{4} - 4 = 0$$

$$\therefore \left(x + \frac{7}{2}\right)^2 = \frac{65}{4}$$

$$\therefore x + \frac{7}{2} = \pm \frac{\sqrt{65}}{2}$$

$$\therefore x = -\frac{7}{2} \pm \frac{\sqrt{65}}{2}$$

$$8 \quad x^2 + 4x + 1 = 0$$

$$\therefore x^2 + 4x + 4 - 4 + 1 = 0$$

$$\therefore (x+2)^2 = 3$$

$$\therefore x+2 = \pm\sqrt{3}$$

$$\therefore x = -2 \pm \sqrt{3}$$

$$9 \quad a \quad x^2 - 7x + 3 = 0$$

$$\text{has } a = 1, \quad b = -7 \quad \text{and} \quad c = 3$$

$$\therefore x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{49 - 12}}{2}$$

$$= \frac{7 \pm \sqrt{37}}{2}$$

$$\text{i.e., } x = \frac{7}{2} \pm \frac{\sqrt{37}}{2}$$

$$b \quad 2x^2 - 5x + 4 = 0$$

$$\text{has } a = 2, \quad b = -5 \quad \text{and} \quad c = 4$$

$$\therefore x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(4)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{25 - 32}}{4}$$

$$\text{i.e., } x = \frac{5 \pm \sqrt{-7}}{4}$$

$$\therefore x \text{ has no real solutions.}$$

REVIEW SET 7B

$$1 \quad y = -x^2 + 2x = x(2 - x)$$

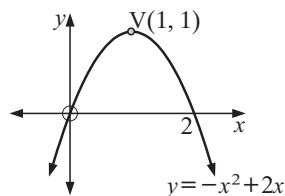
$\therefore$ the graph has x -intercepts 0 and 2, and y -intercept 0

Its axis of symmetry is midway between the x -intercepts,

i.e., at $x = 1$

$$\text{and when } x = 1, \quad y = -1^2 + 2 = 1$$

$\therefore$ the vertex is (1, 1)



$$2 \quad y = -3x^2 + 8x + 7 \quad \text{has} \quad a = -3, \quad b = 8 \quad \text{and} \quad c = 7$$

$$\text{The axis of symmetry is } x = -\frac{b}{2a} = -\frac{8}{2(-3)}$$

$$\text{i.e., } x = \frac{4}{3}$$

$$\text{When } x = \frac{4}{3}, \quad y = -3\left(\frac{4}{3}\right)^2 + 8\left(\frac{4}{3}\right) + 7$$

$$= -\frac{16}{3} + \frac{32}{3} + 7$$

$$= \frac{37}{3}$$

$\therefore$ the axis of symmetry is $x = \frac{4}{3}$ and the vertex is $\left(\frac{4}{3}, \frac{37}{3}\right)$.

$$3 \quad y = 2x^2 + 4x - 3$$

$$\text{has } a = 2, \quad b = 4, \quad c = -3$$

$$\text{The axis of symmetry is } x = -\frac{b}{2a} = -\frac{4}{2(2)} \quad \text{i.e., } x = -1$$

$$\text{When } x = -1, \quad y = 2(-1)^2 + 4(-1) - 3$$

$$= 2 - 4 - 3$$

$$= -5$$

$\therefore$ the axis of symmetry is $x = -1$ and the vertex is $(-1, -5)$.

4 a $3x^2 - 5x + 7 = 0$

has $a = 3$, $b = -5$ and $c = 7$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-5)^2 - 4(3)(7) \\ &= -59\end{aligned}$$

Since $\Delta < 0$, there are no real solutions.

b $-2x^2 - 4x + 3 = 0$

has $a = -2$, $b = -4$ and $c = 3$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-4)^2 - 4(-2)(3) \\ &= 40\end{aligned}$$

Since $\Delta > 0$, there are two real solutions.

5 $5 + 7x + 3x^2$ has $a = 3$, $b = 7$ and $c = 5$ $\therefore \Delta = b^2 - 4ac$

$$\begin{aligned}&= 7^2 - 4(3)(5) \\ &= -11\end{aligned}$$

Since $\Delta < 0$, the graph of $y = 5 + 7x + 3x^2$ never cuts or touches the x -axis.

$\therefore$ since $a > 0$, the graph lies completely above the x -axis,

and $5 + 7x + 3x^2$ is positive definite.

6 $y = -2x^2 + 4x - 3$ has $a = -2$, $b = 4$ and $c = 3$

Since $a < 0$, the graph is  and will have a maximum.

The axis of symmetry is $x = -\frac{b}{2a} = -\frac{4}{2(-2)}$ i.e., $x = 1$

When $x = 1$, $y = -2(1)^2 + 4(1) + 3$

$$= 5$$

$\therefore$ the maximum is 5, and this occurs when $x = 1$.

7 $y = x^2 - 3x$ meets $y = 3x^2 - 5x - 24$

when $x^2 - 3x = 3x^2 - 5x - 24$

$$\therefore 2x^2 - 2x - 24 = 0$$

$$\therefore x^2 - x - 12 = 0$$

$$\therefore (x - 4)(x + 3) = 0$$

$$\therefore x = 4 \text{ or } -3$$

Substituting into $y = x^2 - 3x$,

when $x = 4$, $y = 4^2 - 3 \times 4 = 4$

and when $x = -3$, $y = (-3)^2 - 3(-3)$

$$= 9 + 9 = 18$$

$\therefore$ the graphs meet at $(4, 4)$ and $(-3, 18)$.

8 $y = -2x^2 + 5x + k$

has $a = -2$, $b = 5$ and $c = k$.

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 5^2 - 4(-2)k \\ &= 25 + 8k\end{aligned}$$

The graph does not cut the x -axis

if $\Delta < 0$

$$\therefore 25 + 8k < 0$$

$$\therefore 8k < -25$$

$$\therefore k < -\frac{25}{8}$$

$$\text{i.e., } k < -3\frac{1}{8}$$

9 a The total length of wire for the fence is 60 m.

$$\therefore AB + BC + CD = 60$$

Since the enclosure is rectangular,

$$CD = AB$$

$$\therefore 2AB + x = 60$$

$$\therefore 2AB = 60 - x$$

$$\therefore AB = 30 - \frac{1}{2}x$$

$\therefore$ the area of the rectangle is

$$\begin{aligned}A &= x\left(30 - \frac{1}{2}x\right) \\ &= \left(30x - \frac{1}{2}x^2\right) \text{ m}^2\end{aligned}$$

b $A = 30x - \frac{1}{2}x^2$

has $a = -\frac{1}{2}$ and $b = 30$.

Since $a < 0$, A has a maximum at the axis of symmetry, and this is at

$$x = -\frac{b}{2a} = -\frac{30}{2(-\frac{1}{2})} = 30$$

When $x = 30$, $AB = 30 - \frac{1}{2} \times 30$

$$= 15 \text{ m}$$

$\therefore$ the enclosure is 15 m by 30 m.

REVIEW SET 7C

1 a $x^2 + 5x + 3 = 0$

has $a = 1$, $b = 5$, $c = 3$

$$\therefore x = \frac{-5 \pm \sqrt{5^2 - 4(1)(3)}}{2(1)}$$

$$= -\frac{5}{2} \pm \frac{\sqrt{13}}{2}$$

2 a $x^2 - 5x - 3 = 0$

has $a = 1$, $b = -5$, $c = -3$

$$\therefore x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{5}{2} \pm \frac{\sqrt{37}}{2}$$

3 a $x \div -5.828$ or -0.1716

b $x \div -1.135$ or 1.468

4 a $x \div 0.5858$ or 3.414

b $x \div -0.1861$ or 2.686

5 a $2x^2 - 5x - 7 = 0$

has $a = 2$, $b = -5$, $c = -7$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-5)^2 - 4(2)(-7) \\ &= 25 + 56 \\ &= 81\end{aligned}$$

$$\therefore \Delta > 0 \text{ and } \sqrt{\Delta} = 9$$

$\therefore$ there are two distinct real rational roots

6 $2x^2 - 3x + m = 0$

has $a = 2$, $b = -3$ and $c = m$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-3)^2 - 4(2)m \\ &= 9 - 8m\end{aligned}$$

b There are two distinct real roots if $\Delta > 0$

$$\text{i.e., } 9 - 8m > 0$$

$$\therefore 8m < 9$$

$$\therefore m < \frac{9}{8}$$

7 $3x^2 + 4x + t = 0$

has $a = 3$, $b = 4$ and $c = t$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= 4^2 - 4(3)t \\ &= 16 - 12t\end{aligned}$$

b There are two distinct real roots if $\Delta > 0$

$$\text{i.e., } 16 - 12t > 0$$

$$\therefore 12t < 16$$

$$\therefore t < \frac{4}{3}$$

b $3x^2 + 11x - 2 = 0$

has $a = 3$, $b = 11$, $c = -2$

$$\therefore x = \frac{-11 \pm \sqrt{11^2 - 4(3)(-2)}}{2(3)}$$

$$= -\frac{11}{6} \pm \frac{\sqrt{145}}{6}$$

b $2x^2 - 7x - 3 = 0$

has $a = 2$, $b = -7$, $c = -3$

$$\therefore x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{7}{4} \pm \frac{\sqrt{73}}{4}$$

b $3x^2 - 24x + 48 = 0$

has $a = 3$, $b = -24$, $c = 48$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-24)^2 - 4(3)(48) \\ &= 576 - 576 \\ &= 0\end{aligned}$$

$\therefore$ there is a repeated real root

a There is a repeated root if $\Delta = 0$

$$\text{i.e., } 9 - 8m = 0$$

$$\therefore m = \frac{9}{8}$$

c There are no real roots if $\Delta < 0$

$$\text{i.e., } 9 - 8m < 0$$

$$\therefore 8m > 9$$

$$\therefore m > \frac{9}{8}$$

a There is a repeated root if $\Delta = 0$

$$\text{i.e., } 16 - 12t = 0$$

$$\therefore 12t = 16$$

$$\therefore t = \frac{4}{3}$$

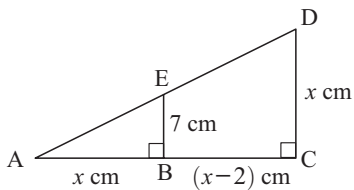
c There are no real roots if $\Delta < 0$

$$\text{i.e., } 16 - 12t < 0$$

$$\therefore 12t > 16$$

$$\therefore t > \frac{4}{3}$$

- 8 Suppose AB is x cm in length. Then, using the information given, we can label the diagram:



Now by similar triangles, $\frac{BE}{AB} = \frac{CD}{AC}$

$$\therefore \frac{7}{x} = \frac{x}{x + (x - 2)}$$

$$\therefore \frac{7}{x} = \frac{x}{2x - 2}$$

$$\therefore 7(2x - 2) = x^2$$

$$\therefore 14x - 14 = x^2$$

$$\therefore x^2 - 14x + 14 = 0$$

which has $a = 1$, $b = -14$ and $c = 14$

$$\therefore x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(1)(14)}}{2(1)} = \frac{14 \pm \sqrt{140}}{2}$$

Now $x > 0$, so $x = \frac{14 + \sqrt{140}}{2} \div 12.92$ cm

$\therefore$ AB is approximately 12.9 cm long.

- 9 Let the hypotenuse have length x cm.

$\therefore$ the longer of the remaining sides has length $(x - 2)$ cm,

and the third side has length $(x - 9)$ cm.

By Pythagoras' theorem,

$$(x - 2)^2 + (x - 9)^2 = x^2$$

$$\therefore x^2 - 4x + 4 + x^2 - 18x + 81 = x^2$$

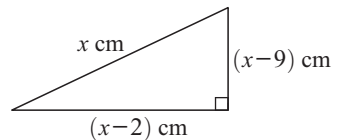
$$\therefore x^2 - 22x + 85 = 0$$

$$\therefore (x - 5)(x - 17) = 0$$

$$\therefore x = 5 \text{ or } 17$$

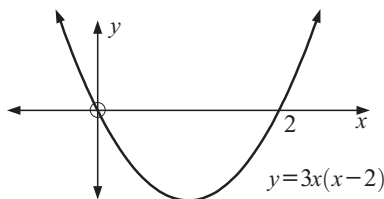
But if x was 5, the shortest side would have negative length.

$\therefore$ the only solution is that the hypotenuse has length 17 cm.



REVIEW SET 7D

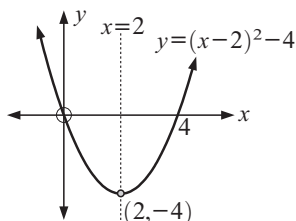
- 1 $y = 3x(x - 2)$ has x -intercepts 0 and 2
and y -intercept 0



- 2 a $y = (x - 2)^2 - 4$ has vertex $(2, -4)$
and axis of symmetry $x = 2$.

When $x = 0$, $y = (-2)^2 - 4 = 0$

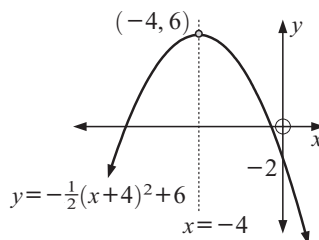
so the y -intercept is 0.



- b $y = -\frac{1}{2}(x + 4)^2 + 6$ has vertex $(-4, 6)$
and axis of symmetry $x = -4$.

When $x = 0$, $y = -\frac{1}{2}(4)^2 + 6 = -2$

so the y -intercept is -2 .



3 a $y = 2x^2 + 4x - 1$

has $a = 2$, $b = 4$ and $c = -1$

The axis of symmetry is $x = -\frac{b}{2a}$

$$\text{i.e., } x = -\frac{4}{2 \times 2}$$

$$\text{i.e., } x = -1$$

c When $x = 0$, $y = -1$,
so the y -intercept is -1

When $y = 0$, $2x^2 + 4x - 1 = 0$

$$\begin{aligned} \therefore x &= \frac{-4 \pm \sqrt{4^2 - 4(2)(-1)}}{2(2)} \\ &= \frac{-4 \pm \sqrt{24}}{4} \\ &= \frac{-4 \pm 2\sqrt{6}}{4} = -1 \pm \frac{1}{2}\sqrt{6} \end{aligned}$$

$\therefore$ the x -intercepts are $-1 \pm \frac{1}{2}\sqrt{6}$

4 a $y = 2x^2 + 3x - 7$

has $a = 2$, $b = 3$ and $c = -7$

$$\begin{aligned} \therefore \Delta &= b^2 - 4ac \\ &= 3^2 - 4(2)(-7) \\ &= 65 \end{aligned}$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

Since $a > 0$, the graph is



5 a $y = -2x^2 + 3x + 2$

has $a = -2$, $b = 3$ and $c = 2$

$$\begin{aligned} \therefore \Delta &= b^2 - 4ac \\ &= 3^2 - 4(-2)(2) \\ &= 25 \end{aligned}$$

Since $\Delta > 0$, the function is neither positive definite nor negative definite.

6 a The graph has x -intercepts ± 3 , so its equation is
 $y = a(x + 3)(x - 3)$ for some $a \neq 0$.

Its y -intercept is -27 , so

$$a(3)(-3) = -27$$

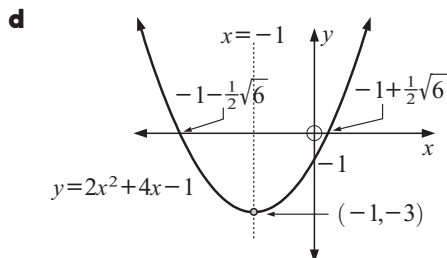
$$\therefore -9a = -27$$

$$\therefore a = 3$$

$\therefore$ the equation is $y = 3(x + 3)(x - 3)$

b When $x = -1$, $y = 2(-1)^2 + 4(-1) - 1$
 $= 2 - 4 - 1$
 $= -3$

$\therefore$ the vertex is $(-1, -3)$



b $y = -3x^2 - 7x + 4$

has $a = -3$, $b = -7$ and $c = 4$

$$\begin{aligned} \therefore \Delta &= b^2 - 4ac \\ &= (-7)^2 - 4(-3)(4) \\ &= 97 \end{aligned}$$

Since $\Delta > 0$, the graph cuts the x -axis twice.

Since $a < 0$, the graph is



b $y = 3x^2 + x + 11$

has $a = 3$, $b = 1$ and $c = 11$

$$\begin{aligned} \therefore \Delta &= b^2 - 4ac \\ &= 1^2 - 4(3)(11) \\ &= -131 \end{aligned}$$

$\therefore \Delta < 0$, and since $a > 0$, the function is positive definite.

b The quadratic has vertex $(2, 25)$
 $\therefore$ its equation is $y = a(x - 2)^2 + 25$
The y -intercept is 1, so

$$a(-2)^2 + 25 = 1$$

$$\therefore 4a = -24$$

$$\therefore a = -6$$

$\therefore$ the equation is $y = -6(x - 2)^2 + 25$

7 Let the number be x .

$\therefore$ its reciprocal is $\frac{1}{x}$.

$$\therefore x + \frac{1}{x} = 2\frac{1}{30} = \frac{61}{30}$$

$$\therefore x^2 + 1 = \frac{61}{30}x$$

$$\therefore 30x^2 + 30 = 61x$$

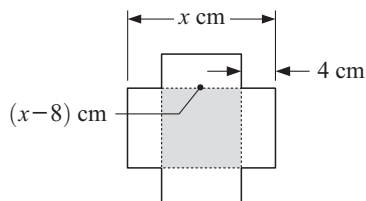
$$\therefore 30x^2 - 61x + 30 = 0$$

$$\therefore (6x - 5)(5x - 6) = 0$$

$$\therefore x = \frac{5}{6} \text{ or } \frac{6}{5}$$

$\therefore$ the number is $\frac{5}{6}$ or $\frac{6}{5}$

8



Since the container has a square base, the original tinplate must have been square.

Suppose its side was x cm long, so the base of the container is $(x - 8)$ cm by $(x - 8)$ cm.

The height of the container is 4 cm, so its capacity is $4(x - 8)(x - 8)$ cm³.

$$\therefore 4(x - 8)^2 = 120$$

$$\therefore (x - 8)^2 = 30$$

$$\therefore x - 8 = \pm\sqrt{30}$$

$$\therefore x = 8 \pm \sqrt{30}$$

Clearly, $x > 8$, so $x = 8 + \sqrt{30} \div 13.48$

$\therefore$ the tinplate was about 13.5 cm by 13.5 cm

9 $y = -x^2 - 5x + 3$ meets $y = x^2 + 3x + 11$

when $-x^2 - 5x + 3 = x^2 + 3x + 11$

$$\therefore 2x^2 + 8x + 8 = 0$$

$$\therefore x^2 + 4x + 4 = 0$$

$$\therefore (x + 2)^2 = 0$$

$$\therefore x = -2$$

$\therefore$ the graphs touch at $(-2, 9)$.

Substituting into $y = x^2 + 3x + 11$,

$$\begin{aligned} \text{when } x = -2, \quad y &= (-2)^2 + 3(-2) + 11 \\ &= 4 - 6 + 11 \\ &= 9 \end{aligned}$$

REVIEW SET 7E

1 a The graph has x -intercepts -3 and 1 , so its equation is

$$y = a(x + 3)(x - 1) \text{ for some } a \neq 0.$$

Its y -intercept is 18 , so

$$a(3)(-1) = 18$$

$$\therefore a = -6$$

So the equation is

$$y = -6(x + 3)(x - 1).$$

2 a Since one x -intercept is 7 and the axis of symmetry is $x = 4$, the other x -intercept is $x = 1$.

$\therefore$ the graph has equation

$$y = a(x - 7)(x - 1) \text{ for some } a \neq 0.$$

The y -intercept is -2

$$\therefore a(-7)(-1) = -2$$

$$\therefore a = -\frac{2}{7}$$

$\therefore$ the equation is $y = -\frac{2}{7}(x - 7)(x - 1)$.

b The graph has vertex $(2, -20)$, so its equation is

$$y = a(x - 2)^2 - 20 \text{ for some } a \neq 0.$$

Now an x -intercept is 5

$$\therefore a(5 - 2)^2 - 20 = 0$$

$$\therefore 9a = 20 \text{ and so } a = \frac{20}{9}$$

So the equation is $y = \frac{20}{9}(x - 2)^2 - 20$.

b The graph has vertex $(-3, 0)$, so its equation is

$$y = a(x + 3)^2 \text{ for some } a \neq 0.$$

The y -intercept is 2

$$\therefore a(3)^2 = 2$$

$$\therefore 9a = 2 \text{ and so } a = \frac{2}{9}$$

So the equation is $y = \frac{2}{9}(x + 3)^2$.

- 3** The x -intercepts are 3 and -2 , so the equation is $y = a(x - 3)(x + 2)$ for some $a \neq 0$.

But the y -intercept is 24 $\therefore a(-3)(2) = 24$

$$\therefore -6a = 24$$

$$\therefore a = -4$$

$$\therefore \text{the equation is } y = -4(x - 3)(x + 2) \quad \text{i.e., } y = -4(x^2 - x - 6)$$

$$\text{i.e., } y = -4x^2 + 4x + 24$$

- 4** The graph touches the x -axis at 4, so its vertex is $(4, 0)$.

$\therefore$ its equation is $y = a(x - 4)^2$ for some $a \neq 0$.

The graph also passes through $(2, 12)$ $\therefore a(2 - 4)^2 = 12$

$$\therefore 4a = 12$$

$$\therefore a = 3$$

$$\therefore \text{the equation is } y = 3(x - 4)^2 \quad \text{i.e., } y = 3(x^2 - 8x + 16)$$

$$\text{i.e., } y = 3x^2 - 24x + 48$$

- 5** The quadratic has vertex $(-4, 1)$, so its equation is $y = a(x + 4)^2 + 1$ for some $a \neq 0$.

The graph also passes through $(1, 11)$

$$\therefore 11 = a(1 + 4)^2 + 1$$

$$\therefore 25a = 10$$

$$\therefore a = \frac{2}{5}$$

$$\therefore \text{the equation is } y = \frac{2}{5}(x + 4)^2 + 1 \quad \text{i.e., } y = \frac{2}{5}(x^2 + 8x + 16) + 1$$

$$\text{i.e., } y = \frac{2}{5}x^2 + \frac{16}{5}x + \frac{37}{5}$$

- 6 a** $y = 3x^2 + 4x + 7$

has $a = 3$, $b = 4$ and $c = 7$

Since $a > 0$,

the graph is



and so has a minimum.

This occurs on the axis of symmetry

$$x = -\frac{b}{2a}$$

$$\text{i.e., } x = -\frac{4}{2(3)} = -\frac{2}{3}$$

When $x = -\frac{2}{3}$,

$$y = 3\left(-\frac{2}{3}\right)^2 + 4\left(-\frac{2}{3}\right) + 7$$

$$= \frac{4}{3} - \frac{8}{3} + 7$$

$$= \frac{17}{3}$$

$\therefore$ the minimum is $\frac{17}{3}$ when $x = -\frac{2}{3}$

- b** $y = -2x^2 - 5x + 2$

has $a = -2$, $b = -5$ and $c = 2$

Since $a < 0$,

the graph is



and so has a maximum.

This occurs on the axis of symmetry

$$x = -\frac{b}{2a}$$

$$\text{i.e., } x = -\frac{(-5)}{2(-2)} = -\frac{5}{4}$$

When $x = -\frac{5}{4}$,

$$y = -2\left(-\frac{5}{4}\right)^2 - 5\left(-\frac{5}{4}\right) + 2$$

$$= -\frac{50}{16} + \frac{25}{4} + 2$$

$$= \frac{-25 + 50 + 16}{8}$$

$$= \frac{41}{8}$$

$\therefore$ the maximum is $\frac{41}{8}$ when $x = -\frac{5}{4}$

7 $y = x^2 - 2x + k$ has $a = 1$, $b = -2$ and $c = k$

$$\begin{aligned}\therefore \Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(1)k \\ &= 4 - 4k\end{aligned}$$

The graph cuts the x -axis twice if $\Delta > 0$

$$\therefore 4 - 4k > 0$$

$$\therefore 4k < 4$$

$$\therefore k < 1$$

8 a The total length of fencing is

$$(8x + 9y) \text{ m}$$

$$\therefore 8x + 9y = 600$$

$$\therefore 9y = 600 - 8x$$

$$\therefore y = \frac{600 - 8x}{9}$$

c $A = x \left(\frac{600 - 8x}{9} \right)$

$$= \frac{600}{9}x - \frac{8}{9}x^2$$

which has $a = -\frac{8}{9}$, $b = \frac{600}{9}$

Since $a < 0$, A is maximised at the axis

of symmetry, which is $x = -\frac{b}{2a}$

$$\text{i.e., } x = -\frac{\frac{600}{9}}{2(-\frac{8}{9})} = \frac{600}{16}$$

$$\text{i.e., } x = \frac{75}{2}$$

When $x = \frac{75}{2}$, $y = \frac{600 - 8(\frac{75}{2})}{9} = 33\frac{1}{3}$

$\therefore$ for maximum area, each pen should be $37\frac{1}{2} \text{ m} \times 33\frac{1}{3} \text{ m}$

b The area of each pen is

$$A = xy$$

$$= x \left(\frac{600 - 8x}{9} \right) \text{ m}^2$$

d The maximum area of each pen is

$$37\frac{1}{2} \times 33\frac{1}{3}$$

$$= \frac{75}{2} \times \frac{100}{3}$$

$$= 1250 \text{ m}^2$$

Chapter 8

COMPLEX NUMBERS AND POLYNOMIALS

EXERCISE 8A

$$\begin{aligned} 1 \quad a \quad & \sqrt{-9} \\ &= \sqrt{9} \times \sqrt{-1} \\ &= 3i \end{aligned}$$

$$\begin{aligned} d \quad & \sqrt{-5} \\ &= \sqrt{5} \times \sqrt{-1} \\ &= i\sqrt{5} \end{aligned}$$

$$\begin{aligned} 2 \quad a \quad & x^2 - 9 \\ &= (x+3)(x-3) \end{aligned}$$

$$\begin{aligned} d \quad & x^2 + 7 \\ &= x^2 + (\sqrt{7}i)^2 \\ &= (x + i\sqrt{7})(x - i\sqrt{7}) \end{aligned}$$

$$\begin{aligned} g \quad & 2x^2 - 9 \\ &= (\sqrt{2}x + 3)(\sqrt{2}x - 3) \end{aligned}$$

$$\begin{aligned} j \quad & x^3 + x \\ &= x(x^2 + 1) \\ &= x(x^2 - i^2) \\ &= x(x+i)(x-i) \end{aligned}$$

$$\begin{aligned} k \quad & x^4 - 1 \\ &= (x^2 + 1)(x^2 - 1) \\ &= (x^2 - i^2)(x^2 - 1) \\ &= (x+i)(x-i)(x+1)(x-1) \end{aligned}$$

$$\begin{aligned} 3 \quad a \quad & x^2 - 25 = 0 \\ & (x+5)(x-5) = 0 \\ & \therefore x = \pm 5 \end{aligned}$$

$$\begin{aligned} c \quad & x^2 - 5 = 0 \\ & (x + \sqrt{5})(x - \sqrt{5}) = 0 \\ & x = \pm\sqrt{5} \end{aligned}$$

$$\begin{aligned} e \quad & 4x^2 - 9 = 0 \\ & (2x+3)(2x-3) = 0 \\ & \therefore x = \pm\frac{3}{2} \end{aligned}$$

$$\begin{aligned} g \quad & x^3 - 4x = 0 \\ & x(x^2 - 4) = 0 \\ & x(x+2)(x-2) = 0 \\ & \therefore x = 0 \text{ or } \pm 2 \end{aligned}$$

$$\begin{aligned} b \quad & \sqrt{-64} \\ &= \sqrt{64} \times \sqrt{-1} \\ &= 8i \end{aligned}$$

$$\begin{aligned} e \quad & \sqrt{-8} \\ &= \sqrt{8} \times \sqrt{-1} \\ &= 2i\sqrt{2} \end{aligned}$$

$$\begin{aligned} b \quad & x^2 + 9 \\ &= x^2 - 9i^2 \\ &= (x+3i)(x-3i) \end{aligned}$$

$$\begin{aligned} e \quad & 4x^2 - 1 \\ &= (2x+1)(2x-1) \end{aligned}$$

$$\begin{aligned} h \quad & 2x^2 + 9 \\ &= 2x^2 - 9i^2 \\ &= (\sqrt{2}x + 3i)(\sqrt{2}x - 3i) \end{aligned}$$

$$\begin{aligned} l \quad & x^4 - 16 \\ &= (x^2 + 4)(x^2 - 4) \\ &= (x^2 - 4i^2)(x^2 - 4) \\ &= (x+2i)(x-2i)(x+2)(x-2) \end{aligned}$$

$$\begin{aligned} b \quad & x^2 + 25 = 0 \\ & x^2 - 25i^2 = 0 \\ & (x+5i)(x-5i) = 0 \\ & \therefore x = \pm 5i \end{aligned}$$

$$\begin{aligned} d \quad & x^2 + 5 = 0 \\ & x^2 - 5i^2 = 0 \\ & (x + i\sqrt{5})(x - i\sqrt{5}) = 0 \\ & \therefore x = \pm i\sqrt{5} \end{aligned}$$

$$\begin{aligned} f \quad & 4x^2 + 9 = 0 \\ & 4x^2 - 9i^2 = 0 \\ & (2x+3i)(2x-3i) = 0 \\ & \therefore x = \pm\frac{3}{2}i \end{aligned}$$

$$\begin{aligned} h \quad & x^3 + 4x = 0 \\ & x(x^2 + 4) = 0 \\ & x(x^2 - 4i^2) = 0 \\ & x(x+2i)(x-2i) = 0 \\ & \therefore x = 0 \text{ or } \pm 2i \end{aligned}$$

$$\begin{aligned} c \quad & \sqrt{-\frac{1}{4}} \\ &= \sqrt{\frac{1}{4}} \times \sqrt{-1} \\ &= \frac{1}{2}i \end{aligned}$$

$$\begin{aligned} c \quad & x^2 - 7 \\ &= (x + \sqrt{7})(x - \sqrt{7}) \end{aligned}$$

$$\begin{aligned} f \quad & 4x^2 + 1 \\ &= 4x^2 - i^2 \\ &= (2x+i)(2x-i) \end{aligned}$$

$$\begin{aligned} i \quad & x^3 - x \\ &= x(x^2 - 1) \\ &= x(x+1)(x-1) \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & x^3 - 3x = 0 \\ & x(x^2 - 3) = 0 \\ & x(x + \sqrt{3})(x - \sqrt{3}) = 0 \\ & \therefore x = 0 \text{ or } \pm\sqrt{3} \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad & x^4 - 1 = 0 \\ & (x^2 + 1)(x^2 - 1) = 0 \\ & (x^2 - i^2)(x^2 - 1) = 0 \\ & (x + i)(x - i)(x + 1)(x - 1) = 0 \\ & \therefore x = \pm i \text{ or } \pm 1 \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & x^3 + 3x = 0 \\ & x(x^2 + 3) = 0 \\ & x(x^2 - 3i^2) = 0 \\ & x(x + i\sqrt{3})(x - i\sqrt{3}) = 0 \\ & \therefore x = 0 \text{ or } \pm i\sqrt{3} \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad & x^4 = 81 \\ & x^4 - 81 = 0 \\ & (x^2 + 9)(x^2 - 9) = 0 \\ & (x^2 - 9i^2)(x^2 - 9) = 0 \\ & (x + 3i)(x - 3i)(x + 3)(x - 3) = 0 \\ & \therefore x = \pm 3i \text{ or } \pm 3 \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad & \text{If } x^2 - 10x + 29 = 0 \\ \text{then } x &= \frac{10 \pm \sqrt{100 - 4 \times 1 \times 29}}{2} \\ &= \frac{10 \pm \sqrt{-16}}{2} \\ &= 5 \pm \sqrt{-4} \\ &= 5 \pm 2i \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \text{If } x^2 + 14x + 50 = 0, \\ \text{then } x &= \frac{-14 \pm \sqrt{14^2 - 4 \times 1 \times 50}}{2} \\ &= \frac{-14 \pm \sqrt{-4}}{2} \\ &= -7 \pm \sqrt{-1} \\ &= -7 \pm i \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \text{If } x^2 - 2\sqrt{3}x + 4 = 0, \\ \text{then } x &= \frac{2\sqrt{3} \pm \sqrt{12 - 4 \times 1 \times 4}}{2} \\ &= \frac{2\sqrt{3} \pm \sqrt{-4}}{2} \\ &= \sqrt{3} \pm \sqrt{-1} \\ &= \sqrt{3} \pm i \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \text{If } x^2 + 6x + 25 = 0 \\ \text{then } x &= \frac{-6 \pm \sqrt{36 - 4 \times 1 \times 25}}{2} \\ &= \frac{-6 \pm \sqrt{-64}}{2} \\ &= -3 \pm \sqrt{-16} \\ &= -3 \pm 4i \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \text{If } 2x^2 + 5 = 6x, \\ \text{then } 2x^2 - 6x + 5 &= 0 \\ x &= \frac{6 \pm \sqrt{36 - 4 \times 2 \times 5}}{4} \\ &= \frac{6 \pm \sqrt{-4}}{4} \\ &= \frac{3 \pm \sqrt{-1}}{2} \\ &= \frac{3}{2} \pm \frac{1}{2}i \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \text{If } 2x + \frac{1}{x} = 1, \\ \text{then } 2x^2 + 1 &= x \\ \therefore 2x^2 - x + 1 &= 0 \\ x &= \frac{1 \pm \sqrt{1 - 4 \times 2 \times 1}}{4} \\ &= \frac{1 \pm \sqrt{-7}}{4} \\ &= \frac{1}{4} \pm i\frac{\sqrt{7}}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad & x^4 + 2x^2 = 3 \\ & x^4 + 2x^2 - 3 = 0 \\ & (x^2 + 3)(x^2 - 1) = 0 \\ & (x^2 - 3i^2)(x^2 - 1) = 0 \\ & (x + i\sqrt{3})(x - i\sqrt{3})(x + 1)(x - 1) = 0 \\ & \therefore x = \pm i\sqrt{3} \text{ or } \pm 1 \end{aligned}$$

b

$$x^4 = x^2 + 6$$

$$x^4 - x^2 - 6 = 0$$

$$(x^2 - 3)(x^2 + 2) = 0$$

$$(x^2 - 3)(x^2 - 2i^2) = 0$$

$$(x + \sqrt{3})(x - \sqrt{3})(x + i\sqrt{2})(x - i\sqrt{2}) = 0 \quad \therefore x = \pm\sqrt{3} \text{ or } \pm i\sqrt{2}$$

c

$$x^4 + 5x^2 = 36$$

$$x^4 + 5x^2 - 36 = 0$$

$$(x^2 + 9)(x^2 - 4) = 0$$

$$(x^2 - 9i^2)(x^2 - 4) = 0$$

$$(x + 3i)(x - 3i)(x + 2)(x - 2) = 0 \quad \therefore x = \pm 3i \text{ or } \pm 2$$

d

$$x^4 + 9x^2 + 14 = 0$$

$$(x^2 + 7)(x^2 + 2) = 0$$

$$(x^2 - 7i^2)(x^2 - 2i^2) = 0$$

$$(x + i\sqrt{7})(x - i\sqrt{7})(x + i\sqrt{2})(x - i\sqrt{2}) = 0 \quad \therefore x = \pm i\sqrt{7} \text{ or } \pm i\sqrt{2}$$

e

$$x^4 + 1 = 2x^2$$

$$x^4 - 2x^2 + 1 = 0$$

$$(x^2 - 1)^2 = 0$$

$$(x + 1)^2(x - 1)^2 = 0$$

$$\therefore x = \pm 1$$

f

$$x^4 + 2x^2 + 1 = 0$$

$$(x^2 + 1)^2 = 0$$

$$(x^2 - i^2)^2 = 0$$

$$(x + i)^2(x - i)^2 = 0$$

$$\therefore x = \pm i$$

EXERCISE 8B.1**1**

z	$\text{Re}(z)$	$\text{Im}(z)$	z	$\text{Re}(z)$	$\text{Im}(z)$
$3 + 2i$	3	2	$-3 + 4i$	-3	4
$5 - i$	5	-1	$-7 - 2i$	-7	-2
3	3	0	$-11i$	0	-11
0	0	0	$i\sqrt{3}$	0	$\sqrt{3}$

2**a**

$$\begin{aligned} z + w &= (5 - 2i) + (2 + i) \\ &= 7 - i \end{aligned}$$

d

$$\begin{aligned} z - w &= (5 - 2i) - (2 + i) \\ &= 5 - 2i - 2 - i \\ &= 3 - 3i \end{aligned}$$

g

$$\begin{aligned} w^2 &= (2 + i)^2 \\ &= 4 + 4i + i^2 \\ &= 3 + 4i \end{aligned}$$

3**a**

$$\begin{aligned} z + 2w &= (1 + i) + 2(-2 + 3i) \\ &= 1 + i - 4 + 6i \\ &= -3 + 7i \end{aligned}$$

d

$$\begin{aligned} iz &= i(1 + i) \\ &= i + i^2 \\ &= -1 + i \end{aligned}$$

b

$$\begin{aligned} 2z &= 2(5 - 2i) \\ &= 10 - 4i \end{aligned}$$

e

$$\begin{aligned} 2z - 3w &= 2(5 - 2i) - 3(2 + i) \\ &= 10 - 4i - 6 - 3i \\ &= 4 - 7i \end{aligned}$$

h

$$\begin{aligned} z^2 &= (5 - 2i)^2 \\ &= 25 - 20i + 4i^2 \\ &= 21 - 20i \end{aligned}$$

c

$$\begin{aligned} iw &= i(2 + i) \\ &= 2i + i^2 \\ &= -1 + 2i \end{aligned}$$

f

$$\begin{aligned} zw &= (5 - 2i)(2 + i) \\ &= 10 - 4i + 5i - 2i^2 \\ &= 12 + i \end{aligned}$$

c

$$\begin{aligned} z^3 &= z^2 \times z \\ &= 2i(1 + i) \\ &= 2i + 2i^2 \\ &= -2 + 2i \end{aligned}$$

f

$$\begin{aligned} zw &= (1 + i)(-2 + 3i) \\ &= -2 + 3i - 2i + 3i^2 \\ &= -5 + i \end{aligned}$$

$$\begin{array}{ll} \mathbf{g} & z^2 w = (1+i)^2(-2+3i) \\ & = 2i(-2+3i) \\ & = -4i + 6i^2 \\ & = -6 - 4i \\ \mathbf{h} & izw = i(1+i)(-2+3i) \\ & = i(-5+i) \\ & = -5i + i^2 \\ & = -1 - 5i \end{array}$$

$$\begin{array}{llll} \mathbf{4} & i^0 = 1 & i^4 = 1 & i^8 = 1 & i^{-1} = -i \\ & i^1 = i & i^5 = i & i^9 = i & i^{-2} = -1 \\ & i^2 = -1 & i^6 = -1 & & i^{-3} = i \\ & i^3 = -i & i^7 = -i & & i^{-4} = 1 \\ & & & & i^{-5} = -i \end{array} \quad \therefore i^{4n+3} = -i \text{ where } n \text{ is any integer}$$

$$\begin{array}{l} \mathbf{5} \quad \mathbf{a} \quad \text{If } z = \cos \theta + i \sin \theta \text{ then} \\ z^2 = (\cos \theta + i \sin \theta)^2 \\ = \cos^2 \theta + 2i \cos \theta \sin \theta + i^2 \sin^2 \theta \\ = (\cos^2 \theta - \sin^2 \theta) + i(2 \sin \theta \cos \theta) \\ = \cos 2\theta + i \sin 2\theta \end{array}$$

$$\begin{array}{l} \mathbf{b} \quad \frac{1}{z} = \frac{1}{\cos \theta + i \sin \theta} \times \frac{\cos \theta - i \sin \theta}{\cos \theta - i \sin \theta} \\ = \frac{\cos \theta - i \sin \theta}{\cos^2 \theta - i^2 \sin^2 \theta} \\ = \frac{\cos \theta - i \sin \theta}{\cos^2 \theta + \sin^2 \theta} \\ = \frac{\cos \theta - i \sin \theta}{1} \\ = \cos \theta - i \sin \theta \end{array}$$

$$\begin{array}{ll} \mathbf{6} & (1+i)^4 = [(1+i)^2]^2 \\ & = (1+2i+i^2)^2 \\ & = (2i)^2 \\ & = -4 \\ & (1+i)^{101} = (1+i)^{100} \times (1+i) \\ & = [(1+i)^4]^{25} \times (1+i) \\ & = [-4]^{25} (1+i) \\ & = -2^{50} (1+i) \end{array}$$

$$\begin{array}{l} \mathbf{7} \quad (a+bi)^2 = -16 - 30i \\ a^2 + 2abi + b^2 i^2 = -16 - 30i \\ \therefore a^2 - b^2 = -16 \quad \text{and} \quad 2ab = -30, \\ \text{i.e., } ab = -15 \quad \text{and} \quad \therefore b = -\frac{15}{a} \end{array}$$

$$\text{so } a^2 - \left(-\frac{15}{a}\right)^2 = -16$$

$$\therefore a^2 - \frac{225}{a^2} = -16$$

$$\therefore a^4 + 16a^2 - 225 = 0$$

$$\therefore (a^2 + 25)(a^2 - 9) = 0$$

$$\therefore a = \pm 3 \quad \text{or} \quad \pm 5i$$

$$\text{But } a \text{ is real and } > 0 \quad \therefore a = 3 \quad \text{and} \quad b = -\frac{15}{3} = -5$$

$$\begin{array}{l} \mathbf{8} \quad \mathbf{a} \quad \frac{z}{w} = \frac{2-i}{1+3i} \times \frac{1-3i}{1-3i} \\ = \frac{2-6i-i+3i^2}{1-9i^2} \\ = \frac{-1-7i}{10} \\ = -\frac{1}{10} - \frac{7}{10}i \end{array}$$

$$\begin{array}{l} \mathbf{b} \quad \frac{i}{z} = \frac{i}{2-i} \times \frac{2+i}{2+i} \\ = \frac{2i+i^2}{4-i^2} \\ = \frac{-1+2i}{5} \\ = -\frac{1}{5} + \frac{2}{5}i \end{array}$$

$$\text{c} \quad \frac{w}{iz} = \frac{1+3i}{i(2-i)}$$

$$= \frac{1+3i}{2i-i^2}$$

$$= \frac{1+3i}{1+2i} \times \frac{1-2i}{1-2i}$$

$$= \frac{1-2i+3i-6i^2}{1-4i^2}$$

$$= \frac{7+i}{5}$$

$$= \frac{7}{5} + \frac{1}{5}i$$

$$\text{d} \quad z^{-2} = \frac{1}{(2-i)^2}$$

$$= \frac{1}{4-4i+i^2}$$

$$= \frac{1}{3-4i} \times \frac{3+4i}{3+4i}$$

$$= \frac{3+4i}{9-16i^2}$$

$$= \frac{3+4i}{25}$$

$$= \frac{3}{25} + \frac{4}{25}i$$

$$\text{9 a} \quad \frac{i}{1-2i}$$

$$= \frac{i}{1-2i} \times \frac{1+2i}{1+2i}$$

$$= \frac{i+2i^2}{1-4i^2}$$

$$= \frac{-2+i}{5}$$

$$= -\frac{2}{5} + \frac{1}{5}i$$

$$\text{b} \quad \frac{i(2-i)}{3-2i}$$

$$= \frac{2i-i^2}{3-2i}$$

$$= \frac{1+2i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{3+2i+6i+4i^2}{9-4i^2}$$

$$= \frac{-1+8i}{13}$$

$$= -\frac{1}{13} + \frac{8}{13}i$$

$$\text{c} \quad \frac{1}{2-i} - \frac{2}{2+i}$$

$$= \frac{1}{2-i} \left(\frac{2+i}{2+i} \right) - \frac{2}{2+i} \left(\frac{2-i}{2-i} \right)$$

$$= \frac{2+i-2(2-i)}{(2-i)(2+i)}$$

$$= \frac{2+i-4+2i}{4-i^2}$$

$$= \frac{-2+3i}{5}$$

$$= -\frac{2}{5} + \frac{3}{5}i$$

$$\begin{aligned} \text{10 a} \quad 4z-3w &= 4(2+i)-3(-1+2i) \\ &= 8+4i+3-6i \\ &= 11-2i \end{aligned}$$

$$\therefore \operatorname{Im}(4z-3w) = -2$$

$$\begin{aligned} \text{c} \quad iz^2 &= i(2+i)^2 \\ &= i(4+4i+i^2) \\ &= i(3+4i) \\ &= 3i+4i^2 \\ &= -4+3i \end{aligned}$$

$$\therefore \operatorname{Im}(iz^2) = 3$$

$$\begin{aligned} \text{b} \quad zw &= (2+i)(-1+2i) \\ &= -2+4i-i+2i^2 \\ &= -4+3i \end{aligned}$$

$$\therefore \operatorname{Re}(zw) = -4$$

$$\begin{aligned} \text{d} \quad \frac{z}{w} &= \frac{2+i}{-1+2i} \times \frac{-1-2i}{-1-2i} \\ &= \frac{-2-4i-i-2i^2}{1-4i^2} \\ &= \frac{0-5i}{5} \\ &= -i \end{aligned}$$

$$\therefore \operatorname{Re}\left(\frac{z}{w}\right) = 0$$

EXERCISE 8B.2

$$\text{1 a} \quad 2x+3iy = -x-6i$$

Equating real and imaginary parts,

$$2x = -x \quad \text{and} \quad 3y = -6$$

$\{x, y \text{ are real}\}$

$$\therefore 3x = 0 \quad \text{and} \quad y = -2$$

$$\therefore x = 0 \quad \text{and} \quad y = -2$$

$$\text{b} \quad x^2+ix = 4-2i$$

Equating real and imaginary parts,

$$x^2 = 4 \quad \text{and} \quad x = -2 \quad \{x \text{ is real}\}$$

$$\therefore x = \pm 2 \quad \text{and} \quad x = -2$$

$$\therefore x = -2$$

$$\text{c } (x + iy)(2 - i) = 8 + i$$

$$\therefore x + iy = \frac{8 + i}{2 - i} \times \frac{2 + i}{2 + i}$$

$$\therefore x + iy = \frac{16 + 8i + 2i + i^2}{4 - i^2}$$

$$\therefore x + iy = \frac{15 + 10i}{5}$$

$$\therefore x + iy = 3 + 2i$$

Equating real and imag. parts, for real x, y

$$x = 3 \quad \text{and} \quad y = 2$$

$$\text{2 a } 2(x + iy) = x - iy$$

$$\therefore 2x + 2iy = x - iy$$

Equating real and imaginary parts,

$$2x = x \quad \text{and} \quad 2y = -y$$

$$\therefore x = 0 \quad \text{and} \quad 3y = 0$$

$$\therefore x = 0 \quad \text{and} \quad y = 0$$

$$\text{c } (x + i)(3 - iy) = 1 + 13i$$

$$3x - ixy + 3i - i^2y = 1 + 13i$$

$$(3x + y) + i(3 - xy) = 1 + 13i$$

Equating real and imaginary parts,

$$3x + y = 1 \quad \text{and} \quad 3 - xy = 13$$

$$y = 1 - 3x \quad \text{and} \quad xy = -10$$

$$x(1 - 3x) = -10$$

$$x - 3x^2 = -10$$

$$0 = 3x^2 - x - 10$$

$$0 = (3x + 5)(x - 2)$$

$$\therefore x = -\frac{5}{3} \quad \text{or} \quad x = 2$$

$$\text{When } x = -\frac{5}{3}, \quad y = 1 - 3(-\frac{5}{3}) = 6$$

$$\text{and when } x = 2, \quad y = 1 - 3 \times 2 = -5$$

$$\therefore x = -\frac{5}{3} \quad \text{and} \quad y = 6$$

$$\text{or } x = 2 \quad \text{and} \quad y = -5$$

$$\text{d } (3 + 2i)(x + iy) = -i$$

$$\therefore x + iy = \frac{-i}{3 + 2i} \times \frac{3 - 2i}{3 - 2i}$$

$$\therefore x + iy = \frac{-3i + 2i^2}{9 - 4i^2}$$

$$\therefore x + iy = \frac{-2 - 3i}{13}$$

Equating real and imag. parts, for real x, y

$$x = -\frac{2}{13} \quad \text{and} \quad y = -\frac{3}{13}$$

$$\text{b } (x + 2i)(y - i) = -4 - 7i$$

$$\therefore xy - ix + 2iy - 2i^2 = -4 - 7i$$

$$\therefore (xy + 2) + i(2y - x) = -4 - 7i$$

Equating real and imaginary parts,

$$xy + 2 = -4 \quad \text{and} \quad 2y - x = -7$$

$$xy = -6 \quad \text{and} \quad x = 2y + 7$$

$$(2y + 7)y = -6$$

$$2y^2 + 7y = -6$$

$$2y^2 + 7y + 6 = 0$$

$$(2y + 3)(y + 2) = 0$$

$$\therefore y = -\frac{3}{2} \quad \text{or} \quad y = -2$$

$$\text{When } y = -2, \quad x = 2(-2) + 7 = 3$$

$$\text{When } y = -\frac{3}{2}, \quad x = 2(-\frac{3}{2}) + 7 = 4$$

$$\therefore x = 3 \quad \text{and} \quad y = -2$$

$$\text{or } x = 4 \quad \text{and} \quad y = -\frac{3}{2}$$

$$\text{d } (x + iy)(2 + i) = 2x - i(y + 1)$$

$$2x + ix + 2iy + yi^2 = 2x + i(-y - 1)$$

$$(2x - y) + i(x + 2y) = 2x + i(-y - 1)$$

Equating real and imaginary parts,

$$2x - y = 2x \quad \text{and} \quad x + 2y = -y - 1$$

$$\therefore -y = 2x - 2x$$

$$\therefore y = 0 \quad \text{and consequently}$$

$$x + 0 = 0 - 1 = -1$$

$$\therefore x = -1 \quad \text{and} \quad y = 0$$

EXERCISE 8B.3

$$\text{1 a } \text{roots } \alpha \text{ and } \beta \text{ are } 3 \pm i \quad \therefore \alpha + \beta = 6 \quad \text{and} \quad \alpha\beta = 9 - i^2 = 10$$

$$\therefore \text{quadratics have form } k(x^2 - 6x + 10) = 0 \quad k \neq 0$$

$$\text{b } \text{roots } \alpha \text{ and } \beta \text{ are } 1 \pm 3i \quad \therefore \alpha + \beta = 2 \quad \text{and} \quad \alpha\beta = 1 - 9i^2 = 10$$

$$\therefore \text{quadratics have form } k(x^2 - 2x + 10) = 0 \quad k \neq 0$$

$$\text{c } \text{roots } \alpha \text{ and } \beta \text{ are } -2 \pm 5i \quad \therefore \alpha + \beta = -4 \quad \text{and} \quad \alpha\beta = 4 - 25i^2 = 29$$

$$\therefore \text{quadratics have form } k(x^2 + 4x + 29) = 0 \quad k \neq 0$$

$$\text{d } \text{roots } \alpha \text{ and } \beta \text{ are } \sqrt{2} \pm i \quad \therefore \alpha + \beta = 2\sqrt{2} \quad \text{and} \quad \alpha\beta = 2 - i^2 = 3$$

$$\therefore \text{quadratics have form } k(x^2 - 2\sqrt{2}x + 3) = 0 \quad k \neq 0$$

e roots α and β are $2 \pm \sqrt{3}$ $\therefore \alpha + \beta = 4$ and $\alpha\beta = 4 - 3 = 1$
 $\therefore$ quadratics have form $k(x^2 - 4x + 1) = 0$ $k \neq 0$

f roots α and β are 0 and $-\frac{2}{3}$ $\therefore$ factors are $x, 3x + 2$
 $\therefore$ quadratics have form $kx(3x + 2) = 0$
 $\therefore k(3x^2 + 2x) = 0, k \neq 0$

g roots α and β are $\pm i\sqrt{2}$ $\therefore \alpha + \beta = 0$ and $\alpha\beta = -2i^2 = +2$
 $\therefore$ quadratics have form $k(x^2 + 2) = 0$ $k \neq 0$

h roots α and β are $-6 \pm i$ $\therefore \alpha + \beta = -12$ and $\alpha\beta = 36 - i^2 = 37$
 $\therefore$ quadratics have form $k(x^2 + 12x + 37) = 0$ $k \neq 0$

2 a If $3 + i$ is a root then so is $3 - i$ (if a and b are real)

$$\therefore \alpha + \beta = 6 \text{ and } \alpha\beta = 9 - i^2 = 10$$

$$\therefore x^2 - 6x + 10 = 0 \text{ and so } a = -6, b = 10$$

b If $1 - \sqrt{2}$ is a root then so is $1 + \sqrt{2}$ if a, b are rational.

$$\therefore \alpha + \beta = 2 \text{ and } \alpha\beta = 1 - 2 = -1$$

$$\therefore x^2 - 2x - 1 = 0$$

$$\therefore a = -2 \text{ and } b = -1$$

c If $a + ai$ is a root then so is $a - ai$ (if a and b are real and $a \neq 0$.)

$$\therefore \alpha + \beta = 2a \text{ and } \alpha\beta = (a + ai)(a - ai)$$

$$= a^2 - (ai^2)$$

$$= 2a^2$$

$$\therefore x^2 - 2ax + 2a^2 = 0$$

$$\therefore -2a = 4 \text{ and } b = 2a^2$$

$$\therefore a = -2 \text{ and } b = 8$$

However if $a = 0$, $a + i = 0$ (i.e., is *not* complex) so the other root could be any real number.

But $\alpha\beta = 0$ $\therefore b = 0$
 $\therefore a = 0, b = 0$ is also a solution.

EXERCISE 8B.4

1 To Prove: $(z_1 - z_2)^* = z_1^* - z_2^*$

Let $z_1 = a + ib$, and $z_2 = c + id$.

$$\therefore (z_1 - z_2)^* = [(a + ib) - (c + id)]^*$$

$$= [(a - c) + i(b - d)]^*$$

$$= (a - c) - i(b - d)$$

$$= a - c - bi + di$$

$$= (a - bi) - (c - di)$$

$$= z_1^* - z_2^*$$

2 $(w^* - z)^* - (w - 2z^*)$

$$= w^{**} - z^* - w + 2z^*$$

$$= w - z^* - w + 2z^*$$

$$= -z^* + 2z^*$$

$$= z^*$$

3 Let $z = a + bi$ $\therefore z^* = a - bi$

If $z^* = -z$, then $a - bi = -a - bi$

$$\therefore a = -a$$

$$\therefore 2a = 0$$

i.e., $a = 0$ and b is any real number

i.e., z is purely imaginary

or $a = 0, b = 0$ i.e., z is zero.

4 a Let $z_1 = a + bi$ $z_2 = c + di$

$$\begin{aligned}
 \therefore \frac{z_1}{z_2} &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di} \\
 &= \frac{(a + bi)(c - di)}{(c + di)(c - di)} \\
 &= \frac{ac - adi + bci - bdi^2}{c^2 - i^2d^2} \\
 &= \frac{(ac + bd) + i(-ad + bc)}{(c^2 + d^2)} \\
 &= \left[\frac{ac + bd}{c^2 + d^2} \right] + \left[\frac{bc - ad}{c^2 + d^2} \right] i
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{z_1^*}{z_2^*} &= \frac{a - bi}{c - di} \times \frac{c + di}{c + di} \\
 &= \frac{(a - bi)(c + di)}{(c - di)(c + di)} \\
 &= \frac{ac + adi - bci - bdi^2}{c^2 - i^2d^2} \\
 &= \frac{(ac + bd) - i(bc - ad)}{c^2 + d^2} \\
 &= \left[\frac{ac + bd}{c^2 + d^2} \right] - \left[\frac{bc - ad}{c^2 + d^2} \right] i \\
 &= \left(\frac{z_1}{z_2} \right)^* \quad \text{for all } z_2 \neq 0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \left(\frac{z_1}{z_2} \right)^* \times z_2^* &= \left(\frac{z_1}{z_2} \times z_2 \right)^* \quad (\text{from Example 13}) \\
 &= z_1^*
 \end{aligned}$$

$$\therefore \left(\frac{z_1}{z_2} \right)^* = \frac{z_1^*}{z_2^*} \quad \{\text{on dividing both sides by } z_2^*\}$$

6 a Let $z = a + bi$ and $w = c + di$

$$\begin{aligned}
 \therefore zw^* + z^*w &= (a + bi)(c - di) + (a - bi)(c + di) \\
 &= ac - adi + bci + bd + ac + adi - bci + bd \\
 &= ac + bd + ac + bd \\
 &= 2ac + 2bd \quad \text{which is a real number}
 \end{aligned}$$

b Let $z = a + bi$ and $w = c + di$

$$\begin{aligned}
 \therefore zw^* - z^*w &= (a + bi)(c - di) - (a - bi)(c + di) \\
 &= ac - adi + bci + bd - [ac + adi - bci + bd] \\
 &= 2bci - 2adi \\
 &= (2bc - 2ad)i \quad \text{which is purely imaginary or zero}
 \end{aligned}$$

7 a If $z = a + bi$

$$\begin{aligned}
 \text{then } z^2 &= (a + bi)(a + bi) \\
 &= a^2 + 2abi + b^2i^2 \\
 &= (a^2 - b^2) + 2abi
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{From a, } (z^2)^* &= (a^2 - b^2) - 2abi \\
 &= (a^2 - 2abi + b^2i^2) \\
 &= (a - bi)^2 \\
 &= (z^*)^2 \quad \text{as required.}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad z^3 &= (z^2)z \\
 &= ((a^2 - b^2) + 2abi)(a + bi) \\
 &= a(a^2 - b^2) + b(a^2 - b^2)i + 2a^2bi + 2ab^2i^2 \\
 &= a^3 - ab^2 + a^2bi - b^3i + 2a^2bi - 2ab^2 \\
 &= (a^3 - 3ab^2) + (3a^2b - b^3)i
 \end{aligned}$$

$$\therefore (z^3)^* = (a^3 - 3ab^2) - (3a^2b - b^3)i$$

$$\begin{aligned}
 (z^*)^3 &= (z^*)^2 z^* \\
 &= [(a^2 - b^2) - 2abi](a - bi) \\
 &= a(a^2 - b^2) - b(a^2 - b^2)i - 2a^2bi + 2ab^2i^2 \\
 &= a^3 - ab^2 - a^2bi + b^3i - 2a^2bi - 2ab^2 \\
 &= (a^3 - 3ab^2) - (3a^2b - b^3)i \quad \text{which is } (z^3)^* \quad \text{as required}
 \end{aligned}$$

$$8 \quad w = \frac{z-1}{z^*+1} \quad \text{where } z = a+bi$$

$$\begin{aligned} \therefore w &= \frac{(a-1)+bi}{(a+1)-bi} \times \frac{(a+1)+bi}{(a+1)+bi} \\ &= \frac{(a^2-1)+(a-1)bi+(a+1)bi+b^2i^2}{(a+1)^2-b^2i^2} \\ &= \frac{(a^2-b^2-1)+2abi}{(a+1)^2+b^2} \end{aligned}$$

a w is real if $2ab = 0$
i.e., if $a = 0$ or $b = 0$, $a \neq 1$
However, if $b = 0$ and $a = -1$,
 w is undefined and hence is not real.
 $\therefore a = 0$ or $(b = 0, a \neq -1)$.

b w is purely imaginary if
 $a^2 - b^2 - 1 = 0$
i.e., $a^2 - b^2 = 1$
and neither a nor b is zero.

EXERCISE 8B.5

$$\begin{aligned} 1 \quad \mathbf{a} \quad (z_1 z_2 z_3)^* &= [z_1 \times (z_2 \times z_3)]^* \\ &= z_1^* (z_2 \times z_3)^* \quad \{\text{as } (zw)^* = z^* w^*\} \\ &= z_1^* \times z_2^* \times z_3^* \quad \{\text{as } (zw)^* = z^* w^* \text{ again}\} \\ \mathbf{b} \quad (z_1 z_2 z_3 z_4)^* &= (z_1 z_2 z_3)^* \times z_4^* \quad \{\text{as } (zw)^* = z^* w^*\} \\ &= z_1^* \times z_2^* \times z_3^* \times z_4^* \quad \{\text{using a}\} \\ \mathbf{c} \quad (z_1 \times z_2 \times z_3 \dots z_n)^* &= z_1^* \times z_2^* \times z_3^* \dots z_n^* \\ \mathbf{d} \quad (z^n)^* &= (z \times z \times z \times \dots \times z)^* \\ &= z^* \times z^* \times z^* \times z^* \times \dots \times z^* \quad \{\text{using c}\} \\ &= (z^*)^n \end{aligned}$$

EXERCISE 8C.1

$$\begin{aligned} 1 \quad \mathbf{a} \quad 3P(x) &= 3(x^2 + 2x + 3) \\ &= 3x^2 + 6x + 9 \\ \mathbf{c} \quad P(x) - 2Q(x) &= (x^2 + 2x + 3) - 2(4x^2 + 5x + 6) \\ &= x^2 + 2x + 3 - 8x^2 - 10x - 12 \\ &= -7x^2 - 8x - 9 \\ 2 \quad \mathbf{a} \quad f(x) + g(x) &= (x^2 - x + 2) + (x^3 - 3x + 5) \\ &= x^3 + x^2 - 4x + 7 \\ \mathbf{c} \quad 2f(x) + 3g(x) &= 2(x^2 - x + 2) + 3(x^3 - 3x + 5) \\ &= 2x^2 - 2x + 4 + 3x^3 - 9x + 15 \\ &= 3x^3 + 2x^2 - 11x + 19 \\ \mathbf{e} \quad f(x)g(x) &= (x^2 - x + 2)(x^3 - 3x + 5) \\ &= x^5 - x^4 + 2x^3 - 3x^3 + 3x^2 \\ &\quad - 6x + 5x^2 - 5x + 10 \\ &= x^5 - x^4 - x^3 + 8x^2 - 11x + 10 \\ \mathbf{b} \quad P(x) + Q(x) &= (x^2 + 2x + 3) + (4x^2 + 5x + 6) \\ &= 5x^2 + 7x + 9 \\ \mathbf{d} \quad P(x)Q(x) &= (x^2 + 2x + 3)(4x^2 + 5x + 6) \\ &= 4x^4 + 8x^3 + 12x^2 + 5x^3 + 10x^2 \\ &\quad + 15x + 6x^2 + 12x + 18 \\ &= 4x^4 + 13x^3 + 28x^2 + 27x + 18 \\ \mathbf{b} \quad g(x) - f(x) &= (x^3 - 3x + 5) - (x^2 - x + 2) \\ &= x^3 - 3x + 5 - x^2 + x - 2 \\ &= x^3 - x^2 - 2x + 3 \\ \mathbf{d} \quad g(x) + xf(x) &= (x^3 - 3x + 5) + x(x^2 - x + 2) \\ &= x^3 - 3x + 5 + x^3 - x^2 + 2x \\ &= 2x^3 - x^2 - x + 5 \\ \mathbf{f} \quad [f(x)]^2 &= (x^2 - x + 2)(x^2 - x + 2) \\ &= x^4 - x^3 + 2x^2 - x^3 + x^2 \\ &\quad - 2x + 2x^2 - 2x + 4 \\ &= x^4 - 2x^3 + 5x^2 - 4x + 4 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & (x^2 - 2x + 3)(2x + 1) \\
 &= 2x^3 - 4x^2 + 6x + x^2 - 2x + 3 \\
 &= 2x^3 - 3x^2 + 4x + 3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & (x + 2)^3 \\
 &= (x + 2)(x + 2)^2 \\
 &= (x + 2)(x^2 + 4x + 4) \\
 &= x^3 + 2x^2 + 4x^2 + 8x + 4x + 8 \\
 &= x^3 + 6x^2 + 12x + 8
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & (2x - 1)^4 \\
 &= (2x - 1)^2(2x - 1)^2 \\
 &= (4x^2 - 4x + 1)(4x^2 - 4x + 1) \\
 &= 16x^4 - 16x^3 + 4x^2 - 16x^3 + 16x^2 \\
 &\quad - 4x + 4x^2 - 4x + 1 \\
 &= 16x^4 - 32x^3 + 24x^2 - 8x + 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & (2x^2 - 3x + 5)(3x - 1) \\
 &= 6x^3 - 11x^2 + 18x - 5 \\
 \text{as} \quad & \begin{array}{r}
 \begin{array}{rrrr}
 & 2 & -3 & 5 \\
 \times & & 3 & -1 \\
 \hline
 & -2 & 3 & -5 \\
 6 & -9 & 15 & \\
 \hline
 6 & -11 & 18 & -5
 \end{array}
 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & (2x^2 + 3x + 2)(5 - x) \\
 &= -2x^3 + 7x^2 + 13x + 10 \\
 \text{as} \quad & \begin{array}{r}
 \begin{array}{rrrr}
 & 2 & 3 & 2 \\
 \times & & -1 & 5 \\
 \hline
 & 10 & 15 & 10 \\
 -2 & -3 & -2 & \\
 \hline
 -2 & 7 & 13 & 10
 \end{array}
 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & (x^2 - 3x + 2)(2x^2 + 4x - 1) \\
 &= 2x^4 - 2x^3 - 9x^2 + 11x - 2 \\
 \text{as} \quad & \begin{array}{r}
 \begin{array}{rrrr}
 & 1 & -3 & 2 \\
 \times & 2 & 4 & -1 \\
 \hline
 & -1 & 3 & -2 \\
 4 & -12 & 8 & \\
 \hline
 2 & -6 & 4 & \\
 2 & -2 & -9 & 11 & -2
 \end{array}
 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & (x^2 - x + 3)^2 \\
 &= x^4 - 2x^3 + 7x^2 - 6x + 9 \\
 \text{as} \quad & \begin{array}{r}
 \begin{array}{rrrr}
 & 1 & -1 & 3 \\
 \times & 1 & -1 & 3 \\
 \hline
 & 3 & -3 & 9 \\
 -1 & 1 & -3 & \\
 \hline
 1 & -1 & 3 & \\
 1 & -2 & 7 & -6 & 9
 \end{array}
 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & (x - 1)^2(x^2 + 3x - 2) \\
 &= (x^2 - 2x + 1)(x^2 + 3x - 2) \\
 &= x^4 - 2x^3 + x^2 + 3x^3 - 6x^2 \\
 &\quad + 3x - 2x^2 + 4x - 2 \\
 &= x^4 + x^3 - 7x^2 + 7x - 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & (2x^2 - x + 3)^2 \\
 &= (2x^2 - x + 3)(2x^2 - x + 3) \\
 &= 4x^4 - 2x^3 + 6x^2 - 2x^3 + x^2 \\
 &\quad - 3x + 6x^2 - 3x + 9 \\
 &= 4x^4 - 4x^3 + 12x^2 - 6x + 9
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & (3x - 2)^2(2x + 1)(x - 4) \\
 &= (9x^2 - 12x + 4)(2x^2 - 7x - 4) \\
 &= 18x^4 - 24x^3 + 8x^2 - 63x^3 + 84x^2 \\
 &\quad - 28x - 36x^2 + 48x - 16 \\
 &= 18x^4 - 87x^3 + 56x^2 + 20x - 16
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & (4x^2 - x + 2)(2x + 5) \\
 &= 8x^3 + 18x^2 - x + 10 \\
 \text{as} \quad & \begin{array}{r}
 \begin{array}{rrrr}
 & 4 & -1 & 2 \\
 \times & & 2 & 5 \\
 \hline
 & 20 & -5 & 10 \\
 8 & -2 & 4 & \\
 \hline
 8 & 18 & -1 & 10
 \end{array}
 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & (x - 2)^2(2x + 1) \\
 &= (x^2 - 4x + 4)(2x + 1) \\
 &= 2x^3 - 7x^2 + 4x + 4 \\
 \text{as} \quad & \begin{array}{r}
 \begin{array}{rrrr}
 & 1 & -4 & 4 \\
 \times & & 2 & 1 \\
 \hline
 & 1 & -4 & 4 \\
 2 & -8 & 8 & \\
 \hline
 2 & -7 & 4 & 4
 \end{array}
 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & (3x^2 - x + 2)(5x^2 + 2x - 3) \\
 &= 15x^4 + x^3 - x^2 + 7x - 6 \\
 \text{as} \quad & \begin{array}{r}
 \begin{array}{rrrr}
 & 3 & -1 & 2 \\
 \times & 5 & 2 & -3 \\
 \hline
 & -9 & 3 & -6 \\
 6 & -2 & 4 & \\
 \hline
 15 & -5 & 10 & \\
 15 & 1 & -1 & 7 & -6
 \end{array}
 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & (2x^2 + x - 4)^2 \\
 &= 4x^4 + 4x^3 - 15x^2 - 8x + 16 \\
 \text{as} \quad & \begin{array}{r}
 \begin{array}{rrrr}
 & 2 & 1 & -4 \\
 \times & 2 & 1 & -4 \\
 \hline
 & -8 & -4 & 16 \\
 2 & 1 & -4 & \\
 \hline
 4 & 2 & -8 & \\
 4 & 4 & -15 & -8 & 16
 \end{array}
 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & (2x+5)^3 \\
 &= (2x+5)^2(2x+5) \\
 &= (4x^2+20x+25)(2x+5) \\
 &= 8x^3+60x^2+150x+125
 \end{aligned}$$

$$\begin{array}{r}
 \text{as} \quad \begin{array}{r}
 \begin{array}{rrrr}
 & 4 & 20 & 25 \\
 \times & & 2 & 5 \\
 \hline
 & 20 & 100 & 125 \\
 8 & 40 & 50 & \\
 \hline
 8 & 60 & 150 & 125
 \end{array}
 \end{array}
 \end{array}$$

$$\begin{aligned}
 \mathbf{j} \quad & (x^3+x^2-2)^2 \\
 &= x^6+2x^5+x^4-4x^3-4x^2+4
 \end{aligned}$$

$$\begin{array}{r}
 \text{as} \quad \begin{array}{rrrrrr}
 & & 1 & 1 & 0 & -2 \\
 & \times & 1 & 1 & 0 & -2 \\
 \hline
 & & -2 & -2 & 0 & 4 \\
 & 0 & 0 & 0 & 0 & \\
 & 1 & 1 & 0 & -2 & \\
 1 & 1 & 0 & -2 & & \\
 \hline
 1 & 2 & 1 & -4 & -4 & 0 & 4
 \end{array}
 \end{array}$$

EXERCISE 8C.2

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \begin{array}{r}
 x \\
 x+2 \overline{) x^2+2x-3} \\
 \underline{x^2+2x} \\
 -3
 \end{array} \\
 \therefore \quad & Q(x) = x \\
 & R = -3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \begin{array}{r}
 2x^2+10x+16 \\
 x-2 \overline{) 2x^3+6x^2-4x+3} \\
 \underline{2x^3-4x^2} \\
 10x^2-4x \\
 \underline{10x^2-20x} \\
 16x+3 \\
 \underline{16x-32} \\
 35
 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \begin{array}{r}
 x-4 \\
 x-1 \overline{) x^2-5x+1} \\
 \underline{x^2-x} \\
 -4x+1 \\
 \underline{-4x+4} \\
 -3
 \end{array} \quad \therefore \quad \begin{array}{l} Q(x) = x-4 \\ R = -3 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \quad & Q(x) = 2x^2+10x+16 \\
 & R = 35
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \begin{array}{r}
 x+1 \\
 x-4 \overline{) x^2-3x+6} \\
 \underline{x^2-4x} \\
 x+6 \\
 \underline{x-4} \\
 10
 \end{array} \\
 \therefore \quad & D(x) = x+1 + \frac{10}{x-4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \begin{array}{r}
 2x-3 \\
 x-2 \overline{) 2x^2-7x+2} \\
 \underline{2x^2-4x} \\
 -3x+2 \\
 \underline{-3x+6} \\
 -4
 \end{array} \\
 \therefore \quad & D(x) = 2x-3 - \frac{4}{x-2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \begin{array}{r}
 x+1 \\
 x+3 \overline{) x^2+4x-11} \\
 \underline{x^2+3x} \\
 x-11 \\
 \underline{x+3} \\
 -14
 \end{array} \\
 \therefore \quad & D(x) = x+1 - \frac{14}{x+3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \begin{array}{r}
 x^2+x-2 \\
 2x+1 \overline{) 2x^3+3x^2-3x-2} \\
 \underline{2x^3+x^2} \\
 2x^2-3x \\
 \underline{2x^2+x} \\
 -4x-2 \\
 \underline{-4x-2} \\
 0
 \end{array} \\
 \therefore \quad & D(x) = x^2+x-2
 \end{aligned}$$

$$\begin{array}{r}
 \text{e} \qquad \qquad \qquad x^2 + 4x + 4 \\
 3x - 1 \overline{) \begin{array}{r} 3x^3 + 11x^2 + 8x + 7 \\ \underline{3x^3 - x^2} \\ 12x^2 + 8x \\ \underline{12x^2 - 4x} \\ 12x + 7 \\ \underline{12x - 4} \\ 11 \end{array}} \\
 \therefore D(x) = x^2 + 4x + 4 + \frac{11}{3x - 1}
 \end{array}$$

$$\begin{array}{r}
 \text{f} \qquad \qquad \qquad x^3 - 2x^2 + \frac{5}{2}x - \frac{1}{4} \\
 2x + 3 \overline{) \begin{array}{r} 2x^4 - x^3 - x^2 + 7x + 4 \\ \underline{2x^4 + 3x^3} \\ -4x^3 - x^2 \\ \underline{-4x^3 - 6x^2} \\ 5x^2 + 7x \\ \underline{5x^2 + \frac{15}{2}x} \\ -\frac{1}{2}x + 4 \\ \underline{-\frac{1}{2}x - \frac{3}{4}} \\ \frac{19}{4} \end{array}} \\
 \therefore D(x) = x^3 - 2x^2 + \frac{5}{2}x - \frac{1}{4} + \frac{\frac{19}{4}}{2x + 3}
 \end{array}$$

$$\begin{array}{r}
 \text{3 a} \qquad \qquad \qquad x + 2 \\
 x - 2 \overline{) \begin{array}{r} x^2 + 0x + 5 \\ \underline{x^2 - 2x} \\ 2x + 5 \\ \underline{2x - 4} \\ 9 \end{array}} \\
 \therefore D(x) = x + 2 + \frac{9}{x - 2}
 \end{array}$$

$$\begin{array}{r}
 \text{c} \qquad \qquad \qquad 3x - 4 \\
 x + 2 \overline{) \begin{array}{r} 3x^2 + 2x - 5 \\ \underline{3x^2 + 6x} \\ -4x - 5 \\ \underline{-4x - 8} \\ +3 \end{array}} \\
 \therefore D(x) = 3x - 4 + \frac{3}{x + 2}
 \end{array}$$

$$\begin{array}{r}
 \text{e} \qquad \qquad \qquad 2x^2 - 8x + 31 \\
 x + 4 \overline{) \begin{array}{r} 2x^3 + 0x^2 - x + 0 \\ \underline{2x^3 + 8x^2} \\ -8x^2 - x \\ \underline{-8x^2 - 32x} \\ 31x + 0 \\ \underline{31x + 124} \\ -124 \end{array}} \\
 \therefore D(x) = 2x^2 - 8x + 31 - \frac{124}{x + 4}
 \end{array}$$

$$\begin{array}{r}
 \text{b} \qquad \qquad \qquad 2x + 1 \\
 x + 1 \overline{) \begin{array}{r} 2x^2 + 3x + 0 \\ \underline{2x^2 + 2x} \\ x + 0 \\ \underline{x + 1} \\ -1 \end{array}} \\
 \therefore D(x) = 2x + 1 - \frac{1}{x + 1}
 \end{array}$$

$$\begin{array}{r}
 \text{d} \qquad \qquad \qquad x^2 + 3x - 2 \\
 x - 1 \overline{) \begin{array}{r} x^3 + 2x^2 - 5x + 2 \\ \underline{x^3 - x^2} \\ 3x^2 - 5x \\ \underline{3x^2 - 3x} \\ -2x + 2 \\ \underline{-2x + 2} \\ 0 \end{array}} \\
 \therefore D(x) = x^2 + 3x - 2
 \end{array}$$

$$\begin{array}{r}
 \text{f} \qquad \qquad \qquad x^2 + 3x + 6 \\
 x - 2 \overline{) \begin{array}{r} x^3 + x^2 + 0x - 5 \\ \underline{x^3 - 2x^2} \\ 3x^2 + 0x \\ \underline{3x^2 - 6x} \\ 6x - 5 \\ \underline{6x - 12} \\ 7 \end{array}} \\
 \therefore D(x) = x^2 + 3x + 6 + \frac{7}{x - 2}
 \end{array}$$

EXERCISE 8C.3

$$\begin{array}{r}
 \text{1 a} \qquad \qquad \qquad x + 1 \\
 x^2 + x + 1 \overline{) \begin{array}{r} x^3 + 2x^2 + x - 3 \\ \underline{x^3 + x^2 + x} \\ x^2 + 0x - 3 \\ \underline{x^2 + x + 1} \\ -x - 4 \end{array}} \\
 \therefore \begin{array}{l} Q(x) = x + 1 \\ R(x) = -x - 4 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \text{b} \qquad \qquad \qquad 3 \\
 x^2 - 1 \overline{) \begin{array}{r} 3x^2 - x + 0 \\ \underline{3x^2} \\ -x + 0 \end{array}} \\
 \therefore \begin{array}{l} Q(x) = 3 \\ R(x) = -x + 3 \end{array}
 \end{array}$$

$$\begin{array}{r} 3x \\ x^2 + 1 \overline{) 3x^3 + 0x^2 + x - 1} \\ \underline{3x^3 + 3x} \\ -2x - 1 \end{array}$$

$$\begin{aligned} \therefore Q(x) &= 3x \\ R(x) &= -2x - 1 \end{aligned}$$

$$\begin{aligned} d \quad Q(x) &= 0 \\ R(x) &= x - 4 \end{aligned}$$

$$\begin{array}{r} 1 \\ x^2 + x + 1 \overline{) x^2 - x + 1} \\ \underline{x^2 + x + 1} \\ -2x \end{array}$$

$$\therefore D(x) = 1 - \frac{2x}{x^2 + x + 1}$$

$$\begin{array}{r} x \\ x^2 + 2 \overline{) x^3 + 0x^2 + 0x + 0} \\ \underline{x^3 + 2x} \\ -2x + 0 \end{array}$$

$$\therefore D(x) = x - \frac{2x}{x^2 + 2}$$

$$\begin{array}{r} x^2 + x + 3 \\ x^2 - x + 1 \overline{) x^4 + 0x^3 + 3x^2 + x - 1} \\ \underline{x^4 - x^3 + x^2} \\ x^3 + 2x^2 + x \\ \underline{x^3 - x^2 + x} \\ 3x^2 + 0x - 1 \\ \underline{3x^2 - 3x + 3} \\ 3x - 4 \end{array}$$

$$\therefore D(x) = x^2 + x + 3 + \frac{3x - 4}{x^2 - x + 1}$$

$$d \quad \frac{2x^3 - x + 6}{(x-1)^2} = \frac{2x^3 - x + 6}{x^2 - 2x + 1}$$

$$\begin{array}{r} 2x + 4 \\ x^2 - 2x + 1 \overline{) 2x^3 + 0x^2 - x + 6} \\ \underline{2x^3 - 4x^2 + 2x} \\ 4x^2 - 3x + 6 \\ \underline{4x^2 - 8x + 4} \\ 5x + 2 \end{array}$$

$$\therefore D(x) = 2x + 4 + \frac{5x + 2}{(x-1)^2}$$

$$e \quad \frac{x^4}{(x+1)^2} = \frac{x^4}{x^2 + 2x + 1}$$

$$\begin{array}{r} x^2 - 2x + 3 \\ x^2 + 2x + 1 \overline{) x^4 + 0x^3 + 0x^2 + 0x + 0} \\ \underline{x^4 + 2x^3 + x^2} \\ -2x^3 - x^2 + 0x \\ \underline{-2x^3 - 4x^2 - 2x} \\ 3x^2 + 2x + 0 \\ \underline{3x^2 + 6x + 3} \\ -4x - 3 \end{array}$$

$$\therefore D(x) = x^2 - 2x + 3 - \frac{4x + 3}{(x+1)^2}$$

$$f \quad \frac{x^4 - 2x^3 + x + 5}{(x-1)(x+2)} = \frac{x^4 - 2x^3 + x + 5}{x^2 + x - 2}$$

$$\begin{array}{r} x^2 - 3x + 5 \\ x^2 + x - 2 \overline{) x^4 - 2x^3 + 0x^2 + x + 5} \\ \underline{x^4 + x^3 - 2x^2} \\ -3x^3 + 2x^2 + x \\ \underline{-3x^3 - 3x^2 + 6x} \\ 5x^2 - 5x + 5 \\ \underline{5x^2 + 5x - 10} \\ -10x + 15 \end{array}$$

$$\therefore D(x) = x^2 - 3x + 5 + \frac{15 - 10x}{(x-1)(x+2)}$$

$$\begin{aligned} 3 \quad \frac{P(x)}{x-2} &= \frac{(x-2)(x^2 + 2x + 3) + 7}{x-2} \quad \therefore \text{quotient is } x^2 + 2x + 3 \\ &= x^2 + 2x + 3 + \frac{7}{x-2}, \text{ etc} \end{aligned}$$

EXERCISE 8C.4

$$1 \quad a \quad +1 \quad \begin{array}{r|rrr} 3 & -2 & -3 \\ 0 & 3 & 1 \\ \hline 3 & 1 & -2 \end{array}$$

$$\therefore D(x) = 3x + 1 - \frac{2}{x-1}$$

$$b \quad -3 \quad \begin{array}{r|rrrr} 1 & 5 & 6 & 5 \\ 0 & -3 & -6 & 0 \\ \hline 1 & 2 & 0 & 5 \end{array}$$

$$\therefore D(x) = x^2 + 2x + \frac{5}{x+3}$$

$$\mathbf{c} \quad -1 \left| \begin{array}{ccc|c} 3 & -1 & 2 & \\ 0 & -3 & 4 & \\ \hline 3 & -4 & 6 & \end{array} \right|$$

$$\therefore D(z) = 3z - 4 + \frac{6}{z+1}$$

$$\mathbf{e} \quad +3 \left| \begin{array}{ccccc|c} 1 & -2 & 1 & 0 & -4 & \\ 0 & 3 & 3 & 12 & 36 & \\ \hline 1 & 1 & 4 & 12 & 32 & \end{array} \right|$$

$$\therefore D(z) = z^3 + z^2 + 4z + 12 + \frac{32}{z-3}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad P(x) &= (x-2)(3x^2 - 2x + 1) + 4 \\ &= 3x^3 - 2x^2 + x - 6x^2 + 4x - 2 + 4 \\ &= 3x^3 - 8x^2 + 5x + 2 \end{aligned}$$

$$\mathbf{d} \quad -3 \left| \begin{array}{cccc|c} 1 & 0 & 0 & 27 & \\ 0 & -3 & 9 & -27 & \\ \hline 1 & -3 & 9 & 0 & \end{array} \right|$$

$$\therefore D(x) = x^2 - 3x + 9$$

$$\mathbf{f} \quad -1 \left| \begin{array}{ccccc|c} 1 & 0 & 1 & -1 & 0 & \\ 0 & -1 & 1 & -2 & 3 & \\ \hline 1 & -1 & 2 & -3 & 3 & \end{array} \right|$$

$$\therefore D(z) = z^3 - z^2 + 2z - 3 + \frac{3}{z+1}$$

$$\begin{aligned} \mathbf{b} \quad P(x) &= (x+3)(x^2 - 7x + 8) - 5 \\ &= x^3 - 7x^2 + 8x + 3x^2 - 21x + 24 - 5 \\ &= x^3 - 4x^2 - 13x + 19 \end{aligned}$$

EXERCISE 8D.1

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad 2x^2 - 5x - 12 \text{ has zeros} \\ x &= \frac{5 \pm \sqrt{25 - 4(2)(-12)}}{4} \\ &= \frac{5 \pm \sqrt{121}}{4} \\ &= \frac{5 \pm 11}{4} \\ &= 4, -\frac{6}{4} \\ \therefore \text{zeros are } 4, -\frac{3}{2} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad z^2 - 6z + 6 \text{ has zeros} \\ z &= \frac{6 \pm \sqrt{36 - 4(1)(6)}}{2} \\ &= \frac{6 \pm \sqrt{12}}{2} \\ &= 3 \pm \sqrt{3} \\ \therefore \text{zeros are } 3 \pm \sqrt{3} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad z^3 + 2z &= z(z^2 + 2) \\ &= z(z^2 - 2i^2) \\ &= z(z + i\sqrt{2})(z - i\sqrt{2}) \\ \therefore \text{zeros are } 0, \pm i\sqrt{2} \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad 5x^2 &= 3x + 2 \\ \therefore 5x^2 - 3x - 2 &= 0 \\ \therefore (5x+2)(x-1) &= 0 \\ \therefore \text{roots are } 1, -\frac{2}{5} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad x^2 + 6x + 10 \text{ has zeros} \\ x &= \frac{-6 \pm \sqrt{36 - 4(1)(10)}}{2} \\ &= \frac{-6 \pm \sqrt{-4}}{2} \\ &= -3 \pm i \\ \therefore \text{zeros are } -3 \pm i \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad x^3 - 4x &= x(x^2 - 4) \\ &= x(x+2)(x-2) \\ \therefore \text{zeros are } 0, \pm 2 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad z^4 + 4z^2 - 5 &= (z^2 + 5)(z^2 - 1) \\ &= (z^2 - 5i^2)(z^2 - 1) \\ &= (z + i\sqrt{5})(z - i\sqrt{5})(z+1)(z-1) \\ \therefore \text{zeros are } \pm i\sqrt{5}, \pm 1 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (2x+1)(x^2+3) &= 0 \\ \therefore (2x+1)(x^2-3i^2) &= 0 \\ \therefore (2x+1)(x+i\sqrt{3})(x-i\sqrt{3}) &= 0 \\ \therefore \text{roots are } -\frac{1}{2}, \pm i\sqrt{3} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & -2z(z^2 - 2z + 2) = 0 \\ z = 0 \text{ or } & \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2} \\ & = 0 \text{ or } \frac{2 \pm \sqrt{-4}}{2} \\ & = 0 \text{ or } 1 \pm i \\ \therefore \text{ roots are } & 0, 1 \pm i \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & z^3 + 5z = 0 \\ & z(z^2 + 5) = 0 \\ & z(z^2 - 5i^2) = 0 \\ & z(z + i\sqrt{5})(z - i\sqrt{5}) = 0 \\ \therefore \text{ roots are } & 0, \pm i\sqrt{5} \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & 2x^2 - 7x - 15 \\ & = (2x + 3)(x - 5) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & x^3 + 2x^2 - 4x \\ & = x(x^2 + 2x - 4) \\ & x^2 + 2x - 4 \text{ is zero when} \\ & x = \frac{-2 \pm \sqrt{4 + 16}}{2} \\ & = -1 \pm \sqrt{5} \\ \therefore \quad & x^3 + 2x^2 - 4x \\ & = x(x + 1 + \sqrt{5})(x + 1 - \sqrt{5}) \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & z^4 - 6z^2 + 5 \\ & = (z^2 - 1)(z^2 - 5) \\ & = (z + 1)(z - 1)(z + \sqrt{5})(z - \sqrt{5}) \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad & P(x) = a(x - \alpha)(x - \beta)(x - \gamma) \\ \therefore \quad & P(\alpha) = a \times 0 \times (\alpha - \beta)(\alpha - \gamma) = 0 \\ \text{and } & P(\beta) = a(\beta - \alpha) \times 0 \times (\beta - \gamma) = 0 \\ \text{and } & P(\gamma) = a(\gamma - \alpha)(\gamma - \beta) \times 0 = 0 \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad & \text{The zeros } \pm 2 \\ & \text{have sum} = 0 \text{ and product} = -4 \\ \therefore \quad & \text{come from quadratic factor } z^2 - 4 \\ & \text{and zero } 3 \text{ comes from } (z - 3) \\ \therefore \quad & P(z) = a(z^2 - 4)(z - 3), \quad a \neq 0 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \text{The zeros } -1 \pm i \\ & \text{have sum} = -2 \text{ and product} = 2 \\ \therefore \quad & \text{come from quadratic factor } z^2 + 2z + 2 \\ & \text{and zero } 3 \text{ comes from } (z - 3) \\ \therefore \quad & P(z) = a(z - 3)(z^2 + 2z + 2), \quad a \neq 0 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & x^3 = 5x \\ \therefore \quad & x^3 - 5x = 0 \\ & x(x^2 - 5) = 0 \\ & x(x + \sqrt{5})(x - \sqrt{5}) = 0 \\ \therefore \quad & \text{roots are } 0, \pm\sqrt{5} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & z^4 = 3z^2 + 10 \\ \therefore \quad & z^4 - 3z^2 - 10 = 0 \\ & (z^2 - 5)(z^2 + 2) = 0 \\ & (z^2 - 5)(z^2 - 2i^2) = 0 \\ & (z + \sqrt{5})(z - \sqrt{5})(z + i\sqrt{2})(z - i\sqrt{2}) = 0 \\ \therefore \quad & \text{roots are } \pm\sqrt{5}, \pm i\sqrt{2} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & z^2 - 6z + 16 \text{ is zero when} \\ & z = \frac{6 \pm \sqrt{36 - 4(1)(16)}}{2} \\ & = 3 \pm i\sqrt{7} \\ \therefore \quad & z^2 - 6z + 16 \\ & = (z - 3 + i\sqrt{7})(z - 3 - i\sqrt{7}) \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & 6z^3 - z^2 - 2z \\ & = z(6z^2 - z - 2) \\ & = z(2z + 1)(3z - 2) \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & z^4 - 2z^2 - 2 \\ & = (z^2 - 2)(z^2 + 1) \\ & = (z + \sqrt{2})(z - \sqrt{2})(z + i)(z - i) \end{aligned}$$

$$\therefore \quad \alpha, \beta \text{ and } \gamma \text{ all satisfy } P(x) = 0$$

$$\therefore \quad \alpha, \beta \text{ and } \gamma \text{ are zeros of } P(x)$$

$$\begin{aligned} \mathbf{b} \quad & \text{The zeros } \pm i \\ & \text{have sum} = 0 \text{ and product} = 1 \\ \therefore \quad & \text{come from quadratic factor } z^2 + 1 \\ & \text{and zero } -2 \text{ comes from } (z + 2) \\ \therefore \quad & P(z) = a(z^2 + 1)(z + 2), \quad a \neq 0 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \text{The zeros } -2 \pm \sqrt{2} \\ & \text{have sum} = -4 \text{ and product} = 2 \\ \therefore \quad & \text{come from quadratic factor } z^2 + 4z + 2 \\ & \text{and zero } -1 \text{ comes from } (z + 1) \\ \therefore \quad & P(z) = a(z + 1)(z^2 + 4z + 2), \quad a \neq 0 \end{aligned}$$

- 6 a** For zeros of ± 1 , sum = 0 and product = $-1 \therefore$ come from $z^2 - 1$
 For zeros of $\pm\sqrt{2}$, sum = 0 and product = $-2 \therefore$ come from $z^2 - 2$
 $\therefore P(z) = a(z^2 - 1)(z^2 - 2), a \neq 0$
- b** For zeros of $\pm i\sqrt{3}$, sum = 0 and product = 3 $\therefore$ come from $z^2 + 3$
 zeros of 2, -1 come from $(z - 2)(z + 1)$
 $\therefore P(z) = a(z - 2)(z + 1)(z^2 + 3), a \neq 0$
- c** For zeros of $\pm\sqrt{3}$, sum = 0 and product = $-3 \therefore$ come from $z^2 - 3$
 For zeros of $1 \pm i$, sum = 2 and product = 2 $\therefore$ come from $z^2 - 2z + 2$
 $\therefore P(z) = a(z^2 - 3)(z^2 - 2z + 2), a \neq 0$
- d** For zeros of $2 \pm \sqrt{5}$, sum = 4 and product = $-1 \therefore$ come from $z^2 - 4z - 1$
 For zeros of $-2 \pm 3i$, sum = -4 and product = 13 $\therefore$ come from $z^2 + 4z + 13$
 $\therefore P(z) = a(z^2 - 4z - 1)(z^2 + 4z + 13), a \neq 0$

EXERCISE 8D.2

- 1 a** $2x^2 + 4x + 5 = ax^2 + [2b - 6]x + c \therefore 2b = 10$
 Equating coefficients gives i.e., $b = 5$
 $a = 2, 2b - 6 = 4, \text{ and } c = 5 \therefore a = 2 \quad b = 5 \quad c = 5$
- b** $2x^3 - x^2 + 6$
 $= (x - 1)^2(2x + a) + bx + c$
 $= (x^2 - 2x + 1)(2x + a) + bx + c$
 $= 2x^3 + [a - 4]x^2 + [2 - 2a]x + a + bx + c$
 $= 2x^3 + [a - 4]x^2 + [2 - 2a + b]x + [a + c]$
 Equating coefficients gives $\therefore a - 4 = -1 \quad 2 - 2a + b = 0 \quad a + c = 6$
 $\therefore a = 3 \quad b = 2a - 2 \quad c = 6 - a$
 $b = 4 \quad c = 3$
- 2 a** $z^4 + 4 = (z^2 + az + 2)(z^2 + bz + 2)$
 $= z^4 + [a + b]z^3 + [4 + ab]z^2 + [2a + 2b]z + 4$
 Equating coefficients gives:
 $a + b = 0 \quad 4 + ab = 0$
 $\therefore a = -b \quad \therefore ab = -4$
 By inspection $a = 2$ and $b = -2$
 or $a = -2$ and $b = 2$
- | | | | | |
|---|----------|----------|-----------|---|
| | $\times$ | 1 | a | 2 |
| | | 1 | b | 2 |
| | | 2 | $2a$ | 4 |
| | b | ab | $2b$ | |
| 1 | a | 2 | | |
| 1 | $a + b$ | $4 + ab$ | $2a + 2b$ | 4 |
- b** $2z^4 + 5z^3 + 4z^2 + 7z + 6$
 $= (z^2 + az + 2)(2z^2 + bz + 3)$
 $= 2z^4 + [2a + b]z^3 + [ab + 7]z^2 + [3a + 2b]z + 6$
 Equating coefficients gives:
 $2a + b = 5 \dots (1)$
 $3a + 2b = 7 \dots (2)$
 $ab + 7 = 4 \dots (3)$
 $\therefore 4a + 2b = 10 \quad \{(1) \times 2\}$
 and $3a + 2b = 7$
 and solving these two equations gives
 $a = 3, b = -1$
 which checks with (3) as $ab + 7 = -3 + 7 = 4 \quad \checkmark$
- | | | | | |
|---|----------|----------|-----------|---|
| | $\times$ | 1 | a | 2 |
| | | 2 | b | 3 |
| | | 3 | $3a$ | 6 |
| | b | ab | $2b$ | |
| 2 | $2a$ | 4 | | |
| 2 | $2a + b$ | $ab + 7$ | $3a + 2b$ | 6 |

3 Consider

$$z^4 + 64 = (z^2 + az + 8)(z^2 + bz + 8)$$

$$= z^4 + [a + b]z^3 + [ab + 16]z^2 + [8a + 8b]z + 64$$

Equating coefficients gives:

$$a + b = 0 \quad \text{and} \quad ab + 16 = 0$$

$$\therefore a = -b \quad \text{i.e.,} \quad ab = -16$$

$$\therefore \text{by inspection} \quad a = 4 \quad \text{and} \quad b = -4$$

$$\text{or} \quad a = -4 \quad \text{and} \quad b = 4$$

$$\therefore z^4 + 64 \text{ can be factorised into } (z^2 + 4z + 8)(z^2 - 4z + 8)$$

Now consider

$$z^4 + 64 = (z^2 + az + 16)(z^2 + bz + 4)$$

$$= z^4 + [a + b]z^3 + [ab + 20]z^2 + [4a + 16b]z + 64$$

Equating coefficients gives:

$$a + b = 0 \dots (1) \quad \text{and} \quad ab + 20 = 0$$

$$4a + 16b = 0 \dots (2) \quad \text{i.e.,} \quad ab = -20 \dots (3)$$

Solution to (1), (2) is $a = b = 0$

But this does not satisfy (3)

 $\therefore$ no values of a and b exist which obey the original assumption $\therefore$ cannot be factorised in this way.**4** Consider

$$x^4 - 4x^2 + 8x - 4$$

$$= (x^2 + ax + 2)(x^2 + bx - 2)$$

$$= x^4 + [a + b]x^3 + [ab]x^2 + [2b - 2a]x - 4$$

Equating coefficients gives:

$$a + b = 0 \quad \text{and} \quad ab = -4 \quad \text{and} \quad -2a + 2b = 8$$

$$\therefore 2a + 2b = 0 \quad \dots (1)$$

$$-2a + 2b = 8 \quad \dots (2)$$

Adding (1) and (2) gives $4b = 8 \quad \therefore b = 2$ and hence $a = -2$

$$\therefore P(x) = (x^2 - 2x + 2)(x^2 + 2x - 2)$$

$$\text{Now if } x^4 + 8x = 4x^2 + 4$$

$$\text{then } x^4 - 4x^2 + 8x - 4 = 0$$

$$\therefore (x^2 - 2x + 2)(x^2 + 2x - 2) = 0$$

$$\therefore x^2 - 2x + 2 = 0 \quad \text{or} \quad x^2 + 2x - 2 = 0$$

$$\therefore x = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i \quad \text{or} \quad x = -\frac{2 \pm \sqrt{4+8}}{2} = -1 \pm \sqrt{3}$$

$$\therefore x = 1 \pm i, -1 \pm \sqrt{3}$$

5 a $P(z) = 2z^3 - z^2 + az - 3$

$$= (2z - 3)(z^2 + bz + 1) \quad \text{for some value } b$$

$$= 2z^3 + [2b - 3]z^2 + [2 - 3b]z - 3$$

Equating coefficients gives:

$$\therefore 2b - 3 = -1 \quad \text{and} \quad 2 - 3b = a$$

$$2b = 2 \quad \therefore a = 2 - 3$$

$$b = 1 \quad a = -1$$

$$\begin{array}{r} \times \quad \begin{array}{r} 1 \quad a \quad 8 \\ 1 \quad b \quad 8 \\ \hline 8 \quad 8a \quad 64 \\ b \quad ab \quad 8b \\ \hline 1 \quad a \quad 8 \\ \hline 1 \quad a+b \quad ab+16 \quad 8a+8b \quad 64 \end{array} \end{array}$$

$$\begin{array}{r} \times \quad \begin{array}{r} 1 \quad a \quad 16 \\ 1 \quad b \quad 4 \\ \hline 4 \quad 4a \quad 64 \\ b \quad ab \quad 16b \\ \hline 1 \quad a \quad 16 \\ \hline 1 \quad a+b \quad ab+20 \quad 4a+16b \quad 64 \end{array} \end{array}$$

$$\begin{array}{r} \times \quad \begin{array}{r} 1 \quad a \quad 2 \\ 1 \quad b \quad -2 \\ \hline -2 \quad -2a \quad -4 \\ b \quad ab \quad 2b \\ \hline 1 \quad a \quad 2 \\ \hline 1 \quad a+b \quad ab \quad 2b-2a \quad -4 \end{array} \end{array}$$

$$\begin{array}{r} \times \quad \begin{array}{r} 1 \quad b \quad 1 \\ 2 \quad -3 \\ \hline -3 \quad -3b \quad -3 \\ 2 \quad 2b \quad 2 \\ \hline 2 \quad 2b-3 \quad 2-3b \quad -3 \end{array} \end{array}$$

$$\therefore P(z) = (2z - 3)(z^2 + z + 1)$$



this quadratic has zeros of $z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

$$\therefore a = -1 \text{ and zeros are } \frac{3}{2}, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\begin{aligned} \mathbf{b} \quad P(z) &= 3z^3 - z^2 + [a+1]z + a \\ &= (3z+2)(z^2 + bz + c) \\ &= 3z^3 + [2+3b]z^2 + [2b+3c]z + 2c \end{aligned}$$

		1	b	c
×			3	2
	2		2b	2c
3	3b		3c	
3	2+3b	2b+3c	2c	

Equating coefficients gives:

$$\therefore 2+3b = -1, \quad 2b+3c = a+1 \quad \text{and} \quad 2c = a$$

$$\text{Now as } 2+3b = -1$$

$$\therefore 3b = -3$$

$$\therefore b = -1$$

$$\text{Substituting } b = -1 \text{ and } a = 2c \text{ into } 2b+3c = a+1 \text{ gives } \begin{aligned} 2(-1)+3c &= 2c+1 \\ \therefore -2+3c &= 2c+1 \end{aligned}$$

$$\begin{aligned} c &= 3 \\ \text{and so } a &= 6 \end{aligned}$$

$$\therefore P(z) = (3z+2)(z^2 - z + 3)$$

$\downarrow$
 with zeros $\frac{1 \pm \sqrt{1-4(3)(1)}}{2} = \frac{1 \pm i\sqrt{11}}{2}$

$$\text{So, the zeros are } -\frac{2}{3}, \frac{1}{2} \pm i\frac{\sqrt{11}}{2} \text{ and } a = 6$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad P(x) &= 2x^4 + ax^3 + bx^2 - 12x - 8 \\ &= (2x+1)(x-2)(x^2 + cx + 4) \\ &= (2x^2 - 3x - 2)(x^2 + cx + 4) \end{aligned}$$

		2	-3	-2
×		1	c	4
	8	-12	-8	
	2c	-3c	-2c	
2	-3	-2		
2	2c-3	6-3c	-2c-12	-8

$$\begin{aligned} \text{Equating coefficients: } \quad 2c-3 &= a, \\ 6-3c &= b \quad \text{and} \quad -2c-12 = -12 \end{aligned}$$

The last equation has solution $c = 0$, and consequently, $a = -3$ and $b = 6$

$$\therefore P(x) = (2x+1)(x-2)(x^2 + 4) = (2x+1)(x-2)(x+2i)(x-2i)$$

$$\therefore \text{zeros are } -\frac{1}{2}, 2 \text{ and } \pm 2i \text{ and } a = -3, b = 6.$$

$$\begin{aligned} \mathbf{b} \quad P(x) &= 2x^4 + ax^3 + bx^2 + ax + 3 \\ &= (x+3)(2x-1)(x^2 + cx - 1) \\ &= (2x^2 + 5x - 3)(x^2 + cx - 1) \end{aligned}$$

		2	5	-3
×		1	c	-1
	-2	-5	3	
	2c	5c	-3c	
2	5	-3		
2	2c+5	5c-5	-5-3c	3

$$\text{Equating coefficients: } a = 2c + 5,$$

$$b = 5c - 5, \quad a = -5 - 3c$$

$$\therefore 2c + 5 = -5 - 3c \quad \{\text{equating } a\text{'s}\}$$

$$\therefore 5c = -10$$

$$c = -2 \text{ and so, } a = 1 \quad b = -15$$

$$\therefore P(x) = (x+3)(2x-1)(x^2 - 2x - 1)$$



this quadratic has zeros $\frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$

$$\therefore \text{zeros are } -3, \frac{1}{2}, 1 \pm \sqrt{2} \text{ and } a = 1, b = -15$$

$$\begin{aligned}
 7 \quad \mathbf{a} \quad & x^3 + 3x^2 - 9x + c \\
 & = (x+a)^2(x+b) \\
 & = (x^2 + 2ax + a^2)(x+b) \\
 & = x^3 + [b+2a]x^2 + [a^2+2ab]x + a^2b
 \end{aligned}$$

Equating coefficients gives

$$2a + b = 3, \quad a^2 + 2ab = -9 \quad \text{and} \quad c = a^2b$$

Substituting $b = 3 - 2a$ into the second equation gives:

$$a^2 + 2a(3 - 2a) = -9$$

$$\therefore a^2 + 6a - 4a^2 = -9$$

$$\therefore -3a^2 + 6a = -9$$

$$\therefore 3a^2 - 6a - 9 = 0$$

$$\therefore a^2 - 2a - 3 = 0$$

$$\therefore (a-3)(a+1) = 0$$

$$\therefore a = 3 \quad \text{or} \quad a = -1$$

$$\therefore b = -3 \quad \text{or} \quad b = 5$$

$$\therefore c = -27 \quad \text{or} \quad c = 5$$

$$\begin{array}{r}
 \begin{array}{rrrr}
 & 1 & 2a & a^2 \\
 \times & & 1 & b \\
 \hline
 & b & 2ab & a^2b \\
 1 & 2a & a^2 & \\
 \hline
 1 & b+2a & a^2+2ab & a^2b
 \end{array}
 \end{array}$$

$$\text{If } c = -27, \quad P(x) = (x+3)^2(x-3)$$

$$\text{If } c = 5, \quad P(x) = (x-1)^2(x+5)$$

$$\begin{aligned}
 \mathbf{b} \quad & 3x^3 + 4x^2 - x + m \\
 & = (x+a)^2(3x+b) \\
 & = (x^2 + 2ax + a^2)(3x+b) \\
 & = x^3 + [6a+b]x^2 + [3a^2+2ab]x + a^2b
 \end{aligned}$$

Equating coefficients gives

$$6a + b = 4, \quad 3a^2 + 2ab = -1 \quad \text{and} \quad a^2b = m$$

Substituting $b = 4 - 6a$ into the second equation gives:

$$3a^2 + 2a(4 - 6a) = -1$$

$$3a^2 + 8a - 12a^2 = -1$$

$$9a^2 - 8a - 1 = 0$$

$$(9a+1)(a-1) = 0$$

$$\therefore a = -\frac{1}{9} \quad \text{or} \quad a = 1$$

$$\text{When } a = 1, \quad b = -2 \quad \text{and} \quad m = -2. \quad \text{So, } P(x) = (x+1)^2(3x-2) \quad \therefore \text{zeros are } -1, \frac{2}{3}$$

$$\text{When } a = -\frac{1}{9}, \quad b = \frac{14}{3} \quad \text{and} \quad m = \frac{14}{243}. \quad \text{So, } P(x) = (x - \frac{1}{9})^2(3x + \frac{14}{3})$$

$$\therefore \text{zeros are } \frac{1}{9}, -\frac{14}{9}$$

EXERCISE 8D.3

- 1 **a** If $P(2) = 7$, then $P(x) = (x-2)Q(x) + 7$ and $P(x)$ divided by $(x-2)$ leaves a remainder of 7.
- b** If $P(x) = (x+3)Q(x) - 8$, then $P(-3) = -8$ and $P(x)$ divided by $(x+3)$ leaves a remainder of -8 .
- c** If $P(x)$ when divided by $(x-5)$ has a remainder of 11, then $P(5) = 11$ and $P(x) = (x-5)Q(x) + 11$.

$$\begin{aligned}
 2 \quad \mathbf{a} \quad & P(x) = x^3 + 2x^2 - 7x + 5 \\
 \therefore R = P(1) \quad \{\text{Remainder thm.}\} \\
 & = 1^3 + 2(1)^2 - 7 + 5 \\
 & = 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & P(x) = x^4 - 2x^2 + 3x - 1 \\
 \therefore R = P(-2) \quad \{\text{Remainder thm.}\} \\
 & = (-2)^4 - 2(-2)^2 + 3(-2) - 1 \\
 & = 16 - 8 - 6 - 1 \\
 & = 1
 \end{aligned}$$

- 8** Suppose $P(x)$ is divided by $(x-a)(x-b)$ and has remainder $Ex + F$

hence $P(x) = Q(x) \times (x-a)(x-b) + Ex + F$

Now $P(a) = Ea + F$ (1) and $P(b) = Eb + F$ (2)

Subtracting (2) and (1), $P(b) - P(a) = Eb - Ea = E(b-a)$

$$\therefore E = \frac{P(b) - P(a)}{b-a}$$

$$\therefore \text{ from (1) } F = P(a) - Ea = P(a) - \left[\frac{P(b) - P(a)}{b-a} \right] a$$

Now $R(x) = Ex + F$

$$\therefore R(x) = \left[\frac{P(b) - P(a)}{b-a} \right] x + P(a) - \left[\frac{P(b) - P(a)}{b-a} \right] a$$

$$\therefore R(x) = \left[\frac{P(b) - P(a)}{b-a} \right] (x-a) + P(a)$$

- 9** Now suppose $P(x)$ is divided by $(x-a)^2$ and has remainder $Ex + F$

$$\therefore P(x) = Q(x) \times (x-a)^2 + Ex + F$$

Hence, $P(a) = Q(a)(0)^2 + Ea + F$

$$\therefore P(a) = Ea + F \text{ (1)}$$

Now $P'(x) = [Q'(x) \times (x-a)^2 + Q(x) \times 2(x-a)] + E$

$$\therefore P'(a) = E \text{ (2)}$$

Now substituting (2) into (1) $P(a) = P'(a) \times a + F$

$$\therefore F = P(a) - aP'(a)$$

Now $R(x) = Ex + F$

$$= xP'(a) + (P(a) - aP'(a))$$

$$= xP'(a) - aP'(a) + P(a)$$

$$\therefore R(x) = (x-a)P'(a) + P(a)$$

EXERCISE 8D.4

1 a $P(x) = 2x^3 + x^2 + kx - 4$

if $x+2$ is a factor then $P(-2) = 0$

$$\therefore -2k - 16 = 0$$

$$\therefore k = -8$$

$$\therefore P(x) = 2x^3 + x^2 - 8x - 4$$

$$= (x+2)(2x^2 - 3x - 2) \quad \{\text{as when } k = -8, k+6 = -2\}$$

$$\therefore P(x) = (x+2)(2x+1)(x-2) \quad \text{and } k = -8$$

$$\begin{array}{ccc|ccc} -2 & 2 & 1 & k & -4 & \\ & 0 & -4 & 6 & -2k-12 & \\ & 2 & -3 & k+6 & -2k-16 & \end{array}$$

b $P(x) = x^4 - 3x^3 - kx^2 + 6x$

if $x-3$ is a factor then $P(3) = 0$

$$\therefore 18 - 9k = 0$$

$$\therefore 9k = 18$$

$$\therefore k = 2$$

$$\therefore P(x) = x^4 - 3x^3 - 2x^2 + 6x$$

$$\therefore P(x) = (x-3)(x^3 - 2x) \quad \{\text{as when } k = 2, -k = -2 \text{ and } 6 - 3k = 0\}$$

$$= x(x-3)(x^2 - 2)$$

$$= x(x-3)(x+\sqrt{2})(x-\sqrt{2}) \quad \text{and } k = 2$$

$$\begin{array}{ccc|ccc} 3 & 1 & -3 & -k & 6 & 0 \\ & 0 & 3 & 0 & -3k & 18-9k \\ & 1 & 0 & -k & 6-3k & 18-9k \end{array}$$

2 $P(x) = 2x^3 + ax^2 + bx + 5$

if $x - 1$ is a factor, $P(1) = 0$

$$\therefore 2(1)^3 + a(1)^2 + b(1) + 5 = 0$$

$$2 + a + b + 5 = 0$$

$$\therefore a + b = -7 \dots\dots (1)$$

Adding (1) and (2) gives: $6a = 42$

$$\therefore a = 7 \text{ and } b = -14$$

if $x + 5$ is a factor, $P(-5) = 0$

$$2(-5)^3 + a(-5)^2 + b(-5) + 5 = 0$$

$$-250 + 25a - 5b + 5 = 0$$

$$25a - 5b = 245$$

$$\therefore 5a - b = 49 \dots\dots (2)$$

3 a $P(z) = z^3 - z^2 + [k - 5]z + [k^2 - 7]$

if 3 is a zero, $R = P(3) = 0$

$$\therefore k^2 + 3k - 4 = 0$$

$$(k + 4)(k - 1) = 0$$

$$\therefore k = -4 \text{ or } k = 1$$

if $k = 1$

$$P(z) = (z - 3)(z^2 + 2z + 2)$$

the quadratic has zeros: $\frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i$

$\therefore$ zeros are 3, $-1 \pm i$

if $k = -4$

$$P(z) = (z - 3)(z^2 + 2z - 3)$$

$$= (z - 3)(z + 3)(z - 1)$$

$\therefore$ zeros are 3, -3 and 1

b $P(z) = z^3 + mz^2 + (3m - 2)z - 10m - 4$

if $z - 2$ is a factor, $P(2) = 0$

since $* = 0$ R_1 is always 0

$\therefore z - 2$ is always a factor

now for $(z - 2)^2$ to be a factor

$$7m + 10 = 0 \quad \{R_2 \text{ is also } 0\} \quad \therefore m = -\frac{10}{7}$$

$$3 \left| \begin{array}{cccc} 1 & -1 & k-5 & k^2-7 \\ 0 & 3 & 6 & 3k+3 \end{array} \right|$$

$$1 \quad 2 \quad k+1 \quad | \quad k^2+3k-4$$

$$2 \left| \begin{array}{cccc} 1 & m & 3m-2 & -10m-4 \\ 0 & 2 & 2m+4 & 10m+4 \end{array} \right|$$

$$2 \left| \begin{array}{ccc} 1 & m+2 & 5m+2 \\ 0 & 2 & 2m+8 \end{array} \right| \quad 0 \dots (*)$$

$$1 \quad m+4 \quad | \quad 7m+10$$

4 a i $P(x) = x^3 - a^3$

$$P(a) = a^3 - a^3$$

$$= 0$$

$\therefore x - a$ is a linear factor of $P(x)$ for all a

ii $a \left| \begin{array}{cccc} 1 & 0 & 0 & -a^3 \\ 0 & a & a^2 & a^3 \end{array} \right|$

$$1 \quad a \quad a^2 \quad | \quad 0$$

$$\therefore P(x) = (x - a)(x^2 + ax + a^2)$$

b i $P(x) = x^3 + a^3$

$$P(-a) = -a^3 + a^3$$

$$= 0$$

i.e. $x + a$ is a factor of $P(x)$ for all a

ii $-a \left| \begin{array}{cccc} 1 & 0 & 0 & a^3 \\ 0 & -a & a^2 & -a^3 \end{array} \right|$

$$1 \quad -a \quad a^2 \quad | \quad 0$$

$$\therefore P(x) = (x + a)(x^2 - ax + a^2)$$

5 a Consider $P(x) = x^n + 1$

if $x + 1$ is a factor then

$$P(-1) = 0$$

$$\therefore (-1)^n + 1 = 0$$

$$\therefore (-1)^n = -1$$

which is only true if n is odd

$\therefore x + 1$ is a factor of $x^n + 1 \Leftrightarrow n$ is odd.

if n is odd $(-1)^n = -1$

$$\therefore (-1)^n + 1 = 0$$

then $P(-1) = 0$ if $P(x) = x^n + 1$

$\therefore x = -1$ is a zero of $P(x)$

$\therefore x + 1$ is a factor of $P(x)$

b $P(x) = x^3 - 3ax - 9$ and if $x - 1 - a$ is a factor then $P(1 + a) = 0$

$$1 + a \left| \begin{array}{cccc} 1 & 0 & -3a & -9 \\ 0 & 1 + a & 1 + 2a + a^2 & a^3 + 1 \end{array} \right|$$

$$1 \quad 1 + a \quad a^2 - a + 1 \quad | \quad a^3 - 8$$

$$\therefore a^3 - 8 = 0$$

$$\therefore a = 2 \quad \{\text{the only real soln.}\}$$

EXERCISE 8E.1

- 1**
- a** A single factor such as $(x - \alpha)$ indicates that the graph *cuts* the x -axis at α .
- b** A squared factor such as $(x - \alpha)^2$ indicates that the graph *touches* the x -axis at α .
- c** A cubed factor such as $(x - \alpha)^3$ indicates that the graph *cuts* the x -axis at α , and at α the graph is horizontal.
- 2**
- a** The x -intercepts are: $-1, 2$ and $3 \quad \therefore y = a(x+1)(x-2)(x-3), a \neq 0$
As the curve passes through $(0, 12), \quad 12 = a(1)(-2)(-3) \quad \therefore a = 2$
 $\therefore y = 2(x+1)(x-2)(x-3)$
- b** The x -intercepts are: $-3, -\frac{1}{2}$ and $\frac{1}{2} \quad \therefore y = a(x+3)(2x+1)(2x-1), a \neq 0$
As the curve passes through $(0, 6), \quad 6 = a(3)(1)(-1) \quad \therefore a = -2$
 $\therefore y = -2(x+3)(2x+1)(2x-1)$
- c** The x -intercepts are: $-4, -4$ and $3 \quad \therefore y = a(x+4)^2(x-3), a \neq 0$
As the curve passes through $(0, -12), \quad -12 = a(4)^2(-3) \quad \therefore a = \frac{1}{4}$
 $\therefore y = \frac{1}{4}(x+4)^2(x-3)$
- d** The x -intercepts are: $-5, -2$ and $5 \quad \therefore y = a(x+5)(x+2)(x-5), a \neq 0$
As the curve passes through $(0, -5), \quad -5 = a(5)(2)(-5) \quad \therefore a = \frac{1}{10}$
 $\therefore y = \frac{1}{10}(x+5)(x+2)(x-5)$
- e** The x -intercepts are: $-4, 3$ and $3 \quad \therefore y = a(x+4)(x-3)^2, a \neq 0$
As the curve passes through $(0, 9), \quad 9 = a(4)(-3)^2 \quad \therefore a = \frac{1}{4}$
 $\therefore y = \frac{1}{4}(x+4)(x-3)^2$
- f** The x -intercepts are: $-3, -2$ and $-\frac{1}{2} \quad \therefore y = a(x+3)(x+2)(2x+1), a \neq 0$
As the curve passes through $(0, -12), \quad -12 = a(3)(2)(1) \quad \therefore a = -2$
 $\therefore y = -2(x+3)(x+2)(2x+1)$
- 3**
- a** $P(x) = a(x-3)(x-1)(x+2)$
Since $P(x)$ passes through $(2, -4)$
 $-4 = a(-1)(1)(4)$
 $\therefore -4 = -4a$
 $\therefore a = 1$
 $\therefore P(x) = (x-3)(x-1)(x+2)$
- b** $P(x) = ax(x+2)(2x-1)$
Since $P(x)$ passes through $(-3, -21)$
 $-21 = -3a(-1)(-7)$
 $\therefore -21 = -21a$
 $\therefore a = 1$
 $\therefore P(x) = x(x+2)(2x-1)$
- c** $P(x) = a(x-1)^2(x+2)$
Since $P(x)$ passes through $(4, 54)$
 $54 = a(9)(6)$
 $\therefore a = 1$
 $\therefore P(x) = (x-1)^2(x+2)$
- d** $P(x) = a(3x+2)^2(x-4)$
Since $P(x)$ passes through $(-1, -5)$
 $-5 = a(1)(-5)$
 $\therefore a = 1$
 $\therefore P(x) = (3x+2)^2(x-4)$
- 4**
- a** $y = 2(x-1)(x+2)(x+4)$
has x -intercepts $1, -2, -4$
has y -intercept $2(-1)(2)(4) = -16$
 $\therefore$ matches graph **F**
- b** $y = -(x+1)(x-2)(x-4)$
has x -intercepts $-1, 2, 4$
has y -intercept $-(1)(-2)(-4) = -8$
 $\therefore$ matches graph **C**
- c** $y = (x-1)(x-2)(x+4)$
has x -intercepts $1, 2, -4$
has y -intercept $(-1)(-2)(4) = 8$
 $\therefore$ matches graph **A**
- d** $y = -2(x-1)(x+2)(x+4)$
has x -intercepts $1, -2, -4$
has y -intercept $-2(-1)(2)(4) = 16$
 $\therefore$ matches graph **E**

- e** $y = -(x-1)(x+2)(x+4)$ has x -intercepts 1, -2, -4
has y -intercept $-(-1)(2)(4) = 8$
 $\therefore$ matches graph **D**
- f** $y = 2(x-1)(x-2)(x+4)$ has x -intercepts 1, 2, -4
has y -intercept $2(-1)(-2)(4) = 16$
 $\therefore$ matches graph **B**

- 5 a** $\frac{1}{2}$ and -3 are zeros, and so $(2x-1)$ and $(x+3)$ are factors
 $\therefore P(x) = (2x-1)(x+3)(ax+b)$
But $P(0) = 30 \quad \therefore b(-1)(3) = 30$ and so $b = -10$
 $\therefore P(x) = (2x-1)(x+3)(ax-10)$
Now $P(1) = -20 \quad (1)(4)(a-10) = -20$
 $\therefore a-10 = -5$ and so $a = 5$
 $\therefore P(x) = (2x-1)(x+3)(5x-10)$
i.e., $P(x) = 5(x-2)(2x-1)(x+3)$

- b** 1 is a zero and so $(x-1)$ is a factor, touches at -2 indicates that $(x+2)^2$ is a factor
 $\therefore P(x) = k(x-1)(x+2)^2$
But $P(0) = 8 \quad \therefore 8 = k(-1)(2)^2$ and so $k = -2$
 $\therefore P(x) = -2(x-1)(x+2)^2$

- c** cuts at (2, 0) and so $(x-2)$ is a factor
 $\therefore P(x) = (x-2)(ax^2+bx+c)$
But $P(0) = -4 \quad \therefore -2c = -4$ and so $c = 2$ (1)
Also $P(1) = -1 \quad \therefore -1(a+b+2) = -1$
 $\therefore a+b+2 = 1 \quad \therefore a+b = -1$ (2)
Also $P(-1) = -21 \quad \therefore -3(a-b+2) = -21$
 $\therefore a-b+2 = 7 \quad \therefore a-b = 5$ (3)

Adding (2) and (3) gives $2a = 4$
 $\therefore a = 2$ and so $b = -3$
 $\therefore P(x) = (x-2)(2x^2-3x+2)$

EXERCISE 8E.2

- 1 a** $P(x) = a(x+1)^2(x-1)^2$
where $a \neq 0$, and passes through (0, 2)
 $2 = a(1)(1)$
 $\therefore a = 2$
 $\therefore P(x) = 2(x+1)^2(x-1)^2$
- b** $P(x) = a(x+3)(x+1)^2(3x-2)$
where $a \neq 0$, and passes through (0, -6)
 $-6 = a(3)(1)(-2)$
 $\therefore a = 1$
 $\therefore P(x) = (x+3)(x+1)^2(3x-2)$
- c** $P(x) = a(x+2)(x+1)(x-2)^2$
where $a \neq 0$, and passes through (0, -16)
 $-16 = a(2)(1)(4)$
 $\therefore a = -2$
 $\therefore P(x) = -2(x+2)(x+1)(x-2)^2$
- d** $P(x) = a(x+3)(x+1)(2x-3)(x-3)$
where $a \neq 0$, and passes through (0, -9)
 $-9 = a(3)(1)(-3)(-3)$
 $\therefore a = -\frac{1}{3}$
 $\therefore P(x) = -\frac{1}{3}(x+3)(x+1)(2x-3)(x-3)$
- e** $P(x) = a(x+1)(x-4)^3$
where $a \neq 0$, and passes through (0, -16)
 $-16 = a(1)(-4)^3$
 $\therefore a = \frac{1}{4}$
 $\therefore P(x) = \frac{1}{4}(x+1)(x-4)^3$
- f** $P(x) = ax^2(x+2)(x-3)$
where $a \neq 0$, and passes through (-3, 54)
 $54 = a(9)(-1)(-6)$
 $\therefore 54 = 54a$
 $\therefore a = 1$
 $\therefore P(x) = x^2(x+2)(x-3)$

- 2 a** $y = (x-1)^2(x+1)(x+3)$
has x -intercepts $-1, -3$, touches at 1
has y -intercept $(-1)^2(1)(3) = 3 (> 0)$
 $\therefore$ matches graph **C**
- c** $y = (x-1)(x+1)^2(x+3)$
has x -intercepts $1, -3$, touches at -1
has y -intercept $(-1)(1)^2(3) = -3 (< 0)$
 $\therefore$ matches graph **A**
- e** $y = -\frac{1}{3}(x-1)(x+1)(x+3)^2$
has x -intercepts $1, -1$, touches at -3
has y -intercept $-\frac{1}{3}(-1)(1)(3)^2 = 3 (> 0)$
 $\therefore$ matches graph **B**
- b** $y = -2(x-1)^2(x+1)(x+3)$
has x -intercepts $-1, -3$, touches at 1
has y -intercept $-2(-1)^2(1)(3) = -6 (< 0)$
 $\therefore$ matches graph **F**
- d** $y = (x-1)(x+1)^2(x-3)$
has x -intercepts $1, 3$, touches at -1
has y -intercept $(-1)(1)^2(-3) = 3 (> 0)$
 $\therefore$ matches graph **E**
- f** $y = -(x-1)(x+1)(x-3)^2$
has x -intercepts $1, -1$, touches at 3
has y -intercept $-(-1)(1)(3)^2 = 9 (> 0)$
 $\therefore$ matches graph **D**

- 3 a** $P(x) = a(x+4)(2x-1)(x-2)^2$
where $a \neq 0$, and passes through $(1, 5)$
 $5 = a \times 5 \times 1 \times 1$
 $\therefore a = 1$
 $\therefore P(x) = (x+4)(2x-1)(x-2)^2$
- c** $P(x) = a(2x+1)(2x-1)(x+2)(x-2)$
where $a \neq 0$, and passes through $(1, -18)$
 $-18 = a(3)(1)(3)(-1)$
 $\therefore a = 2$
 $\therefore P(x) = 2(2x+1)(2x-1)(x+2)(x-2)$
- b** $P(x) = a(3x-2)^2(x+3)^2$
where $a \neq 0$, and passes through $(-4, 49)$
 $49 = a(-14)^2(1)$
 $\therefore a = \frac{1}{4}$
 $\therefore P(x) = \frac{1}{4}(3x-2)^2(x+3)^2$
- d** $P(x) = (x-1)^2(ax^2+bx+c)$
where $a \neq 0$, and cuts y -axis at $(0, -1)$
 $-1 = 1 \times (0+0+c)$
 $\therefore c = -1$
 $\therefore P(x) = (x-1)^2(ax^2+bx-1)$

But $P(-1) = -4$

$$\therefore -4 = 4(a-b-1)$$

$$\therefore a-b = 0 \dots\dots (1)$$

Also $P(2) = 15$

$$\therefore 15 = 1(4a+2b-1)$$

$$\therefore 16 = 4a+2b$$

$$\therefore 2a+b = 8 \dots\dots (2)$$

Adding (1) and (2) we get:

$$\therefore a = \frac{8}{3} \text{ and so } b = \frac{8}{3} \text{ also}$$

$$\therefore P(x) = (x-1)^2\left(\frac{8}{3}x^2 + \frac{8}{3}x - 1\right)$$

$$\therefore P(x) = \frac{1}{3}(x-1)^2(8x^2 + 8x - 3)$$

EXERCISE 8E.3

- 1 a** $P(x) = x^3 - 3x^2 - 3x + 1$
From a calculator -1 is a zero.
Check: $P(-1) = -1 - 3 + 3 + 1 = 0$
 $\therefore x+1$ is a factor

From the division process $x^2 - 4x + 1$ is a quadratic factor

and it has zeros of $\frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$

$\therefore$ zeros are $-1, 2 \pm \sqrt{3}$

$$\begin{array}{r|rrrr} -1 & 1 & -3 & -3 & 1 \\ & 0 & -1 & 4 & -1 \\ \hline & 1 & -4 & 1 & 0 \end{array}$$

b $P(x) = x^3 - 3x^2 + 4x - 2$

From a calculator 1 is a zero.

Check: $P(1) = 1 - 3 + 4 - 2 = 0$

$\therefore x - 1$ is a factor

From the division process $x^2 - 4x + 1$ is a quadratic factor

and it has zeros of $\frac{2 \pm \sqrt{4 - 4 \times 1}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$

$\therefore$ zeros are $1, 1 \pm i$

$$1 \left| \begin{array}{cccc|c} 1 & -3 & 4 & -2 & \\ 0 & 1 & -2 & 2 & \\ \hline 1 & -2 & 2 & 0 & \end{array} \right|$$

c $P(x) = 2x^3 - 3x^2 - 4x - 35$

From a calculator $\frac{7}{2}$ is a zero.

Check: $P(\frac{7}{2}) = \frac{343}{4} - \frac{147}{4} - 14 - 35$
 $= \frac{343 - 147 - 56 - 140}{4}$
 $= 0$

From the division process $2x^2 + 4x + 10$ is a quadratic factor

$\therefore P(x) = (x - \frac{7}{2})(2x^2 + 4x + 10)$
 $= (2x - 7)(x^2 + 2x + 5)$

where the quadratic has zeros $\frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i$

$\therefore$ zeros are $\frac{7}{2}, -1 \pm 2i$

$$\frac{7}{2} \left| \begin{array}{cccc|c} 2 & -3 & -4 & -35 & \\ 0 & 7 & 14 & 35 & \\ \hline 2 & 4 & 10 & 0 & \end{array} \right|$$

d $P(x) = 2x^3 - x^2 + 20x - 10$

From a calculator $\frac{1}{2}$ is a zero.

Check: $P(\frac{1}{2}) = \frac{1}{4} - \frac{1}{4} + 10 - 10 = 0$

$\therefore P(x) = (x - \frac{1}{2})(2x^2 + 20)$
 $= (2x - 1)(x^2 + 10)$

$\therefore$ zeros are $\frac{1}{2}, \pm i\sqrt{10}$

$$\frac{1}{2} \left| \begin{array}{cccc|c} 2 & -1 & 20 & -10 & \\ 0 & 1 & 0 & 10 & \\ \hline 2 & 0 & 20 & 0 & \end{array} \right|$$

e $P(x) = 4x^4 - 4x^3 - 25x^2 + x + 6$

From a calculator -2 and 3 are zeros

Check: $P(-2) = 64 + 32 - 100 - 2 + 6 = 0$

$P(3) = 324 - 108 - 225 + 3 + 6 = 0$

$\therefore P(x) = (x + 2)(x - 3)(4x^2 - 1)$

$\therefore$ zeros are $-2, 3, \pm \frac{1}{2}$

$$\begin{array}{r} -2 \\ 3 \end{array} \left| \begin{array}{cccc|c} 4 & -4 & -25 & 1 & 6 \\ 0 & -8 & 24 & 2 & -6 \\ \hline 4 & -12 & -1 & 3 & 0 \\ 0 & 12 & 0 & -3 & \\ \hline 4 & 0 & -1 & 0 & \end{array} \right|$$

f $P(x) = x^4 - 6x^3 + 22x^2 - 48x + 40$

From a calculator 2 seems to be a double zero.

{Graph touches the x -axis at 2}

Check: $P(2) = 16 - 48 + 88 - 96 + 40 = 0$

$\therefore P(x) = (x - 2)^2(x^2 - 2x + 10)$

where the quadratic has zeros of $\frac{2 \pm \sqrt{4 - 40}}{2}$
 $= 1 \pm 3i$

$\therefore$ zeros are $2, 2, 1 \pm 3i$

$$\begin{array}{r} 2 \\ 2 \end{array} \left| \begin{array}{cccc|c} 1 & -6 & 22 & -48 & 40 \\ 0 & 2 & -8 & 28 & -40 \\ \hline 1 & -4 & 14 & -20 & 0 \\ 0 & 2 & -4 & 20 & \\ \hline 1 & -2 & 10 & 0 & \end{array} \right|$$

2 a

$$P(x) = x^3 + 2x^2 + 3x + 6$$

From a calculator -2 is a zero.

$$\text{Check: } P(-2) = -8 + 8 - 6 + 6 = 0$$

$$\therefore P(x) = (x + 2)(x^2 + 3)$$

$$\therefore P(x) = (x + 2)(x + i\sqrt{3})(x - i\sqrt{3})$$

$$\therefore \text{ roots of } P(x) = 0 \text{ are } x = -2 \text{ and } x = \pm i\sqrt{3}$$

$$-2 \left| \begin{array}{cccc} 1 & 2 & 3 & 6 \\ 0 & -2 & 0 & -6 \\ 1 & 0 & 3 & 0 \end{array} \right|$$

b

$$P(x) = 2x^3 + 3x^2 - 3x - 2$$

From a calculator 1 is a zero.

$$\text{Check: } P(1) = 2 + 3 - 3 - 2 = 0$$

$$\therefore P(x) = (x - 1)(2x^2 + 5x + 2)$$

$$= (x - 1)(2x + 1)(x + 2)$$

$$\therefore \text{ roots of } P(x) = 0 \text{ are } 1, -\frac{1}{2}, -2$$

$$1 \left| \begin{array}{cccc} 2 & 3 & -3 & -2 \\ 0 & 2 & 5 & 2 \\ 2 & 5 & 2 & 0 \end{array} \right|$$

c

$$P(x) = x^3 - 6x^2 + 12x - 8$$

From a calculator 2 is a zero.

$$\text{Check: } P(2) = 8 - 24 + 24 - 8 = 0$$

$$\therefore P(x) = (x - 2)(x^2 - 4x + 4)$$

$$= (x - 2)(x - 2)(x - 2)$$

$$\therefore \text{ only root is } x = 2, \text{ (a treble root)}$$

$$2 \left| \begin{array}{cccc} 1 & -6 & 12 & -8 \\ 0 & 2 & -8 & 8 \\ 1 & -4 & 4 & 0 \end{array} \right|$$

d

$$P(x) = 2x^3 - 5x^2 - 9x + 18$$

From a calculator 3 is a zero.

$$\text{Check: } P(3) = 54 - 45 - 27 + 18 = 0$$

$$\therefore P(x) = (x - 3)(2x^2 + x - 6)$$

$$= (x - 3)(2x - 3)(x + 2)$$

$$\therefore \text{ roots of } 2x^3 - 5x^2 - 9x + 18 = 0 \text{ are } 3, \frac{3}{2} \text{ and } -2$$

$$3 \left| \begin{array}{cccc} 2 & -5 & -9 & 18 \\ 0 & 6 & 3 & -18 \\ 2 & 1 & -6 & 0 \end{array} \right|$$

e

$$P(x) = x^4 - x^3 - 9x^2 + 11x + 6 = 0$$

From a calculator 2 and -3 are zeros.

$$\text{Check: } P(2) = 16 - 8 - 36 + 22 + 6 = 0$$

$$P(-3) = 81 + 27 - 81 - 33 + 6 = 0$$

$$\therefore P(x) = (x - 2)(x + 3)(x^2 - 2x - 1)$$

$$\text{where the quadratic has zeros of } \frac{2 \pm \sqrt{4 + 4}}{2} = 1 \pm \sqrt{2}$$

$$\therefore \text{ roots are } 2, -3, 1 \pm \sqrt{2}$$

$$2 \left| \begin{array}{ccccc} 1 & -1 & -9 & 11 & 6 \\ 0 & 2 & 2 & -14 & -6 \\ -3 & 1 & -7 & -3 & 0 \\ 0 & -3 & 6 & 3 & 0 \\ 1 & -2 & -1 & 0 & 0 \end{array} \right|$$

f

$$P(x) = 2x^4 - 13x^3 + 27x^2 - 13x - 15$$

From a calculator $-\frac{1}{2}$ and 3 are zeros.

Check:

$$P(-\frac{1}{2}) = \frac{1}{8} + \frac{13}{8} + \frac{27}{4} + \frac{13}{2} - 15 = 0$$

$$P(3) = 162 - 351 + 243 - 39 - 15 = 0$$

$$\therefore P(x) = (x + \frac{1}{2})(x - 3)(2x^2 - 8x + 10)$$

$$= (2x + 1)(x - 3)(x^2 - 4x + 5)$$

$$\text{where the quadratic has zeros of } \frac{4 \pm \sqrt{16 - 20}}{2} = 2 \pm i$$

$$\therefore x = -\frac{1}{2}, x = 3 \text{ or } x = 2 \pm i$$

$$3 \left| \begin{array}{ccccc} 2 & -13 & 27 & -13 & -15 \\ 0 & -1 & 7 & -17 & 15 \\ 2 & -14 & 34 & -30 & 0 \\ 0 & 6 & -24 & 30 & 0 \\ 2 & -8 & 10 & 0 & 0 \end{array} \right|$$

3 a Consider $P(x) = x^3 - 3x^2 + 4x - 2$

From a calculator 1 is a zero.

(Check: $P(1) = 1 - 3 + 4 - 2 = 0$)

Now $P(x) = (x - 1)(x^2 - 2x + 2)$

where the quadratic has zeros of $\frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i$

$\therefore P(x) = (x - 1)(x - [1 + i])(x - [1 - i])$

$$1 \left| \begin{array}{ccc|c} 1 & -3 & 4 & -2 \\ 0 & 1 & -2 & 2 \\ \hline 1 & -2 & 2 & 0 \end{array} \right|$$

b Consider $P(x) = x^3 + 3x^2 + 4x + 12$

From a calculator -3 is a zero.

(Check: $P(-3) = -27 + 27 - 12 + 12 = 0$)

Now $P(x) = (x + 3)(x^2 + 4)$

$\therefore P(x) = (x + 3)(x - 2i)(x + 2i)$

$$-3 \left| \begin{array}{ccc|c} 1 & 3 & 4 & 12 \\ 0 & -3 & 0 & -12 \\ \hline 1 & 0 & 4 & 0 \end{array} \right|$$

c Consider $P(x) = 2x^3 - 9x^2 + 6x - 1$

From a calculator $\frac{1}{2}$ is a zero.

(Check: $P(\frac{1}{2}) = \frac{2}{8} - \frac{9}{4} + 3 - 1 = 0$)

Now $P(x) = (x - \frac{1}{2})(2x^2 - 8x + 2)$

$= (2x - 1)(x^2 - 4x + 1)$

where the quadratic has zeros of $\frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$

$\therefore P(x) = (2x - 1)(x - [2 + \sqrt{3}])(x - [2 - \sqrt{3}])$

$$\frac{1}{2} \left| \begin{array}{ccc|c} 2 & -9 & 6 & -1 \\ 0 & 1 & -4 & 1 \\ \hline 2 & -8 & 2 & 0 \end{array} \right|$$

d $P(x) = x^3 - 4x^2 + 9x - 10$

From a calculator 2 is a zero.

(Check: $P(2) = 8 - 16 + 18 - 10 = 0$)

Now $P(x) = (x - 2)(x^2 - 2x + 5)$

where the quadratic has zeros of $\frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$

$\therefore P(x) = (x - 2)(x - [1 + 2i])(x - [1 - 2i])$

$$2 \left| \begin{array}{ccc|c} 1 & -4 & 9 & -10 \\ 0 & 2 & -4 & 10 \\ \hline 1 & -2 & 5 & 0 \end{array} \right|$$

e $P(x) = 4x^3 - 8x^2 + x + 3$

From a calculator 1 is a zero.

(Check: $P(1) = 4 - 8 + 1 + 3 = 0$)

Now $P(x) = (x - 1)(4x^2 - 4x - 3)$

$= (x - 1)(2x - 3)(2x + 1)$

$\therefore P(x) = (x - 1)(2x + 1)(2x - 3)$

$$1 \left| \begin{array}{ccc|c} 4 & -8 & 1 & 3 \\ 0 & 4 & -4 & -3 \\ \hline 4 & -4 & -3 & 0 \end{array} \right|$$

f $P(x) = 3x^4 + 4x^3 + 5x^2 + 12x - 12$

From a calculator -2 and $\frac{2}{3}$ are zeros.

(Check: $P(-2) = 48 - 32 + 20 - 24 - 12 = 0$)

$P(\frac{2}{3}) = \frac{16}{27} + \frac{32}{27} + \frac{20}{9} + 8 - 12 = 0$

Now $P(x) = (x + 2)(x - \frac{2}{3})(3x^2 + 9)$

$= (x + 2)(3x - 2)(x^2 + 3)$

i.e., $P(x) = (x + 2)(3x - 2)(x + i\sqrt{3})(x - i\sqrt{3})$

$$\begin{array}{c} -2 \\ \frac{2}{3} \end{array} \left| \begin{array}{cccc|c} 3 & 4 & 5 & 12 & -12 \\ 0 & -6 & 4 & -18 & 12 \\ \hline 3 & -2 & 9 & -6 & 0 \\ 0 & 2 & 0 & 6 & \\ \hline 3 & 0 & 9 & 0 & \end{array} \right|$$

g $P(x) = 2x^4 - 3x^3 + 5x^2 + 6x - 4$

From a calculator -1 and $\frac{1}{2}$ are zeros.)

(Check: $P(-1) = 2 + 3 + 5 - 6 - 4 = 0$

$$P\left(\frac{1}{2}\right) = \frac{1}{8} - \frac{3}{8} + \frac{5}{4} + 3 - 4 = 0)$$

Now $P(x) = (x+1)\left(x - \frac{1}{2}\right)(2x^2 - 4x + 8)$

$$= (x+1)(2x-1)(x^2 - 2x + 4)$$

where the quadratic has zeros $\frac{2 \pm \sqrt{4-16}}{2} = 1 \pm i\sqrt{3}$

$$\therefore P(x) = (x+1)(2x-1)(x - [1 + i\sqrt{3}])(x - [1 - i\sqrt{3}])$$

h $P(x) = 2x^3 + 5x^2 + 8x + 20$

From a calculator $-\frac{5}{2}$ is a zero.

(Check: $P(-\frac{5}{2}) = -\frac{125}{4} + \frac{125}{4} - 20 + 20 = 0$)

Now $P(x) = (x + \frac{5}{2})(2x^2 + 8)$

$$= (2x+5)(x^2+4)$$

$$\therefore P(x) = (2x+5)(x-2i)(x+2i)$$

$$\begin{array}{r|rrrrr} -1 & 2 & -3 & 5 & 6 & -4 \\ & 0 & -2 & 5 & -10 & 4 \\ \hline \frac{1}{2} & 2 & -5 & 10 & -4 & 0 \\ & 0 & 1 & -2 & 4 & \\ \hline & 2 & -4 & 8 & 0 & \end{array}$$

$$\begin{array}{r|rrrr} -\frac{5}{2} & 2 & 5 & 8 & 20 \\ & 0 & -5 & 0 & -20 \\ \hline & 2 & 0 & 8 & 0 \end{array}$$

4 a Using technology, $x^3 + 2x^2 - 6x - 6$ has zeros of -0.8596 , 2.1326 and -3.2731

b Using technology, $x^3 + x^2 - 7x - 8$ has zeros of -2.5182 , -1.1782 and 2.6964

EXERCISE 8F.1

1 Since it is a real polynomial, zeros must be $-\frac{1}{2}$, $1 - 3i$ and $1 + 3i$.

For $1 \pm 3i$, $\alpha + \beta = 2$ and $\alpha\beta = 1 - 9i^2 = 10$

$\therefore$ factors are $(2x+1)$ and $(x^2 - 2x + 10)$

$\therefore P(x) = k(2x+1)(x^2 - 2x + 10)$ $k \neq 0$

2 $p(1) = p(2+i) = 0$

Hence zeros of $p(x)$ must be $1, 2 \pm i$ {as $p(x)$ is real}

For $2 \pm i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - i^2 = 5$

$\therefore$ factors must be $(x-1)$ and $(x^2 - 4x + 5)$

$$\therefore p(x) = k(x-1)(x^2 - 4x + 5)$$

Since $p(0) = -20$ then $-20 = k(-1)(5)$

$$\therefore k = 4$$

$$\therefore p(x) = 4(x-1)(x^2 - 4x + 5)$$

$$\therefore p(x) = 4x^3 - 20x^2 + 36x - 20$$

$$\begin{array}{r|rrr} & 1 & -4 & 5 \\ \times & & 4 & -4 \\ \hline & -4 & 16 & -20 \\ 4 & -16 & 20 & \\ \hline 4 & -20 & 36 & -20 \end{array}$$

3 $2 - 3i$ is a zero of $z^3 + pz + q$ and as the cubic has real coefficients, $2 + 3i$ is also a zero.

For $2 \pm 3i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - 9i^2 = 13$

$\therefore x^2 - 4x + 13$ is a factor

$\therefore z^3 + pz + q = (x^2 - 4x + 13)(x + a)$ for some a .

Equating coefficients:

$$a - 4 = 0, \quad 13 - 4a = p \quad \text{and} \quad 13a = q$$

$$\therefore a = 4, \quad p = -3, \quad q = 52$$

$\therefore$ the other zeros are -4 and $2 + 3i$

$$\begin{array}{r|rrr} & 1 & -4 & 13 \\ \times & & 1 & a \\ \hline & a & -4a & 13a \\ 1 & -4 & 13 & \\ \hline 1 & a-4 & 13-4a & 13a \end{array}$$

Check: Since $P(2 - 3i) = 0$, $(2 - 3i)^3 + p(2 - 3i) + q = 0$

$$\text{Expanding, } (-46 - 9i) + p(2 - 3i) + q = 0$$

$$\therefore (-46 + 2p + q) + (-9 - 3p)i = 0$$

$$\text{Equating real and imaginary parts, } -46 + 2p + q = 0 \quad \dots(1)$$

$$\text{and } -9 - 3p = 0 \quad \dots(2)$$

$$\text{From (2), } p = -3, \text{ so in (1), } -46 - 6 + q = 0 \quad \therefore p = -3, q = 52$$

- 4** $3 + i$ is a root of $z^4 - 2z^3 + az^2 + bz + 10 = 0$ where the coefficients are real.

$\therefore 3 - i$ is also a root

$$\text{For } 3 \pm i, \alpha + \beta = 6 \text{ and } \alpha\beta = 9 - i^2 = 10$$

$\therefore z^2 - 6z + 10$ is a factor

$$\begin{aligned} \therefore z^4 - 2z^3 + az^2 + bz + 10 \\ = (z^2 - 6z + 10)(z^2 + sz + 1) \text{ for some } s. \end{aligned}$$

Equating coefficients:

$$s - 6 = -2, \quad 11 - 6s = a$$

$$\therefore s = 4$$

$$a = 11 - 6(4) = -13$$

$$\text{and } 10s - 6 = b$$

$$b = 10(4) - 6 = 34$$

$$\therefore \text{ the other factor is } z^2 + 4z + 1 \text{ which has zeros } \frac{-4 \pm \sqrt{16 - 4}}{2} = -2 \pm \sqrt{3}$$

$$\therefore a = -13 \quad b = 34 \text{ and the other roots are } 3 - i, -2 \pm \sqrt{3}.$$

- 5** Let the purely imaginary zero be bi . Since $P(z)$ is real, another zero is $-bi$, (b is real)

$\therefore z^2 + b^2$ is a factor of $P(z)$

$$\begin{aligned} \therefore z^3 + az^2 + 3z + 9 &= (z^2 + b^2)(z + c) \\ &= z^3 + cz^2 + b^2z + b^2c \end{aligned}$$

$$\text{Equating coefficients } b^2 = 3, \quad b^2c = 9 \text{ and } a = c$$

$$\therefore c = 3 \text{ and } a = 3 \text{ and } b = \pm\sqrt{3}$$

$$\therefore P(z) = (z + 3)(z^2 + 3) \text{ i.e., } P(z) = (z + 3)(z + i\sqrt{3})(z - i\sqrt{3}), a = 3$$

- 6** Let ai be the purely imaginary zero of $3x^3 + kx^2 + 15x + 10$

$\therefore$ as $P(x)$ is real, $-ai$ is also a zero

$$\text{For } \pm ai, \alpha + \beta = 0 \text{ and } \alpha\beta = -a^2i^2 = a^2$$

$\therefore x^2 + a^2$ is a factor

$$\begin{aligned} \therefore 3x^3 + kx^2 + 15x + 10 &= (x^2 + a^2)(3x + b) \\ &= 3x^3 + bx^2 + 3a^2x + a^2b \end{aligned}$$

$$\text{Equating coefficients } k = b$$

$$\text{and } 3a^2 = 15 \quad \therefore a^2 = 5$$

$$\text{and } a^2b = 10 \quad \therefore b = 2 \quad \therefore k = 2$$

$$\therefore P(x) = (x^2 + 5)(3x + 2)$$

$$\therefore P(x) = (3x + 2)(x - i\sqrt{5})(x + i\sqrt{5}), k = 2$$

	1	0	a^2
$\times$		3	b
	b	0	a^2b
	3	0	$3a^2$
	3	b	a^2b

EXERCISE 8F.2

- 1 a** $f(t) = kt(t - a)^2$

From the graph a is the t -value at the point where the graph touches the t -axis.

$$\therefore a = 700 \text{ milliseconds}$$

i.e., the time the barrier has returned to its original position

- b** when $t = 100 \text{ ms}$ $f(t) = 85 \text{ mm}$

$$\therefore 85 = k \times 100(100 - 700)^2$$

$$85 = 100 \times k \times 360\,000$$

$$k = \frac{85}{360\,000}$$

$$\therefore f(t) = \frac{85t}{360\,000}(t - 700)^2$$

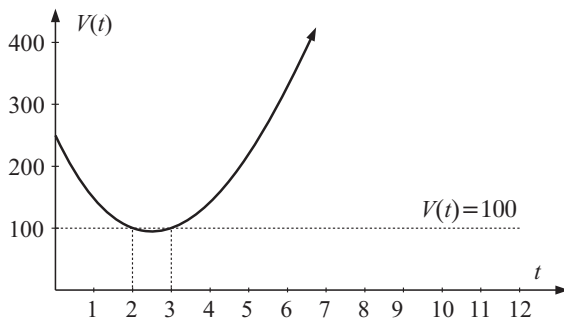
- c Using technology to find the maximum, on $0 \leq t \leq 700$
 the maximum occurs when $t = 233$ ms i.e., when $f(t) = 119.98 \div 120$ mm

- 2 $V(t) = -t^3 + 30t^2 - 131t + 250$ we
 graph $V(t)$ against t and add the
 graph of $V(t) = 100$.

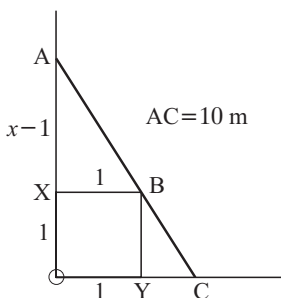
From the graph, the level drops below
 100 ML when $t = 2$ and rises above
 100 ML again when $t = 3$.

Now as $t = 0$ is Jan 1st,
 $0 \leq t < 1$ is January.

$\therefore$ as irrigation is prohibited for
 $2 < t < 3$, it is banned during March.



3



Let the height of the wall where the ladder
 touches be x m.

Using similar triangles AXB , AOC :

$$\frac{x-1}{1} = \frac{x}{OC}$$

$$\therefore OC = \frac{x}{x-1}$$

$$\text{but } x^2 + OC^2 = 10^2$$

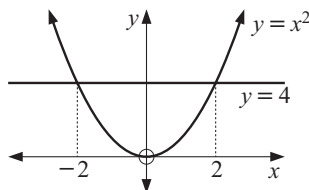
$$x^2 + \left(\frac{x}{x-1}\right)^2 = 100$$

Using technology to find the intersection of $y = x^2 + \left(\frac{x}{x-1}\right)^2$ and $y = 100$
 $x \div 1.112$ or 9.938

i.e., distance $\div 9.938$ m or 1.112 m

EXERCISE 8G.1

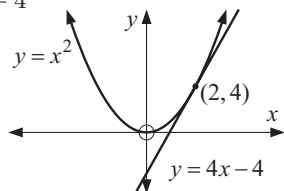
- 1 a $x^2 > 4$



So, $x^2 > 4$ when the graph of $y = x^2$
 is above the graph of $y = 4$.

This is when $x < -2$ or $x > 2$
 i.e., $x \in]-\infty, -2[$ or $x \in]2, \infty[$

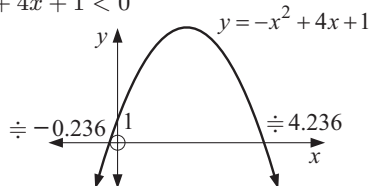
- c $x^2 \geq 4x - 4$



$x^2 \geq 4x - 4$ when $y = x^2$ is above
 or on $y = 4x - 4$.

This is true for all x in R .

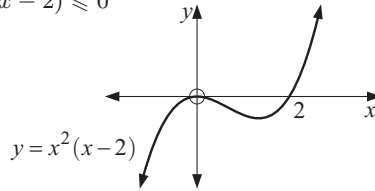
- b $-x^2 + 4x + 1 < 0$



So, if $-x^2 + 4x + 1 < 0$, the graph will
 be below the x -axis.

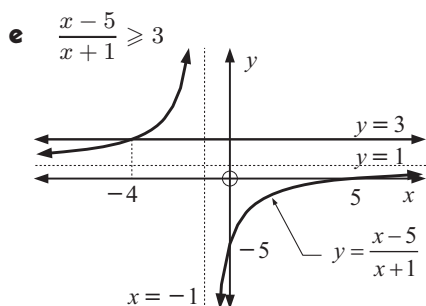
$\therefore x < -0.236$ or $x > 4.236$
 i.e., $x \in]-\infty, -0.236[$ or $x \in]4.236, \infty[$

- d $x^2(x - 2) \leq 0$



$x^2(x - 2) \leq 0$ when $y = x^2(x - 2)$
 is below or on the x -axis.

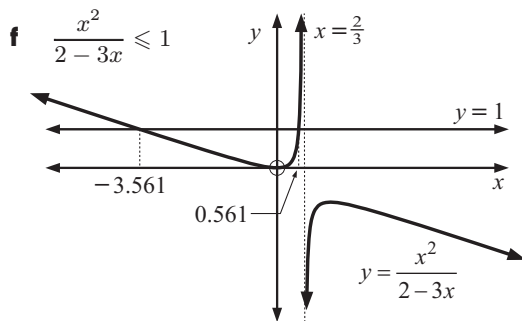
$\therefore x \leq 2$ i.e., $x \in]-\infty, 2]$



$\frac{x-5}{x+1} \geq 3$ when the graph of

$y = \frac{x-5}{x+1}$ is on or above

$y = 3$, $\therefore x \in [-4, -1[$



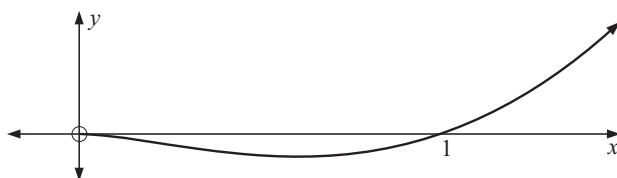
$\frac{x^2}{2-3x} \leq 1$ when the graph of $y = \frac{x^2}{2-3x}$

is on or below the graph of $y = 1$.

$\therefore x \in [-3.561, 0.561]$ or $x \in]\frac{2}{3}, \infty[$

2 f is defined when $\ln x$ is defined. This is when $x > 0$.

So, the domain is $x \in]0, \infty[$



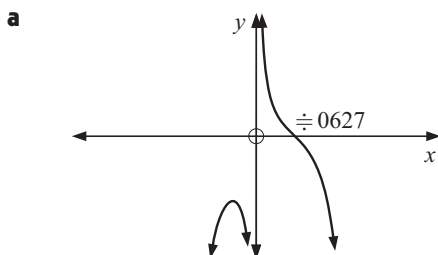
If $f(x) \leq 0$ then $x^2 \ln x \leq 0$

$\therefore$ the graph of $y = x^2 \ln x$ is on or below $y = 0$.

$\therefore 0 < x \leq 1$

i.e., $x \in]0, 1]$

3 $f(x) = \frac{2}{x} - e^{2x^2-x+1}$



b domain is $\{x : x \in \mathbb{R}, x \neq 0\}$
range is $\{y : y \in \mathbb{R}\}$

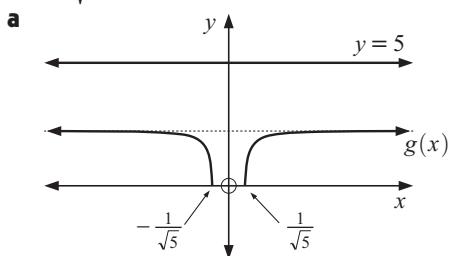
c If $e^{2x^2-x+1} > \frac{2}{x}$
then $\frac{2}{x} - e^{2x^2-x+1} < 0$

So, we want x such that $f(x) < 0$.

This is for $x < 0$ or $x > 0.627$

i.e., $x \in]-\infty, 0[$ or $x \in]0.627, \infty[$

4 $g(x) = \sqrt{5 - \frac{1}{x^2}}$



b g is finite is $x \neq 0$

and real if $5 - \frac{1}{x^2} \geq 0$

i.e., $5 \geq \frac{1}{x^2}$

$\therefore x \leq -\frac{1}{\sqrt{5}}$ or $x \geq \frac{1}{\sqrt{5}}$

So, $5 \geq \frac{1}{x^2}$ when the graph exists i.e., $x \in]-\infty, -\frac{1}{\sqrt{5}}]$ or $x \in [\frac{1}{\sqrt{5}}, \infty[$

EXERCISE 8G.2

1 a $4 - x^2$
 $= (2 + x)(2 - x)$ which
 has sign diagram:

c $x^2 - x - 12$
 $= (x + 4)(x - 3)$ which
 has sign diagram:

e $-x^2 + 4x + 1$ has zeros of

$$\frac{-4 \pm \sqrt{16 - 4(-1)(1)}}{-2}$$

$$= \frac{-4 \pm \sqrt{20}}{-2} = 2 \pm \sqrt{5}$$

g

i

k

m $2x^3 - 5x^2 + 10x$
 $= x(2x^2 - 5x + 10)$
 The quadratic has “a” > 0
 and $\Delta = 25 - 4(2)(10) = -55$
 and as $\Delta < 0$ the quadratic
 factor is always positive.

o $x^4 - 5x^2 + 4$
 $= (x^2 - 1)(x^2 - 4)$
 $= (x + 1)(x - 1)(x + 2)(x - 2)$

q $\frac{x-5}{x+1} + 3$
 $= \frac{x-5}{x+1} + 3\left(\frac{x+1}{x+1}\right)$
 $= \frac{4x-2}{x+1}$

b $3x^2 + x$
 $= x(3x + 1)$ which
 has sign diagram:

d $x^2 + 2x - 2$ has zeros of

$$\frac{-2 \pm \sqrt{4 - 4(1)(-2)}}{2} = -1 \pm \sqrt{3}$$

f $-2x^2 + x - 2$
 has $\Delta = 1^2 - 4(-2)(-2)$
 $= 1 - 16$
 $= -15$ i.e., < 0
 and “a” = -2 is < 0
 $\therefore$ is negative for all x

h

j $4x - 4 - x^2$
 $= -(x^2 - 4x + 4)$
 $= -(x - 2)^2$

l

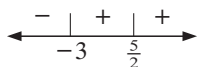
n $(3 - x)(x^2 + 2)$
 has $x^2 + 2$ always positive.

p $\frac{x^2}{x+3}$

r $\frac{x^2}{2-3x} + 1$
 $= \frac{x^2}{2-3x} + \frac{2-3x}{2-3x}$
 $= \frac{x^2 - 3x + 2}{2-3x}$
 $= \frac{(x-1)(x-2)}{2-3x}$

EXERCISE 8G.3

1 a $(2x - 5)^2(x + 3) < 0$



$$\therefore x < -3$$

$$\text{i.e., } x \in]-\infty, -3[$$

b $x^2 \geq 4x + 7$

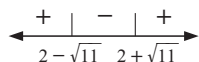
$$\therefore x^2 - 4x - 7 \geq 0$$

$$\text{But } x^2 - 4x - 7 = 0$$

$$\text{when } x = \frac{4 \pm \sqrt{16 - 4(1)(-7)}}{2}$$

$$= \frac{4 \pm \sqrt{44}}{2}$$

$$= 2 \pm \sqrt{11}$$



$$\therefore x < 2 - \sqrt{11} \text{ or } x > 2 + \sqrt{11}$$

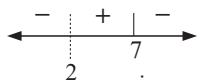
$$\text{So, } x \in]-\infty, 2 - \sqrt{11}[\text{ or }]2 + \sqrt{11}, \infty[$$

c $\frac{x+3}{x-2} > 2$

$$\therefore \frac{x+3}{x-2} - 2 > 0$$

$$\therefore \frac{x+3}{x-2} - 2 \left(\frac{x-2}{x-2} \right) > 0$$

$$\therefore \frac{-x+7}{x-2} > 0$$



$$\text{i.e., } 2 < x < 7$$

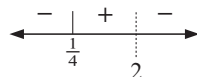
$$\text{i.e., } x \in]2, 7[$$

d $\frac{3x+1}{2-x} \leq 1$

$$\therefore \frac{3x+1}{2-x} - 1 \leq 0$$

$$\therefore \frac{3x+1}{2-x} - 1 \left(\frac{2-x}{2-x} \right) \leq 0$$

$$\therefore \frac{4x-1}{2-x} \leq 0$$



$$\therefore x \leq \frac{1}{4} \text{ or } x > 2$$

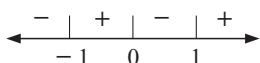
$$\text{i.e., } x \in]-\infty, \frac{1}{4}] \text{ or }]2, \infty[$$

e $x^3 \geq x$

$$\therefore x^3 - x \geq 0$$

$$\therefore x(x^2 - 1) \geq 0$$

$$\therefore x(x+1)(x-1) \geq 0$$



$$\therefore -1 \leq x \leq 0 \text{ or } x \geq 1$$

$$\therefore x \in [-1, 0] \text{ or } [1, \infty[$$

f

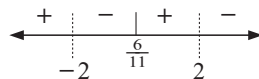
$$\frac{2x-3}{x+2} < \frac{2x}{x-2}$$

$$\therefore \frac{2x-3}{x+2} - \frac{2x}{x-2} < 0$$

$$\therefore \left(\frac{2x-3}{x+2} \right) \left(\frac{x-2}{x-2} \right) - \left(\frac{2x}{x-2} \right) \left(\frac{x+2}{x+2} \right) < 0$$

$$\therefore \frac{2x^2 - 7x + 6 - [2x^2 + 4x]}{(x+2)(x-2)} < 0$$

$$\therefore \frac{-11x + 6}{(x+2)(x-2)} < 0$$



$$\therefore -2 < x < \frac{6}{11} \text{ or } x > 2$$

$$\text{i.e., } x \in]-2, \frac{6}{11}[\text{ or }]2, \infty[$$

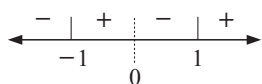
$$\mathbf{g} \quad x \geq \frac{1}{x}$$

$$\therefore x - \frac{1}{x} \geq 0$$

$$\therefore \frac{x^2}{x} - \frac{1}{x} \geq 0$$

$$\therefore \frac{x^2 - 1}{x} \geq 0$$

$$\therefore \frac{(x+1)(x-1)}{x} \geq 0$$



$$\therefore -1 \leq x < 0 \text{ or } x \geq 1$$

$$\text{i.e., } x \in [-1, 0[\text{ or } [1, \infty[$$

$$\mathbf{i} \quad \frac{x^2}{3x-2} \leq 1$$

$$\therefore \frac{x^2}{3x-2} - 1 \leq 0$$

$$\therefore \frac{x^2}{3x-2} - \left(\frac{3x-2}{3x-2} \right) \leq 0$$

$$\therefore \frac{x^2 - 3x + 2}{(3x-2)} \leq 0$$

$$\therefore \frac{(x-1)(x-2)}{3x-2} \leq 0$$

$$\mathbf{2} \quad kx^2 + 2x - (k+1) = 0$$

$$\text{has } \Delta = 4 + 4k(k+1)$$

$$= 4 + 4k^2 + 4k$$

$$= 4(k^2 + k + 1)$$

$$\Delta = 0$$

$$\text{when } k = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2}$$

but k is real, so no values for k exist.

$$\mathbf{h} \quad x^2 \leq \frac{8}{x}$$

$$\therefore x^2 - \frac{8}{x} \leq 0$$

$$\therefore \frac{x^3}{x} - \frac{8}{x} \leq 0$$

$$\therefore \frac{x^3 - 8}{x} \leq 0$$

$$\therefore \frac{(x-2)(x^2 + 2x + 4)}{x} \leq 0$$

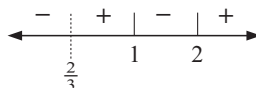
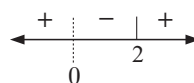
But the quadratic has “ a ” = 1 which is > 0

$$\text{and } \Delta = 2^2 - 4(1)(4) = -12$$

i.e., always positive



$$\therefore 0 < x \leq 2 \text{ i.e., } x \in]0, 2]$$



$$\therefore x < \frac{2}{3} \text{ or } 1 \leq x \leq 2$$

$$\therefore x \in]-\infty, \frac{2}{3}[\text{ or } [1, 2]$$

$$\mathbf{3} \quad e^{2x} + 2e^x \geq 6 + 3e^x$$

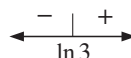
$$\therefore e^{2x} - e^x - 6 \geq 0$$

$$\therefore (e^x - 3)(e^x + 2) \geq 0$$

$$\text{Now } (e^x - 3)(e^x + 2) = 0$$

$$\text{when } e^x = 3 \text{ or } e^x = -2$$

$$\text{i.e., when } x = \ln 3 \text{ as } e^x > 0$$



$$\therefore e^{2x} + 2e^x \geq 6 + 3e^x \text{ when } x \geq \ln 3$$

EXERCISE 8G.4

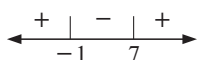
$$\mathbf{1} \quad \mathbf{a} \quad |x-3| \leq 4$$

$$\therefore (x-3)^2 \leq 4^2$$

$$\therefore (x-3)^2 - 4^2 \leq 0$$

$$\therefore (x-3+4)(x-3-4) \leq 0$$

$$\therefore (x+1)(x-7) \leq 0$$



$$\therefore -1 \leq x \leq 7$$

$$\text{i.e., } x \in [-1, 7]$$

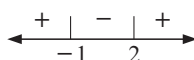
$$\mathbf{b} \quad |2x-1| \leq 3$$

$$\therefore (2x-1)^2 \leq 3^2$$

$$\therefore (2x-1)^2 - 3^2 \leq 0$$

$$\therefore (2x-1+3)(2x-1-3) \leq 0$$

$$\therefore (2x+2)(2x-4) \leq 0$$



$$\therefore -1 \leq x \leq 2$$

$$\text{i.e., } x \in [-1, 2]$$

c

$$|3x+1| > 2$$

$$\therefore (3x+1)^2 > 2^2$$

$$\therefore (3x+1)^2 - 2^2 > 0$$

$$\therefore (3x+1+2)(3x+1-2) > 0$$

$$\therefore (3x+3)(3x-1) > 0$$

$$\begin{array}{c} + \quad | \quad - \quad | \quad + \\ \leftarrow -1 \quad \frac{1}{3} \rightarrow \end{array}$$

$$\therefore x < -1 \text{ or } x > \frac{1}{3}$$

i.e., $x \in]-\infty, -1[\text{ or }]\frac{1}{3}, \infty[$

d

$$|5-2x| \geq 7$$

$$\therefore (5-2x)^2 \geq 7^2$$

$$\therefore (5-2x)^2 - 7^2 \geq 0$$

$$\therefore (5-2x+7)(5-2x-7) \geq 0$$

$$\therefore (-2x+12)(-2x-2) \geq 0$$

$$\begin{array}{c} + \quad | \quad - \quad | \quad + \\ \leftarrow -1 \quad 6 \rightarrow \end{array}$$

$$\therefore x \leq -1 \text{ or } x \geq 6$$

i.e., $x \in]-\infty, -1] \text{ or } [6, \infty[$

e

$$|x| \geq |2-x|$$

$$\therefore x^2 \geq (2-x)^2$$

$$\therefore x^2 - (2-x)^2 \geq 0$$

$$\therefore [x+(2-x)][x-(2-x)] \geq 0$$

$$\therefore 2(2x-2) \geq 0$$

$$\begin{array}{c} - \quad | \quad + \\ \leftarrow 1 \rightarrow \end{array}$$

$$\therefore x \geq 1$$

i.e., $x \in [1, \infty[$

f

$$3|x| \leq |1-2x|$$

$$\therefore 9|x|^2 \leq (1-2x)^2$$

$$\therefore 9x^2 - (1-2x)^2 \leq 0$$

$$\therefore (3x+[1-2x])(3x-[1-2x]) \leq 0$$

$$\therefore (x+1)(5x-1) \leq 0$$

$$\begin{array}{c} + \quad | \quad - \quad | \quad + \\ \leftarrow -1 \quad \frac{1}{5} \rightarrow \end{array}$$

$$\therefore -1 \leq x \leq \frac{1}{5}$$

i.e., $x \in [-1, \frac{1}{5}]$

g

$$\left| \frac{x}{x-2} \right| \geq 3$$

$$\therefore \left(\frac{x}{x-2} \right)^2 \geq 3^2$$

$$\therefore \left(\frac{x}{x-2} \right)^2 - 3^2 \geq 0$$

$$\therefore \left(\frac{x}{x-2} + 3 \right) \left(\frac{x}{x-2} - 3 \right) \geq 0$$

$$\therefore \left(\frac{x+3x-6}{x-2} \right) \left(\frac{x-3x+6}{x-2} \right) \geq 0$$

$$\therefore \frac{(4x-6)(-2x+6)}{(x-2)^2} \geq 0$$

$$\begin{array}{c} - \quad | \quad + \quad | \quad - \quad | \quad + \quad | \quad - \\ \leftarrow 1\frac{1}{2} \quad 2 \quad 3 \rightarrow \end{array}$$

$\therefore x \in [1\frac{1}{2}, 3]$, but $x \neq 2$

h

$$\left| \frac{2x+3}{x-1} \right| \geq 2$$

$$\therefore \left(\frac{2x+3}{x-1} \right)^2 \geq 2^2$$

$$\therefore \left(\frac{2x+3}{x-1} \right)^2 - 2^2 \geq 0$$

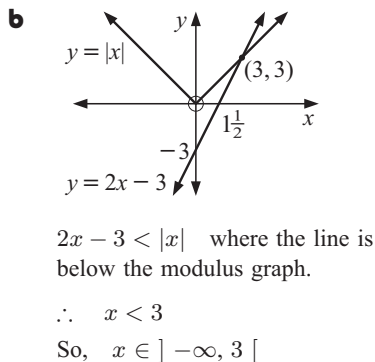
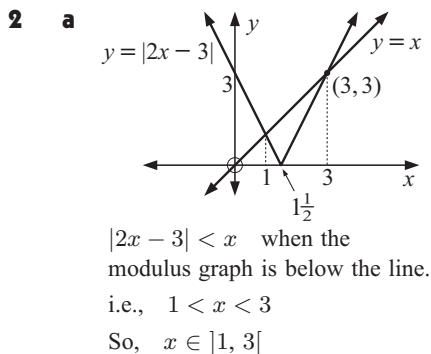
$$\therefore \left(\frac{2x+3}{x-1} + 2 \right) \left(\frac{2x+3}{x-1} - 2 \right) \geq 0$$

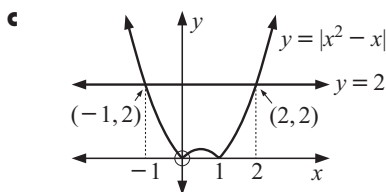
$$\therefore \left(\frac{2x+3+2x-2}{x-1} \right) \left(\frac{2x+3-2x+2}{x-1} \right) \geq 0$$

$$\therefore \frac{(4x+1)(5)}{(x-1)^2} \geq 0$$

$$\begin{array}{c} - \quad | \quad + \quad | \quad + \\ \leftarrow -\frac{1}{4} \quad 1 \rightarrow \end{array}$$

$\therefore x \in [-\frac{1}{4}, \infty[$, but $x \neq 1$

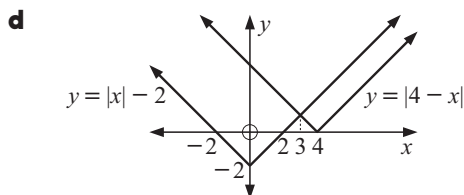




$|x^2 - x| > 2$ when the modulus graph is above the line.

$$\therefore x < -1 \text{ or } x > 2$$

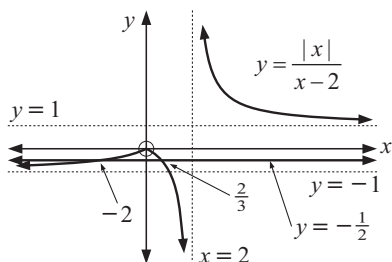
$$\text{So, } x \in]-\infty, -1[\text{ or }]2, \infty[$$



$|x| - 2 \geq |4 - x|$ when the graph of $y = |x| - 2$ is above or on the graph of $y = |4 - x|$.

$$\therefore x \geq 3 \text{ So, } x \in [3, \infty[$$

3



$$\text{If } \frac{|x|}{x-2} \geq -\frac{1}{2}$$

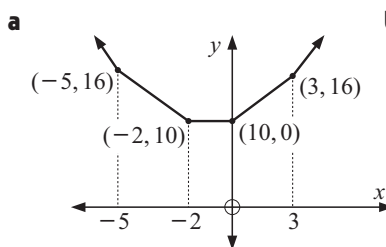
then the graph of $y = f(x)$ is above or on $y = -\frac{1}{2}$.

They intersect at -2 and $\frac{2}{3}$

$$\therefore -2 \leq x \leq \frac{2}{3} \text{ or } x > 2$$

$$\text{i.e., } x \in [-2, \frac{2}{3}] \text{ or }]2, \infty[$$

4



b i Suppose x is anywhere on AB, then

$$XP = |x - (-5)| = |x + 5|$$

$$XQ = |x - (-2)| = |x + 2|$$

$$XO = |x - 0| = |x|$$

$$XR = |x - 3|$$

So total length is

$$|x + 5| + |x + 2| + |x| + |x - 3|$$

ii The minimum length is 10 km when $-2 \leq x \leq 0$ i.e., anywhere between O and Q.

iii We need to graph $y = |x + 5| + |x + 2| + |x| + |x - 3| + |x - 7|$

From technology, the minimum cable length is 17 km when $x = 0$ i.e., at O.

REVIEW SET 8A

1 a $a + bi = 4 = 4 + 0i$, $\therefore a = 4$, $b = 0$

b $(1 - 2i)(a + bi) = -5 - 10i$

c $(a + 2i)(1 + bi) = 17 - 19i$

$$\therefore a + 2i + abi + 2i^2b = 17 - 19i$$

$$\therefore (a - 2b) + i(ab + 2) = 17 - 19i$$

Equating real and imaginary parts,

$$a - 2b = 17 \quad \text{and} \quad ab + 2 = -19$$

$$\therefore a = 2b + 17 \quad ab = -21$$

$$\therefore b(2b + 17) = -21$$

$$\therefore 2b^2 + 17b + 21 = 0$$

$$\therefore (2b + 3)(b + 7) = 0$$

$$\therefore b = -\frac{3}{2} \text{ or } b = -7$$

When $b = -7$, $a = 3$ and when $b = -\frac{3}{2}$, $a = 14$

$$\begin{aligned} \therefore a + bi &= \frac{-5 - 10i}{1 - 2i} \times \frac{1 + 2i}{1 + 2i} \\ &= \frac{-5 - 10i - 10i - 20i^2}{1 - 4i^2} \\ &= \frac{15 - 20i}{5} \\ &= 3 - 4i \\ \therefore a &= 3 \quad b = -4 \end{aligned}$$

7 Let $P(z) = z^2 + az + [3 + a]$

if $-2 + bi$ is a zero then

$$P(-2 + bi) = 0$$

$$\therefore (-2 + bi)^2 + a(-2 + bi) + 3 + a = 0$$

$$4 - 4bi + b^2i^2 - 2a + abi + 3 + a = 0$$

$$(4 - b^2 - 2a + 3 + a) + i(-4b + ab) = 0$$

$$\therefore 4 - b^2 - 2a + 3 + a = 0 \quad \text{and} \quad -4b + ab = 0$$

$$a = 7 - b^2 \quad \therefore b(a - 4) = 0$$

$$\text{If } b = 0 \text{ then } a = 7 - 0 = 7.$$

$$\therefore b = 0 \text{ or } a = 4$$

$$\text{If } a = 4 \text{ then } b^2 = 3 \text{ and so } b = \pm\sqrt{3}.$$

8 Let $P(z) = 2z^4 - 5z^3 + 13z^2 - 4z - 6$

From a calculator 1 and $-\frac{1}{2}$ are zeros.

$$(\text{Check: } P(1) = 2 - 5 + 13 - 4 - 6 = 0$$

$$P(-\frac{1}{2}) = \frac{1}{8} + \frac{5}{8} + \frac{13}{4} + 2 - 6 = 0)$$

$$\therefore P(z) = (z - 1)(z + \frac{1}{2})(2z^2 - 4z + 12) \\ = (z - 1)(2z + 1)(z^2 - 2z + 6)$$

$$\text{where the quadratic has zeros of } \frac{2 \pm \sqrt{4 - 24}}{2} = 1 \pm i\sqrt{5}. \quad \therefore \text{zeros are } 1, -\frac{1}{2}, 1 \pm i\sqrt{5}.$$

$$\begin{array}{r|rrrrr} 1 & 2 & -5 & 13 & -4 & -6 \\ & 0 & 2 & -3 & 10 & 6 \\ \hline -\frac{1}{2} & 2 & -3 & 10 & 6 & 0 \\ & 0 & -1 & 2 & -6 & \\ \hline & 2 & -4 & 12 & 0 & \end{array}$$

9 Let $P(z) = z^4 + 2z^3 - 2z^2 + 8$

From a calculator -2 seems to be a double zero.

$$(\text{Check: } P(-2) = 16 - 16 - 8 + 8 = 0$$

$$\therefore P(z) = (z + 2)^2(z^2 - 2z + 2))$$

$$\text{where the quadratic has zeros of } \frac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i$$

$$\therefore P(z) = (z + 2)^2(z - [1 + i])(z - [1 - i])$$

$$\begin{array}{r|rrrrr} -2 & 1 & 2 & -2 & 0 & 8 \\ & 0 & -2 & 0 & 4 & -8 \\ \hline -2 & 1 & 0 & -2 & 4 & 0 \\ & 0 & -2 & 4 & -4 & \\ \hline & 1 & -2 & 2 & 0 & \end{array}$$

10 $2 - i\sqrt{3}$ and $\sqrt{2} + 1$

Since the quartic has real rational coefficients, $2 - i\sqrt{3}$ and $2 + i\sqrt{3}$ are zeros
 $\sqrt{2} + 1$ and $-\sqrt{2} + 1$ are zeros

$\therefore$ the four zeros are: $2 \pm i\sqrt{3}$ and $\pm\sqrt{2} + 1$

$$\text{For } 2 \pm i\sqrt{3}, \quad \alpha + \beta = 4$$

$$\alpha\beta = 4 - 3i^2 = 7$$

$$\text{For } \pm\sqrt{2} + 1, \quad \alpha + \beta = 2$$

$$\alpha\beta = -2 + 1 = -1$$

$$\therefore P(x) = (x^2 - 4x + 7)(x^2 - 2x - 1)$$

$$\therefore P(x) = x^4 - 6x^3 + 14x^2 - 10x - 7$$

$$\begin{array}{r} \times \begin{array}{rrr} 1 & -4 & 7 \\ 1 & -2 & -1 \\ \hline -1 & 4 & -7 \\ -2 & 8 & -14 \\ 1 & -4 & 7 \\ \hline 1 & -6 & 14 & -10 & -7 \end{array} \end{array}$$

11 $f(x) = x^3 - 3x^2 - 9x + b \dots\dots (1)$

$$= (x - k)^2(x + a)$$

$$= (x^2 - 2kx + k^2)(x + a)$$

$$= x^3 + [a - 2k]x^2 + [k^2 - 2ak]x + ak^2 \dots\dots (2)$$

$$\begin{array}{r} \times \begin{array}{rrrr} 1 & -2k & k^2 & \\ a & -2ak & ak^2 & \\ \hline 1 & -2k & k^2 & \\ 1 & a - 2k & k^2 - 2ak & ak^2 \end{array} \end{array}$$

Equating coefficients of (1) and (2) gives

$$\therefore a - 2k = -3, \quad k^2 - 2ak = -9 \quad \text{and} \quad ak^2 = b$$

Since $a = 2k - 3$, then as $k^2 - 2ak = -9$

$$k^2 - 2k(2k - 3) = -9$$

$$k^2 - 4k^2 + 6k = -9$$

$$\therefore 3k^2 - 6k - 9 = 0$$

$$k^2 - 2k - 3 = 0$$

$$(k - 3)(k + 1) = 0 \quad \therefore k = -1 \text{ or } k = 3$$

If $k = -1$, $a = -5$ and $b = ak^2 = -5$ and so $f(x) = (x + 1)^2(x - 5)$

and the roots of $f(x) = 0$ are k and $-a$, i.e., $-1, 5$

If $k = 3$, $a = 3$ and $b = ak^2 = 3 \times 9 = 27$ and so $f(x) = (x - 3)^2(x + 3)$

and the roots of $f(x) = 0$ are $3, -3$

12 a

$$x^2 + 2x \geq 5$$

$$\therefore x^2 + 2x - 5 \geq 0$$

$$\therefore x^2 + 2x - 5 = 0 \text{ when}$$

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(-5)}}{2}$$

$$= \frac{-2 \pm 2\sqrt{6}}{2}$$

$$= -1 \pm \sqrt{6}$$

$$\begin{array}{c} + \quad | \quad - \quad | \quad + \\ \hline -1 - \sqrt{6} \quad -1 + \sqrt{6} \end{array}$$

$$\therefore x \leq -1 - \sqrt{6} \text{ or } x \geq -1 + \sqrt{6}$$

$$\text{i.e., } x \in] -\infty, -1 - \sqrt{6}] \text{ or}$$

$$x \in [-1 + \sqrt{6}, \infty [$$

c

$$\left| \frac{x}{8-x} \right| \leq 2$$

$$\therefore \left(\frac{x}{8-x} \right)^2 \leq 2^2$$

$$\therefore \left(\frac{x}{8-x} \right)^2 - 2^2 \leq 0$$

$$\therefore \left(\frac{x}{8-x} + 2 \right) \left(\frac{x}{8-x} - 2 \right) \leq 0$$

b

$$x < \frac{9}{x}$$

$$\therefore x - \frac{9}{x} < 0$$

$$\therefore \frac{x^2 - 9}{x} < 0$$

$$\therefore \frac{(x+3)(x-3)}{x} < 0$$

$$\begin{array}{c} - \quad | \quad + \quad | \quad - \quad | \quad + \\ \hline -3 \quad 0 \quad 3 \end{array}$$

$$\therefore x < -3 \text{ or } 0 < x < 3$$

$$\therefore x \in] -\infty, -3 [$$

$$\text{or } x \in] 0, 3 [$$

$$\therefore \frac{(16-x)(3x-16)}{(8-x)^2} \leq 0$$

$$\begin{array}{c} - \quad | \quad + \quad | \quad + \quad | \quad - \\ \hline 5\frac{1}{3} \quad 8 \quad 16 \end{array}$$

$$\therefore x \leq 5\frac{1}{3} \text{ or } x \geq 16$$

$$\text{i.e., } x \in] -\infty, 5\frac{1}{3}] \text{ or } x \in [16, \infty [$$

13 They meet where $x^2 + (2x + k)^2 + 8x - 4(2x + k) + 2 = 0$

$$\therefore x^2 + 4x^2 + 4kx + k^2 + 8x - 8x - 4k + 2 = 0$$

$$\text{i.e., } 5x^2 + 4kx + [k^2 - 4k + 2] = 0$$

$$\text{Now } \Delta = (4k)^2 - 4(5)(k^2 - 4k + 2)$$

$$= 16k^2 - 20k^2 + 80k - 40$$

$$= -4k^2 + 80k - 40$$

$$\text{and } \Delta < 0 \text{ when } -4k^2 + 80k - 40 < 0$$

$$\text{i.e., } k^2 - 20k + 10 > 0$$

$$\text{i.e., } k^2 - 20k + 10^2 + 10 - 10^2 > 0$$

$$(k - 10)^2 - 90 > 0$$

$$\therefore [k - 10 + 3\sqrt{10}][k - 10 - 3\sqrt{10}] > 0$$

$$\begin{array}{c} + \quad | \quad - \quad | \quad + \\ \hline 10 - 3\sqrt{10} \quad 10 + 3\sqrt{10} \end{array}$$

$$\therefore k \in] -\infty, 10 - 3\sqrt{10}] \text{ or } k \in [10 + 3\sqrt{10}, \infty [$$

14 $P(x) = x^n + 3x^2 + kx + 6$

$$P(-1) = 12 \quad \{\text{Remainder Theorem}\} \quad \therefore 12 = (-1)^n + 3 - k + 6$$

$$\therefore k = (-1)^n - 3 \quad \dots (1)$$

$$P(1) = 8 \quad \{\text{Remainder Theorem}\} \quad \therefore 8 = 1^n + 3 + k + 6$$

$$\therefore 8 = 10 + k$$

$$\therefore k = -2$$

$$\therefore (1) \text{ becomes } -2 = (-1)^n - 3$$

$$(-1)^n = 1 \quad \therefore n \text{ is even}$$

If $34 < n < 38$, then $n = 36$.

15 Let $x^3 - x + 1 = (x - \alpha)(x - \beta)(x - \gamma)$

$$= (x^2 - [\alpha + \beta]x + \alpha\beta)(x - \gamma)$$

$$= x^3 - [\alpha + \beta + \gamma]x^2 + [\alpha\beta + \beta\gamma + \alpha\gamma] - \alpha\beta\gamma$$

$$\alpha + \beta + \gamma = 0$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = -1$$

$$\therefore \alpha\beta\gamma = -1$$

Now $\gamma = \frac{-1}{\alpha\beta} \dots (1)$ and

$$\alpha\beta + \gamma(\alpha + \beta) = -1 \dots (2)$$

$$\therefore \alpha\beta + \gamma(-\gamma) = -1$$

$$\therefore \alpha\beta - \gamma^2 = -1$$

$$\therefore \alpha\beta - \frac{1}{(\alpha\beta)^2} = -1 \quad \{\text{using (1)}\}$$

$$\therefore (\alpha\beta)^3 - 1 = -(\alpha\beta)^2$$

$$\therefore (\alpha\beta)^3 + (\alpha\beta)^2 - 1 = 0$$

$$\therefore \alpha\beta \text{ is a root of } x^3 + x^2 - 1 = 0$$

REVIEW SET 8B

1 If $z = a + bi$

Also, $zz^* = (a + bi)(a - bi)$

then $z + z^* = (a + bi) + (a - bi)$

$$= 2a \quad \text{which is real.}$$

$$= a^2 + abi - abi - b^2i^2$$

$$= a^2 + b^2 \quad \text{which is real.}$$

2 $z = 4 + i \quad w = 3 - 2i \quad 2w^* - iz = 2(3 + 2i) - i(4 + i)$

$$= 6 + 4i - 4i - i^2$$

$$= 7$$

3 If $\frac{2 - 3i}{2a + bi} = 3 + 2i$ then $2a + bi = \frac{2 - 3i}{3 + 2i} \times \frac{3 - 2i}{3 - 2i}$

$$\therefore 2a + bi = \frac{6 - 4i - 9i + 6i^2}{9 - 4i^2} = \frac{-13i}{13}$$

$$\therefore 2a + bi = 0 - i \quad \therefore a = 0 \quad \text{and} \quad b = -1$$

4 If $a + ai$ is a root of $x^2 - 6x + b = 0$ then $(a + ai)^2 - 6(a + ai) + b = 0$

$$\therefore a^2 + 2a^2i - a^2 - 6a - 6ai + b = 0$$

$$\therefore (b - 6a) + (2a^2 - 6a)i = 0$$

$$\therefore b = 6a \quad \text{and} \quad 2a^2 - 6a = 0$$

$$\therefore b = 6a \quad \text{and} \quad 2a(a - 3) = 0$$

$$\therefore a = 0 \text{ or } 3, \quad \text{and when } a = 0, \quad b = 0$$

$$\quad \quad \quad \text{and when } a = 3, \quad b = 18$$

- 5** Let $P(x) = x^{47} - 3x^{26} + 5x^3 + 11$ $\therefore R = P(-1)$ {Remainder theorem}
 $= (-1)^{47} - 3(-1)^{26} + 5(-1)^3 + 11$
 $= -1 - 3 - 5 + 11$
 $\therefore \text{remainder} = 2$

- 6** Touches the x -axis at $(-2, 0)$ and cuts it at $(1, 0)$.

$$\therefore P(x) = (x+2)^2(x-1)(ax+b)$$

But $P(0) = 12 \therefore 4(-1)b = 12 \therefore b = -3$

$$\therefore P(x) = (x+2)^2(x-1)(ax-3)$$

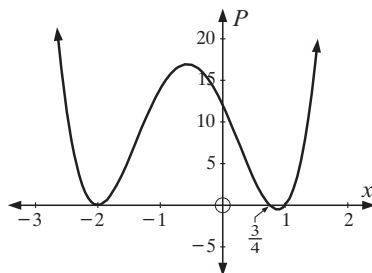
Also $P(2) = 80 \therefore 80 = 16(1)(2a-3)$

$$\therefore 2a-3 = 5$$

$$\therefore 2a = 8$$

$$\therefore a = 4$$

$$\therefore P(x) = (x+2)^2(x-1)(4x-3)$$



- 7** As $(x-3)(x+2) = x^2 - x - 6$,
 $= Q(x)[x^2 - x - 6] + (Ax + B)$, where $Q(x)$ is the quotient and $Ax + B$ is the remainder.
 Now $P(x)$ has remainder 2 when divided by $x-3$ and so $P(3) = 2$ {Rem.Theorem}
 $\therefore Q(3)[9 - 3 - 6] + (3A + B) = 2$
 $\therefore 3A + B = 2$ (1)

Also $P(x)$ has remainder -13 when divided by $x+2$ and so $P(-2) = -13$

$$\therefore Q(-2)[4 + 2 - 6] + [-2A + B] = -13$$

$$\therefore -2A + B = -13$$
 (2)

Solving (1) and (2):

$$5A = 15$$

$$\therefore A = 3 \text{ and } B = -7$$

$$\therefore R(x) = 3x - 7$$

- 8** Since the coefficients are rational, $3 + i\sqrt{2}$ and $1 + \sqrt{2}$ also have to be zeros.

For zeros of $3 \pm i\sqrt{2}$, $\alpha + \beta = 6$ and $\alpha\beta = 9 - 2i^2 = 11$

For zeros of $1 \pm \sqrt{2}$, $\alpha + \beta = 2$ and $\alpha\beta = 1 - 2 = -1$

$$\therefore P(x) = a(x^2 - 6x + 11)(x^2 - 2x - 1), \quad a \neq 0$$

$$\therefore P(x) = a(x^4 - 8x^3 + 22x^2 - 16x - 11), \quad a \neq 0$$

		1	-6	11	
	$\times$	1	-2	-1	
		-1	6	-11	
		-2	12	-22	
	1	-6	11		
	1	-8	22	-16	-11

- 9** $P(z) = 2z^3 + z^2 + 10z + 5$

From a calculator, $-\frac{1}{2}$ is a zero.

Check: $P(-\frac{1}{2}) = -\frac{1}{4} + \frac{1}{4} - 5 + 5 = 0$

$$\therefore P(z) = (z + \frac{1}{2})(2z^2 + 10)$$

$$P(z) = (2z + 1)(z^2 + 5)$$

$$P(z) = (2z + 1)(z - i\sqrt{5})(z + i\sqrt{5})$$

$$-\frac{1}{2} \left| \begin{array}{cccc} 2 & 1 & 10 & 5 \\ 0 & -1 & 0 & -5 \\ 2 & 0 & 10 & 0 \end{array} \right|$$

- 10** Zeros are $2 + i$ and $-1 + 3i$.

Since we have a real polynomial, the other zeros are $2 - i$ and $-1 - 3i$.

For zeros of $2 \pm i$, $\alpha + \beta = 4$ and $\alpha\beta = 4 - i^2 = 5$

For zeros of $-1 \pm 3i$, $\alpha + \beta = -2$ and $\alpha\beta = 1 - 9i^2 = 10$

$$\therefore P(x) = a(x^2 - 4x + 5)(x^2 + 2x + 10), \quad a \neq 0$$

REVIEW SET 8C

- 1 Since $3 - 2i$ is a zero, so is $3 + 2i$. These have $\alpha + \beta = 6$ and $\alpha\beta = 9 - 4i^2 = 13$.

$\therefore z^2 - 6z + 13$ is a factor

$$\begin{array}{r} \therefore P(z) = (z^2 - 6z + 13)(z^2 + Az + B) \\ = z^4 + kz^3 + 32z + 3k - 1 \end{array} \quad \begin{array}{r} \times \quad \begin{array}{r} 1 \quad -6 \quad 13 \\ 1 \quad A \quad B \\ \hline B \quad -6B \quad 13B \\ A \quad -6A \quad 13A \end{array} \end{array}$$

Equating coefficients gives

$$\begin{aligned} A - 6 &= k \\ B - 6A + 13 &= 0 \\ 13A - 6B &= 32 \\ 3k - 1 &= 13B \end{aligned}$$

$$-6A + B = -13 \quad \dots (1)$$

$$13A - 6B = 32 \quad \dots (2)$$

$$(1) \times 6 \quad -36A + 6B = -78 \quad \dots (3)$$

Adding (2) and (3) gives: $-23A = -46 \quad \therefore A = 2$

$$\therefore k = A - 6 = 2 - 6 = -4 \quad \text{and} \quad B = 6A - 13 = 12 - 13 = -1$$

$$\therefore P(z) = (z^2 - 6z + 13)(z^2 + 2z - 1)$$

$\uparrow$
this quadratic has zeros $\frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$

$$\therefore \text{zeros are } 3 \pm 2i, -1 \pm \sqrt{2}$$

- 2 Consider $P(z) = z^4 + 2z^3 + 6z^2 + 8z + 8$.

If one zero is purely imaginary, for example, ai , then $-ai$ is also a zero (a is real).

$$\pm ai \text{ have } \alpha + \beta = 0 \quad \text{and} \quad \alpha\beta = -a^2i^2 = a^2$$

$\therefore z^2 + a^2$ is a factor, i.e., $z^2 + A$ is a factor

$$\begin{aligned} \therefore P(z) &= (z^2 + A)(z^2 + Bz + C) \\ &= z^4 + Bz^3 + [A + C]z^2 + ABz + AC \end{aligned}$$

$$\begin{array}{r} \times \quad \begin{array}{r} 1 \quad B \quad C \\ 1 \quad 0 \quad A \\ \hline A \quad AB \quad AC \\ 1 \quad B \quad C \\ \hline 1 \quad B \quad A+C \quad AB \quad AC \end{array} \end{array}$$

Equating coefficients gives:

$$B = 2, \quad A + C = 6, \quad AB = 8 \quad \text{and} \quad AC = 8$$

$$\therefore B = 2 \quad \therefore A = 4 \quad \therefore C = 2$$

$$\therefore P(z) = (z^2 + 4)(z^2 + 2z + 2)$$

$\downarrow$
has zeros of $\frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$$\therefore \text{zeros are } \pm 2i, -1 \pm i$$

$$3 \quad \frac{P(x)}{x^2 - 3x + 2} = Q(x) + \frac{2x + 3}{x^2 - 3x + 2}$$

$$\therefore P(x) = Q(x)(x^2 - 3x + 2) + 2x + 3$$

$$\text{and } P(2) = Q(2)[4 - 6 + 2] + 4 + 3$$

$$= Q(2) \times [0] + 7$$

$$= 7, \quad \text{so the remainder is 7.}$$

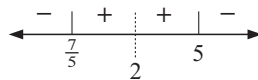
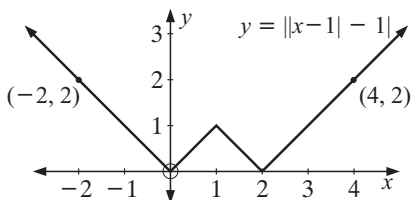
$$4 \quad \left| \frac{2x-1}{2-x} \right| \geq 3$$

$$\therefore \left(\frac{2x-1}{2-x} \right)^2 \geq 3^2$$

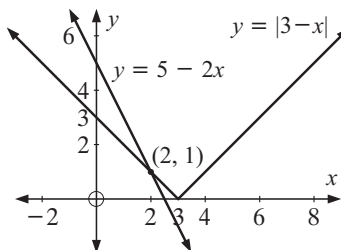
$$\therefore \left(\frac{2x-1}{2-x} \right)^2 - 3^2 \geq 0$$

$$\begin{aligned}\therefore \left(\frac{2x-1}{2-x} + 3\right) \left(\frac{2x-1}{2-x} - 3\right) &\geq 0 \\ \therefore \left(\frac{2x-1+6-3x}{2-x}\right) \left(\frac{2x-1-6+3x}{2-x}\right) &\geq 0 \\ \therefore \frac{(5-x)(5x-7)}{(2-x)^2} &\geq 0\end{aligned}$$

$$\therefore x \in \left[\frac{7}{5}, 5\right], x \neq 2$$

**5 a**

$$\begin{aligned}\mathbf{b} \quad ||x-1| - 1| &< 2 \\ \text{for } -2 &\leq x \leq 4 \\ \text{i.e., } x &\in [-2, 4]\end{aligned}$$

6 a**b**

$$\begin{aligned}|3-x| &> 5-2x \\ \text{If } x &\leq 3, \quad 3-x > 0 \\ \therefore 3-x &> 5-2x \\ \therefore x &> 2 \\ \text{If } x &> 3, \quad 3-x < 0 \\ \text{then } -(3-x) &> 5-2x \\ \therefore -3+x &> 5-2x \\ \therefore 3x &> 8 \\ \therefore \text{over all } x &> \frac{8}{3} \text{ which is true} \\ \{\text{as } |3-x| &> 5-2x \text{ when } x > 2\} \\ \therefore \text{over all } x &> 2\end{aligned}$$

7 a If $x + iy = 0$ then $x = 0$ and $y = 0$ {equating real and imaginary parts}

$$\begin{aligned}\mathbf{b} \quad (3-2i)(x+i) &= 17+yi \\ 3x+3i-2xi-2i^2 &= 17+yi \\ (3x+2)+i(3-2x) &= 17+yi \\ \therefore 3x+2 &= 17 \quad \text{and} \quad y=3-2x \\ \therefore 3x &= 15 \quad \text{and so} \quad \therefore y=3-10 \\ \therefore x &= 5 \quad \therefore y=-7\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad (x+iy)^2 &= x-iy \\ x^2+i^2y^2+2xyi &= x-iy \\ x^2-y^2 &= x \quad \text{and} \quad 2xy = -y\end{aligned}$$

$$\begin{aligned}\text{Now if } 2xy+y &= 0 \\ \text{then } y(2x+1) &= 0 \\ \therefore y &= 0 \quad \text{or} \quad x = -\frac{1}{2}\end{aligned}$$

$$\text{If } y = 0, \text{ then } x^2 = x \text{ and so } x = 0 \text{ or } 1$$

$$\begin{aligned}\text{If } x = -\frac{1}{2}, \text{ then } \frac{1}{4} - y^2 &= -\frac{1}{2}, \quad \therefore y^2 = \frac{3}{4} \\ \text{and so } y &= \pm \frac{\sqrt{3}}{2}\end{aligned}$$

Possible solutions are:

x	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$
y	0	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$

8 Let $z = a + bi$ and $w = c + di$ where $b \neq 0$ and $d \neq 0$ (1)

$$\begin{aligned}\text{Now } z + w &= (a + c) + (b + d)i \quad \text{and} \quad zw = (a + bi)(c + di) \\ &= (ac - bd) + i(bc + ad)\end{aligned}$$

As $z + w$ is real, $b + d = 0 \quad \therefore b = -d$ (2)

As zw is real, $bc + ad = 0$ (3)

Substituting (2) into (3) $-dc + ad = 0$

$$\therefore d(a - c) = 0$$

Since $d \neq 0$ {from (1)}

$$\therefore a = c \quad \text{and} \quad b = -d$$

$$\therefore z^* = a - bi = c + di = w$$

9 If $\sqrt{z} = \frac{2}{3 - 2i} + 2 + 5i$, using technology $z = \frac{-3737}{169} + \frac{4416}{169}i$.

10 Let $P(x) = 2x^{17} + 5x^{10} - 7x^3 + 6$

$$\begin{aligned}\text{Now } R &= P(2) \quad \{\text{Remainder theorem}\} \\ &= 2^{18} + 5 \times 2^{10} - 7 \times 2^3 + 6 \\ &= 267\,214\end{aligned}$$

REVIEW SET 8D

1 Let $P(z) = 2z^3 + az^2 + 62z + [a - 5]$

If $5 - i$ is a zero, then so is $5 + i$ {as $P(z)$ is a real polynomial}

and for these zeros $\alpha + \beta = 10$

$$\alpha\beta = 25 - i^2 = 26$$

$$\therefore P(z) = (z^2 - 10z + 26)(2z + b)$$

Equating coefficients gives:

$$a = b - 20, \quad 52 - 10b = 62 \quad \text{and} \quad a - 5 = 26b$$

$$\text{From } 52 - 10b = 62$$

$$-10b = 10$$

$$\therefore b = -1$$

$$\text{and as } a = b - 20 \quad \text{then } a = -21$$

$$(\text{Check: } a - 5 = -26 \quad \text{and} \quad 26b = -26 \quad \checkmark)$$

$$\begin{aligned}\therefore P(z) &= 2z^3 - 21z^2 + 62z - 26 \\ &= (z^2 - 10z + 26)(2z - 1)\end{aligned}$$

$$\therefore \text{other two zeros are } \frac{1}{2}, 5 + i \quad \text{and} \quad a = -21$$

2 a zeros are $i\sqrt{2}$ and $\frac{1}{2}$, $\therefore$ another zero must be $-i\sqrt{2}$

$$\text{Now } \pm i\sqrt{2} \text{ have } \alpha + \beta = 0 \quad \text{and} \quad \alpha\beta = -2i^2 = 2$$

$$\therefore x^2 + 2 \text{ is a factor}$$

$$\therefore P(x) = k(2x - 1)(x^2 + 2), \quad k \neq 0$$

b zeros are $1 - i$ and $-3 - i$

$$\therefore \text{other zeros must be } 1 + i \quad \text{and} \quad -3 + i$$

$$1 \pm i \text{ have } \alpha + \beta = 2 \quad \text{and} \quad -3 \pm i \text{ have } \alpha + \beta = -6$$

$$\text{and } \alpha\beta = 1 - i^2 = 2$$

$$\text{and } \alpha\beta = 9 - i^2 = 10$$

$$\therefore P(x) = k(x^2 - 2x + 2)(x^2 + 6x + 10), \quad k \neq 0$$

	1	-10	26
$\times$		2	b
	b	-10b	26b
2	-20	52	
2	b - 20	52 - 10b	26b

$$\begin{aligned}
 \mathbf{3} \quad P(x) &= 2x^3 + 7x^2 + kx - k \dots\dots (1) \\
 &= (x+a)^2(2x+b) \\
 &= (x^2 + 2ax + a^2)(2x+b) \\
 &= x^3[b+4a]x^2 + [2ab+2a^2]x + a^2b \dots\dots (2)
 \end{aligned}$$

	1	$2a$	a^2
$\times$	b	2	b
	$2ab$	a^2b	
2	$4a$	$2a^2$	
2	$b+4a$	$2ab+2a^2$	a^2b

Equating coefficients gives in (1) and (2):

$$\begin{aligned}
 b+4a &= 7, \quad 2ab+2a^2 = k \quad \text{and} \quad a^2b = -k \\
 \therefore \quad 2ab+2a^2 &= -a^2b \quad \{\text{equating } k\text{'s}\} \\
 \therefore \quad 2a(7-4a)+2a^2 &= -a^2(7-4a) \\
 \therefore \quad 14a-8a^2+2a^2+7a^2-4a^3 &= 0 \\
 \therefore \quad 4a^3-a^2-14a &= 0 \\
 \therefore \quad a(4a^2-a-14) &= 0 \\
 \therefore \quad a(4a+7)(a-2) &= 0 \\
 \therefore \quad a &= 0, -\frac{7}{4} \text{ or } 2
 \end{aligned}$$

If $a = 0$, $b = 7$ and $k = 0$.

If $a = 2$, $b = -1$ and $k = 4$.

If $a = -\frac{7}{4}$, $b = 14$ and $k = -\frac{343}{8}$

$\therefore$ largest value of k is $k = 4$ and $P(x) = (x+2)^2(2x-1)$.

$$\mathbf{4} \quad 2z^4 - 3z^3 + 2z^2 = 6z + 4$$

$$\therefore 2z^4 - 3z^3 + 2z^2 - 6z - 4 = 0$$

From a calculator we see that 2 and $-\frac{1}{2}$ are solutions

(Check: $P(2) = 32 - 24 + 8 - 12 - 4 = 0$

$$P(-\frac{1}{2}) = \frac{1}{8} + \frac{3}{8} + \frac{1}{2} + 3 - 4 = 0)$$

$$\therefore P(z) = (z-2)(z+\frac{1}{2})(2z^2+4)$$

$$= (z-2)(2z+1)(z^2+2) \quad \therefore \text{ roots are } 2, -\frac{1}{2}, \pm i\sqrt{2}$$

$$\begin{array}{r|rrrrr}
 2 & 2 & -3 & 2 & -6 & -4 \\
 & 0 & 4 & 2 & 8 & 4 \\
 \hline
 -\frac{1}{2} & 2 & 1 & 4 & 2 & 0 \\
 & 0 & -1 & 0 & -2 & \\
 \hline
 & 2 & 0 & 4 & 0 &
 \end{array}$$

$$\begin{aligned}
 \mathbf{5} \quad P(z) &= z^3 + az^2 + kz + ka \\
 P(-a) &= -a^3 + a^3 - ka + ka = 0
 \end{aligned}$$

$\therefore z+a$ is a factor

$$\therefore P(z) = (z+a)(z^2+k)$$

$$\begin{array}{r|rrrr}
 -a & 1 & a & k & ka \\
 & 0 & -a & 0 & -ka \\
 \hline
 & 1 & 0 & k & 0
 \end{array}$$

a $\therefore P(z) = 0$ has one real root if $k > 0$ $a \in R$, or $P(z) = z^3 + az^2 + kz + ka$
b and 3 real roots if $k \leq 0$ $a \in R$
 $= z^2(z+a) + k(z+a)$
 $= (z+a)(z^2+k)$, etc

$$\mathbf{6} \quad \text{Let } P(x) = 6x^3 + ax^2 - 4ax + b$$

$3x+2$ and $x-2$ are factors

$$\begin{aligned}
 \therefore 6x^3 + ax^2 - 4ax + b &= (3x+2)(x-2)(2x+c) \\
 &= (3x^2-4x-4)(2x+c) \\
 &= 6x^3 - 8x^2 - 8x + 3cx^2 - 4cx - 4c \\
 &= 6x^3 + [3c-8]x^2 + [-8-4c]x - 4c
 \end{aligned}$$

Equating coefficients gives: $a = 3c - 8$, $-4a = -8 - 4c$ and $b = -4c$

Equating a 's gives: $3c - 8 = 2 + c$

$$\therefore 2c = 10$$

$$\therefore c = 5$$

Consequently, $a = 3(5) - 8 = 7$ and $b = -20$.

7 a

$$\begin{aligned}
 |1 - 2x| &\leq 5 \\
 \therefore (1 - 2x)^2 &\leq 5^2 \\
 \therefore (1 - 2x)^2 - 5^2 &\leq 0 \\
 \therefore [(1 - 2x) + 5][1 - 2x - 5] &\leq 0 \\
 \therefore [6 - 2x][-2x - 4] &\leq 0 \\
 \begin{array}{c} + \quad | \quad - \quad | \quad + \\ \leftarrow \quad \quad \quad \quad \quad \rightarrow \\ \quad -2 \quad \quad 3 \end{array} \\
 \therefore -2 \leq x \leq 3 \\
 \text{i.e., } x \in [-2, 3]
 \end{aligned}$$

b

$$\begin{aligned}
 \frac{x^2 - 1}{x} &\geq \frac{8}{3} \\
 \therefore \frac{x^2 - 1}{x} - \frac{8}{3} &\geq 0 \\
 \therefore \frac{3(x^2 - 1) - 8x}{3x} &\geq 0 \\
 \therefore \frac{3x^2 - 8x - 3}{3x} &\geq 0 \\
 \therefore \frac{(3x + 1)(x - 3)}{3x} &\geq 0 \\
 \begin{array}{c} - \quad | \quad + \quad \cdots \quad - \quad | \quad + \\ \leftarrow \quad \quad \quad \quad \quad \rightarrow \\ \quad -\frac{1}{3} \quad \quad 0 \quad \quad 3 \end{array} \\
 \therefore -\frac{1}{3} \leq x < 0 \quad \text{or} \quad x \geq 3 \\
 \text{i.e., } x \in [-\frac{1}{3}, 0[\quad \text{or} \quad [3, \infty[
 \end{aligned}$$

8 $y = x - k$ meets $(x - 2)^2 + (y + 3)^2 = 4$ where

$$\begin{aligned}
 (x - 2)^2 + (x - k + 3)^2 &= 4 \\
 \therefore x^2 - 4x + 4 + (x - k)^2 + 6(x - k) + 9 - 4 &= 0 \\
 \therefore x^2 - 4x + 4 + x^2 - 2kx + k^2 + 6x - 6k + 5 &= 0 \\
 \therefore 2x^2 + [-4 - 2k + 6]x + [4 + k^2 - 6k + 5] &= 0 \\
 \text{i.e., } 2x^2 + [2 - 2k]x + [k^2 - 6k + 9] &= 0 \\
 \text{which is a quadratic in } x \text{ and in the tangent case } \Delta &= 0 \\
 \therefore [2 - 2k]^2 - 4(2)(k^2 - 6k + 9) &= 0 \\
 \therefore 4 - 8k + 4k^2 - 8k^2 + 48k - 72 &= 0 \\
 \therefore -4k^2 + 40k - 68 &= 0 \\
 \text{i.e., } k^2 - 10k + 17 &= 0 \\
 \therefore k &= 5 \pm 2\sqrt{2}
 \end{aligned}$$

9

$$\begin{array}{r|rrrrr}
 & & x^2 & + 3x & - 9 & \\
 x^2 + 2 & x^4 & + 3x^3 & - 7x^2 & + 11x & - 1 \\
 & x^4 & & + 2x^2 & & \\
 \hline
 & & 3x^3 & - 9x^2 & + 11x & \\
 & & 3x^3 & & + 6x & \\
 \hline
 & & & - 9x^2 & + 5x & - 1 \\
 & & & - 9x^2 & & - 18 \\
 \hline
 & & & & 5x & + 17
 \end{array}$$

$\therefore Q(x) = x^2 + 3x - 9$ $R(x) = 5x + 17$ and the new function would be divisible by $x^2 + 2$ if $x^4 + 3x^3 - 7x^2 + (2 + a)x + b = P(x) - R(x)$

$$\begin{aligned}
 \text{and as } P(x) - R(x) &= x^4 + 3x^3 - 7x^2 + 11x - 1 - (5x + 17) \\
 &= x^4 + 3x^3 - 7x^2 + 6x - 18
 \end{aligned}$$

then $2 + a = 6$ and $b = -18$ {equating coefficients}

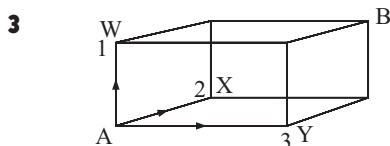
$$\therefore a = 4 \quad \text{and} \quad b = -18$$

Chapter 9

COUNTING AND BINOMIAL THEOREM

EXERCISE 9A

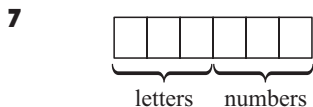
- 1** There are 3 paths from P to Q
4 paths from Q to R
2 paths from R to S
 $\therefore$ number of routes possible
= $3 \times 4 \times 2$ {product principle}
= 24



From A there are 3 possible first leg paths, to W, X or Y. Then there are 2 second leg paths to B

$\therefore$ total number = $3 \times 2 = 6$ paths.

- 5** Any of the 8 teams could be 'top'.
Any of the remaining 7 could be second.
Any of the remaining 6 could be third.
Any of the remaining 5 could be fourth
 $\therefore$ there are $8 \times 7 \times 6 \times 5$
= 1680 ways.



Repetitions are allowed.

$\therefore$ total number of ways
= $26 \times 26 \times 26 \times 10 \times 10 \times 10$
= 17 576 000

- 2 a** There are 4 choices for A. But once A is located, there is 1 choice for B, 1 for C and 1 for D
 $\therefore$ there are $4 \times 1 \times 1 \times 1 = 4$ ways.

- b** There are 4 choices for A. But once A is located there are 2 choices for B. Once B is located there is 1 choice for C and 1 for D
 $\therefore$ there are $4 \times 2 \times 1 \times 1 = 8$ ways.



- c** There are 4 choices for A. Once A is located there are 3 choices for B. Once B is located there are 2 choices for C and then 1 for D
 $\therefore$ there are $4 \times 3 \times 2 \times 1 = 24$ ways.

- 4** Any of the 7 teams could be in 'top' position. Then there are 6 left which could be in the 'second' position.
So, there are $7 \times 6 = 42$ possible ways.

- 6** There are 5 digits to choose from.

- a** Number of ways = $5 \times 5 \times 5 = 125$
b Number of ways = $5 \times 4 \times 3 = 60$

- 8 a** The 1st letter could go into either box, i.e., 2 ways and the second could go into either box, i.e., 2 ways $\therefore$ there are $2 \times 2 = 4$ ways.
These are:

Box X	Box Y
A, B	-
A	B
B	A
-	A, B

- b** There are $3 \times 3 = 9$ ways.
c There are $3 \times 3 \times 3 \times 3 = 81$ ways.

EXERCISE 9B

- 1 a** There are $2 \times 2 + 3 \times 3$
= 13 different paths
c There are $2 + 4 \times 2 + 3 \times 3$
= 19 different paths

- b** There are $4 \times 2 + 3 \times 2 \times 2$
= 20 different paths
d There are $2 \times 2 + 2 \times 2 + 2 \times 3 \times 4$
= 32 different paths

EXERCISE 9C

$$\begin{array}{ll}
 \mathbf{1} & 0! = 1 \\
 & 1! = 1 \\
 & 2! = 2 \times 1 = 2 \\
 & 3! = 3 \times 2 \times 1 = 6 \\
 & 4! = 4 \times 3 \times 2 \times 1 = 24 \\
 & 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120 \\
 & 6! = 6 \times 5! = 6 \times 120 = 720 \\
 & 7! = 7 \times 6! = 7 \times 720 = 5040 \\
 & 8! = 8 \times 7! = 8 \times 5040 = 40\,320 \\
 & 9! = 9 \times 8! = 9 \times 40\,320 = 362\,880 \\
 & 10! = 10 \times 9! = 10 \times 362\,880 = 3\,628\,800
 \end{array}$$

$$\begin{array}{lllll}
 \mathbf{2} & \mathbf{a} & \frac{6!}{5!} & \mathbf{b} & \frac{6!}{4!} & \mathbf{c} & \frac{6!}{7!} & \mathbf{d} & \frac{4!}{6!} & \mathbf{e} & \frac{100!}{99!} \\
 & & = \frac{6 \times 5!}{5!} & & = \frac{6 \times 5 \times 4!}{4!} & & = \frac{6!}{7 \times 6!} & & = \frac{4!}{6 \times 5 \times 4!} & & = \frac{100 \times 99!}{99!} \\
 & & = 6 & & = 30 & & = \frac{1}{7} & & = \frac{1}{30} & & = 100
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{f} & \frac{7!}{5! \times 2!} & \mathbf{3} & \mathbf{a} & \frac{n!}{(n-1)!} & \mathbf{b} & \frac{(n+2)!}{n!} & \mathbf{c} & \frac{(n+1)!}{(n-1)!} \\
 & = \frac{7 \times 6 \times 5!}{5! \times 2} & & = \frac{n \times (n-1)!}{(n-1)!} & & = \frac{(n+2)(n+1)n!}{n!} & & = \frac{(n+1)(n)(n-1)!}{(n-1)!} \\
 & = 21 & & = n & & = (n+2)(n+1) & & = n(n+1)
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{4} & \mathbf{a} & \frac{7 \times 6 \times 5}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \\
 & & = \frac{7!}{4!} \\
 & \mathbf{b} & \frac{10 \times 9}{10 \times 9 \times 8!} \\
 & & = \frac{10!}{8!} \\
 & \mathbf{c} & \frac{11 \times 10 \times 9 \times 8 \times 7}{11 \times 10 \times 9 \times 8 \times 7 \times 6!} \\
 & & = \frac{11!}{6!}
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{d} & \frac{13 \times 12 \times 11}{3 \times 2 \times 1} & \mathbf{e} & \frac{1}{6 \times 5 \times 4} \\
 & = \frac{13 \times 12 \times 11 \times 10!}{10! \times 3 \times 2 \times 1} & & = \frac{3!}{6 \times 5 \times 4 \times 3!} \\
 & = \frac{13!}{10!3!} & & = \frac{3!}{6!} \\
 \mathbf{f} & \frac{4 \times 3 \times 2 \times 1}{20 \times 19 \times 18 \times 17} & & = \frac{4! \times 16!}{20 \times 19 \times 18 \times 17 \times 16!} \\
 & = \frac{4! \times 16!}{20!}
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{5} & \mathbf{a} & 5! + 4! & \mathbf{b} & 11! - 10! & \mathbf{c} & 6! + 8! & \mathbf{d} & 12! - 10! \\
 & & = 5 \times 4! + 4! & & = 11 \times 10! - 10! & & = 6! + 8 \times 7 \times 6! & & = 12 \times 11 \times 10! - 10! \\
 & & = 4![5 + 1] & & = 10![11 - 1] & & = 6![1 + 8 \times 7] & & = 10![12 \times 11 - 1] \\
 & & = 6 \times 4! & & = 10 \times 10! & & = 6! \times 57 & & = 10! \times 131 \\
 & & & & & & = 57 \times 6! & & = 131 \times 10!
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{e} & 9! + 8! + 7! & \mathbf{f} & 7! - 6! + 8! & \mathbf{g} & 12! - 2 \times 11! \\
 & = 9 \times 8 \times 7! + 8 \times 7! + 7! & & = 7 \times 6! - 6! + 8 \times 7 \times 6! & & = 12 \times 11! - 2 \times 11! \\
 & = 7![72 + 8 + 1] & & = 6![7 - 1 + 56] & & = 11![12 - 2] \\
 & = 7! \times 81 & & = 6! \times 62 & & = 10 \times 11! \\
 & = 81 \times 7! & & = 62 \times 6!
 \end{array}$$

$$\begin{array}{l}
 \mathbf{h} \quad 3 \times 9! + 5 \times 8! \\
 = 3 \times 9 \times 8! + 5 \times 8! \\
 = 8![27 + 5] \\
 = 32 \times 8!
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{6} \quad \mathbf{a} & \frac{12! - 11!}{11} & \mathbf{b} & \frac{10! + 9!}{11} & \mathbf{c} & \frac{10! - 8!}{89} & \mathbf{d} & \frac{10! - 9!}{9!} \\
 & = \frac{12 \times 11! - 11!}{11} & & = \frac{10 \times 9! + 9!}{11} & & = \frac{10 \times 9 \times 8! - 8!}{89} & & = \frac{10 \times 9! - 9!}{9!} \\
 & = \frac{11![12 - 1]}{11} & & = \frac{9![10 + 1]}{11} & & = \frac{8![90 - 1]}{89} & & = \frac{9!(10 - 1)}{9!} \\
 & = \frac{11! \times 11}{11} & & = \frac{9! \times 11}{11} & & = \frac{8! \times 89}{89} & & = 9 \\
 & = 11! & & = 9! & & = 8! & &
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{e} & \frac{6! + 5! - 4!}{4!} & \mathbf{f} & \frac{n! + (n-1)!}{(n-1)!} & \mathbf{g} & \frac{n! - (n-1)!}{n-1} \\
 & = \frac{6 \times 5 \times 4! + 5 \times 4! - 4!}{4!} & & = \frac{n \times (n-1)! + (n-1)!}{(n-1)!} & & = \frac{n \times (n-1)! - (n-1)!}{n-1} \\
 & = \frac{4![30 + 5 - 1]}{4!} & & = \frac{(n-1)![n + 1]}{(n-1)!} & & = \frac{(n-1)![n - 1]}{n-1} \\
 & = 34 & & = n + 1 & & = (n-1)!
 \end{array}$$

$$\begin{aligned}
 \mathbf{h} \quad \frac{(n+2)! + (n+1)!}{n+3} &= \frac{(n+2)(n+1)! + (n+1)!}{n+3} \\
 &= \frac{(n+1)![n+2+1]}{n+3} \\
 &= (n+1)!
 \end{aligned}$$

EXERCISE 9D

- 1** **a** W, X, Y, Z
b WX, WY, WZ, XW, XY, XZ, YW, YX, YZ, ZW, ZX, ZY
c WXY, WXZ, WYX, WYZ, WZX, WZY, XWY, XWZ, XYW, XYZ, XZW, XZY, YWX, YWZ, YXW, YXZ, YZX, YZW, ZWX, ZWY, ZXW, ZXY, ZYW, ZYX
- 2** **a** AB, AC, AD, AE, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EA, EB, EC, ED
b ABC, ABD, ABE, ACB, ACD, ACE, ADB, ADC, ADE, AEB, AEC, AED, BAC, BAD, BAE, BCA, BCD, BCE, BDA, BDC, BDE, BEA, BEC, BED, CAB, CAD, CAE, CBA, CBD, CBE, CDA, CDB, CDE, CEA, CEB, CED, DAB, DAC, DAE, DBA, DBC, DBE, DCA, DCB, DCE, DEA, DEB, DEC, EAB, EAC, EAD, EBA, EBC, EBD, ECA, ECB, ECD, EDA, EDB, EDC
 2 at a time: 20 3 at a time: 60

- 3** **a** There are $5! = 120$ different orderings.
b There are $8 \times 7 \times 6 = 336$ different orderings.
c There are $10 \times 9 \times 8 \times 7 = 5040$ different signals.

- 4** **a**

4	3
---	---

 i.e., $4 \times 3 = 12$ different signals
b

4	3	2
---	---	---

 i.e., $4 \times 3 \times 2 = 24$ different signals
c $12 + 24 = 36$ different signals {using **a** and **b**}

- 5** There are 6 different letters $\therefore 6! = 720$ permutations.

a

4	3	2	1	1	1
---	---	---	---	---	---

 i.e., $4 \times 3 \times 2 \times 1 \times 1 \times 1 = 24$ permutations

 $\begin{array}{c} \uparrow \\ \text{E} \end{array}$
 $\begin{array}{c} \uparrow \\ \text{D} \end{array}$

- 2** $C_{n-r}^n = \frac{n!}{(n-r)!(n-[n-r])!} = \frac{n!}{(n-r)!r!} = C_r^n$
- 3** ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE and $C_3^5 = 10$ ✓
- 4** There are $C_{11}^{17} = 12\,376$ different teams.
- 5** There are $C_5^9 = 126$ different possible selections.
If question 1 is compulsory there are $C_1^1 C_4^8 = 1 \times 70 = 70$ possible selections.
- 6** If no restrictions, there are $C_3^{13} = 286$ different committees.
 $C_1^1 C_2^{12}$ consist of the president and two others, i.e., 66 of them.
- 7** If no restrictions, there are $C_5^{12} = 792$ different teams.
a Those containing the captain and vice-captain number $C_2^2 C_3^{10} = 1 \times 120 = 120$.
b Those containing exactly one of the captain and vice captain number $C_1^1 C_4^{10} = 2 \times 210 = 420$.
- 8** Number of different teams $= C_3^3 C_0^1 C_6^{11} = 1 \times 1 \times 462 = 462$.
- 9** **a** If 1 person is always in the selection, number of ways $= C_1^1 C_3^9 = 84$
b If 2 are always excluded, the number of ways $= C_0^2 C_4^8 = 70$
c If 1 is always 'in' and 2 are always 'out', the number of ways is $C_1^1 C_0^2 C_3^7 = 35$
- 10** **a** If there are no restrictions the number of ways $= C_5^{16} = 4368$
b The three men can be chosen in C_3^{10} ways and the 2 women in C_2^6 ways.
 $\therefore$ total number of ways $= C_3^{10} \times C_2^6 = 120 \times 15 = 1800$ ways.
c If it contains all men, the number of ways $= C_5^{10} \times C_0^6 = 252$
d If it contains at least 3 men it would contain
 3 men and 2 women or 4 men and 1 woman or 5 men and 0 women
 and this can be done in $C_3^{10} C_2^6 + C_4^{10} C_1^6 + C_5^{10} C_0^6$ ways $= 3312$ ways.
e If it contains at least one of each sex, the total number of ways
 $= C_1^{10} C_4^6 + C_2^{10} C_3^6 + C_3^{10} C_2^6 + C_4^{10} C_1^6 = 4110$ or $C_5^{16} - C_0^{10} C_5^6 - C_5^{10} C_0^6 = 4110$
- 11** **a** The 2 doctors can be chosen in C_2^6 ways
 The 1 dentist can be chosen in C_1^3 ways
 The 2 others can be chosen in C_2^7 ways
 $\therefore$ the total number of ways $= C_2^6 \times C_1^3 \times C_2^7 = 945$
b If it contains 2 doctors, 3 must be chosen from the other 10 i.e., in $C_2^6 C_3^{10} = 1800$ ways
c If it contains at least one of the two professions this can be done in
 $C_1^9 C_4^7 + C_2^9 C_3^7 + C_3^9 C_2^7 + C_4^9 C_1^7 = 4347$ or $C_5^{16} - C_0^9 C_5^7 = 4347$
- 12** There are 20 points (for vertices) to choose from and any 2 form a line.
 This can be done in C_2^{20} ways. But this count includes the 20 lines joining the vertices.
 $\therefore$ the number of diagonals $= C_2^{20} - 20 = 190 - 20 = 170$
- 13** **a** **i** $C_2^{12} = 66$ lines can be determined.
ii Of the lines in **ai** $C_1^1 C_1^{11} = 11$ pass through B.
b **i** $C_3^{12} = 220$ triangles can be determined.
ii Of the triangles in **bi** $C_1^1 C_2^{11} = 55$ have vertex B.

- 14** If the digits are in ascending order they must be from 1 to 9 i.e., 9 of them, and we want any 4. This can be done in $C_4^9 = 126$ ways.
Once they have been selected they can be put in one ascending order
 $\therefore$ total number = $126 \times 1 = 126$.

- 15 a** The different committees of 4, consisting of selections from 5 men and 6 women *in all possible ways* are
(4 men, 0 women) or (3 men, 1 woman) or (2 men, 2 women) or (1 man, 3 women) or (0 men, 4 women)
i.e., $C_4^5 C_0^6 + C_3^5 C_1^6 + C_2^5 C_2^6 + C_1^5 C_3^6 + C_0^5 C_4^6 = C_4^{11}$ ← total number unrestricted.

- b** The generalisation is:

$$C_0^m C_r^n + C_1^m C_{r-1}^n + C_2^m C_{r-2}^n + \dots + C_{r-2}^m C_2^n + C_{r-1}^m C_1^n + C_r^m C_0^n = C_r^{m+n}$$

- 16 a** Consider a simpler case of 4 people (A, B, C and D) going into two equal groups.

AB	CD	(1)	(1) and (6) are the same division.
AC	BD	(2)	(2) and (5) are the same division
AD	BC	(3)	(3) and (4) are the same division
BC	AD	(4)	
BD	AC	(5)	So, the number of ways is $\frac{1}{2}$ of C_3^6 .
CD	AB	(6)	

$$\begin{aligned} \text{So, for 2 equal groups of 6, the number of ways} &= \frac{1}{2} \text{ of } C_6^{12} C_6^6 \\ &= \frac{1}{2} \times 924 \\ &= 462 \end{aligned}$$

- b** For 3 equal groups of 4, the number of ways $= \frac{1}{3!} \times C_4^{12} \times C_4^8 \times C_4^4$
 $= 5775$

EXERCISE 9F

- 1 a** $(x+1)^3$
 $= x^3 + 3x^2(1)^1 + 3x(1)^2 + (1)^3$
 $= x^3 + 3x^2 + 3x + 1$
- c** $(x-4)^3$
 $= x^3 + 3x^2(-4)^1 + 3x(-4)^2 + (-4)^3$
 $= x^3 - 12x^2 + 48x - 64$
- e** $(2x-1)^3$
 $= (2x)^3 + 3(2x)^2(-1) + 3(2x)(-1)^2 + (-1)^3$
 $= 8x^3 - 12x^2 + 6x - 1$
- f** $(3x-1)^3$
 $= (3x)^3 + 3(3x)^2(-1) + 3(3x)(-1)^2 + (-1)^3$
 $= 27x^3 - 27x^2 + 9x - 1$
- g** $(2x+5)^3$
 $= (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3$
 $= 8x^3 + 60x^2 + 150x + 125$
- b** $(x+2)^3$
 $= x^3 + 3x^2(2)^1 + 3x(2)^2 + (2)^3$
 $= x^3 + 6x^2 + 12x + 8$
- d** $(2x+1)^3$
 $= (2x)^3 + 3(2x)^2(1) + 3(2x)(1)^2 + (1)^3$
 $= 8x^3 + 12x^2 + 6x + 1$
- h** $\left(2x + \frac{1}{x}\right)^3$
 $= (2x)^3 + 3(2x)^2\left(\frac{1}{x}\right) + 3(2x)\left(\frac{1}{x}\right)^2 + \left(\frac{1}{x}\right)^3$
 $= 8x^3 + 12x + \frac{6}{x} + \frac{1}{x^3}$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad (x+2)^4 &= x^4 + 4x^3(2)^1 + 6x^2(2)^2 + 4x(2)^3 + 2^4 \\ &= x^4 + 8x^3 + 24x^2 + 32x + 16 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (x-2)^4 &= x^4 + 4x^3(-2)^1 + 6x^2(-2)^2 + 4x(-2)^3 + (-2)^4 \\ &= x^4 - 8x^3 + 24x^2 - 32x + 16 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad (2x+3)^4 &= (2x)^4 + 4(2x)^3(3)^1 + 6(2x)^2(3)^2 + 4(2x)(3)^3 + (3)^4 \\ &= 16x^4 + 12 \times 8x^3 + 54 \times 4x^2 + 108 \times 2x + 81 \\ &= 16x^4 + 96x^3 + 216x^2 + 216x + 81 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad (3x-1)^4 &= (3x)^4 + 4(3x)^3(-1) + 6(3x)^2(-1)^2 + 4(3x)(-1)^3 + (-1)^4 \\ &= 81x^4 - 4 \times 27x^3 + 6 \times 9x^2 - 4 \times 3x + 1 \\ &= 81x^4 - 108x^3 + 54x^2 - 12x + 1 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \left(x + \frac{1}{x}\right)^4 &= x^4 + 4x^3\left(\frac{1}{x}\right) + 6x^2\left(\frac{1}{x}\right)^2 + 4x\left(\frac{1}{x}\right)^3 + \left(\frac{1}{x}\right)^4 \\ &= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \left(2x - \frac{1}{x}\right)^4 &= (2x)^4 + 4(2x)^3\left(-\frac{1}{x}\right) + 6(2x)^2\left(-\frac{1}{x}\right)^2 + 4(2x)\left(-\frac{1}{x}\right)^3 + \left(-\frac{1}{x}\right)^4 \\ &= 16x^4 - 32x^2 + 24 - \frac{8}{x^2} + \frac{1}{x^4} \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad (x+2)^5 &= x^5 + 5x^4(2) + 10x^3(2)^2 + 10x^2(2)^3 + 5x(2)^4 + (2)^5 \\ &= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad (x-2)^5 &= x^5 + 5x^4(-2) + 10x^3(-2)^2 + 10x^2(-2)^3 + 5x(-2)^4 + (-2)^5 \\ &= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad (2x+1)^5 &= (2x)^5 + 5(2x)^4(1) + 10(2x)^3(1)^2 + 10(2x)^2(1)^3 + 5(2x)(1)^4 + (1)^5 \\ &= 32x^5 + 5 \times 16x^4 + 10 \times 8x^3 + 10 \times 4x^2 + 10x + 1 \\ &= 32x^5 + 80x^4 + 80x^3 + 40x^2 + 10x + 1 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \left(2x - \frac{1}{x}\right)^5 &= (2x)^5 + 5(2x)^4\left(-\frac{1}{x}\right) + 10(2x)^3\left(-\frac{1}{x}\right)^2 + 10(2x)^2\left(-\frac{1}{x}\right)^3 + 5(2x)\left(-\frac{1}{x}\right)^4 + \left(-\frac{1}{x}\right)^5 \\ &= 32x^5 - 80x^3 + 80x - \frac{40}{x} + \frac{10}{x^3} - \frac{1}{x^5} \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad & \begin{array}{cccccc} 1 & 5 & 10 & 10 & 5 & 1 \end{array} \longleftarrow \text{the 5th row} \\ & \begin{array}{cccccc} 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{array} \longleftarrow \text{the 6th row} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \mathbf{i} \quad (x+2)^6 &= x^6 + 6x^5(2) + 15x^4(2)^2 + 20x^3(2)^3 + 15x^2(2)^4 + 6x(2)^5 + (2)^6 \\ &= x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64 \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad (2x-1)^6 &= (2x)^6 + 6(2x)^5(-1) + 15(2x)^4(-1)^2 + 20(2x)^3(-1)^3 + 15(2x)^2(-1)^4 \\ &\quad + 6(2x)(-1)^5 + (-1)^6 \\ &= 64x^6 - 6 \times 32x^5 + 15 \times 16x^4 - 20 \times 8x^3 + 15 \times 4x^2 - 6 \times 2x + 1 \\ &= 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1 \end{aligned}$$

$$\begin{aligned} \mathbf{iii} \quad \left(x + \frac{1}{x}\right)^6 &= x^6 + 6x^5\left(\frac{1}{x}\right) + 15x^4\left(\frac{1}{x}\right)^2 + 20x^3\left(\frac{1}{x}\right)^3 + 15x^2\left(\frac{1}{x}\right)^4 + 6x\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6 \\ &= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6} \end{aligned}$$

- 5 a** $(1 + \sqrt{2})^3 = (1)^3 + 3(1)^2(\sqrt{2}) + 3(1)(\sqrt{2})^2 + (\sqrt{2})^3$
 $= 1 + 3\sqrt{2} + 3 \times 2 + 2 \times \sqrt{2}$
 $= 1 + 3\sqrt{2} + 6 + 2\sqrt{2}$
 $= 7 + 5\sqrt{2}$
- b** $(1 + \sqrt{5})^4 = (1)^4 + 4(1)^3(\sqrt{5}) + 6(1)^2(\sqrt{5})^2 + 4(1)(\sqrt{5})^3 + (\sqrt{5})^4$
 $= 1 + 4\sqrt{5} + 30 + 20\sqrt{5} + 25$
 $= 56 + 24\sqrt{5}$
- c** $(2 - \sqrt{2})^5$
 $= (2)^5 + 5(2)^4(-\sqrt{2}) + 10(2)^3(-\sqrt{2})^2 + 10(2)^2(-\sqrt{2})^3 + 5(2)^1(-\sqrt{2})^4 + (-\sqrt{2})^5$
 $= 32 - 80\sqrt{2} + 160 - 80\sqrt{2} + 40 - 4\sqrt{2}$
 $= 232 - 164\sqrt{2}$
- 6 a** $(2 + x)^6 = (2)^6 + 6(2)^5x + 15(2)^4x^2 + 20(2)^3x^3 + 15(2)^2x^4 + 6(2)x^5 + x^6$
 $= 64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$
- b** $(2.01)^6$ is obtained by letting $x = 0.01$
 $\therefore (2.01)^6 = 64 + 192 \times (0.01) + 240 \times (0.01)^2 + 160 \times (0.01)^3 + 60 \times (0.01)^4$
 $+ 12 \times (0.01)^5 + (0.01)^6$
 $= 65.944\,160\,601\,201$
- 7** $(2x + 3)(x + 1)^4$
 $= (2x + 3)(x^4 + 4x^3 + 6x^2 + 4x + 1)$
 $= 2x^5 + 8x^4 + 12x^3 + 8x^2 + 2x$
 $+ 3x^4 + 12x^3 + 18x^2 + 12x + 3$
 $= 2x^5 + 11x^4 + 24x^3 + 26x^2 + 14x + 3$
- 8 a** $(3a + b)^5 = (3a)^5 + 5(3a)^4b + 10(3a)^3b^2 + \dots$
 $\therefore$ the coefficient of a^3b^2 is $10 \times 3^3 = 270$
- b** $(2a + 3b)^6 = (2a)^6 + 6(2a)^5(3b) + 15(2a)^4(3b)^2 + 20(2a)^3(3b)^3 + \dots$
 $\therefore$ the coefficient of a^3b^3 is $20 \times 2^3 \times 3^3 = 4320$

EXERCISE 9G

- 1 a** $(1 + 2x)^{11} = 1^{11} + \binom{11}{1}1^{10}(2x)^1 + \binom{11}{2}1^9(2x)^2 + \dots + \binom{11}{10}1^1(2x)^{10} + \binom{11}{11}(2x)^{11}$
 $= 1 + \binom{11}{1}(2x) + \binom{11}{2}(2x)^2 + \dots + \binom{11}{10}(2x)^{10} + \binom{11}{11}(2x)^{11}$
- b** $\left(3x + \frac{2}{x}\right)^{15}$
 $= (3x)^{15} + \binom{15}{1}(3x)^{14}\left(\frac{2}{x}\right) + \binom{15}{2}(3x)^{13}\left(\frac{2}{x}\right)^2 + \dots + \binom{15}{14}(3x)\left(\frac{2}{x}\right)^{14} + \binom{15}{15}\left(\frac{2}{x}\right)^{15}$
- c** $\left(2x - \frac{3}{x}\right)^{20}$
 $= (2x)^{20} + \binom{20}{1}(2x)^{19}\left(-\frac{3}{x}\right) + \binom{20}{2}(2x)^{18}\left(-\frac{3}{x}\right)^2 + \dots + \binom{20}{19}(2x)\left(-\frac{3}{x}\right)^{19} + \binom{20}{20}\left(-\frac{3}{x}\right)^{20}$

- 2 a** For $(2x + 5)^{15}$, $a = (2x)$, $b = 5$ and $n = 15$

Now $T_{r+1} = \binom{n}{r} a^{n-r} b^r$ and letting $r = 5$ gives $T_6 = \binom{15}{5} (2x)^{10} 5^5$.

- b** For $\left(x^2 + \frac{5}{x}\right)^9$, $a = (x^2)$, $b = \left(\frac{5}{x}\right)$ and $n = 9$

Now $T_{r+1} = \binom{n}{r} a^{n-r} b^r$ and letting $r = 3$ gives $T_4 = \binom{9}{3} (x^2)^6 \left(\frac{5}{x}\right)^3$.

- c** For $\left(x - \frac{2}{x}\right)^{17}$, $a = x$, $b = \left(-\frac{2}{x}\right)$ and $n = 17$

Now $T_{r+1} = \binom{n}{r} a^{n-r} b^r$ and letting $r = 9$ gives $T_{10} = \binom{17}{9} x^8 \left(-\frac{2}{x}\right)^9$.

- d** For $\left(2x^2 - \frac{1}{x}\right)^{21}$, $a = (2x^2)$, $b = \left(-\frac{1}{x}\right)$ and $n = 21$

Now $T_{r+1} = \binom{n}{r} a^{n-r} b^r$ and letting $r = 8$ gives $T_9 = \binom{21}{8} (2x^2)^{13} \left(-\frac{1}{x}\right)^8$.

- 3 a** In $(3 + 2x^2)^{10}$, $a = 3$, $b = (2x^2)$ and $n = 10$

$$\begin{aligned} \text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{10}{r} 3^{10-r} (2x^2)^r \\ &= \binom{10}{r} 3^{10-r} 2^r x^{2r} \end{aligned}$$

We now let $2r = 10$

$$\therefore r = 5$$

So, $T_6 = \binom{10}{5} 3^5 2^5 x^{10}$

$\therefore$ the coefficient is $\binom{10}{5} 3^5 2^5$.

- b** In $\left(2x^2 - \frac{3}{x}\right)^6$, $a = (2x^2)$, $b = \left(-\frac{3}{x}\right)$ and $n = 6$

$$\begin{aligned} \text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{6}{r} (2x^2)^{6-r} \left(-\frac{3}{x}\right)^r \\ &= \binom{6}{r} 2^{6-r} x^{12-2r} \frac{(-3)^r}{x^r} \\ &= \binom{6}{r} 2^{6-r} (-3)^r x^{12-3r} \end{aligned}$$

We now let $12 - 3r = 3$

$$\therefore 3r = 9$$

$$\therefore r = 3$$

So, $T_4 = \binom{6}{3} 2^3 (-3)^3 x^3$

$\therefore$ the coefficient is $\binom{6}{3} 2^3 (-3)^3$.

- c** In $\left(2x^2 - \frac{1}{x}\right)^{12}$, $a = (2x^2)$, $b = \left(-\frac{1}{x}\right)$ and $n = 12$

$$\begin{aligned} \text{Now } T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{12}{r} (2x^2)^{12-r} \left(-\frac{1}{x}\right)^r \\ &= \binom{12}{r} 2^{12-r} x^{24-2r} \frac{(-1)^r}{x^r} \\ &= \binom{12}{r} 2^{12-r} (-1)^r x^{24-3r} \end{aligned}$$

We now let $24 - 3r = 12$

$$\therefore 3r = 12$$

$$\therefore r = 4$$

So, $T_5 = \binom{12}{4} 2^8 (-1)^4 x^{12}$

$\therefore$ the coefficient is $\binom{12}{4} 2^8$.

- 4 a** For $\left(x + \frac{2}{x^2}\right)^{15}$, $a = x$, $b = \frac{2}{x^2}$ and $n = 15$

Now $T_{r+1} = \binom{n}{r} a^{n-r} b^r = \binom{15}{r} x^{15-r} \left(\frac{2}{x^2}\right)^r = \binom{15}{r} x^{15-r} \frac{2^r}{x^{2r}} = \binom{15}{r} 2^r x^{15-3r}$

The constant term does not contain x . $\therefore 15 - 3r = 0 \therefore r = 5$

$\therefore$ the constant term is $\binom{15}{5} 2^5$.

b For $\left(x - \frac{3}{x^2}\right)^9$, $a = x$, $b = \left(-\frac{3}{x^2}\right)$ and $n = 9$

Now $T_{r+1} = \binom{n}{r} a^{n-r} b^r$ The constant term does not contain x .

$$= \binom{9}{r} x^{9-r} \left(-\frac{3}{x^2}\right)^r \quad \therefore 9 - 3r = 0$$

$$= \binom{9}{r} x^{9-r} \frac{(-3)^r}{x^{2r}} \quad \therefore r = 3$$

$$= \binom{9}{r} (-3)^r x^{9-3r}$$

and $T_4 = \binom{9}{3} (-3)^3 x^0$

$\therefore$ the constant term is $\binom{9}{3} (-3)^3$.

5 a, b

Row 1	1 1	←	sum = 1 + 1 = 2	= 2 ¹
Row 2	1 2 1	←	sum = 1 + 2 + 1 = 4	= 2 ²
Row 3	1 3 3 1	←	sum = 1 + 3 + 3 + 1 = 8	= 2 ³
Row 4	1 4 6 4 1	←	sum = 1 + 4 + 6 + 4 + 1 = 16	= 2 ⁴
Row 5	1 5 10 10 5 1	←	sum = 1 + 5 + 10 + 10 + 5 + 1 = 32	= 2 ⁵

c It seems that the sum of the numbers in row n of Pascal's triangle is 2^n .

d $(1+x)^n = \binom{n}{0} 1^n + \binom{n}{1} 1^{n-1} x + \binom{n}{2} 1^{n-2} x^2 + \binom{n}{3} 1^{n-3} x^3 + \dots + \binom{n}{n-1} 1^1 x^{n-1} + \binom{n}{n} x^n$

$$= \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \dots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n$$

{as all powers of 1 are 1}

Now letting $x = 1$ gives LHS = $(1+1)^n = 2^n$

and RHS = $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n-1} + \binom{n}{n}$

$\therefore \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n} = 2^n$

6 a $(x+2)(x^2+1)^8$

$$= (x+2) \left[(x^2)^8 + \binom{8}{1} (x^2)^7 1 + \binom{8}{2} (x^2)^6 1^2 + \dots + \binom{8}{6} (x^2)^2 1^6 + \binom{8}{7} (x^2)^1 1^7 + \binom{8}{8} 1^8 \right]$$

↑
only terms which when multiplied give an x^5

$\therefore$ coefficient of x^5 is $1 \times \binom{8}{6} = \binom{8}{6} = 28$.

b $(2-x)(3x+1)^9$

$$= (2-x) \left[(3x)^9 + \binom{9}{1} (3x)^8 + \binom{9}{2} (3x)^7 + \binom{9}{3} (3x)^6 + \binom{9}{4} (3x)^5 + \dots \right]$$

↑
↑
↑

$\therefore$ coefficient of x^6 is $2 \times \binom{9}{3} \times 3^6 + (-1) \times \binom{9}{4} \times 3^5 = 2 \binom{9}{3} 3^6 - \binom{9}{4} 3^5 = 91\,854$

7 a $\binom{n}{1} = C_1^n = \frac{n}{1} = n$ and $\binom{n}{2} = C_2^n = \frac{n(n-1)}{2 \times 1} = \frac{n(n-1)}{2}$

b $(1+x)^n$ has $T_3 = \binom{n}{2} 1^{n-2} x^2 = \binom{n}{2} x^2$ and $n \geq 2$

But this term is $36x^2 \quad \therefore \binom{n}{2} = 36$

$\therefore \frac{n(n-1)}{2} = 36$

$\therefore n(n-1) = 72$

$\therefore n^2 - n - 72 = 0$

$\therefore (n-9)(n+8) = 0$

$\therefore n = 9$ or -8

But $n \geq 2 \quad \therefore n = 9$

and $T_4 = \binom{n}{3} 1^{n-3} x^3$

$$= \binom{9}{3} x^3$$

$$= 84x^3$$

$$\begin{aligned} \text{c } (1+kx)^n &= 1^n + \binom{n}{1}1^{n-1}(kx)^1 + \binom{n}{2}1^{n-2}(kx)^2 + \dots \\ &= 1 + \binom{n}{1}kx + \binom{n}{2}k^2x^2 + \dots \end{aligned}$$

$$\therefore \binom{n}{1}k = -12 \quad \text{and} \quad \binom{n}{2}k^2 = 60$$

$$\therefore nk = -12 \quad \text{and} \quad \frac{n(n-1)}{2}k^2 = 60$$

$$\therefore n(n-1)k^2 = 120$$

$$\text{But } k = -\frac{12}{n} \quad \therefore n(n-1)\frac{144}{n^2} = 120$$

$$\therefore 144(n-1) = 120n \quad \{n \geq 2\}$$

$$\therefore 144n - 120n = 144$$

$$\therefore 24n = 144$$

$$\therefore n = 6$$

$$\text{and so, as } k = -\frac{12}{n}, \quad k = -2$$

$$8 \quad T_{r+1} = \binom{n}{r}a^{n-r}b^r \quad \text{where } n=10, \quad a=(x^2), \quad b=\left(\frac{1}{ax}\right)$$

$$= \binom{10}{r}(x^2)^{10-r}\left(\frac{1}{ax}\right)^r$$

$$= \binom{10}{r}x^{20-2r} \times \frac{1}{a^r x^r}$$

$$= \binom{10}{r}x^{20-3r} \times \frac{1}{a^r}$$

$$\text{We let } 20-3r=11 \quad \therefore 3r=9 \quad \therefore r=3$$

$$\text{and } T_4 = \binom{10}{3}x^{11} \times \frac{1}{a^3} = \frac{\binom{10}{3}}{a^3}x^{11}$$

$$\text{Thus } \frac{\binom{10}{3}}{a^3} = 15 \quad \text{i.e.,} \quad \frac{120}{a^3} = 15$$

$$\therefore a^3 = 8$$

$$\therefore a = 2$$

REVIEW SET 9A

$$1 \quad \text{a } \begin{array}{|c|c|c|c|c|c|} \hline 26 & 26 & 10 & 10 & 10 & 10 \\ \hline \end{array} \quad \text{i.e., } 26^2 \times 10^4 = 6\,760\,000 \quad \text{if no restrictions}$$

$$\text{b } \begin{array}{|c|c|c|c|c|c|} \hline 5 & 26 & 10 & 10 & 10 & 10 \\ \hline \end{array} \quad \text{i.e., } 5 \times 26 \times 10^4 = 1\,300\,000$$

↑
a vowel

if the first letter is a vowel.

$$\text{c } \begin{array}{|c|c|c|c|c|c|} \hline 26 & 25 & 10 & 9 & 8 & 7 \\ \hline \end{array} \quad \text{i.e., } 26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3\,276\,000, \quad \text{if no repetitions}$$

2



a To form a line we need to select any two points from the 10.

$$\therefore \text{ total is } C_2^{10} = 45 \quad \text{lines.}$$

b To form a triangle we need to select any three points from the 10.

$$\therefore \text{ total is } C_3^{10} = 120 \quad \text{triangles.}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad \frac{n!}{(n-1)!} &= \frac{n(n-1)(n-2)!}{(n-2)!} & \mathbf{b} \quad \frac{n! + (n+1)!}{n!} &= \frac{n! + (n+1)n!}{n!} \\
 &= n(n-1) & &= \frac{n![1 + n + 1]}{n!} \\
 & & &= n + 2
 \end{aligned}$$

$$\mathbf{4} \quad \text{Total number, unrestricted} = C_5^{8+7} = C_5^{15} = 3003$$

$$\mathbf{a} \quad \text{Total with 2 men and 3 women} = C_2^8 C_3^7 = 980$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Total with at least one man} &= \text{total unrestricted} - \text{total with all women} \\
 &= 3003 - C_0^8 C_5^7 \\
 &= 2982
 \end{aligned}$$

$$\mathbf{5} \quad \text{There are } C_2^8 = 28 \text{ handshakes possible.}$$

$$\mathbf{6} \quad \mathbf{a} \quad \text{With no restrictions there are } C_5^{10} = 252 \text{ different committees.}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Those consisting of at least one of each sex} \\
 &= C_5^{10} - C_5^6 C_0^4 \quad \{\text{there are no committees consisting of 5 women}\} \\
 &= 246
 \end{aligned}$$

$$\mathbf{7} \quad \mathbf{a} \quad \begin{array}{|c|c|c|c|c|} \hline 4 & 3 & 2 & 1 & 1 \\ \hline \end{array} \quad \text{i.e., } 4 \times 3 \times 2 \times 1 \times 1 = 24 \text{ end in T}$$

$$\mathbf{b} \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 2 & 1 & 1 \\ \hline \end{array} \quad \text{i.e., } 1 \times 3 \times 2 \times 1 \times 1 = 6 \text{ begin with P and end with T}$$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad (x - 2y)^3 &= x^3 + 3x^2(-2y) + 3x(-2y)^2 + (-2y)^3 \\
 &= x^3 - 6x^2y + 12xy^2 - 8y^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad (3x + 2)^4 &= (3x)^4 + 4(3x)^3(2) + 6(3x)^2(2)^2 + 4(3x)(2)^3 + (2)^4 \\
 &= 81x^4 + 216x^3 + 216x^2 + 96x + 16
 \end{aligned}$$

$$\mathbf{9} \quad \text{In the expansion of } (2x + 5)^6, \quad a = (2x), \quad b = 5, \quad n = 6$$

$$\begin{aligned}
 T_{r+1} &= \binom{n}{r} a^{n-r} b^r & \text{For the coefficient of } x^3 \text{ we let } 6 - r &= 3 \\
 &= \binom{6}{r} (2x)^{6-r} 5^r & & \therefore r = 3 \\
 &= \binom{6}{r} 2^{6-r} x^{6-r} 5^r & \text{and } T_4 &= \binom{6}{3} 2^3 5^3 x^3 \\
 & & & \therefore \text{the coefficient is } \binom{6}{3} 2^3 5^3 = 20\,000.
 \end{aligned}$$

$$\mathbf{10} \quad \text{The three sisters can sit together in } 3! \text{ ways. They as a group plus the other 5 people make 6 items which can be permuted in } 6! \text{ ways.}$$

$$\begin{aligned}
 \therefore \text{total number of orderings} &= 3! \times 6! \\
 &= 4320
 \end{aligned}$$

$$\mathbf{11} \quad \mathbf{a} \quad \begin{array}{|c|c|c|} \hline 9 & 10 & 10 \\ \hline \end{array} \quad \text{if no restrictions i.e., } 9 \times 10 \times 10 = 900$$

$$\mathbf{b} \quad \text{To be divisible by 5 the last digit must be 0 or 5.}$$

$$\begin{aligned}
 \text{i.e., } \begin{array}{|c|c|c|} \hline 9 & 10 & 1 \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|c|c|} \hline 9 & 10 & 1 \\ \hline \end{array} \quad \text{i.e., } 9 \times 10 \times 1 + 9 \times 10 \times 1 = 180 \text{ of them.}
 \end{aligned}$$

REVIEW SET 9B

- 1 a If no restrictions there are $C_8^{18} = 43\,758$ different teams possible.
 b If 4 of each sex are needed there are $C_4^{11} \times C_4^7 = 11\,550$ different teams.
 c If at least 2 women are needed, total number $= C_8^{18} - C_8^{11}C_0^7 - C_7^{11}C_1^7$
 $= 41\,283$ different teams

- 2 a

9	9	8	7
---	---	---	---

 i.e., $9 \times 9 \times 8 \times 7 = 4536$ numbers are possible
 $\uparrow$
 can't use 0

- b Either they end in 0 or in 5

9	8	7	1
---	---	---	---

 or

8	8	7	1
---	---	---	---

 i.e., $9 \times 8 \times 7 \times 1 + 8 \times 8 \times 7 \times 1$
 $\uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow$
 0 can't use 0 5
 $= 952$ numbers are possible.

- 3 The sixth row of Pascal's triangle is 1 6 15 20 15 6 1

$$\therefore (a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

a $(x-3)^6 = x^6 + 6x^5(-3) + 15x^4(-3)^2 + 20x^3(-3)^3 + 15x^2(-3)^4 + 6x(-3)^5 + (-3)^6$
 $= x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729$

b $\left(1 + \frac{1}{x}\right)^6$
 $= (1)^6 + 6(1)^5\left(\frac{1}{x}\right) + 15(1)^4\left(\frac{1}{x}\right)^2 + 20(1)^3\left(\frac{1}{x}\right)^3 + 15(1)^2\left(\frac{1}{x}\right)^4 + 6(1)\left(\frac{1}{x}\right)^5 + \left(\frac{1}{x}\right)^6$
 $= 1 + \frac{6}{x} + \frac{15}{x^2} + \frac{20}{x^3} + \frac{15}{x^4} + \frac{6}{x^5} + \frac{1}{x^6}$

4 $(\sqrt{3} + 2)^5 = (\sqrt{3})^5 + 5(\sqrt{3})^4(2) + 10(\sqrt{3})^3(2)^2 + 10(\sqrt{3})^2(2)^3 + 5(\sqrt{3})^1(2)^4 + 2^5$
 $= 9\sqrt{3} + 90 + 120\sqrt{3} + 240 + 80\sqrt{3} + 32$
 $= 362 + 209\sqrt{3}$

5 $(4+x)^3 = 4^3 + 3(4)^2x^1 + 3(4)^1x^2 + x^3$
 $= 64 + 48x + 12x^2 + x^3$

Letting $x = 0.02$ gives $(4.02)^3 = 64 + 48(0.02) + 12(0.02)^2 + (0.02)^3$
 $= 64 + 0.96 + 0.0048 + 0.000\,008$
 $= 64.964\,808$

6 For $\left(3x^2 + \frac{1}{x}\right)^8$, $a = (3x^2)$, $b = \left(\frac{1}{x}\right)$, $n = 8$

$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} b^r \\ &= \binom{8}{r} (3x^2)^{8-r} \left(\frac{1}{x}\right)^r \\ &= \binom{8}{r} 3^{8-r} x^{16-2r-r} \\ &= \binom{8}{r} 3^{8-r} x^{16-3r} \end{aligned}$$

Now a constant term does not contain x

$$\therefore 16 - 3r = 0$$

$$\therefore 3r = 16$$

$$\therefore r = 5\frac{1}{3}$$

which is impossible as r is in $\mathbf{Z}$

$\therefore$ no constant term exists.

$$\mathbf{7} \quad \text{In } \left(2x - \frac{3}{x^2}\right)^{12}, \quad a = (2x), \quad b = \left(-\frac{3}{x^2}\right), \quad n = 12$$

$$\begin{aligned} T_{r+1} &= \binom{n}{r} a^{n-r} b^r && \text{For the coefficient of } x^{-6} \text{ we let } 12 - 3r = -6 \\ &= \binom{12}{r} (2x)^{12-r} \left(-\frac{3}{x^2}\right)^r && \therefore 3r = 18 \\ &= \binom{12}{r} 2^{12-r} x^{12-r} \frac{(-3)^r}{x^{2r}} && \therefore r = 6 \\ &= \binom{12}{r} 2^{12-r} (-3)^r x^{12-3r} && \text{So, } T_7 = \underbrace{\binom{12}{6} 2^6 (-3)^6}_{\text{the coefficient is } \binom{12}{6} 2^6 (-3)^6} x^{-6} \end{aligned}$$

$$\mathbf{8} \quad (2x+3)(x-2)^6$$

$$= (2x+3) [x^6 + 6x^5(-2) + 15x^4(-2)^2 + \dots]$$

$$\therefore \text{coefficient of } x^5 \text{ is } 2 \times 15 \times (-2)^2 + 3 \times 6 \times (-2) = 120 - 36 = 84$$

$$\mathbf{9} \quad T_{r+1} = C_r^9 (2x)^{9-r} \left(\frac{1}{ax^2}\right)^r$$

Letting $r = 2$, $T_3 = \underbrace{C_2^9 2^7 a^{-2}}_{\text{coefficient is } C_2^9 2^7 a^{-2}} x^3$

$$\begin{aligned} &= C_r^9 2^{9-r} x^{9-r} \times \frac{1}{a^r x^{2r}} && \therefore \frac{C_2^9 2^7}{2} = 288 \\ &= C_r^9 2^{9-r} a^{-r} x^{9-3r} && \therefore a^2 = \frac{C_2^9 2^7}{288} = 16 \\ &&& \therefore a = \pm 4 \end{aligned}$$

$$\mathbf{10} \quad T_{r+1} = C_r^6 (3x)^{6-r} \left(\frac{-2}{x^2}\right)^r$$

$$\begin{aligned} &= C_r^6 3^{6-r} x^{6-r} (-2)^r x^{-2r} \\ &= C_r^6 3^{6-r} x^{6-3r} (-2)^r \quad \text{and we let } 6 - 3r = 0 \\ &&& \therefore r = 2 \end{aligned}$$

$$\therefore T_3 = \underbrace{C_2^6 3^4 (-2)^2}_{\text{coefficient of } x^3 = C_2^6 3^4 2^2 = 4860} x^3$$

$$\mathbf{11} \quad \left(x^3 + \frac{q}{x^3}\right)^8 \quad \text{has } T_{r+1} = C_r^8 (x^3)^{8-r} \left(\frac{q}{x^3}\right)^r$$

$$\begin{aligned} &= C_r^8 x^{24-3r} \frac{q^r}{x^{3r}} \\ &= C_r^8 x^{24-6r} q^r \end{aligned}$$

$$\text{which has constant term } C_4^8 q^4 \quad \{24 - 6r = 0 \text{ when } r = 4\}$$

$$\left(x^3 + \frac{q}{x^3}\right)^4 \quad \text{has } T_{r+1} = C_r^4 (x^3)^{4-r} \left(\frac{q}{x^3}\right)^r$$

$$\begin{aligned} &= C_r^4 x^{12-3r} q^r x^{-3r} \\ &= C_r^4 x^{12-6r} q^r \end{aligned}$$

$$\text{which has constant term } C_2^4 q^2 \quad \{12 - 6r = 0 \text{ when } r = 2\}$$

$$\begin{aligned} &\therefore C_4^8 q^4 = C_2^4 q^2 \\ &\text{i.e., } 70q^4 - 6q^2 = 0 \\ &\therefore 2q^2(35q^2 - 3) = 0 \\ &\therefore 35q^2 - 3 = 0 \quad \text{as } q \neq 0 \\ &\therefore q^2 = \frac{3}{35} \\ &\therefore q = \pm \sqrt{\frac{3}{35}} \end{aligned}$$

Chapter 10

MATHEMATICAL INDUCTION

EXERCISE 10A

- 1 a** $3 = 4 \times 1 - 1$ Our proposition is:
 $7 = 4 \times 2 - 1$ The n th term of the sequence $3, 7, 11, 15, 19, \dots$ is $4n - 1$
 $11 = 4 \times 3 - 1$ for $n = 1, 2, 3, 4, \dots$
 $15 = 4 \times 4 - 1$
 $19 = 4 \times 5 - 1$ etc.

- b** $3^1 = 3$ $1 + 2(1) = 3$ Our proposition is:
 $3^2 = 9$ $1 + 2(2) = 5$ $3^n > 1 + 2n$ for $n = 2, 3, 4, 5, \dots$
 $3^3 = 27$ $1 + 2(3) = 7$ i.e., $n \geq 2, n \in \mathbb{Z}^+$
 $3^4 = 81$ $1 + 2(4) = 9$

- c** $11^1 - 1 = 10$ Our proposition is:
 $11^2 - 1 = 121 - 1 = 120$ $11^n - 1$ is divisible by 10 for all $n \in \mathbb{Z}^+$
 $11^3 - 1 = 1331 - 1 = 1330$
 $11^4 - 1 = 14641 - 1 = 14640$

- d** $2 = 2 = 1 \times 2$ Our proposition is:
 $2 + 4 = 6 = 2 \times 3$ $2 + 4 + 6 + 8 + 10 + \dots + 2n = n(n+1)$ for all $n \in \mathbb{Z}^+$
 $2 + 4 + 6 = 12 = 3 \times 4$ $\uparrow$
 $2 + 4 + 6 + 8 = 20 = 4 \times 5$ n th term
 $2 + 4 + 6 + 8 + 10 = 30 = 5 \times 6$

- e** $1! = 1$
 $1! + 2 \times 2! = 1 + 2(2) = 5$
 $1! + 2 \times 2! + 3 \times 3! = 1 + 4 + 18 = 23$
 $1! + 2 \times 2! + 3 \times 3! + 4 \times 4! = 1 + 4 + 18 + 96 = 119$

where each number result is 1 less than a factorial number

- $1 = 2! - 1$ Our proposition is:
 $5 = 3! - 1$ $1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \dots + n \times n! = (n+1)! - 1$
 $23 = 4! - 1$ for all $n \in \mathbb{Z}^+$
 $119 = 5! - 1$

- f** $\frac{1}{2!} = \frac{1}{2} = \frac{2! - 1}{2!}$
 $\frac{1}{2!} + \frac{2}{3!} = \frac{1}{2} + \frac{2}{6} = \frac{5}{6} = \frac{3! - 1}{3!}$
 $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{1}{2} + \frac{2}{6} + \frac{3}{24} = \frac{23}{24} = \frac{4! - 1}{4!}$
 $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} = \frac{1}{2} + \frac{2}{6} + \frac{3}{24} + \frac{4}{120} = \frac{119}{120} = \frac{5! - 1}{5!}$

Our proposition is: $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \dots + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}, n \in \mathbb{Z}^+$

- g** $7^1 + 2 = 7 + 2 = 9 = 3 \times 3$ Our proposition is:
 $7^2 + 2 = 49 + 2 = 51 = 3 \times 17$ $7^n + 2$ is divisible by 3 for all $n \in \mathbb{Z}^+$
 $7^3 + 2 = 343 + 2 = 345 = 3 \times 115$
 $7^4 + 2 = 2401 + 2 = 2403 = 3 \times 801$

$$\mathbf{h} \quad (1 - \frac{1}{2}) = \frac{1}{2}$$

$$(1 - \frac{1}{2})(1 - \frac{1}{3}) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$$

$$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$$

$$(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4})(1 - \frac{1}{5}) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5}$$

Our proposition is: $(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) \dots (1 - \frac{1}{n+1}) = \frac{1}{n+1}$, $n \in \mathbb{Z}^+$

$$\mathbf{i} \quad \frac{1}{2 \times 5} = \frac{1}{10}$$

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} = \frac{1}{10} + \frac{1}{40} = \frac{5}{40} = \frac{1}{8} = \frac{2}{16}$$

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} = \frac{1}{10} + \frac{1}{40} + \frac{1}{88} = \frac{3}{22}$$

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \frac{1}{11 \times 14} = \frac{1}{7} = \frac{4}{28}$$

10, 16, 22, 28 is arithmetic
with $u_1 = 10$, $d = 6$

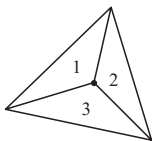
$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 10 + 6(n-1) \\ &= 6n + 4 \end{aligned}$$

Our proposition is:

$$\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \frac{1}{11 \times 14} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4} \text{ for all } n \in \mathbb{Z}^+$$

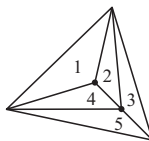
$\{2, 5, 8, 11 \dots\}$ are arithmetic with $u_1 = 2$, $d = 3 \therefore u_n = 2 + (n-1)3 = 3n - 1$

2 For $n = 1$



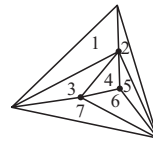
$$\begin{aligned} T_1 &= 3 \\ &= 2 \times 1 + 1 \end{aligned}$$

For $n = 2$



$$\begin{aligned} T_2 &= 5 \\ &= 2 \times 2 + 1 \end{aligned}$$

For $n = 3$



$$\begin{aligned} T_3 &= 7 \\ &= 2 \times 3 + 1 \end{aligned}$$

Our proposition is:

The number of triangular partitions for n dots within the triangle is $T_n = 2n + 1$, $n \in \mathbb{Z}^+$.

EXERCISE 10B

1 a P_n is “ $1 + 2 + 3 + 4 + \dots + n = \frac{n(n+1)}{2}$ ” for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = 1 and RHS = $\frac{1(2)}{2} = 1$, $\therefore P_1$ is true

(2) If P_k is true then $1 + 2 + 3 + 4 + \dots + k = \frac{k(k+1)}{2}$ (*)

$$\begin{aligned} \text{Thus } & 1 + 2 + 3 + 4 + \dots + k + (k+1) \\ &= \frac{k(k+1)}{2} + k + 1 \quad \{\text{using (*)}\} \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \quad \{\text{to equalise denominators}\} \\ &= \frac{(k+1)[k+2]}{2} \quad \left\{ \text{common factor of } \frac{(k+1)}{2} \right\} \\ &= \frac{(k+1)[(k+1)+1]}{2} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

- b** P_n is “ $1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $1 \times 2 = 2$ and RHS = $\frac{1(2)(3)}{3} = 2$, $\therefore P_1$ is true

(2) If P_k is true then

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3} \quad \dots (*)$$

$$\begin{aligned} \therefore 1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + k(k+1) + (k+1)(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \{\text{using } (*)\} \\ &= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \quad \{\text{to equalise denominators}\} \\ &= \frac{(k+1)(k+2)[k+3]}{3} \quad \{\text{common factor of } (k+1)(k+2)\} \\ &= \frac{[k+1]([k+1]+1)([k+1]+2)}{3} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

- c** P_n is “ $3 \times 5 + 6 \times 6 + 9 \times 7 + 12 \times 8 + \dots + 3n(n+4) = \frac{n(n+1)(2n+13)}{2}$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $3 \times 5 = 15$, RHS = $\frac{1 \times 2 \times (2+13)}{2} = 15$, $\therefore P_1$ is true

(2) If P_k is true, then

$$3 \times 5 + 6 \times 6 + 9 \times 7 + \dots + 3k(k+4) = \frac{k(k+1)(2k+13)}{2} \quad \dots (*)$$

$$\begin{aligned} \text{Now } 3 \times 5 + 6 \times 6 + 9 \times 7 + \dots + 3k(k+4) + 3(k+1)(k+5) \\ &= \frac{k(k+1)(2k+13)}{2} + 3(k+1)(k+5) \quad \{\text{using } (*)\} \\ &= \frac{k(k+1)(2k+13)}{2} + \frac{6(k+1)(k+5)}{2} \quad \{\text{to equalise denominators}\} \\ &= \frac{(k+1)[k(2k+13) + 6(k+5)]}{2} \quad \{\text{common factor}\} \\ &= \frac{(k+1)[2k^2 + 19k + 30]}{2} \\ &= \frac{(k+1)(k+2)(2k+15)}{2} \\ &= \frac{(k+1)([k+1]+1)(2[k+1]+13)}{2} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

- d** P_n is “ $1^3 + 2^3 + 3^3 + 4^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $1^3 = 1$, RHS = $\frac{1^2(2)^2}{4} = 1$ $\therefore P_1$ is true

$$(2) \text{ If } P_k \text{ is true, then } 1^3 + 2^3 + 3^3 + \dots + k^3 = \frac{k^2(k+1)^2}{4} \dots (*)$$

$$\begin{aligned} \text{Now } & 1^3 + 2^3 + 3^3 + \dots + k^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \{\text{using } (*)\} \\ &= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \quad \{\text{equalising denominators}\} \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \quad \{\text{common factor}\} \\ &= \frac{(k+1)^2[k^2 + 4k + 4]}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

e P_n is “ $1 + 2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \dots + n \times 2^{n-1} = (n-1) \times 2^n + 1$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = 1, RHS = $(0)2^0 + 1 = 1$, $\therefore P_1$ is true

(2) If P_k is true, then

$$1 + 2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \dots + k \times 2^{k-1} = (k-1)2^k + 1 \dots (*)$$

$$\begin{aligned} \text{Now } & 1 + 2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \dots + k \times 2^{k-1} + (k+1)2^k \\ &= (k-1)2^k + 1 + (k+1)2^k \quad \{\text{using } (*)\} \\ &= 2^k[k-1+k+1] + 1 \\ &= 2^k[2k] + 1 \\ &= k \cdot 2^{k+1} + 1 \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

2 a P_n is “ $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $\frac{1}{1 \times 2} = \frac{1}{2}$, RHS = $\frac{1}{1+1} = \frac{1}{2}$ $\therefore P_1$ is true

(2) If P_k is true, then $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1} \dots (*)$

$$\begin{aligned} \text{Now } & \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad \{\text{using } (*)\} \\ &= \frac{k}{k+1} \left(\frac{k+2}{k+2} \right) + \frac{1}{(k+1)(k+2)} \quad \{\text{equalising denominators}\} \\ &= \frac{k(k+2) + 1}{(k+1)(k+2)} \\ &= \frac{k^2 + 2k + 1}{(k+1)(k+2)} \\ &= \frac{(k+1)^2}{(k+1)(k+2)} \end{aligned}$$

$$= \frac{k+1}{k+2}$$

$$= \frac{k+1}{[k+1]+1}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true
 $\therefore P_n$ is true {Principle of Mathematical Induction}

$$\begin{aligned} \text{and } \frac{1}{10 \times 11} + \frac{1}{11 \times 12} + \frac{1}{12 \times 13} + \dots + \frac{1}{20 \times 21} &= S_{20} - S_9 \\ &= \frac{20}{21} - \frac{9}{10} \\ &= \frac{11}{210} \end{aligned}$$

b P_n is “ $\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $\frac{1}{1 \times 2 \times 3} = \frac{1}{6}$, RHS = $\frac{1(4)}{4(2)(3)} = \frac{1}{6} \therefore P_1$ is true

(2) If P_k is true, then

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \dots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)} \dots (*)$$

$$\begin{aligned} \text{Now } \frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \dots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \\ &= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \quad \{\text{using } (*)\} \\ &= \frac{k(k+3)}{4(k+1)(k+2)} \left(\frac{k+3}{k+3} \right) + \frac{4}{4(k+1)(k+2)(k+3)} \quad \{\text{equalising denominators}\} \\ &= \frac{k(k+3)^2 + 4}{4(k+1)(k+2)(k+3)} \\ &= \frac{k(k^2 + 6k + 9) + 4}{4(k+1)(k+2)(k+3)} \\ &= \frac{k^3 + 6k^2 + 9k + 4}{4(k+1)(k+2)(k+3)} \\ &= \frac{(k+1)(k^2 + 5k + 4)}{4(k+1)(k+2)(k+3)} \\ &= \frac{(k+1)(k+4)}{4(k+2)(k+3)} \end{aligned}$$

$$-1 \left| \begin{array}{ccc|c} 1 & 6 & 9 & 4 \\ 0 & -1 & -5 & -4 \\ 1 & 5 & 4 & 0 \end{array} \right|$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true
 $\therefore P_n$ is true {Principle of Mathematical Induction}

3 a P_n is “ $1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! = (n+1)! - 1$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $1 \times 1!$ RHS = $2! - 1$
 $= 1 \times 1$ $= 2 - 1$
 $= 1$ $= 1 \therefore P_1$ is true

(2) If P_k is true, then $1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! = (k+1)! - 1 \dots (*)$

$$\begin{aligned} \text{Now } 1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + k \times k! + (k+1)(k+1)! \\ &= (k+1)! - 1 + (k+1)(k+1)! \quad \{\text{using } (*)\} \\ &= (k+1)![1 + (k+1)] - 1 \\ &= (k+1)!(k+2) - 1 \quad \text{Thus } P_{k+1} \text{ is true whenever } P_k \text{ is true and } P_1 \text{ is true} \\ &= (k+2)! - 1 \therefore P_n \text{ is true } \{\text{Principle of Mathematical Induction}\} \end{aligned}$$

b P_n is “ $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $\frac{1}{2!} = \frac{1}{2}$, RHS = $\frac{2! - 1}{2!} = \frac{2 - 1}{2} = \frac{1}{2} \therefore P_1$ is true

(2) If P_k is true, then

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} = \frac{(k+1)! - 1}{(k+1)!} \quad \dots (*)$$

$$\begin{aligned} \text{Now} \quad & \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} \\ &= \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!} \quad \{\text{using } (*)\} \\ &= \left(\frac{k+2}{k+2}\right) \left[\frac{(k+1)! - 1}{(k+1)!}\right] + \frac{k+1}{(k+2)!} \quad \{\text{equalising denominators}\} \\ &= \frac{(k+2)! - (k+2) + k+1}{(k+2)!} \\ &= \frac{(k+2)! - k - 2 + k + 1}{(k+2)!} \\ &= \frac{(k+2)! - 1}{(k+2)!} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

and $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{9}{10!} = \frac{10! - 1}{10!} = \frac{3\,628\,799}{3\,628\,800}$

4 P_n is “ $1 \times n + 2 \times (n-1) + 3 \times (n-2) + \dots + (n-1) \times 2 + n \times 1 = \frac{n(n+1)(n+2)}{6}$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $1 \times 1 = 1$, RHS = $\frac{1(2)(3)}{6} = 1 \therefore P_1$ is true

(2) If P_k is true, then

$$1 \times k + 2(k-1) + 3(k-2) + \dots + (k-1)2 + k \times 1 = \frac{k(k+1)(k+2)}{6} \quad \dots (*)$$

$$\begin{aligned} \text{Now} \quad & 1(k+1) + 2(k) + 3(k-1) + \dots + k2 + (k+1)1 \\ &= 1(k) + 2(k-1) + 3(k-2) + \dots + k1 + 1 + 2 + 3 + \dots + k + k + 1 \\ & \quad \quad \quad \{\text{using the hint}\} \\ &= \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2} \quad \{\text{using } (*) \text{ and the sum of an Arith. series}\} \\ &= \frac{k(k+1)(k+2)}{6} + \frac{3(k+1)(k+2)}{6} \quad \{\text{equalising denominators}\} \\ &= \frac{(k+1)(k+2)[k+3]}{6} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

5 a P_n is “ $n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

- (1) If $n = 1$, $1^3 + 2(1) = 3$ which is divisible by 3
- (2) If P_k is true, then $k^3 + 2k = 3A$ where $A \in \mathbb{Z}$ (*)

$$\begin{aligned}
 \text{Now } (k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
 &= (k^3 + 2k) + 3k^2 + 3k + 3 \\
 &= 3A + 3k^2 + 3k + 3 \quad \{\text{using (*)}\} \\
 &= 3[A + k^2 + k + 1] \quad \text{where } A + k^2 + k + 1 \text{ is an integer} \\
 &\quad \text{as } A \text{ and } k \text{ are integers}
 \end{aligned}$$

$\therefore (k+1)^3 + 2(k+1)$ is divisible by 3

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

b P_n is “ $n(n^2 + 5)$ is divisible by 6 for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

- (1) If $n = 1$, $1(1^2 + 5) = 1 \times 6 = 6$ which is divisible by 6 $\therefore P_1$ is true
- (2) If P_k is true, then $k(k^2 + 5) = 6A$ where A is an integer (*)

$$\begin{aligned}
 \text{Now } (k+1)[(k+1)^2 + 5] &= (k+1)(k^2 + 2k + 1 + 5) \\
 &= (k+1)(k^2 + 2k + 6) \\
 &= k^3 + 2k^2 + 6k + k^2 + 2k + 6 \\
 &= k^3 + 5k + [3k^2 + 3k + 6] \\
 &= k(k^2 + 5) + 3k(k+1) + 6 \\
 &= 6A + 6 + 3k(k+1)
 \end{aligned}$$

We notice that $k(k+1)$ is the product of consecutive integers, one of which must be even $\therefore k(k+1) = 2B$ where $B \in \mathbb{Z}$

$$\begin{aligned}
 \therefore (k+1)[(k+1)^2 + 5] &= 6A + 6 + 3(2B) \\
 &= 6(A + 1 + B) \quad \text{where } A + 1 + B \in \mathbb{Z}
 \end{aligned}$$

$\therefore (k+1)[(k+1)^2 + 5]$ is divisible by 6

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

c P_n is “ $6^n - 1$ is divisible by 5 for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

- (1) If $n = 1$, $6^1 - 1 = 5$ which is divisible by 5 $\therefore P_1$ is true
- (2) If P_k is true, then $6^k - 1 = 5A$ where $A \in \mathbb{Z}$ (*)

$$\begin{aligned}
 \text{Now } 6^{k+1} - 1 &= 6^k \times 6 - 1 \\
 &= 6[5A + 1] - 1 \quad \{\text{using (*)}\} \\
 &= 30A + 6 - 1 \\
 &= 30A + 5 \\
 &= 5(6A + 1) \quad \text{where } 6A + 1 \in \mathbb{Z}
 \end{aligned}$$

i.e., $6^{k+1} - 1$ is divisible by 5

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

d P_n is “ $7^n - 4^n - 3^n$ is divisible by 12 for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $7^1 - 4^1 - 3^1 = 0$ which is divisible by 12 $\therefore P_1$ is true

(2) If P_k is true, then $7^k - 4^k - 3^k = 12A$ where $A \in \mathbb{Z}$ (*)

$$\begin{aligned} \text{Now } 7^{k+1} - 4^{k+1} - 3^{k+1} &= 7(7^k) - 4(4^k) - 3(3^k) \\ &= 7[12A + 4^k + 3^k] - 4(4^k) - 3(3^k) \quad \{\text{using (*)}\} \\ &= 84A + 7(4^k) + 7(3^k) - 4(4^k) - 3(3^k) \\ &= 84A + 3(4^k) + 4(3^k) \\ &= 84A + 3 \times 4 \times 4^{k-1} + 4 \times 3 \times 3^{k-1} \\ &= 12(7A + 4^{k-1} + 3^{k-1}) \quad \text{where } k \geq 2, k \in \mathbb{Z}^+ \\ &= 12 \times \text{an integer} \quad \{\text{as } 4^{k-1} \text{ and } 3^{k-1} \text{ are integers}\} \end{aligned}$$

$\therefore 7^{k+1} - 4^{k+1} - 3^{k+1}$ is divisible by 12

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

6 P_n is “ $\frac{2^n - (-1)^n}{3}$ is an odd number for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $\frac{2^1 - (-1)^1}{3} = \frac{3}{3} = 1$ which is odd $\therefore P_1$ is true

(2) If P_k is true, then $\frac{2^k - (-1)^k}{3} = 2A + 1$ where $A \in \mathbb{Z}$ (*)

$$\begin{aligned} \text{Now } \frac{2^{k+1} - (-1)^{k+1}}{3} &= \frac{2(2^k) - (-1)^{k+1}}{3} \\ &= \frac{2[6A + 3 + (-1)^k] - (-1)^{k+1}}{3} \\ &= \frac{12A + 6 + 2(-1)^k - (-1)(-1)^k}{3} \\ &= \frac{12A + 6 + 2(-1)^k + (-1)^k}{3} \\ &= \frac{12A + 6 + 3(-1)^k}{3} \\ &= 4A + 2 + (-1)^k \end{aligned}$$

Now $4A + 2$ is always even and $(-1)^k$ is either +1 or -1

$\therefore 4A + 2 + (-1)^k$ is odd

$\therefore \frac{2^{k+1} - (-1)^{k+1}}{3}$ is odd

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

7 a P_n is “ $\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{n+1}\right) = \frac{1}{n+1}$ for all $n \in \mathbb{Z}^+$ ”

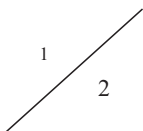
Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $\left(1 - \frac{1}{2}\right) = \frac{1}{2}$, RHS = $\frac{1}{1+1} = \frac{1}{2}$ $\therefore P_1$ is true

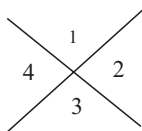
(2) If P_k is true, then $\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{k+1}\right) = \frac{1}{k+1} \dots (*)$

$$\begin{aligned} \therefore & \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{k+1}\right) \left(1 - \frac{1}{k+2}\right) \\ &= \frac{1}{k+1} \left(1 - \frac{1}{k+2}\right) \quad \{\text{using } (*)\} \\ &= \frac{1}{k+1} \left(\frac{k+2-1}{k+2}\right) \\ &= \frac{1}{k+1} \left(\frac{k+1}{k+2}\right) \\ &= \frac{1}{k+2} \end{aligned}$$

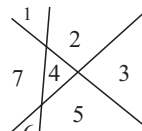
Thus P_{k+1} is true whenever P_k is true and P_1 is true
 $\therefore P_n$ is true {Principle of Mathematical Induction}

b

1 line, $R_1 = 2$



2 lines, $R_2 = 4$



3 lines, $R_3 = 7$, etc.

P_n is “For n lines as described, $R_n = \frac{n(n+1)}{2} + 1$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

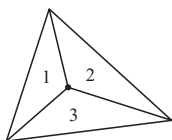
(1) If $n = 1$, $R_1 = \frac{1(2)}{2} + 1 = 1 + 1 = 2 \therefore P_1$ is true

(2) If P_k is true, then $R_k = \frac{k(k+1)}{2} + 1 \dots (*)$

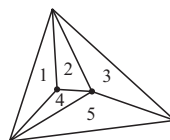
Now the addition of another line creates another $k+1$ regions

$$\begin{aligned} \therefore R_{k+1} &= \frac{k(k+1)}{2} + 1 + k + 1 \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} + 1 \\ &= \frac{k^2 + k + 2k + 2}{2} + 1 \\ &= \frac{k^2 + 3k + 2}{2} + 1 \\ &= \frac{(k+1)(k+2)}{2} + 1 \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true
 $\therefore P_n$ is true {Principle of Mathematical Induction}

c

$n = 1$
 $T_1 = 3$



$n = 2$
 $T_2 = 5$

P_n is “For n points inside the triangle (as described) there are $T_n = 2n + 1$ triangular partitions for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

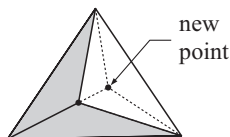
(1) If $n = 1$, $T_1 = 2(1) + 1 = 3 \therefore P_1$ is true

- (2) If P_k is true, then $T_k = 2k + 1 \dots (*)$

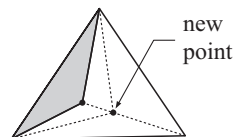
Adding an extra point within the triangle gives the $k + 1^{\text{th}}$ case.

This point could be either

- in an existing triangle or • on an existing line between 2 triangles



So, 1 triangle becomes 3,
a net increase of 2.



So, 2 triangles become 4,
a net increase of 2.

In each case 2 triangles are added

$$\begin{aligned} \therefore T_{k+1} &= T_k + 2 \\ &= 2k + 1 + 2 \quad \{\text{using } (*)\} \\ &= 2(k + 1) + 1 \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

- d** P_n is “ $\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \dots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$ for all $n \geq 2, n \in \mathbb{Z}$ ”

Proof: (By the Principle of Mathematical Induction)

$$\begin{aligned} (1) \text{ If } n = 2, \quad \text{LHS} &= 1 - \frac{1}{2^2} & \text{RHS} &= \frac{2+1}{2(2)} \\ &= 1 - \frac{1}{4} & &= \frac{3}{4} \quad \therefore P_2 \text{ is true} \\ &= \frac{3}{4} \end{aligned}$$

- (2) If P_k is true, then $\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \dots \left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k} \dots (*)$

$$\begin{aligned} \text{Now} \quad & \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \dots \left(1 - \frac{1}{k^2}\right) \left(1 - \frac{1}{(k+1)^2}\right) \\ &= \frac{k+1}{2k} \left(1 - \frac{1}{(k+1)^2}\right) \quad \{\text{using } (*)\} \\ &= \frac{k+1}{2k} \left(\frac{(k+1)^2 - 1}{(k+1)^2}\right) \\ &= \frac{k+1}{2k} \left(\frac{k^2 + 2k + 1 - 1}{(k+1)^2}\right) \\ &= \frac{k^2 + 2k}{2k(k+1)} \\ &= \frac{k(k+2)}{2k(k+1)} \\ &= \frac{k+2}{2(k+1)} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_2 is true
 $\therefore P_n$ is true {Principle of Mathematical Induction}

EXERCISE 10C**1 Proof** (by contradiction)

Let x be the positive number, so we need to prove that $x + 9\left(\frac{1}{x}\right) \geq 6$

So, we suppose the opposite, i.e., $x + \frac{9}{x} < 6$ for all x

$$\therefore x\left(x + \frac{9}{x}\right) < 6x \quad \text{as } x > 0$$

$$\therefore x^2 + 9 < 6x$$

$$\therefore x^2 - 6x + 9 < 0$$

$$\therefore (x - 3)^2 < 0$$

which is false as no perfect square can be negative

Hence, the supposition is false and its opposite is true i.e., $x + 9\left(\frac{1}{x}\right) \geq 6$ for all x

i.e., the sum of a positive number and 9 times its reciprocal is at least 6.

2 Proof (by contradiction)

Suppose the solution of $3^x = 4$ is rational, i.e., $x = \frac{p}{q}$, $q \neq 0$ where p and q are integers having no common factors.

$$\therefore 3^{\frac{p}{q}} = 4$$

$$\therefore (3^{\frac{p}{q}})^q = 4^q$$

$$\therefore 3^p = 4^q$$

which is a contradiction as 3^p is odd and 4^q is even

$\therefore$ the supposition is false and so the solution of $3^x = 4$ is irrational.

3 Proof (by contradiction)

Suppose $\log_2 5$ is rational, i.e., $\log_2 5 = \frac{p}{q}$ where p and q are integers, $q \neq 0$ and p and q have no common factors.

$$\therefore 5 = 2^{\frac{p}{q}}$$

$$\therefore 5^q = (2^{\frac{p}{q}})^q$$

$$\therefore 5^q = 2^p$$

which is a contradiction as 5^q is odd and 2^p is even

$\therefore$ the supposition is false $\therefore \log_2 5$ is irrational.

4 $\sqrt{2}$ is irrational **Proof** (by contradiction)

Suppose $\sqrt{2}$ is rational $\therefore \sqrt{2} = \frac{p}{q}$ where p and q are $\in \mathbb{Z}$, $q \neq 0$ and p and q have no common factors. (1)

$$\therefore 2 = \frac{p^2}{q^2} \quad \text{and so } p^2 = 2q^2 \text{ (2)}$$

$$\therefore p^2 \text{ is even}$$

$$\therefore p \text{ is even } \{\text{the only even perfect squares come from squaring evens}\}$$

$$\therefore p = 2a \quad \text{where } a \in \mathbb{Z}$$

$$\text{Back into (2)} \quad (2a)^2 = 2q^2$$

$$\therefore 4a^2 = 2q^2$$

$$\therefore q^2 = 2a^2 \quad \therefore q^2 \text{ is even}$$

$$\therefore q \text{ is even}$$

So, on the supposition, p and q are even and so have a common factor of 2.

This contradicts (1), so the supposition is false $\therefore \sqrt{2}$ is irrational.

REVIEW SET 10A

- 1** P_n is “ $1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

- (1) If $n = 1$, LHS = 1 and RHS = $1^2 = 1$ $\therefore P_1$ is true
 (2) If P_k is true, then $1 + 3 + 5 + 7 + \dots + (2k - 1) = k^2$ (*)
 $\therefore 1 + 3 + 5 + 7 + \dots + (2k - 1) + (2[k + 1] - 1)$
 $= k^2 + 2k + 2 - 1$ {using (*)}
 $= k^2 + 2k + 1$ Thus P_{k+1} is true whenever P_k is true and P_1 is true
 $= (k + 1)^2$ $\therefore P_n$ is true {Principle of Mathematical Induction}

- 2** P_n is “ $7^n + 2$ is divisible by 3 for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

- (1) If $n = 1$, $7^1 + 2 = 9$ which is divisible by 3 $\therefore P_1$ is true
 (2) If P_k is true, then $7^k + 2 = 3A$ where $A \in \mathbb{Z}$
 $\therefore 7^{k+1} + 2 = 7 \times 7^k + 2$
 $= 7[3A - 2] + 2$
 $= 21A - 14 + 2$
 $= 21A - 12$
 $= 3(7A - 4)$ where $7A - 4$ is an integer as A is an integer
 $\therefore 7^{k+1} + 2$ is divisible by 3
 Thus P_{k+1} is true whenever P_k is true and P_1 is true
 $\therefore P_n$ is true {Principle of Mathematical Induction}

- 3** P_n is “ $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots + n(n + 1)(n + 2) = \frac{n(n + 1)(n + 2)(n + 3)}{4}$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

- (1) If $n = 1$, LHS = $1 \times 2 \times 3 = 6$, RHS = $\frac{1 \times 2 \times 3 \times 4}{4} = 6$ $\therefore P_1$ is true
 (2) If P_k is true, then
 $1 \times 2 \times 3 + 2 \times 3 \times 4 + \dots + k(k + 1)(k + 2) = \frac{k(k + 1)(k + 2)(k + 3)}{4}$ (*)
 $\therefore 1 \times 2 \times 3 + 2 \times 3 \times 4 + \dots + k(k + 1)(k + 2) + (k + 1)(k + 2)(k + 3)$
 $= \frac{k(k + 1)(k + 2)(k + 3)}{4} + (k + 1)(k + 2)(k + 3)$ {using (*)}
 $= \frac{k(k + 1)(k + 2)(k + 3)}{4} + \frac{4(k + 1)(k + 2)(k + 3)}{4}$ {equalising denominators}
 $= \frac{(k + 1)(k + 2)(k + 3)[k + 4]}{4}$

Thus P_{k+1} is true whenever P_k is true and P_1 is true
 $\therefore P_n$ is true {Principle of Mathematical Induction}

- 4** P_n is “ $1 + r + r^2 + r^3 + \dots + r^{n-1} = \frac{1 - r^n}{1 - r}$ for all $n \in \mathbb{Z}^+$, $r \neq 1$ ”

Proof: (By the Principle of Mathematical Induction)

- (1) If $n = 1$, LHS = 1 and RHS = $\frac{1 - r}{1 - r} = 1$ as $r \neq 1$ $\therefore P_1$ is true

(2) If P_k is true, then $1 + r + r^2 + r^3 + \dots + r^{k-1} = \frac{1 - r^k}{1 - r}$ (*)

Now $1 + r + r^2 + r^3 + \dots + r^{k-1} + r^k$

$$= \frac{1 - r^k}{1 - r} + r^k \quad \{\text{using (*)}\}$$

$$= \frac{1 - r^k}{1 - r} + r^k \left(\frac{1 - r}{1 - r} \right) \quad \{\text{equalising denominators}\}$$

$$= \frac{1 - r^k + r^k - r^{k+1}}{1 - r}$$

$$= \frac{1 - r^{k+1}}{1 - r}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

5 P_n is “ $5^{2n} - 1$ is divisible by 24 for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $5^2 - 1 = 25 - 1 = 24$ is divisible by 24 $\therefore P_1$ is true

(2) If P_k is true, then $5^{2k} - 1 = 24A$ where $A \in \mathbb{Z}$

Now $5^{2(k+1)} - 1 = 5^{2k}5^2 - 1$

$$= 25[24A + 1] - 1$$

$$= 25 \times 24A + 25 - 1$$

$$= 25 \times 24A + 24$$

$$= 24(25A + 1) \quad \text{where } 25A + 1 \text{ is an integer}$$

$\therefore 5^{2(k+1)} - 1$ is divisible by 24

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

REVIEW SET 10B

1 P_n is “ $1^2 + 3^2 + 5^2 + 7^2 + \dots + (2n - 1)^2 = \frac{n(2n + 1)(2n - 1)}{3}$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $1^2 = 1$, RHS = $\frac{1 \times 3 \times 1}{3} = 1$ $\therefore P_1$ is true

(2) If P_k is true, then $1^2 + 3^2 + 5^2 + \dots + (2k - 1)^2 = \frac{k(2k + 1)(2k - 1)}{3}$ (*)

$$\therefore 1^2 + 3^2 + 5^2 + \dots + (2k - 1)^2 + (2k + 1)^2$$

$$= \frac{k(2k + 1)(2k - 1)}{3} + (2k + 1)^2 \quad \{\text{using (*)}\}$$

$$= \frac{k(2k + 1)(2k - 1)}{3} + \frac{3(2k + 1)^2}{3} \quad \{\text{equalising denominators}\}$$

$$= \frac{(2k + 1)[k(2k - 1) + 3(2k + 1)]}{3}$$

$$= \frac{(2k + 1)[2k^2 - k + 6k + 3]}{3}$$

$$= \frac{(2k + 1)(2k^2 + 5k + 3)}{3}$$

$$= \frac{(2k + 1)(k + 1)(2k + 3)}{3}$$

$$= \frac{[k + 1](2[k + 1] + 1)(2[k + 1] - 1)}{3}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true $\therefore P_n$ is true {Princ. of Math. Induction}

2 P_n is “ $3^{2n+2} - 8n - 9$ is divisible by 64 for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $3^4 - 8 - 9 = 81 - 17 = 64$ which is divisible by 64 $\therefore P_1$ is true

(2) If P_k is true, then $3^{2k+2} - 8k - 9 = 64A$ where $A \in \mathbb{Z}$

$$\begin{aligned}\text{Now } 3^{2(k+1)+2} - 8(k+1) - 9 &= 3^{2k+2} \times 3^2 - 8k - 8 - 9 \\ &= 9[64A + 8k + 9] - 8k - 17 \\ &= 9 \times 64A + 72k + 81 - 8k - 17 \\ &= 9 \times 64A + 64k + 64 \\ &= 64(9A + k + 1) \text{ where } 9A + k + 1 \text{ is } \in \mathbb{Z} \text{ as } A, k \in \mathbb{Z}\end{aligned}$$

$\therefore 3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 64

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

3 P_n is “ $3 + 5 \times 2 + 7 \times 2^2 + 9 \times 2^3 + \dots + (2n+1)2^{n-1} = 1 + (2n-1) \times 2^n$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = 3 and RHS = $1 + 1 \times 2^1 = 1 + 2 = 3$ $\therefore P_1$ is true

(2) If P_k is true, then

$$\begin{aligned}3 + 5 \times 2 + 7 \times 2^2 + 9 \times 2^3 + \dots + (2k+1)2^{k-1} &= 1 + (2k-1) \times 2^k \dots (*) \\ \therefore 3 + 5 \times 2 + 7 \times 2^2 + 9 \times 2^3 + \dots + (2k+1)2^{k-1} &+ (2k+3)2^k \\ &= 1 + (2k-1)2^k + (2k+3)2^k \text{ {using (*)}} \\ &= 1 + 2^k[2k-1 + 2k+3] \\ &= 1 + 2^k[4k+2] \\ &= 1 + 2^k(2)(2k+1) \\ &= 1 + (2k+1)2^{k+1} \\ &= 1 + (2[k+1] - 1)2^{[k+1]}\end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

4 P_n is “ $5^n + 3$ is divisible by 4 for all $n \in \mathbb{Z}$, $n \geq 0$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 0$, $5^0 + 3 = 4$ which is divisible by 4 $\therefore P_0$ is true

(2) If P_k is true, then $5^k + 3 = 4A$ where A is in $\mathbb{Z}$ (*)

$$\begin{aligned}\text{Now } 5^{k+1} + 3 &= 5 \times 5^k + 3 \\ &= 5[4A - 3] + 3 \text{ {using (*)}} \\ &= 20A - 15 + 3 \\ &= 20A - 12 \\ &= 4(5A - 3) \text{ where } 5A - 3 \text{ is in } \mathbb{Z}, \text{ as } A \text{ is in } \mathbb{Z}\end{aligned}$$

So, $5^{k+1} + 3$ is divisible by 4

Thus P_{k+1} is true whenever P_k is true and P_0 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

5 P_n is “ $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \dots + n(n+1)^2 = \frac{n(n+1)(n+2)(3n+5)}{12}$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $1 \times 2^2 = 4$ and RHS = $\frac{1 \times 2 \times 3 \times 8}{12} = \frac{48}{12} = 4 \therefore P_1$ is true

(2) If P_k is true, then

$$\begin{aligned}
 1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \dots + k(k+1)^2 &= \frac{k(k+1)(k+2)(3k+5)}{12} \dots (*) \\
 \therefore 1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \dots + k(k+1)^2 + (k+1)(k+2)^2 \\
 &= \frac{k(k+1)(k+2)(3k+5)}{12} + (k+1)(k+2)^2 \quad \{\text{using } (*)\} \\
 &= \frac{k(k+1)(k+2)(3k+5)}{12} + \frac{12(k+1)(k+2)^2}{12} \quad \{\text{equalising denominators}\} \\
 &= \frac{(k+1)(k+2)[k(3k+5) + 12(k+2)]}{12} \\
 &= \frac{(k+1)(k+2)[3k^2 + 5k + 12k + 24]}{12} \\
 &= \frac{(k+1)(k+2)(3k^2 + 17k + 24)}{12} \\
 &= \frac{(k+1)(k+2)(k+3)(3k+8)}{12} \\
 &= \frac{[k+1]([k+1]+1)([k+1]+2)(3[k+1]+5)}{12}
 \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

REVIEW SET 10C

1 P_n is “ $1 \times 3 + 2 \times 4 + 3 \times 5 + 4 \times 6 + \dots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $1 \times 3 = 3$ and RHS = $\frac{1 \times 2 \times 9}{6} = \frac{18}{6} = 3 \therefore P_1$ is true

(2) If P_k is true, then

$$\begin{aligned}
 1 \times 3 + 2 \times 4 + 3 \times 5 + \dots + k(k+2) &= \frac{k(k+1)(2k+7)}{6} \dots (*) \\
 \therefore 1 \times 3 + 2 \times 4 + 3 \times 5 + \dots + k(k+2) + (k+1)(k+3) \\
 &= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) \quad \{\text{using } (*)\} \\
 &= \frac{k(k+1)(2k+7)}{6} + \frac{6(k+1)(k+3)}{6} \quad \{\text{equalising denominators}\} \\
 &= \frac{(k+1)[k(2k+7) + 6(k+3)]}{6} \\
 &= \frac{(k+1)[2k^2 + 13k + 18]}{6} \\
 &= \frac{(k+1)(k+2)(2k+9)}{6} \\
 &= \frac{[k+1]([k+1]+1)(2[k+1]+7)}{6}
 \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

2 P_n is “ $7^n - 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $7^1 - 1 = 6$ which is divisible by 6 $\therefore P_1$ is true

(2) If P_k is true, then $7^k - 1 = 6A$ where $A \in \mathbb{Z}$

$$\begin{aligned}\text{Now } 7^{k+1} - 1 &= 7 \times 7^k - 1 \\ &= 7[6A + 1] - 1 \\ &= 42A + 7 - 1 \\ &= 42A + 6 \\ &= 6(7A + 1) \text{ where } 7A + 1 \text{ is in } \mathbb{Z}\end{aligned}$$

Thus $7^{k+1} - 1$ is divisible by 6

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

3 P_n is “ $1^3 + 3^3 + 5^3 + \dots + (2n - 1)^3 = n^2(2n^2 - 1)$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $1^3 = 1$ and RHS = $1^2(2 - 1) = 1 \times 1 = 1 \therefore P_1$ is true

(2) If P_k is true, then

$$1^3 + 3^3 + 5^3 + \dots + (2k - 1)^3 = k^2(2k^2 - 1) \dots (*)$$

$$\therefore 1^3 + 3^3 + 5^3 + \dots + (2k - 1)^3 + (2k + 1)^3$$

$$= k^2(2k^2 - 1) + (2k + 1)^3 \text{ {using (*)}}$$

$$= 2k^4 - k^2 + (2k)^3 + 3(2k)^2 \cdot 1 + 3(2k) \cdot 1^2 + 1^3$$

$$= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1$$

$$= 2k^4 + 8k^3 + 11k^2 + 6k + 1$$

$$= (k + 1)^2(2k^2 + 4k + 1)$$

$$= (k + 1)^2(2[k^2 + 2k + 1] - 1)$$

$$= (k + 1)^2(2[k + 1]^2 - 1)$$

$$\begin{array}{r|rrrrr} -1 & 2 & 8 & 11 & 6 & 1 \\ & 0 & -2 & -6 & -5 & -1 \\ -1 & 2 & 6 & 5 & 1 & 0 \\ & 0 & -2 & -4 & -1 & \\ \hline & 2 & 4 & 1 & 0 & \end{array}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

4 P_n is “ $3^n - 1 - 2n$ is divisible by 4 for all n in $\mathbb{Z}^+$, $n \geq 0$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 0$, $3^0 - 1 - 2(0) = 1 - 1 - 0 = 0$ which is divisible by 4 $\therefore P_0$ is true

(2) If P_k is true, then $3^k - 1 - 2k = 4A$ where $A \in \mathbb{Z} \dots (*)$

$$\begin{aligned}\text{Now } 3^{k+1} - 1 - 2(k + 1) &= 3 \times 3^k - 1 - 2k - 2 \\ &= 3[4A + 1 + 2k] - 1 - 2k - 2 \text{ {using (*)}} \\ &= 12A + 3 + 6k - 1 - 2k - 2 \\ &= 12A + 4k \\ &= 4(3A + k) \text{ where } 3A + k \text{ is in } \mathbb{Z}\end{aligned}$$

$\therefore 3^{k+1} - 1 - 2(k + 1)$ is divisible by 4

Thus P_{k+1} is true whenever P_k is true and P_0 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

5 P_n is “ $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$ for all $n \in \mathbb{Z}^+$ ”

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, LHS = $\frac{1}{1 \times 3} = \frac{1}{3}$, RHS = $\frac{1}{2+1} = \frac{1}{3} \therefore P_1$ is true

(2) If P_k is true, then

$$\begin{aligned} & \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1} \dots (*) \\ \therefore & \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} \\ &= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \quad \{\text{using } (*)\} \\ &= \frac{k}{2k+1} \left(\frac{2k+3}{2k+3} \right) + \frac{1}{(2k+1)(2k+3)} \quad \{\text{equalising denominators}\} \\ &= \frac{2k^2 + 3k + 1}{(2k+1)(2k+3)} \\ &= \frac{(k+1)(2k+1)}{(2k+1)(2k+3)} \\ &= \frac{[k+1]}{2[k+1]+1} \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true

$\therefore P_n$ is true {Principle of Mathematical Induction}

Background knowledge

TRIGONOMETRY WITH RIGHT ANGLED TRIANGLES

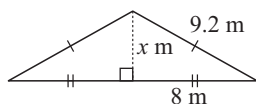
EXERCISE A

1 a $1^2 + x^2 = 1.2^2$ {Pythagoras}
 $\therefore x^2 = 1.2^2 - 1^2$
 $\therefore x = \sqrt{1.2^2 - 1^2}$
 $\therefore x \doteq 0.663$

c $x^2 = 1.8^2 + 1.32^2$ {Pythagoras}
 $\therefore x = \sqrt{1.8^2 + 1.32^2}$
 $\therefore x \doteq 2.23$

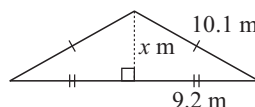
b $x^2 = 3.8^2 + 2.1^2$ {Pythagoras}
 $\therefore x = \sqrt{3.8^2 + 2.1^2}$
 $\therefore x \doteq 4.34$

2 a



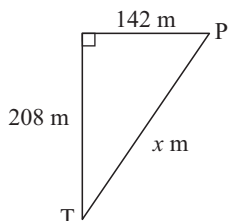
$x^2 + 8^2 = 9.2^2$ {Pythagoras}
 $\therefore x^2 = 9.2^2 - 8^2$
 $\therefore x = \sqrt{9.2^2 - 8^2}$
 $\therefore x \doteq 4.54$
 $\therefore$ the height is 4.54 m

b



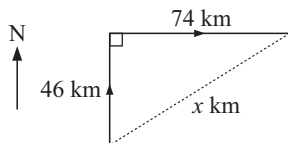
$x^2 + 9.2^2 = 10.1^2$ {Pythagoras}
 $\therefore x^2 = 10.1^2 - 9.2^2$
 $\therefore x = \sqrt{10.1^2 - 9.2^2}$
 $\therefore x \doteq 4.17$
 $\therefore$ the height is 4.17 m

3



$x^2 = 208^2 + 142^2$ {Pythagoras}
 $\therefore x = \sqrt{208^2 + 142^2}$
 $\therefore x \doteq 252$ m
 $\therefore$ must hit the ball (in the air)
 $(252 - 15)$ m
 $= 237$ m

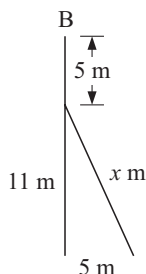
4 a



b $x^2 = 46^2 + 74^2$ {Pythagoras}
 $\therefore x = \sqrt{46^2 + 74^2}$
 $\therefore x \doteq 87.1$
 $\therefore$ is 87.1 km from the start.

EXERCISE B

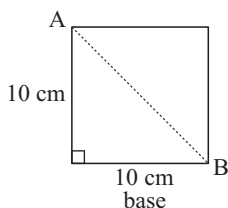
1



$x^2 = 5^2 + 11^2$ {Pythagoras}
 $\therefore x = \sqrt{5^2 + 11^2}$
 $\therefore x \doteq 12.083$

But $4x + 2$ m is needed
i.e., $(4 \times 12.083 + 2)$ m
 $\doteq 50.3$ m

2



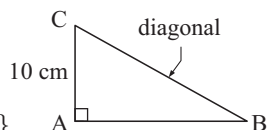
$$\begin{aligned} AB^2 &= 10^2 + 10^2 \quad \{\text{Pythagoras}\} \\ &= 100 + 100 \\ &= 200 \end{aligned}$$

$$\text{But } BC^2 = AB^2 + AC^2 \quad \{\text{Pythagoras, again}\}$$

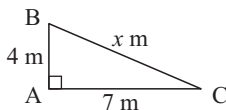
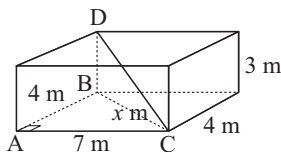
$$\therefore BC^2 = 200 + 10^2$$

$$\therefore BC = \sqrt{300} \div 17.3$$

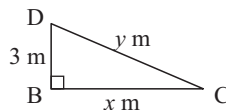
$\therefore$ the diagonal is 17.3 cm long.



3



$$x^2 = 4^2 + 7^2 \quad \{\text{Pythagoras}\}$$



$$y^2 = x^2 + 3^2 \quad \{\text{Pythagoras}\}$$

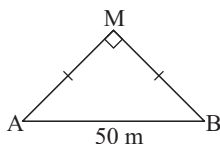
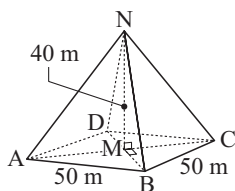
$$\therefore y^2 = 4^2 + 7^2 + 3^2$$

$$\therefore y = \sqrt{4^2 + 7^2 + 3^2}$$

$$\therefore y \div 8.60$$

So, the distance is about 8.60 m

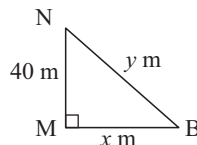
4



$$x^2 + x^2 = 50^2 \quad \{\text{Pythagoras}\}$$

$$\therefore 2x^2 = 50^2$$

$$\therefore x^2 = \frac{50^2}{2}$$



$$y^2 = 40^2 + x^2 \quad \{\text{Pythagoras}\}$$

$$\therefore y^2 = 40^2 + \frac{50^2}{2}$$

$$\therefore y = \sqrt{40^2 + \frac{50^2}{2}} \div 53.4$$

$\therefore$ each slant edge is 53.4 m long.

5

$$a^2 + 10^2 = 22.5^2 \quad \text{and} \quad b^2 + 10^2 = 40.8^2 \quad \{\text{Pythagoras}\}$$

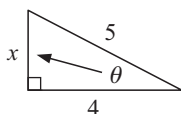
$$\therefore a^2 + b^2 = 22.5^2 - 10^2 + 40.8^2 - 10^2$$

$$\therefore \sqrt{a^2 + b^2} = \sqrt{22.5^2 + 40.8^2 - 200} \div 44.4$$

$\therefore AB \div 44.4$ km, i.e., 44.4 km apart.

EXERCISE C

1 a



$$x^2 + 4^2 = 5^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x = \sqrt{5^2 - 4^2}$$

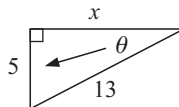
$$\therefore x = 3$$

$$\text{So, } \sin \theta = \frac{3}{5}$$

$$\cos \theta = \frac{4}{5}$$

$$\tan \theta = \frac{3}{4}$$

b



$$x^2 + 5^2 = 13^2 \quad \{\text{Pythagoras}\}$$

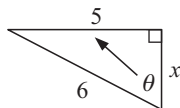
$$\therefore x = \sqrt{13^2 - 5^2}$$

$$\therefore x = 12$$

$$\text{So, } \sin \theta = \frac{5}{13}$$

$$\cos \theta = \frac{12}{13}$$

$$\tan \theta = \frac{5}{12}$$

c


$$x^2 + 5^2 = 6^2 \quad \{\text{Pythagoras}\}$$

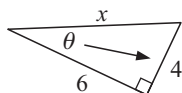
$$\therefore x = \sqrt{6^2 - 5^2}$$

$$\therefore x = \sqrt{11}$$

$$\text{So, } \sin \theta = \frac{5}{6}$$

$$\cos \theta = \frac{\sqrt{11}}{6}$$

$$\tan \theta = \frac{5}{\sqrt{11}}$$

e


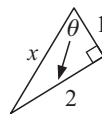
$$x^2 = 4^2 + 6^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x = \sqrt{52}$$

$$\text{So, } \sin \theta = \frac{4}{\sqrt{52}}$$

$$\cos \theta = \frac{6}{\sqrt{52}}$$

$$\tan \theta = \frac{4}{6} = \frac{2}{3}$$

d


$$x^2 = 1^2 + 2^2 \quad \{\text{Pythagoras}\}$$

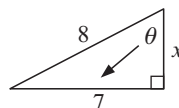
$$\therefore x = \sqrt{1 + 4}$$

$$\therefore x = \sqrt{5}$$

$$\text{So, } \sin \theta = \frac{2}{\sqrt{5}}$$

$$\cos \theta = \frac{1}{\sqrt{5}}$$

$$\tan \theta = \frac{2}{1} = 2$$

f


$$x^2 + 7^2 = 8^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x^2 = 8^2 - 7^2$$

$$\therefore x = \sqrt{64 - 49}$$

$$\therefore x = \sqrt{15}$$

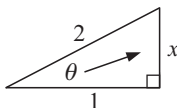
$$\text{So, } \sin \theta = \frac{7}{8}$$

$$\cos \theta = \frac{\sqrt{15}}{8}$$

$$\tan \theta = \frac{7}{\sqrt{15}}$$

2

a as $\cos \theta = \frac{1}{2}$



$$x^2 + 1^2 = 2^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x^2 = 4 - 1$$

$$\therefore x = \sqrt{3}$$

$$\text{So, } \sin \theta = \frac{x}{2} = \frac{\sqrt{3}}{2}$$

$$\tan \theta = \frac{x}{1} = \sqrt{3}$$

b

as $\sin \alpha = \frac{2}{3}$

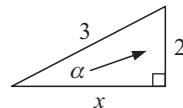
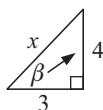
$$\therefore x^2 + 2^2 = 3^2$$

$$\therefore x^2 = 9 - 4$$

$$\therefore x = \sqrt{5}$$

$$\text{So, } \cos \alpha = \frac{x}{3} = \frac{\sqrt{5}}{3}$$

$$\tan \alpha = \frac{2}{x} = \frac{2}{\sqrt{5}}$$


c


as $\tan \beta = \frac{4}{3}$

$$x = 5 \quad \{3\text{-}4\text{-}5 \triangle\}$$

$$\text{So, } \sin \beta = \frac{4}{x} = \frac{4}{5}$$

$$\cos \beta = \frac{3}{x} = \frac{3}{5}$$

3

a $\sin \theta = \frac{b}{c}$

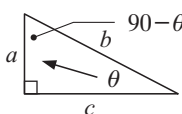
$$\cos \theta = \frac{a}{c}$$

$$\tan \theta = \frac{b}{a}$$

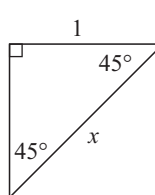
b

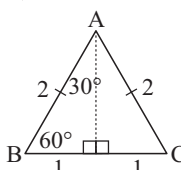
$$\frac{\sin \theta}{\cos \theta} = \frac{\frac{b}{c}}{\frac{a}{c}} = \frac{b}{c} \times \frac{c}{a} = \frac{b}{a}$$

$$\therefore \frac{\sin \theta}{\cos \theta} = \tan \theta$$

4 a 
i $\sin \theta = \frac{a}{c}$
ii $\cos \theta = \frac{b}{c}$
iii $\sin(90 - \theta) = \frac{b}{c}$
iv $\cos(90 - \theta) = \frac{a}{c}$

- b i** The sine of an angle is the cosine of its complement.
ii The cosine of an angle is the sine of its complement.

5 
a $x^2 = 1^2 + 1^2 = 2$
 $\therefore x = \sqrt{2}$ {Pythagoras}
 b $\sin 45^\circ = \frac{1}{x} = \frac{1}{\sqrt{2}}$
 $\cos 45^\circ = \frac{1}{x} = \frac{1}{\sqrt{2}}$
 $\tan 45^\circ = \frac{1}{1} = 1$

6 
a $\angle ABN = 60^\circ$
 $\angle BAN = 30^\circ$
b $\triangle ABC$ is equilateral and AN is perpendicular to BC.
 $\therefore$ N is the midpoint of BC.
 $\therefore BN = 1$
 $\therefore AN^2 = 2^2 - 1^2$ {Pythagoras}
 $\therefore AN = \sqrt{3}$

c i $\sin 60^\circ = \frac{AN}{2} = \frac{\sqrt{3}}{2}$
 $\cos 60^\circ = \frac{1}{2}$
 $\tan 60^\circ = \frac{AN}{1} = \sqrt{3}$
ii $\sin 30^\circ = \frac{1}{2}$
 $\cos 30^\circ = \frac{AN}{2} = \frac{\sqrt{3}}{2}$
 $\tan 30^\circ = \frac{1}{AN} = \frac{1}{\sqrt{3}}$

EXERCISE D

1 a $\sin 35^\circ = \frac{x}{30}$
 $\therefore 30 \times \sin 35^\circ = x$
 $\therefore x \div 17.2$
b $\cos 50^\circ = \frac{x}{400}$
 $\therefore 400 \times \cos 50^\circ = x$
 $\therefore x \div 257$
c $\tan 60^\circ = \frac{x}{8.7}$
 $\therefore \tan 60^\circ \times 8.7 = x$
 $\therefore x \div 15.1$

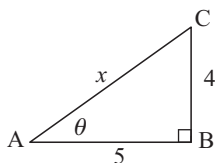
d $\cos 65^\circ = \frac{3}{x}$
 $\therefore x = \frac{3}{\cos 65^\circ}$
 $\therefore x \div 7.10$
e $\tan 36.7^\circ = \frac{413}{x}$
 $\therefore x = \frac{413}{\tan 36.7^\circ}$
 $\therefore x \div 554$
f $\sin 53.9^\circ = \frac{369}{x}$
 $\therefore x = \frac{369}{\sin 53.9^\circ}$
 $\therefore x \div 457$

2 a $\sin \theta = 0.9364$
 $\therefore \theta = \sin^{-1}(0.9364)$
 $\therefore \theta \div 69.5^\circ$
b $\cos \theta = 0.2381$
 $\therefore \theta = \cos^{-1}(0.2381)$
 $\therefore \theta \div 76.2^\circ$
c $\tan \theta = 1.7321$
 $\therefore \theta = \tan^{-1}(1.7321)$
 $\therefore \theta \div 60.0^\circ$

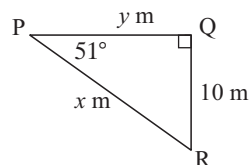
d $\cos \theta = \frac{2}{7}$
 $\therefore \theta = \cos^{-1}\left(\frac{2}{7}\right)$
 $\therefore \theta \div 73.4^\circ$
e $\sin \theta = \frac{1}{3}$
 $\therefore \theta = \sin^{-1}\left(\frac{1}{3}\right)$
 $\therefore \theta \div 19.5^\circ$
f $\tan \theta = \frac{14}{3}$
 $\therefore \theta = \tan^{-1}\left(\frac{14}{3}\right)$
 $\therefore \theta \div 77.9^\circ$

g $\sin \theta = \frac{\sqrt{3}}{11}$
 $\therefore \theta = \sin^{-1}\left(\frac{\sqrt{3}}{11}\right)$
 $\therefore \theta \div 9.06^\circ$
h $\cos \theta = \frac{5}{\sqrt{37}}$
 $\therefore \theta = \cos^{-1}\left(\frac{5}{\sqrt{37}}\right)$
 $\therefore \theta \div 34.7^\circ$

3 a $\sin \theta = \frac{5}{6}$
 $\therefore \theta = \sin^{-1}\left(\frac{5}{6}\right)$
 $\therefore \theta \div 56.4^\circ$
b $\tan \alpha = \frac{1}{12}$
 $\therefore \alpha = \tan^{-1}\left(\frac{1}{12}\right)$
 $\therefore \alpha \div 4.76^\circ$
c $\cos \beta = \frac{4}{6}$
 $\therefore \beta = \cos^{-1}\left(\frac{2}{3}\right)$
 $\therefore \beta \div 48.2^\circ$

4 a


$$\begin{aligned}
 x^2 &= 4^2 + 5^2 \quad \{\text{Pythagoras}\} & \tan \theta &= \frac{4}{5} \\
 \therefore x &= \sqrt{4^2 + 5^2} & \therefore \theta &= \tan^{-1}\left(\frac{4}{5}\right) \\
 \therefore x &= \sqrt{41} & \therefore \theta &\doteq 38.7 \\
 \therefore AC &= \sqrt{41} \text{ m} & \therefore \angle A &\doteq 38.7^\circ, \angle C \doteq 51.3^\circ \\
 &\text{or } \doteq 6.40 \text{ m}
 \end{aligned}$$

b


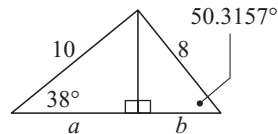
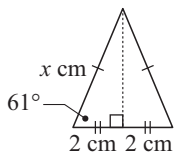
$$\begin{aligned}
 \angle R &= 90^\circ - 51^\circ = 39^\circ & \sin 51^\circ &= \frac{10}{x} & \tan 51^\circ &= \frac{10}{y} \\
 \therefore x &= \frac{10}{\sin 51^\circ} & \therefore y &= \frac{10}{\tan 51^\circ} \\
 \therefore x &\doteq 12.9 & \therefore y &\doteq 8.10 \\
 \text{So, PR} &\doteq 12.9 \text{ m} & \text{So, PQ} &\doteq 8.10 \text{ m}
 \end{aligned}$$

5 a

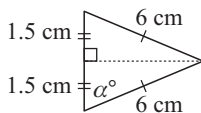
$$\begin{aligned}
 x^2 + 3^2 &= 4^2 \quad \{\text{Pythagoras}\} & \tan \theta &= \frac{2}{x} = \frac{2}{\sqrt{7}} \\
 \therefore x &= \sqrt{4^2 - 3^2} & \therefore \theta &= \tan^{-1}\left(\frac{2}{\sqrt{7}}\right) \text{ and so } \theta \doteq 37.1 \\
 \therefore x &= \sqrt{7} \quad (\text{or } 2.65)
 \end{aligned}$$

b

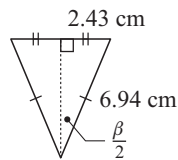
$$\begin{aligned}
 \sin 38^\circ &= \frac{x}{10} & \sin \theta &= \frac{x}{8} \\
 \therefore 10 \times \sin 38^\circ &= x & \therefore \sin \theta &\doteq \frac{6.1566}{8} \\
 \therefore x &\doteq 6.16 & \therefore \theta &\doteq \sin^{-1}\left(\frac{6.1566}{8}\right) \\
 & & \therefore \theta &\doteq 50.3 \\
 \cos 38^\circ &= \frac{a}{10} \text{ and } \cos 50.3157^\circ = \frac{b}{8} \\
 \text{Now } y &= a + b \\
 &= 10 \cos 38^\circ + 8 \cos 50.3157^\circ \\
 &\doteq 13.0
 \end{aligned}$$


6 a


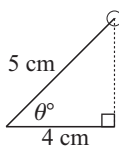
$$\begin{aligned}
 \cos 61^\circ &= \frac{2}{x} \\
 \therefore x &= \frac{2}{\cos 61^\circ} \\
 \therefore x &\doteq 4.13
 \end{aligned}$$

b


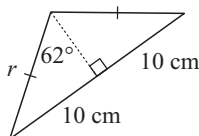
$$\begin{aligned}
 \cos \alpha &= \frac{1.5}{6} \\
 \therefore \alpha &= \cos^{-1}\left(\frac{1.5}{6}\right) \\
 \therefore \alpha &\doteq 75.5
 \end{aligned}$$

c


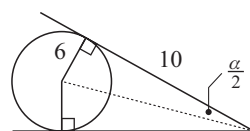
$$\begin{aligned}
 \sin\left(\frac{\beta}{2}\right) &= \frac{2.43}{6.94} \\
 \therefore \beta &= 2 \times \sin^{-1}\left(\frac{2.43}{6.94}\right) \\
 \therefore \beta &\doteq 41.0
 \end{aligned}$$

7 a


$$\begin{aligned}
 \cos \theta &= \frac{4}{5} \\
 \therefore \theta &= \cos^{-1}\left(\frac{4}{5}\right) \\
 \therefore \theta &\doteq 36.9
 \end{aligned}$$

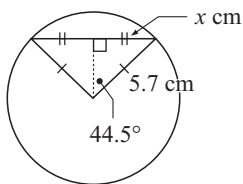
b


$$\begin{aligned}
 \sin 62^\circ &= \frac{10}{r} \\
 \therefore r &= \frac{10}{\sin 62^\circ} \\
 \therefore r &\doteq 11.3
 \end{aligned}$$

c


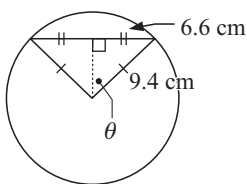
$$\begin{aligned}
 \tan\left(\frac{\alpha}{2}\right) &= \frac{6}{10} = 0.6 \\
 \therefore \alpha &= 2 \times \tan^{-1}(0.6) \\
 \therefore \alpha &\doteq 61.9
 \end{aligned}$$

8



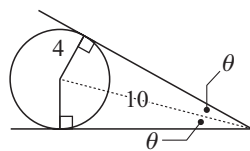
$$\begin{aligned}\sin 44.5^\circ &= \frac{x}{5.7} \\ \therefore 5.7 \times \sin 44.5^\circ &= x \\ \therefore x &\div 3.995 \\ \therefore 2x &\div 7.99 \\ \therefore \text{chord is } 7.99 \text{ cm long.}\end{aligned}$$

9



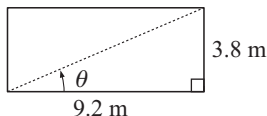
$$\begin{aligned}\sin \theta &= \frac{6.6}{9.4} \\ \therefore \theta &= \sin^{-1}\left(\frac{6.6}{9.4}\right) \\ \therefore \theta &\div 44.6 \\ \therefore 2\theta &\div 89.2 \\ \therefore \text{angle is } 89.2^\circ\end{aligned}$$

10



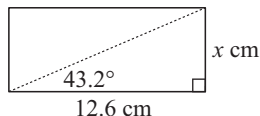
$$\begin{aligned}\sin \theta &= \frac{4}{10} = 0.4 \\ \therefore \theta &= \sin^{-1}(0.4) \\ \therefore 2\theta &= 2 \times \sin^{-1}(0.4) \\ &\div 47.2 \\ \therefore \text{angle is } 47.2^\circ\end{aligned}$$

11



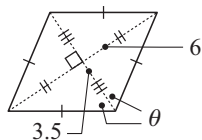
$$\begin{aligned}\tan \theta &= \frac{3.8}{9.2} \\ \therefore \theta &= \tan^{-1}\left(\frac{3.8}{9.2}\right) \\ \therefore \theta &\div 22.4 \\ \therefore \text{the angle is } 22.4^\circ\end{aligned}$$

12



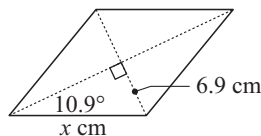
$$\begin{aligned}\tan 43.2^\circ &= \frac{x}{12.6} \\ \therefore 12.6 \times \tan 43.2^\circ &= x \\ \therefore x &\div 11.8 \\ \therefore \text{the shorter side is } 11.8 \text{ cm long}\end{aligned}$$

13



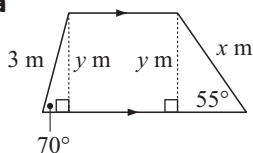
$$\begin{aligned}\tan \theta &= \frac{6}{3.5} \\ \therefore \theta &= \tan^{-1}\left(\frac{6}{3.5}\right) \\ \therefore 2\theta &= 2 \times \tan^{-1}\left(\frac{6}{3.5}\right) \div 119.49 \\ \text{So, the larger angle is } &\div 119^\circ\end{aligned}$$

14



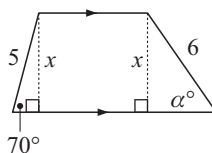
$$\begin{aligned}\sin 10.9^\circ &= \frac{6.9}{x} \\ \therefore x &= \frac{6.9}{\sin 10.9^\circ} \div 36.5 \\ \therefore \text{sides are } 36.5 \text{ cm long}\end{aligned}$$

15 a



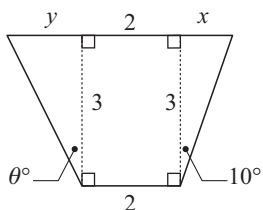
$$\begin{aligned}\sin 70^\circ &= \frac{y}{3} \\ \therefore y &= 3 \times \sin 70^\circ \\ \text{But } \sin 55^\circ &= \frac{y}{x} \\ \therefore \sin 55^\circ &= \frac{3 \times \sin 70^\circ}{x} \\ \therefore x &= \frac{3 \times \sin 70^\circ}{\sin 55^\circ} \div 3.44\end{aligned}$$

b



$$\begin{aligned}\sin 70^\circ &= \frac{x}{5} \\ \therefore 5 \times \sin 70^\circ &= x \\ \text{Now } \sin \alpha &= \frac{x}{6} \\ \therefore \sin \alpha &= \frac{5 \times \sin 70^\circ}{6} \\ \therefore \alpha &= \sin^{-1}\left(\frac{5 \times \sin 70^\circ}{6}\right) \div 51.5\end{aligned}$$

16



$$\tan 10^\circ = \frac{x}{3}$$

$$\text{Now } \tan \theta = \frac{y}{3}$$

$$\therefore 3 \times \tan 10^\circ = x$$

$$\therefore \tan \theta = 0.8237$$

$$\therefore x \doteq 0.5290$$

$$\therefore \theta \doteq 39.48^\circ$$

$$\text{But } y + 2 + x = 5$$

$$\text{So, } \beta \doteq 90 + 39.48^\circ$$

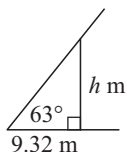
$$\therefore y \doteq 3 - 0.5290$$

$$\therefore \beta \doteq 129^\circ$$

$$\therefore y \doteq 2.471$$

EXERCISE E

1



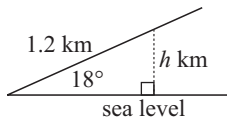
$$\tan 63^\circ = \frac{h}{9.32}$$

$$\therefore \tan 63^\circ \times 9.32 = h$$

$$\therefore h \doteq 18.3$$

$$\therefore \text{the height is } 18.3 \text{ m}$$

2 a



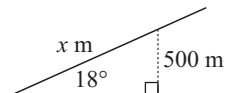
$$\sin 18^\circ = \frac{h}{1.2}$$

$$\therefore 1.2 \times \sin 18^\circ = h$$

$$\therefore h \doteq 0.371$$

$$\therefore \text{height is } 371 \text{ m above sea level.}$$

b



$$\sin 18^\circ = \frac{500}{x}$$

$$\therefore x = \frac{500}{\sin 18^\circ}$$

$$\therefore x \doteq 1618$$

$$\therefore \text{have walked } 1.62 \text{ km up the hill.}$$

3

$$\text{Let } AB = x \text{ m}$$

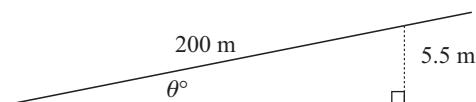
$$\text{Now } \tan 37^\circ = \frac{120}{x}$$

$$\therefore x = \frac{120}{\tan 37^\circ}$$

$$\therefore x \doteq 159$$

$$\therefore \text{the canal is } 159 \text{ m wide}$$

4



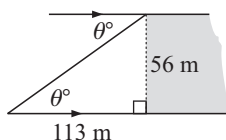
$$\sin \theta = \frac{5.5}{200}$$

$$\therefore \theta = \sin^{-1} \left(\frac{5.5}{200} \right)$$

$$\therefore \theta \doteq 1.58$$

$$\text{i.e., an incline of } 1.58^\circ$$

5



$$\tan \theta = \frac{56}{113}$$

$$\therefore \theta \doteq \tan^{-1} \left(\frac{56}{113} \right)$$

$$\therefore \theta \doteq 26.4$$

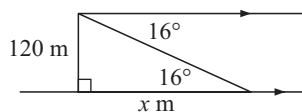
$$\therefore \text{angle of elevation is about } 26.4^\circ$$

$$\text{The angle of depression is also } \theta^\circ,$$

$$\text{i.e., about } 26.4^\circ$$

$$\{\text{equal alternate angles}\}$$

6



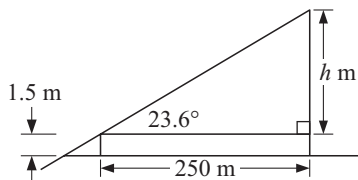
$$\tan 16^\circ = \frac{120}{x}$$

$$\therefore x = \frac{120}{\tan 16^\circ}$$

$$\therefore x \doteq 418$$

$$\therefore \text{boat is } 418 \text{ m out from the base of the cliff.}$$

7



$$\tan 23.6^\circ = \frac{h}{250}$$

$$\therefore 250 \times \tan 23.6^\circ = h$$

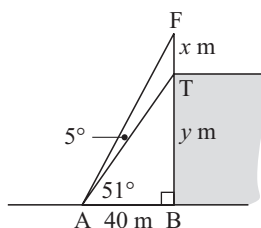
$$\therefore h = 109.2$$

$$\therefore \text{tree height} \div 109.2 + 1.5$$

$$\div 110.7 \text{ m}$$

$$\div 111 \text{ m}$$

9



$$\tan 51^\circ = \frac{y}{40}$$

$$\therefore 40 \times \tan 51^\circ = y$$

$$\therefore y \div 49.396$$

$$\tan 56^\circ = \frac{x+y}{40}$$

$$\therefore 40 \times \tan 56^\circ = x+y$$

$$\therefore x+y \div 59.302$$

$$\therefore x \div 59.302 - 49.396$$

$$\therefore x \div 9.91$$

i.e., it is 9.91 m high

11



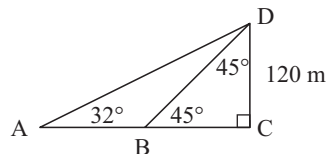
$$7.5 + 0.6 = 8.1 \text{ m}$$

$$\cos \theta = \frac{8.1}{8.3}$$

$$\therefore \theta = \cos^{-1} \left(\frac{8.1}{8.3} \right)$$

$$\therefore \theta \div 12.6$$

8



$\triangle BCD$ is right-angled isosceles

$$\therefore BC = 120 \text{ m also}$$

$$\text{In } \triangle ACD, \tan 32^\circ = \frac{120}{AC}$$

$$\therefore AC = \frac{120}{\tan 32^\circ}$$

$$\therefore AC \div 192$$

$$\therefore \text{she walks } (192 - 120) \text{ m}$$

$$\div 72 \text{ m}$$

$$10 \text{ a } v = \sqrt{gr \tan \theta}$$

$$= \sqrt{9.8 \times 100 \times \tan 15^\circ}$$

$$\div 16.2 \text{ m/s}$$

b

$$v = \sqrt{gr \tan \theta}$$

$$\therefore 20 = \sqrt{9.8 \times 200 \times \tan \theta}$$

$$\therefore 400 = 9.8 \times 200 \times \tan \theta$$

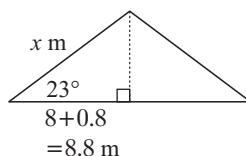
$$\therefore \frac{2}{9.8} = \tan \theta$$

$$\therefore \theta = \tan^{-1} \left(\frac{2}{9.8} \right)$$

$$\therefore \theta \div 11.5$$

i.e., a banked angle of 11.5°

12



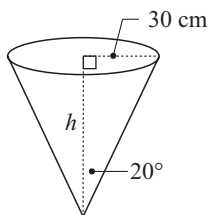
$$\cos 23^\circ = \frac{8.8}{x}$$

$$\therefore x = \frac{8.8}{\cos 23^\circ}$$

$$\therefore x \div 9.56$$

$\therefore$ a beam is 9.56 m long

13



$$\tan 20^\circ = \frac{30}{h}$$

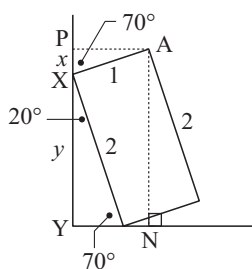
$$\therefore h = \frac{30}{\tan 20^\circ}$$

$$\therefore h \doteq 82.42 \text{ cm}$$

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi \times 30^2 \times 82.42 \\ &\doteq 77\,683 \text{ cm}^3 \end{aligned}$$

$\therefore$ capacity is 77 683 mL
i.e., $\doteq 77.7 \text{ L}$

14



Height $AN = PY$ also

$$\text{Now } \cos 70^\circ = \frac{x}{1}$$

$$\therefore x = \cos 70^\circ$$

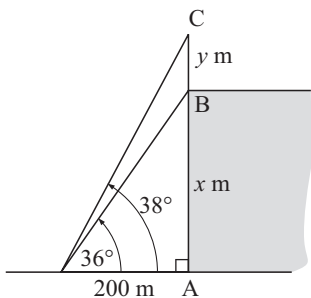
$$\text{and } \sin 70^\circ = \frac{y}{2}$$

$$\therefore y = 2 \sin 70^\circ$$

$$\begin{aligned} \text{Now } PY &= x + y \\ &= \cos 70^\circ + 2 \sin 70^\circ \\ &\doteq 2.22 \text{ m} \end{aligned}$$

$\therefore$ A is 2.22 m above the floor

15



$$\tan 36^\circ = \frac{x}{200}$$

$$\therefore 200 \times \tan 36^\circ = x$$

$$\therefore x \doteq 145.31 \quad \dots\dots (1)$$

$$\tan 38^\circ = \frac{x+y}{200}$$

$$\therefore 200 \times \tan 38^\circ = x + y$$

$$\therefore x + y \doteq 156.26 \quad \dots\dots (2)$$

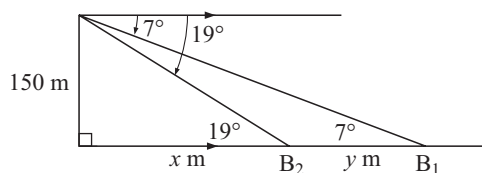
Using (1) and (2)

$$y \doteq 156.26 - 145.31$$

$$\therefore y \doteq 10.9$$

$\therefore$ the pole is about 10.9 m long.

16



$$\tan 19^\circ = \frac{150}{x}$$

$$\therefore x = \frac{150}{\tan 19^\circ} \doteq 435.63 \quad \dots\dots (1)$$

$$\tan 7^\circ = \frac{150}{x+y}$$

$$\therefore x + y = \frac{150}{\tan 7^\circ} \doteq 1221.65 \quad \dots\dots (2)$$

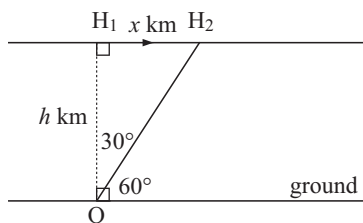
From (1) and (2)

$$y \doteq 1221.65 - 435.63$$

$$\therefore y \doteq 786$$

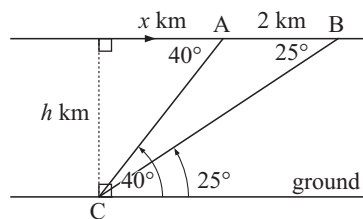
i.e., needs to be 786 m closer in.

17



$$\begin{aligned} \text{speed} &= \frac{\text{distance}}{\text{time}} \\ \therefore 100 &= \frac{x}{\frac{20}{3600} \text{ hours}} \\ \therefore x &= 100 \times \frac{20}{3600} \\ \therefore x &= \frac{20}{36} = \frac{5}{9} \\ \text{Now } \tan 30^\circ &= \frac{\frac{5}{9}}{h} \\ \therefore h &= \frac{\frac{5}{9}}{\tan 30^\circ} \\ \therefore h &\div 0.962 \text{ km} \\ \therefore &\text{ is about 962 m above ground level.} \end{aligned}$$

18



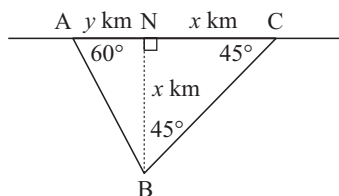
Notice that

$$\tan 40^\circ = \frac{h}{x}$$

and $\tan 25^\circ = \frac{h}{x+2}$

$$\begin{aligned} \therefore h &= x \tan 40^\circ \\ \text{and } h &= (x+2) \tan 25^\circ \\ \therefore x \tan 40^\circ &= (x+2) \tan 25^\circ \\ \therefore \frac{x+2}{x} &= \frac{\tan 40^\circ}{\tan 25^\circ} \\ \therefore \frac{x+2}{x} &\div 1.79945 \\ \therefore x+2 &\div 1.79945x \\ \therefore 2 &\div 0.79945x \\ \therefore x &\div 2.502 \\ \text{So } h &= 2.502 \times \tan 40^\circ \\ \therefore h &\div 2.10 \\ \text{i.e., } &2.10 \text{ km about ground level} \end{aligned}$$

19



$\triangle NBC$ is right angled isosceles

$$\therefore BN = x \text{ km}$$

Now $\tan 60^\circ = \frac{x}{y}$

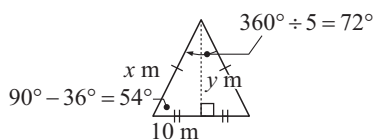
$$\therefore \sqrt{3} = \frac{x}{5-x}$$

{as $x+y=5$ }

$$\begin{aligned} \therefore 5\sqrt{3} - x\sqrt{3} &= x \\ \therefore 5\sqrt{3} &= x(1+\sqrt{3}) \\ \therefore \frac{5\sqrt{3}}{1+\sqrt{3}} &= x \\ \therefore x &\div 3.17 \\ \therefore &\text{ is 3.17 km from the shore} \end{aligned}$$

20

Each triangle is:



$$\begin{aligned} \cos 54^\circ &= \frac{10}{x} & \therefore x &= \frac{10}{\cos 54^\circ} \\ & & \therefore x &\div 17.01 \\ \tan 54^\circ &= \frac{y}{10} & \therefore y &= 10 \tan 54^\circ \\ & & \therefore y &\div 13.76 \\ \text{But } d &= x+y \\ & & &\div 17.01 + 13.76 \\ & & &\div 30.8 \end{aligned}$$

i.e., width of land is 30.8 m.

EXERCISE F

$$1 \quad a \quad \tan \theta = \frac{4-5}{-1-2}$$

$$\therefore \tan \theta = \frac{-1}{-3} = \frac{1}{3}$$

$$\therefore \theta = \tan^{-1}\left(\frac{1}{3}\right)$$

$$\therefore \theta \doteq 18.4^\circ$$

$$b \quad \tan \theta = \frac{-4-2}{-1-3}$$

$$= \frac{-2}{-4}$$

$$= \frac{1}{2}$$

$$\therefore \theta = \tan^{-1}\left(\frac{1}{2}\right)$$

$$\therefore \theta \doteq 26.6^\circ$$

$$c \quad \tan \theta = \frac{-5-1}{1-2}$$

$$= \frac{-6}{-1}$$

$$= 6$$

$$\therefore \theta = 180^\circ - \tan^{-1}(6)$$

$$\therefore \theta \doteq 116.6^\circ$$

$$d \quad \tan \theta = \frac{-1-4}{-2-7}$$

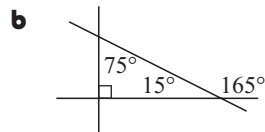
$$= \frac{-5}{-9}$$

$$= \frac{5}{9}$$

$$\therefore \theta = 29.1^\circ$$

$$2 \quad m = \tan 60^\circ$$

$$\doteq 1.73 \text{ or } \sqrt{3}$$



$$m = \tan 165^\circ$$

$$\doteq -0.268$$

$$3 \quad a \quad \theta = 60^\circ$$

$$\therefore \tan \theta = \sqrt{3} = m$$

$$\therefore y = \sqrt{3}x + 2$$

$$b \quad \theta = 120^\circ$$

$$\therefore m = \tan 120^\circ$$

$$\therefore m = -\sqrt{3}$$

$$\therefore y = -\sqrt{3}x$$

$$c \quad \theta = 30^\circ$$

$$\therefore \tan \theta = \frac{1}{\sqrt{3}}$$

$$\text{So, } y = \frac{1}{\sqrt{3}}x + c$$

But $(2\sqrt{3}, 0)$ lies on the line

$$\therefore 0 = \frac{1}{\sqrt{3}}(2\sqrt{3}) + c$$

$$\therefore 0 = 2 + c$$

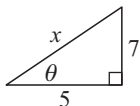
$$\therefore c = -2$$

$$\text{i.e., line is } y = \frac{1}{\sqrt{3}}x - 2$$

REVIEW SET A

$$1 \quad \sin \theta = \frac{7}{11}$$

2



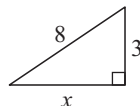
$$x^2 = 5^2 + 7^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x = \sqrt{5^2 + 7^2}$$

$$\therefore x = \sqrt{74}$$

$$\text{So, } \sin \theta = \frac{7}{\sqrt{74}}, \quad \cos \theta = \frac{5}{\sqrt{74}}$$

3



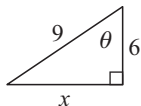
$$x^2 + 3^2 = 8^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x = \sqrt{8^2 - 3^2}$$

$$\therefore x = \sqrt{55}$$

$$\text{So, } \tan \theta = \frac{3}{\sqrt{55}}$$

4



$$x^2 + 6^2 = 9^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x = \sqrt{9^2 - 6^2}$$

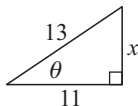
$$\therefore x = \sqrt{45}$$

$$\text{So, } \sin \theta = \frac{\sqrt{45}}{9}$$

$$\cos \theta = \frac{6}{9} = \frac{2}{3}$$

$$\tan \theta = \frac{\sqrt{45}}{6}$$

5



$$x^2 + 11^2 = 13^2$$

$$\therefore x = \sqrt{13^2 - 11^2}$$

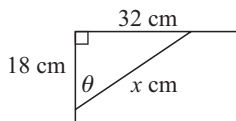
$$\therefore x = \sqrt{48}$$

$$\text{So, } \sin \theta = \frac{\sqrt{48}}{13}$$

$$\cos \theta = \frac{11}{13}$$

$$\tan \theta = \frac{\sqrt{48}}{11}$$

6



$$a \quad x^2 = 18^2 + 32^2$$

$$\therefore x = \sqrt{18^2 + 32^2}$$

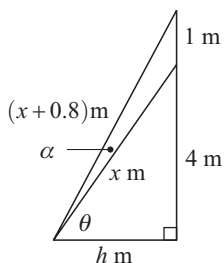
$$\therefore x \doteq 36.715$$

So, the support is 36.7 cm long

$$b \quad \tan \theta = \frac{32}{18}$$

$$\therefore \theta = \tan^{-1}\left(\frac{32}{18}\right) \doteq 60.6^\circ$$

$\therefore$ makes 60.6° with the wall.

7

a $x^2 = h^2 + 4^2 \dots (1)$

and $(x + 0.8)^2 = h^2 + 5^2$

$\therefore (x + 0.8)^2 - 25 = x^2 - 16$

$\therefore x^2 + 1.6x + 0.64 - 25 = x^2 - 16$

$\therefore 1.6x = 8.36$

$\therefore x \div 5.225$

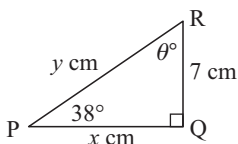
and $x + 0.8 \div 6.025$

$\therefore$ the extended ladder is 6.025 m long

b $\sin \theta = \frac{4}{5.225}$ $\sin(\theta + \alpha) = \frac{5}{6.025}$

$\therefore \theta \div 49.956$ $\therefore \theta + \alpha \div 56.086$

$\therefore \alpha \div 56.086 - 49.956 \div 6.13$ $\therefore$ angle increases by 6.13°

9

$\theta = 90 - 38 = 52$

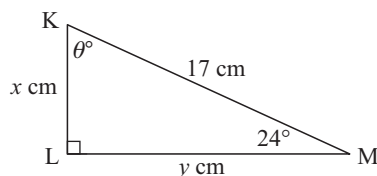
$\therefore \angle PRQ = 52^\circ$

$\tan 38^\circ = \frac{7}{x}$ and $\sin 38^\circ = \frac{7}{y}$

$\therefore x = \frac{7}{\tan 38^\circ}$ $\therefore y = \frac{7}{\sin 38^\circ}$

$\therefore x \div 8.96$ $\therefore y \div 11.4$

So $PQ \div 8.96$ cm $PR \div 11.4$ cm

8

$\theta = 90^\circ - 24^\circ = 66^\circ$

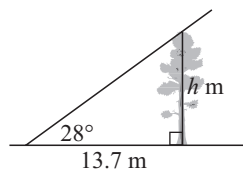
$\cos 24^\circ = \frac{y}{17}$, $\sin 24^\circ = \frac{x}{17}$

$\therefore y = 17 \cos 24^\circ$ $\therefore x = 17 \sin 24^\circ$

$\therefore y \div 15.5$ $\therefore x \div 6.91$

So, $KL \div 6.91$ cm

$LM \div 15.5$ cm

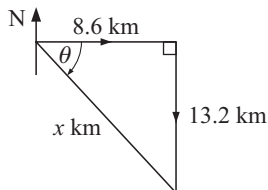
10

$\tan 28^\circ = \frac{h}{13.7}$

$\therefore 13.7 \times \tan 28^\circ = h$

$\therefore h \div 7.28$

$\therefore$ the height is 7.28 m

REVIEW SET B**1**

$x^2 = 8.6^2 + 13.2^2$

$\therefore x = \sqrt{8.6^2 + 13.2^2}$

$\therefore x \div 15.75$

So, distance is 15.75 km

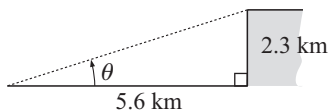
$\tan \theta = \frac{13.2}{8.6}$

$\therefore \theta = \tan^{-1} \left(\frac{13.2}{8.6} \right)$

$\therefore \theta \div 56.9^\circ$

and $90^\circ + 56.9^\circ \div 147^\circ$

$\therefore$ bearing is 147°

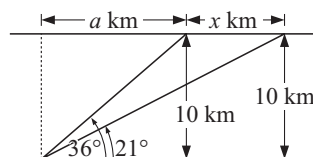
2


$$\tan \theta = \frac{2.3}{5.6} \quad \{\text{assuming a vertical mountain}\}$$

$$\therefore \theta = \tan^{-1} \left(\frac{2.3}{5.6} \right)$$

$$\therefore \theta \doteq 22.3^\circ$$

So, the angle of elevation is 22.3°

3


$$\tan 36^\circ = \frac{10}{a} \quad \text{and} \quad \tan 21^\circ = \frac{10}{a+x}$$

$$\therefore a = \frac{10}{\tan 36^\circ} \quad \text{and} \quad a+x = \frac{10}{\tan 21^\circ}$$

$$\therefore a \doteq 13.764 \quad \text{and} \quad a+x \doteq 26.051$$

$$\therefore x \doteq 26.051 - 13.764$$

$$\therefore x \doteq 12.287$$

$$\text{speed} = \frac{12.287 \text{ km}}{\frac{2}{60} \text{ hours}} \doteq 369 \text{ km/h}$$

4

a $\sin \theta = 0.8147$
 $\therefore \theta = \sin^{-1}(0.8147)$
 $\therefore \theta \doteq 54.6^\circ$

b

$\cos \theta = 0.0917$
 $\therefore \theta = \cos^{-1}(0.0917)$
 $\therefore \theta \doteq 84.7^\circ$

c

$\tan \theta = 5.23$
 $\therefore \theta = \tan^{-1}(5.23)$
 $\therefore \theta \doteq 79.2^\circ$

5

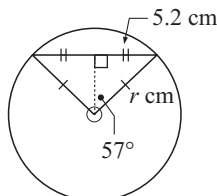
a $\sin \theta = \frac{\sqrt{11}}{5}$
 $\therefore \theta = \sin^{-1} \left(\frac{\sqrt{11}}{5} \right)$
 $\therefore \theta \doteq 41.6^\circ$

b

$\cos \theta = \frac{5}{7}$
 $\therefore \theta = \cos^{-1} \left(\frac{5}{7} \right)$
 $\therefore \theta \doteq 44.4^\circ$

c

$\tan \theta = 0.7452$
 $\therefore \theta = \tan^{-1}(0.7452)$
 $\therefore \theta \doteq 36.7^\circ$

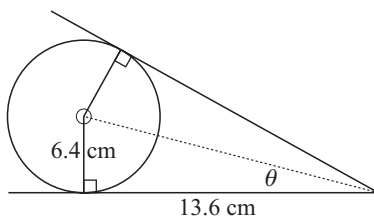
6


$$\sin 57^\circ = \frac{5.2}{r}$$

$$\therefore r = \frac{5.2}{\sin 57^\circ}$$

$$\therefore r \doteq 6.20$$

$\therefore$ the radius is 6.20 cm

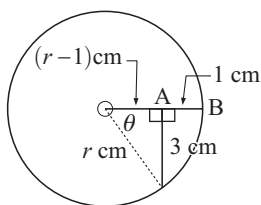
7


$$\tan \theta = \frac{6.4}{13.6}$$

$$\therefore \theta = \tan^{-1} \left(\frac{6.4}{13.6} \right)$$

$$\therefore \theta \doteq 25.2$$

So, the angle measures 25.2°

8


a If the radius OC is r cm

then $OA = (r-1)$ cm
 $\therefore (r-1)^2 + 3^2 = r^2 \quad \{\text{Pythagoras}\}$
 $\therefore r^2 - 2r + 1 + 9 = r^2$
 $\therefore -2r + 10 = 0$
 $\therefore r = 5$

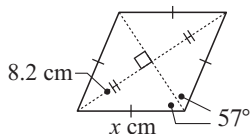
So, the radius is 5 cm

b $\sin \theta = \frac{3}{r} = \frac{3}{5}$

$\therefore \theta = \sin^{-1}(0.6)$
 $\therefore \theta \doteq 36.9$

So, BC subtends 36.9° at O.

9

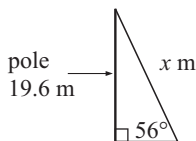


$$\sin 57^\circ = \frac{8.2}{x}$$

$$\therefore x = \frac{8.2}{\sin 57^\circ} \div 9.78$$

i.e., sides are 9.78 cm long

10



$$\sin 56^\circ = \frac{19.6}{x}$$

$$\therefore x = \frac{19.6}{\sin 56^\circ}$$

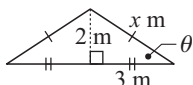
$$\therefore x \div 23.64$$

$$\text{and } 3x \div 70.9$$

So, the total length is 70.9 m

REVIEW SET C

1



$$\mathbf{a} \quad x^2 = 2^2 + 3^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x = \sqrt{2^2 + 3^2}$$

$$\therefore x \div 3.61$$

$\therefore$ beam AB is 3.61 m long.

$$\mathbf{b} \quad \tan \theta = \frac{2}{3}$$

$$\therefore \theta = \tan^{-1}\left(\frac{2}{3}\right)$$

$$\therefore \theta \div 33.7$$

$\therefore$ beam is inclined at 33.7° to the horizontal.

2

$$\tan \theta = \frac{6 - (-4)}{1 - 7} = \frac{10}{-6} = -\frac{5}{3}$$

$$\therefore \theta = 180^\circ - \tan^{-1}\left(\frac{5}{3}\right)$$

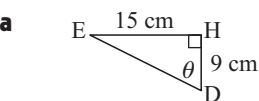
$$\therefore \theta \div 121^\circ$$

3

$$m = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\therefore \text{equation is } y = \frac{1}{\sqrt{3}}x - 3$$

4



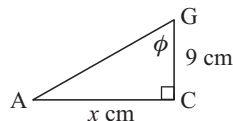
$$\mathbf{b} \quad \tan \theta = \frac{15}{9} = \frac{5}{3}$$

$$\therefore \theta = \tan^{-1}\left(\frac{5}{3}\right)$$

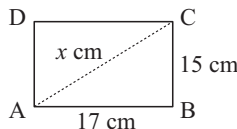
$$\therefore \theta \div 59.0$$

So, $\angle HDE$ is 59.0°

c



d



$$\text{By Pythagoras } x^2 = 15^2 + 17^2$$

$$\therefore x = \sqrt{15^2 + 17^2}$$

$$\therefore x \div 22.67$$

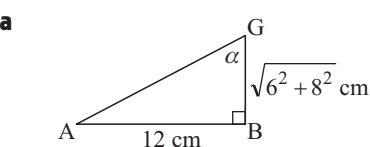
$$\text{and } \tan \phi = \frac{x}{9}$$

$$\therefore \phi = \tan^{-1}\left(\frac{22.67}{9}\right)$$

$$\therefore \phi \div 68.3$$

i.e., $\angle AGC$ is 68.3°

5



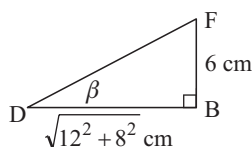
$$\tan \alpha = \frac{12}{\sqrt{6^2 + 8^2}}$$

$$\therefore \tan \alpha = \frac{12}{10}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{6}{5}\right) \div 50.2$$

So, the angle measures 50.2°

b

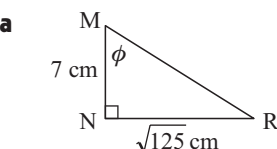


$$\tan \beta = \frac{6}{\sqrt{12^2 + 8^2}} = \frac{6}{\sqrt{208}}$$

$$\therefore \beta = \tan^{-1}\left(\frac{6}{\sqrt{208}}\right) \div 22.6$$

So, the angle measures 22.6°

6



$$\mathbf{b} \quad RN = \sqrt{5^2 + 10^2}$$

$$= \sqrt{125}$$

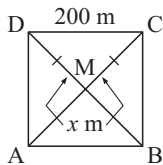
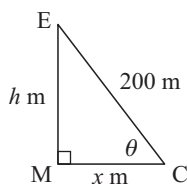
$$\div 11.180 \dots$$

$$\div 11.2 \text{ cm}$$

$$\mathbf{c} \quad \tan \phi = \frac{\sqrt{125}}{7}$$

$$\therefore \phi = \tan^{-1}\left(\frac{\sqrt{125}}{7}\right) \div 57.9$$

i.e., $\angle RMN$ is 57.9°

7 a


$$x^2 + x^2 = 200^2 \quad \{\text{Pythagoras}\}$$

$$\therefore 2x^2 = 40\,000$$

$$\therefore x^2 = 20\,000$$

Also $h^2 + x^2 = 200^2$

$$\therefore h^2 = 200^2 - 20\,000$$

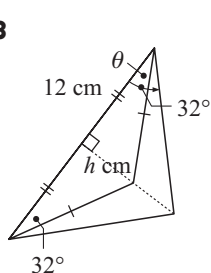
$$\therefore h = \sqrt{200^2 - 20\,000} \div 141$$

i.e., 141 m high

b

$$\sin \theta = \frac{h}{200} \div \frac{141.42}{200}$$

$$\therefore \theta = \sin^{-1} \left(\frac{141.42}{200} \right) \div 45^\circ$$

8


$$\tan 32^\circ = \frac{h}{12}$$

$$\therefore 12 \times \tan 32^\circ = h$$

$$\therefore h \div 7.4984$$

$$\text{Area} = \frac{1}{2} \text{ base} \times \text{height}$$

$$= \frac{1}{2} \times 24 \times 7.4984$$

$$\div 89.981$$

$$\therefore \text{new area} = 2 \times \text{old area}$$

$$\div 179.96 \text{ cm}^2$$

 Now if the new height is H cm

$$\frac{1}{2} \times 24 \times H \div 179.96$$

$$\therefore 12H \div 179.96$$

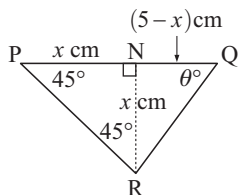
$$\therefore H \div 14.997$$

$$\therefore \tan \theta = \frac{14.997}{12}$$

$$\therefore \theta = \tan^{-1} \left(\frac{14.997}{12} \right)$$

$$\div 51.3$$

 So, the new base angles are 51.3° .

9


$$\text{Let } NR = x \text{ cm}$$

 Now $\triangle PNR$ is right angled isosceles

$$\therefore PN = x \text{ cm} \text{ also}$$

$$\text{and so } QN = (5 - x) \text{ cm}$$

$$\text{Now } \tan \theta = \frac{x}{5-x} \quad \{\text{in } \triangle RQN\}$$

$$\therefore (5-x) \tan \theta = x$$

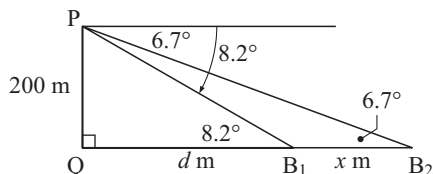
$$\therefore 5 \tan \theta - x \tan \theta = x$$

$$\therefore 5 \tan \theta = x + x \tan \theta$$

$$\therefore 5 \tan \theta = x(1 + \tan \theta)$$

$$\therefore \frac{5 \tan \theta}{1 + \tan \theta} = x$$

$$\text{i.e., } RN = \frac{5 \tan \theta}{1 + \tan \theta} \text{ cm}$$

10


$$\text{Let } QB_1 = d \text{ m} \text{ and } B_1B_2 = x \text{ m}$$

$$\therefore \tan 8.2^\circ = \frac{200}{d} \text{ and } \tan 6.7^\circ = \frac{200}{d+x}$$

$$\therefore d = \frac{200}{\tan 8.2} \div 1387.9$$

$$\text{and } d+x = \frac{200}{\tan 6.7} \div 1702.5$$

$$\text{So } x = 1702.5 - 1387.9 \div 315$$

i.e., the boats are 315 m apart.

Chapter 11

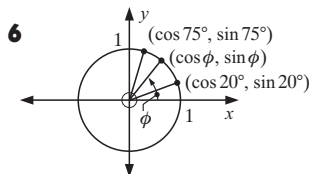
THE UNIT CIRCLE AND RADIAN MEASURE

EXERCISE 11A

- 1** **a** The y -coordinate at 0° is 0, $\therefore \sin 0^\circ = 0$
b The y -coordinate at 15° is $\div 0.26$, $\therefore \sin 15^\circ = 0.26$
c The y -coordinate at 25° is $\div 0.42$, $\therefore \sin 25^\circ = 0.42$
d The y -coordinate at 30° is $\div 0.5$, $\therefore \sin 30^\circ = 0.50$
e The y -coordinate at 45° is $\div 0.71$, $\therefore \sin 45^\circ = 0.71$
f The y -coordinate at 60° is $\div 0.87$, $\therefore \sin 60^\circ = 0.87$
g The y -coordinate at 75° is $\div 0.97$, $\therefore \sin 75^\circ = 0.97$
h The y -coordinate at 90° is 1, $\therefore \sin 90^\circ = 1$

- 3** **a** The x -coordinate at 0° is 1, $\therefore \cos 0^\circ = 1$
b The x -coordinate at 15° is $\div 0.97$, $\therefore \cos 15^\circ \div 0.97$
c The x -coordinate at 25° is $\div 0.91$, $\therefore \cos 25^\circ \div 0.91$
d The x -coordinate at 30° is $\div 0.87$, $\therefore \cos 30^\circ \div 0.87$
e The x -coordinate at 45° is $\div 0.71$, $\therefore \cos 45^\circ \div 0.71$
f The x -coordinate at 60° is $\div 0.5$, $\therefore \cos 60^\circ \div 0.50$
g The x -coordinate at 75° is $\div 0.26$, $\therefore \cos 75^\circ \div 0.26$
h The x -coordinate at 90° is 0, $\therefore \cos 90^\circ = 0$

- 5** The coordinates
 $(\cos 55^\circ, \sin 55^\circ)$
 i.e., $(0.57, 0.82)$

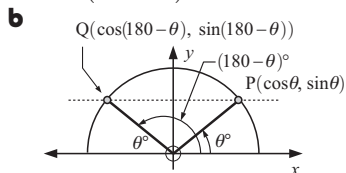


- 7** **a** As $\cos^2 \theta + \sin^2 \theta = 1$,
 then $\sin \theta = \sqrt{1 - \cos^2 \theta}$ as $\sin \theta > 0$
 $= \sqrt{1 - (0.8)^2}$
 $= 0.6$
- b** As $\cos^2 \theta + \sin^2 \theta = 1$,
 then $\cos \theta = \sqrt{1 - \sin^2 \theta}$ as $\cos \theta > 0$
 $= \sqrt{1 - (0.7)^2}$
 $\div 0.714$

EXERCISE 11B

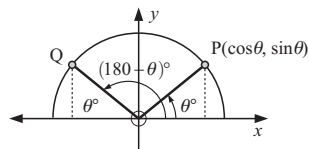
- 1** **a** 0.98 **b** 0.98 **c** 0.87 **d** 0.87 **e** 0.5 **f** 0.5 **g** 0 **h** 0

- 2** **a** It appears that
 $\sin(180 - \theta)^\circ = \sin \theta^\circ$



P and Q have the same
 y -coordinate.
 $\therefore \sin(180 - \theta)^\circ = \sin \theta^\circ$

- 3** **a** -0.34 **4** **a** $\cos(180 - \theta)^\circ = -\cos \theta^\circ$
b 0.34
c -0.64
d 0.64
e -0.77
f 0.77
g -1
h 1

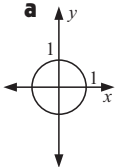
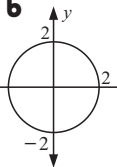
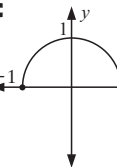
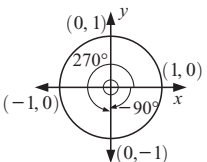


Q is
 $(\cos(180 - \theta)^\circ, \sin(180 - \theta)^\circ)$
 The x -coordinate of Q is the
 negative of the x -coordinate of P.
 $\therefore \cos(180 - \theta)^\circ = -\cos \theta^\circ$

- 5** **a** $\sin 45^\circ$
 $= \sin(180 - 45)^\circ$
 $= \sin 135^\circ$
 i.e., $\theta = 135^\circ$
- b** $\sin 51^\circ$
 $= \sin(180 - 51)^\circ$
 $= \sin 129^\circ$
 i.e., $\theta = 129^\circ$
- c** $\sin 74^\circ$
 $= \sin(180 - 74)^\circ$
 $= \sin 106^\circ$
 i.e., $\theta = 106^\circ$
- d** $\sin 82^\circ$
 $= \sin(180 - 82)^\circ$
 $= \sin 98^\circ$
 i.e., $\theta = 98^\circ$

- 6 a** $\sin 130^\circ$
 $= \sin(180 - 130)^\circ$
 $= \sin 50^\circ$
 i.e., $\theta = 50^\circ$
- b** $\sin 146^\circ$
 $= \sin(180 - 146)^\circ$
 $= \sin 34^\circ$
 i.e., $\theta = 34^\circ$
- c** $\sin 162^\circ$
 $= \sin(180 - 162)^\circ$
 $= \sin 18^\circ$
 i.e., $\theta = 18^\circ$
- d** $\sin 171^\circ$
 $= \sin(180 - 171)^\circ$
 $= \sin 9^\circ$
 i.e., $\theta = 9^\circ$
- 7 a** $\sin 137^\circ$
 $= \sin(180 - 137)^\circ$
 $= \sin 43^\circ$
 $\div 0.6820$
- b** $\sin 59^\circ$
 $= \sin(180 - 59)^\circ$
 $= \sin 121^\circ$
 $\div 0.8572$
- c** $\cos 143^\circ$
 $= -\cos(180 - 143)^\circ$
 $= -\cos 37^\circ$
 $\div -0.7986$
- d** $\cos 24^\circ$
 $= -\cos(180 - 24)^\circ$
 $= -\cos 156^\circ$
 $= 0.9135$
- e** $\sin 115^\circ$
 $= \sin(180 - 115)^\circ$
 $= \sin 65^\circ$
 $\div 0.9063$
- f** $\cos 132^\circ$
 $= -\cos(180 - 132)^\circ$
 $= -\cos 48^\circ$
 $= -0.6691$
- 8 a** $\angle AOQ + \angle BOQ = 180^\circ$
 $\therefore \theta^\circ + \angle BOQ = 180^\circ$
 $\angle BOQ = (180 - \theta)^\circ$
- b** OQ is a reflection of OP in the y -axis
 and so Q has coordinates $(-\cos \theta, \sin \theta)$
- c** $\cos(180 - \theta)^\circ = -\cos \theta^\circ$, $\sin(180 - \theta)^\circ = \sin \theta^\circ$

EXERCISE 11C

- 1 a** 
- b** 
- c** 
- 2 a i** A($\cos 26^\circ$, $\sin 26^\circ$) B($\cos 146^\circ$, $\sin 146^\circ$)
 C($\cos 199^\circ$, $\sin 199^\circ$)
- ii** A(0.899, 0.438) B(-0.829, 0.559)
 C(-0.946, -0.326)
- b i** A($\cos 123^\circ$, $\sin 123^\circ$) B($\cos 251^\circ$, $\sin 251^\circ$)
 C($\cos(-35^\circ)$, $\sin(-35^\circ)$)
- ii** A(-0.545, 0.839) B(-0.326, -0.946)
 C(0.819, -0.574)
- 3** 
- a** $\cos 0^\circ = 1$, $\sin 0^\circ = 0$
- b** $\cos 90^\circ = 0$, $\sin 90^\circ = 1$
- c** $\cos 180^\circ = -1$, $\sin 180^\circ = 0$
- d** $\cos 270^\circ = 0$, $\sin 270^\circ = -1$
- e** $\cos(-90^\circ) = 0$, $\sin(-90^\circ) = -1$
- f** $\cos 450^\circ = 0$, $\sin 450^\circ = 1$

EXERCISE 11D.1

- 1 a** $\frac{\pi}{4}$
 $= \frac{180^\circ}{4}$
 $= 45^\circ$
- b** $\frac{\pi}{6}$
 $= \frac{180^\circ}{6}$
 $= 30^\circ$
- c** $\frac{2\pi}{3}$
 $= \frac{2 \times 180^\circ}{3}$
 $= 120^\circ$
- d** $\frac{3\pi}{2}$
 $= \frac{3 \times 180^\circ}{2}$
 $= 270^\circ$
- e** $\frac{5\pi}{3}$
 $= \frac{5 \times 180^\circ}{3}$
 $= 300^\circ$
- 2 a** 30°
 $= \frac{1}{6}$ of 180°
 $= \frac{\pi}{6}$
 $\therefore$ arc length = $\frac{\pi}{6}$ units
- b** 60°
 $= \frac{1}{3}$ of 180°
 $= \frac{\pi}{3}$
 $\therefore$ arc length = $\frac{\pi}{3}$ units
- c** 90°
 $= \frac{1}{2}$ of 180°
 $= \frac{\pi}{2}$
 $\therefore$ arc length = $\frac{\pi}{2}$ units
- d** 120°
 $= \frac{2}{3}$ of 180°
 $= \frac{2\pi}{3}$
 $\therefore$ arc length = $\frac{2\pi}{3}$ units
- e** 135°
 $= \frac{3}{4}$ of 180°
 $= \frac{3\pi}{4}$
 $\therefore$ arc length = $\frac{3\pi}{4}$ units
- f** 150°
 $= \frac{5}{6}$ of 180°
 $= \frac{5\pi}{6}$
 $\therefore$ arc length = $\frac{5\pi}{6}$ units

g 225° $= 5 \times 45^\circ$ $= 5 \times \frac{1}{4} \text{ of } 180^\circ$ $= \frac{5}{4} \times \pi$ $= \frac{5\pi}{4} \quad \therefore \text{ arc length} = \frac{5\pi}{4} \text{ units}$	h 270° $= 3 \times 90^\circ$ $= 3 \times \frac{\pi}{2}$ $= \frac{3\pi}{2}$ $\therefore \text{ arc length} = \frac{3\pi}{2} \text{ units}$
--	---

- 3 a** $\triangle OAP$ is isosceles, so $\angle AOP = 60^\circ \therefore \theta = 60^\circ$.
- b** 60° is $\frac{1}{6}$ of 360° , $\therefore AP = \frac{1}{6} \times 2\pi$ i.e., $AP = \frac{\pi}{3}$.
- c** As $AP < \text{arc } AP$, then $AP < 1 \therefore \theta < 60^\circ$, i.e., θ decreases.
- d** When $AP = \pi$, $\theta = 180^\circ$ So, when $AP = 1$, $\theta = \frac{180^\circ}{\pi} \div 57.3$

EXERCISE 11D.2

- | | | |
|--|---|---|
| 1 a $180^\circ = \pi \text{ radians}$
$\therefore 90^\circ = \frac{\pi}{2} \text{ radians}$ | b $180^\circ = \pi \text{ radians}$
$\therefore 60^\circ = \frac{\pi}{3} \text{ radians}$ | c $180^\circ = \pi \text{ radians}$
$\therefore 30^\circ = \frac{\pi}{6} \text{ radians}$ |
| d $180^\circ = \pi \text{ radians}$
$\therefore 18^\circ = \frac{\pi}{10} \text{ radians}$ | e $180^\circ = \pi \text{ radians}$
$\therefore 9^\circ = \frac{\pi}{20} \text{ radians}$ | f $180^\circ = \pi \text{ radians}$
$\therefore 45^\circ = \frac{\pi}{4} \text{ radians}$
$\therefore 135^\circ = \frac{3\pi}{4} \text{ radians}$ |
| g $180^\circ = \pi \text{ radians}$
$\therefore 45^\circ = \frac{\pi}{4} \text{ radians}$
$\therefore 225^\circ = \frac{5\pi}{4} \text{ radians}$ | h $180^\circ = \pi \text{ radians}$
$\therefore 90^\circ = \frac{\pi}{2} \text{ radians}$
$\therefore 270^\circ = \frac{3\pi}{2} \text{ radians}$ | i $360^\circ = 2 \times 180^\circ$
$= 2\pi \text{ radians}$ |
| j $720^\circ = 4 \times 180^\circ$
$= 4\pi \text{ radians}$ | k $180^\circ = \pi \text{ radians}$
$\therefore 45^\circ = \frac{\pi}{4} \text{ radians}$
$\therefore 315^\circ = \frac{7\pi}{4} \text{ radians}$ | l $180^\circ = \pi \text{ radians}$
$\therefore 540^\circ = 3\pi \text{ radians}$ |
| m $180^\circ = \pi \text{ radians}$
$\therefore 36^\circ = \frac{\pi}{5} \text{ radians}$ | n $180^\circ = \pi \text{ radians}$
$\therefore 10^\circ = \frac{\pi}{18} \text{ radians}$
$\therefore 80^\circ = \frac{8\pi}{18} \text{ radians}$
$= \frac{4\pi}{9} \text{ radians}$ | o $180^\circ = \pi \text{ radians}$
$\therefore 10^\circ = \frac{\pi}{18} \text{ radians}$
$\therefore 230^\circ = \frac{23\pi}{18} \text{ radians}$ |
-
- | | | |
|--|---|---|
| 2 a 36.7°
$= 36.7 \times \frac{\pi}{180} \text{ radians}$
$\div 0.641 \text{ radians}$ | b 137.2°
$= 137.2 \times \frac{\pi}{180} \text{ radians}$
$\div 2.39 \text{ radians}$ | c 317.9°
$= 317.9 \times \frac{\pi}{180} \text{ radians}$
$\div 5.55 \text{ radians}$ |
| d 219.6°
$= 219.6 \times \frac{\pi}{180} \text{ radians}$
$\div 3.83 \text{ radians}$ | e 396.7°
$= 396.7 \times \frac{\pi}{180} \text{ radians}$
$\div 6.92 \text{ radians}$ | |
-
- | | | | | |
|--|--|---|--|--|
| 3 a $\frac{\pi}{5}$
$= \frac{180^\circ}{5}$
$= 36^\circ$ | b $\frac{3\pi}{5}$
$= \frac{3 \times 180^\circ}{5}$
$= 108^\circ$ | c $\frac{3\pi}{4}$
$= \frac{3 \times 180^\circ}{4}$
$= 135^\circ$ | d $\frac{\pi}{18}$
$= \frac{180^\circ}{18}$
$= 10^\circ$ | e $\frac{\pi}{9}$
$= \frac{180^\circ}{9}$
$= 20^\circ$ |
| f $\frac{7\pi}{9}$
$= \frac{7 \times 180^\circ}{9}$
$= 140^\circ$ | g $\frac{\pi}{10}$
$= \frac{180^\circ}{10}$
$= 18^\circ$ | h $\frac{3\pi}{20}$
$= \frac{3 \times 180^\circ}{20}$
$= 27^\circ$ | i $\frac{5\pi}{6}$
$= \frac{5 \times 180^\circ}{6}$
$= 150^\circ$ | j $\frac{\pi}{8}$
$= \frac{180^\circ}{8}$
$= 22\frac{1}{2}^\circ$ |

- 4 a** 2°
 $= 2 \times \frac{180}{\pi}$ degrees
 $\doteq 114.59^\circ$
- b** 1.53°
 $= 1.53 \times \frac{180}{\pi}$ degrees
 $\doteq 87.66^\circ$
- c** 0.867°
 $= 0.867 \times \frac{180}{\pi}$ degrees
 $\doteq 49.68^\circ$
- d** 3.179°
 $= 3.179 \times \frac{180}{\pi}$ degrees
 $\doteq 182.14^\circ$
- e** 5.267°
 $= 5.267 \times \frac{180}{\pi}$ degrees
 $\doteq 301.78^\circ$

5 a

Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

b

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

EXERCISE 11E.1

- 1 a** $\cos(\frac{\pi}{2}) = 0$
 $\sin(\frac{\pi}{2}) = 1$
- b** $\cos 2\pi = 1$
 $\sin 2\pi = 0$
- c** $\cos(-\frac{\pi}{2}) = 0$
 $\sin(-\frac{\pi}{2}) = -1$
- d** $\cos(\frac{7\pi}{2}) = 0$
 $\sin(\frac{7\pi}{2}) = -1$

- 2 a** $\cos^2 \theta + \sin^2 \theta = 1$
 $\therefore \cos^2 \theta + \frac{1}{4} = 1$
 $\therefore \cos^2 \theta = \frac{3}{4}$
 $\therefore \cos \theta = \pm \frac{\sqrt{3}}{2}$
- b** $\cos^2 \theta + \sin^2 \theta = 1$
 $\therefore \cos^2 \theta + \frac{1}{9} = 1$
 $\therefore \cos^2 \theta = \frac{8}{9}$
 $\therefore \cos \theta = \pm \frac{\sqrt{8}}{3}$
 $= \pm \frac{2\sqrt{2}}{3}$
- c** $\cos^2 \theta + \sin^2 \theta = 1$
 $\therefore \cos^2 \theta + 0 = 1$
 $\therefore \cos \theta = \pm 1$
- d** $\cos^2 \theta + \sin^2 \theta = 1$
 $\therefore \cos^2 \theta + 1 = 1$
 $\therefore \cos \theta = 0$

- 3 a** $\cos^2 \theta + \sin^2 \theta = 1$
 $\therefore \frac{16}{25} + \sin^2 \theta = 1$
 $\therefore \sin^2 \theta = \frac{9}{25}$
 $\therefore \sin \theta = \pm \frac{3}{5}$
- b** $\cos^2 \theta + \sin^2 \theta = 1$
 $\therefore \frac{9}{16} + \sin^2 \theta = 1$
 $\therefore \sin^2 \theta = \frac{7}{16}$
 $\therefore \sin \theta = \pm \frac{\sqrt{7}}{4}$
- c** $\cos^2 \theta + \sin^2 \theta = 1$
 $\therefore 1 + \sin^2 \theta = 1$
 $\therefore \sin^2 \theta = 0$
 $\therefore \sin \theta = 0$
- d** $\cos^2 \theta + \sin^2 \theta = 1$
 $\therefore 0 + \sin^2 \theta = 1$
 $\therefore \sin \theta = \pm 1$

4 a

Quadrant	Degree measure	Radian measure	$\cos \theta$	$\sin \theta$
1	$0 < \theta < 90$	$0 < \theta < \frac{\pi}{2}$	+ve	+ve
2	$90 < \theta < 180$	$\frac{\pi}{2} < \theta < \pi$	-ve	+ve
3	$180 < \theta < 270$	$\pi < \theta < \frac{3\pi}{2}$	-ve	-ve
4	$270 < \theta < 360$	$\frac{3\pi}{2} < \theta < 2\pi$	+ve	-ve

- b i** 1 and 4
ii 2 and 3
iii 3
iv 2

5 a $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \frac{4}{9} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{5}{9}$$

$$\therefore \sin \theta = \pm \frac{\sqrt{5}}{3}$$

But θ is in quadrant 1

where $\sin \theta > 0$

$$\therefore \sin \theta = \frac{\sqrt{5}}{3}$$

d $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \frac{25}{169} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{144}{169}$$

$$\therefore \sin \theta = \pm \frac{12}{13}$$

b $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \frac{4}{25} = 1$$

$$\therefore \cos^2 \theta = \frac{21}{25}$$

$$\therefore \cos \theta = \pm \frac{\sqrt{21}}{5}$$

But θ is in quadrant 2

where $\cos \theta < 0$

$$\therefore \cos \theta = -\frac{\sqrt{21}}{5}$$

But θ is in quadrant 3

where $\sin \theta < 0$

$$\therefore \sin \theta = -\frac{12}{13}$$

c $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \cos^2 \theta + \frac{9}{25} = 1$$

$$\therefore \cos^2 \theta = \frac{16}{25}$$

$$\therefore \cos \theta = \pm \frac{4}{5}$$

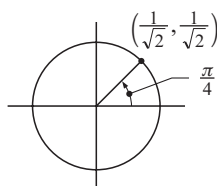
But θ is in quadrant 4

where $\cos \theta > 0$

$$\therefore \cos \theta = \frac{4}{5}$$

EXERCISE 11E.2

1



So $\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$$\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

Draw separate unit circle diagrams for each case.

	a	b	c	d	e
$\sin \theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$\cos \theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$

2 Likewise, draw separate unit circle diagrams for each angle.

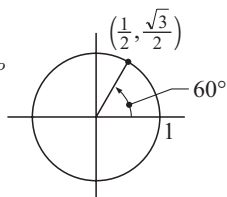
	a	b	c	d	e
$\sin \beta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\cos \beta$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

3 a $\sin^2 60^\circ$

$$= \sin 60^\circ \times \sin 60^\circ$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}$$

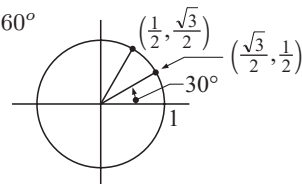
$$= \frac{3}{4}$$



b $\sin 30^\circ \cos 60^\circ$

$$= \frac{1}{2} \times \frac{1}{2}$$

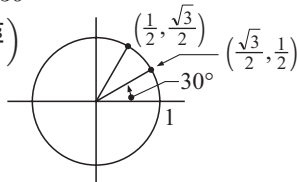
$$= \frac{1}{4}$$



c $4 \sin 60^\circ \cos 30^\circ$

$$= 4 \left(\frac{\sqrt{3}}{2} \right) \left(\frac{\sqrt{3}}{2} \right)$$

$$= 3$$

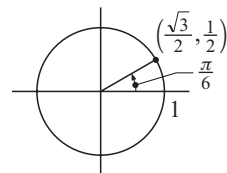


d $1 - \cos^2(\frac{\pi}{6})$

$$= 1 - \left(\frac{\sqrt{3}}{2} \right)^2$$

$$= 1 - \frac{3}{4}$$

$$= \frac{1}{4}$$

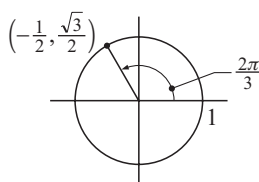


e $\sin^2(\frac{2\pi}{3}) - 1$

$$= \left(\frac{\sqrt{3}}{2} \right)^2 - 1$$

$$= \frac{3}{4} - 1$$

$$= -\frac{1}{4}$$

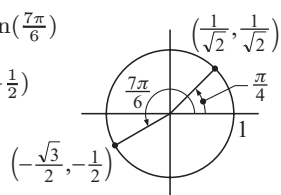


f $\cos^2(\frac{\pi}{4}) - \sin(\frac{7\pi}{6})$

$$= \left(\frac{1}{\sqrt{2}} \right)^2 - \left(-\frac{1}{2} \right)$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

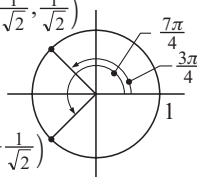


g $\sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right) \quad \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$= \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} \text{ or } \sqrt{2} \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

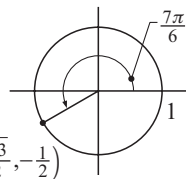


h $1 - 2\sin^2\left(\frac{7\pi}{6}\right)$

$$= 1 - 2\left(-\frac{1}{2}\right)^2$$

$$= 1 - 2 \times \frac{1}{4}$$

$$= \frac{1}{2} \quad \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

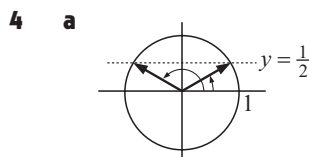
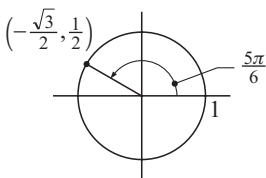


i $\cos^2\left(\frac{5\pi}{6}\right) - \sin^2\left(\frac{5\pi}{6}\right)$

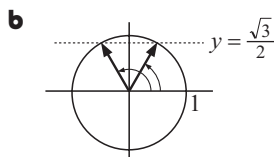
$$= \left(-\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$= \frac{3}{4} - \frac{1}{4}$$

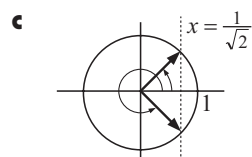
$$= \frac{1}{2}$$



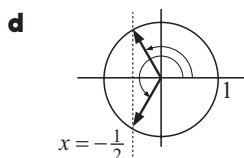
$$\theta = 30^\circ \text{ or } 150^\circ$$



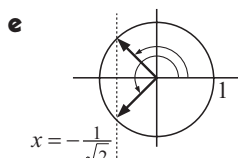
$$\theta = 60^\circ \text{ or } 120^\circ$$



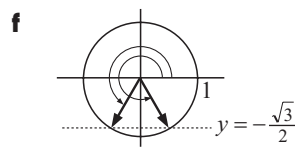
$$\theta = 45^\circ \text{ or } 315^\circ$$



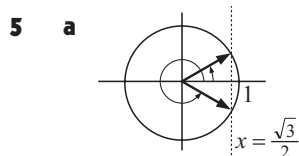
$$\theta = 120^\circ \text{ or } 240^\circ$$



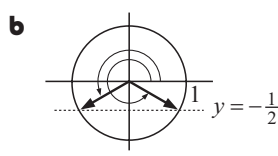
$$\theta = 135^\circ \text{ or } 225^\circ$$



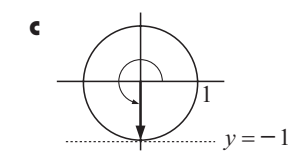
$$\theta = 240^\circ \text{ or } 300^\circ$$



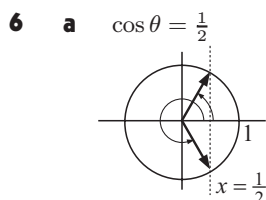
$$\theta = 30^\circ, 330^\circ, 390^\circ, 690^\circ$$



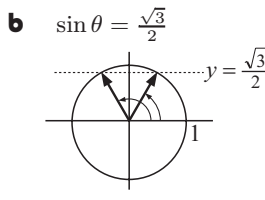
$$\theta = 210^\circ, 330^\circ, 570^\circ, 690^\circ$$



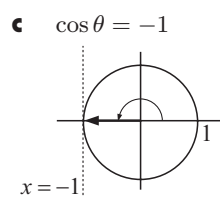
$$\theta = 270^\circ, 630^\circ$$



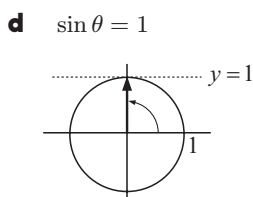
$$\therefore \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$



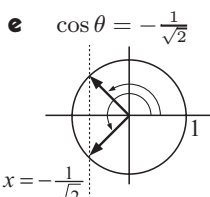
$$\therefore \theta = \frac{\pi}{3}, \frac{2\pi}{3}$$



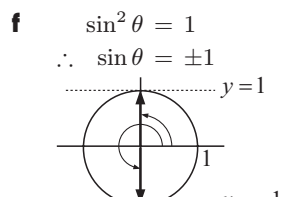
$$\therefore \theta = \pi$$



$$\therefore \theta = \frac{\pi}{2}$$

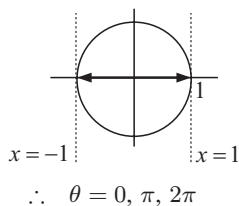


$$\therefore \theta = \frac{3\pi}{4}, \frac{5\pi}{4}$$

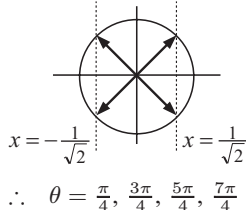


$$\therefore \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

g $\cos^2 \theta = 1$
 $\therefore \cos \theta = \pm 1$



h $\cos^2 \theta = \frac{1}{2} \therefore \cos \theta = \pm \frac{1}{\sqrt{2}}$



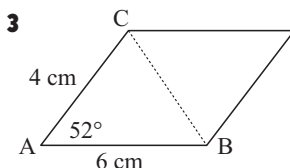
EXERCISE 11F

1 a Area
 $= \frac{1}{2} \times 9 \times 10 \times \sin 40^\circ$
 $\div 28.9 \text{ cm}^2$

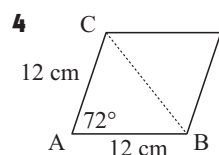
b Area
 $= \frac{1}{2} \times 25 \times 31 \times \sin 82^\circ$
 $\div 384 \text{ km}^2$

c Area
 $= \frac{1}{2} \times 10.2 \times 6.4 \times \sin 125^\circ$
 $\div 26.7 \text{ cm}^2$

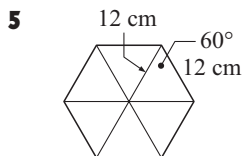
2 Area = 150 cm^2
 $\therefore \frac{1}{2} \times 17 \times x \times \sin 68^\circ = 150$
 $\therefore x = \frac{2 \times 150}{17 \times \sin 68^\circ}$
 $\therefore x \div 19.0$



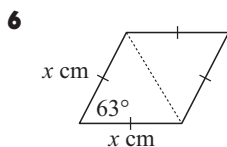
Area
 $= 2 \times \text{area } \triangle ABC$
 $= 2 \times \frac{1}{2} \times 4 \times 6 \times \sin 52^\circ$
 $\div 18.9 \text{ cm}^2$



Area = $2 \times \text{area } \triangle ABC$
 $= 2 \times \frac{1}{2} \times 12^2 \times \sin 72^\circ$
 $\div 137 \text{ cm}^2$

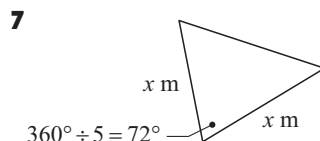


Area
 $= 6 \times \text{area of } \triangle$
 $= 6 \times \frac{1}{2} \times 12^2 \times \sin 60^\circ$
 $\div 374 \text{ cm}^2$



Area = $2 \times \frac{1}{2} x^2 \sin 63^\circ$
 $\therefore x^2 \sin 63^\circ = 50$
 $\therefore x^2 = \frac{50}{\sin 63^\circ}$
 $\therefore x = \sqrt{\frac{50}{\sin 63^\circ}}$
 $\therefore x \div 7.49$

So, sides are 7.49 cm long.



Area of $\triangle = \frac{338}{5}$
 $\therefore \frac{1}{2} x^2 \sin 72^\circ = \frac{338}{5}$
 $\therefore x^2 = \frac{2 \times 338}{5 \times \sin 72^\circ}$
 $\therefore x = \sqrt{\frac{2 \times 338}{5 \times \sin 72^\circ}}$
 $\therefore x \div 11.9$

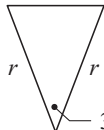
So, OA $\div 11.9$ m long.

8 a If the included angle is θ
then $\frac{1}{2} \times 5 \times 8 \times \sin \theta = 15$
 $\therefore 20 \sin \theta = 15$

$\therefore \sin \theta = \frac{3}{4}$
 $\therefore \theta = \sin^{-1} \left(\frac{3}{4} \right)$
 $\therefore \theta \div 48.6^\circ$ or $(180 - 48.6)^\circ$
i.e., $\theta \div 48.6^\circ$ or 131.4°

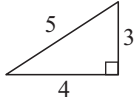
b Likewise,
 $\frac{1}{2} \times 45 \times 53 \times \sin \theta = 800$
 $\therefore \sin \theta = \frac{800 \times 2}{45 \times 53}$

$\therefore \theta = \sin^{-1} \left(\frac{1600}{45 \times 53} \right)$
 $\therefore \theta \div 42.1^\circ$ or $(180 - 42.1)^\circ$
 $\therefore \theta \div 42.1^\circ$ or 137.9°

9 
 Total area of 8 coins = $8 \times 12 \times \frac{1}{2} r^2 \sin 30^\circ$
 $= 48r^2 \left(\frac{1}{2}\right)$
 $= 24r^2$

Area of \$10 note = $8r \times 4r$
 $= 32r^2$

Fraction covered = $\frac{24r^2}{32r^2} = \frac{3}{4}$ $\therefore \frac{1}{4}$ is uncovered

10 a i 
 Area = $\frac{1}{2} \text{ base} \times \text{alt}$
 $= \frac{1}{2} \times 4 \times 3$
 $= 6 \text{ cm}^2$

ii $s = \frac{3+4+5}{2} = 6$
 $\therefore \text{area} = \sqrt{6(6-3)(6-4)(6-5)}$
 $= \sqrt{6 \times 3 \times 2 \times 1}$
 $= 6 \text{ cm}^2$

b i $s = \frac{6+8+12}{2} = 13$ $\therefore A = \sqrt{13(13-6)(13-8)(13-12)}$
 $= \sqrt{13 \times 7 \times 5 \times 1}$
 $\div 21.3 \text{ cm}^2$

ii $s = \frac{7.2+8.9+9.7}{2} = 12.9$ $\therefore A = \sqrt{12.9(12.9-7.2)(12.9-8.9)(12.9-9.7)}$
 $= \sqrt{12.9 \times 5.7 \times 4 \times 3.2}$
 $\div 30.7 \text{ cm}^2$

EXERCISE 11G

1 a i arc length = $\left(\frac{41.6}{360}\right) \times 2\pi \times 9$
 $\div 6.53 \text{ cm}$

ii area = $\left(\frac{41.6}{360}\right) \times \pi \times 9^2$
 $\div 29.4 \text{ cm}^2$

b i arc length = $\left(\frac{122}{360}\right) \times 2\pi \times 4.93$
 $\div 10.5 \text{ cm}$

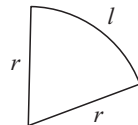
ii area = $\left(\frac{122}{360}\right) \times \pi \times 4.93^2$
 $\div 25.9 \text{ cm}^2$

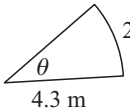
2 a $\theta = 107.9^\circ$, $l = 5.92$
 $\therefore \left(\frac{107.9}{360}\right) \times 2\pi \times r = 5.92$
 $\therefore r = \frac{5.92 \times 360}{107.9 \times 2 \times \pi}$
 $\therefore r \div 3.14 \text{ m}$

b Area = $\left(\frac{107.9}{360}\right) \times \pi \times (3.1436)^2$
 $\div 9.30 \text{ m}^2$

3 a Area = $\left(\frac{\theta}{360}\right) \times \pi r^2$
 $\therefore 20.8 = \left(\frac{68.2}{360}\right) \times \pi r^2$
 $\therefore \frac{20.8 \times 360}{68.2 \times \pi} = r^2$
 $\therefore r = \sqrt{\frac{20.8 \times 360}{68.2 \times \pi}}$
 $\therefore r \div 5.91 \text{ cm}$

b Perimeter = $l + 2r$
 $= \left(\frac{68.2}{360}\right) \times 2\pi \times 5.912 + 2 \times 5.912$
 $\div 18.9 \text{ cm}$



4 a 
 $l = \left(\frac{\theta}{360}\right) \times 2\pi \times r$
 $\therefore 2.95 = \left(\frac{\theta}{360}\right) \times 2\pi \times 4.3$
 $\therefore \frac{2.95 \times 360}{2 \times \pi \times 4.3} = \theta$
 $\therefore \theta \div 39.3^\circ$

b Area = $\left(\frac{\theta}{360}\right) \times \pi r^2$
 $\therefore 30 = \left(\frac{\theta}{360}\right) \times \pi \times 10^2$
 $\therefore \frac{30 \times 360}{\pi \times 100} = \theta$
 $\therefore \theta \div 34.4^\circ$

5 a $l = r\theta$

$$\therefore 6 = 8\theta$$

$$\therefore \theta = \frac{3}{4}^\circ$$

b $l = r\theta$

$$\therefore 8.4 = 5\theta$$

$$\therefore \theta = \frac{8.4}{5} = 1.68^\circ$$

c $l = r(2\pi - \theta)$

$$\therefore 31.7 = 8(2\pi - \theta)$$

$$\therefore 2\pi - \theta = \frac{31.7}{8}$$

$$\therefore \theta = 2\pi - \frac{31.7}{8}$$

$$\therefore \theta \div 2.32^\circ$$

6 a Area $= \frac{1}{2}r^2\theta$

$$= \frac{1}{2} \times 5^2 \times 0.7$$

$$= 8.75 \text{ cm}^2$$

b Shaded area = area of sector – area of Δ

$$= \frac{1}{2} \times 12^2 \times 1.5 - \frac{1}{2} \times 12 \times 12 \times \sin 1.5^\circ$$

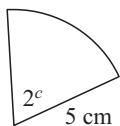
$$\div 36.2 \text{ cm}^2$$

c Area = area of Δ – area of sector

$$= \frac{1}{2} \times 12 \times 30 \times \sin(0.66) - \frac{1}{2} \times 12^2 \times 0.66$$

$$\div 62.8 \text{ cm}^2$$

7



a Arc length

$$= r\theta$$

$$= 5 \times 2$$

$$= 10 \text{ cm}$$

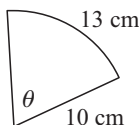
b Area

$$= \frac{1}{2}r^2\theta$$

$$= \frac{1}{2} \times 5^2 \times 2$$

$$= 25 \text{ cm}^2$$

8



a Arc length $= r\theta$

$$\therefore 13 = 10\theta$$

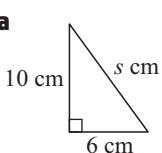
$$\therefore 1.3 = \theta$$

and area $= \frac{1}{2}r^2\theta$

$$= \frac{1}{2} \times 10^2 \times 1.3$$

$$= 65 \text{ cm}^2$$

9 a



$$s^2 = 6^2 + 10^2 \text{ \{Pythagoras\}}$$

$$\therefore s = \sqrt{6^2 + 10^2}$$

$$\therefore s \div 11.6619$$

$$\therefore s \div 11.7 \text{ cm}$$

d arc length $= \left(\frac{\theta}{360}\right) \times 2\pi r$

$$\therefore \frac{\theta}{360} \times 2 \times \pi \times 11.6619 \div 37.6991$$

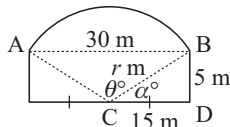
$$\therefore \frac{37.6991 \times 360}{2 \times \pi \times 11.6619} \div \theta$$

$$\therefore \theta \div 185^\circ$$

b $r = s \div 11.7 \text{ cm}$

c arc length $= 2\pi \times 6 \div 37.6991 \dots \div 37.7 \text{ cm}$

10



a $\tan \alpha = \frac{5}{15}$

$$\therefore \alpha = \tan^{-1}\left(\frac{1}{3}\right)$$

$$\therefore \alpha \div 18.43$$

b $\theta + 2\alpha = 180$ {angles on a line}

$$\therefore \theta = 180 - 2 \times 18.43$$

$$\therefore \theta \div 143.1$$

Note: $r^2 = 5^2 + 15^2$

$$\therefore r^2 = 250$$

c Area $= 2 \times \text{area } \Delta CDB + \text{area sector}$

$$= 2 \times \frac{1}{2} \times 15 \times 5 + \left(\frac{143.1}{360}\right) \times \pi \times 250$$

$$\div 387.3 \text{ m}^2$$

11 a $l = \left(\frac{\theta}{360}\right) \times 2\pi r$

$$= \frac{1}{360} \times 2 \times \pi \times 6370 \text{ km}$$

$$\div 1.852957 \dots \text{ km}$$

$$\div 1.853 \text{ km}$$

b speed $= \frac{\text{distance}}{\text{time}} \therefore \text{time} = \frac{\text{distance}}{\text{speed}}$

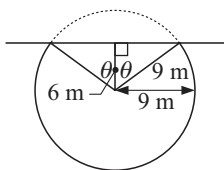
$$= \frac{2130 \text{ km}}{480 \text{ n miles/h}}$$

$$= \frac{2130 \text{ km}}{480 \times 1.853 \text{ km/h}}$$

$$\div 2.3947 \dots \text{ hours}$$

$$\div 2 \text{ hours } 24 \text{ min}$$

12



$$\cos \theta = \frac{6}{9} = \frac{2}{3}$$

$$\therefore \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

$$\therefore \theta \doteq 48.19^\circ$$

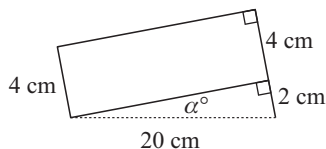
$$\text{So, } 360 - 2\theta \doteq 263.62^\circ$$

Area

 $= \text{area of } \Delta + \text{area of sector}$

$$= \frac{1}{2} \times 9^2 \times \sin 96.38^\circ + \left(\frac{263.62}{360}\right) \times \pi \times 9^2 \doteq 227 \text{ m}^2$$

13



$$\text{a } \sin \alpha = \frac{2}{20} = 0.1$$

$$\therefore \alpha = \sin^{-1}(0.1)$$

$$\therefore \alpha \doteq 5.7392^\circ$$

$$\therefore \alpha \doteq 5.739^\circ$$

$$\text{c } \phi + \theta = 360$$

$$\therefore \phi \doteq 360 - 168.5$$

$$\therefore \phi \doteq 191.5$$

$$\text{b } \theta + 90 + 90 + 2\alpha = 360$$

$$\therefore \theta = 180 - 2\alpha$$

$$\doteq 180 - 2 \times 5.739$$

$$\doteq 168.5$$

$$\text{d } \text{length of belt}$$

$$= 2 \times \sqrt{20^2 - 2^2}$$

$$+ \frac{\theta}{360} \times 2\pi \times 4$$

$$+ \frac{\phi}{360} \times 2\pi \times 6$$

$$\doteq 71.62 \text{ cm}$$

REVIEW SET 11A

$$1 \quad \text{a } \sin 70^\circ \doteq 0.94 \quad \text{b } \cos 35^\circ \doteq 0.82$$

$$2 \quad M(\cos 73^\circ, \sin 73^\circ) \doteq (0.292, 0.956) \quad N(\cos 190^\circ, \sin 190^\circ) \doteq (-0.985, -0.174)$$

$$P(\cos 307^\circ, \sin 307^\circ) \doteq (0.602, -0.799)$$

$$3 \quad \text{The } x\text{-coordinate of A} = -0.222$$

$$\therefore \cos \theta = -0.222$$

$$\therefore \theta = \cos^{-1}(-0.222)$$

$$\therefore \theta \doteq 102.8^\circ$$

$$4 \quad \text{a } \sin 120^\circ = \sin(180 - 120)^\circ = \sin 60^\circ$$

$$\therefore \theta = 60^\circ$$

$$\text{b } \sin 165^\circ = \sin(180 - 165)^\circ = \sin 15^\circ$$

$$\therefore \theta = 15^\circ$$

$$\text{c } \sin 95^\circ = \sin(180 - 95)^\circ = \sin 85^\circ$$

$$\therefore \theta = 85^\circ$$

$$\begin{aligned} 5 \quad \text{a } & \sin 47^\circ \\ &= \sin(180 - 47)^\circ \\ &= \sin 133^\circ \\ \therefore \theta &= 133^\circ \end{aligned}$$

$$\begin{aligned} \text{b } & \sin 8^\circ \\ &= \sin(180 - 8)^\circ \\ &= \sin 172^\circ \\ \therefore \theta &= 172^\circ \end{aligned}$$

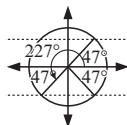
$$\begin{aligned} \text{c } & \sin 86^\circ \\ &= \sin(180 - 86)^\circ \\ &= \sin 94^\circ \\ \therefore \theta &= 94^\circ \end{aligned}$$

$$\begin{aligned} 6 \quad \text{a } & \sin 159^\circ \\ &= \sin(180 - 159)^\circ \\ &= \sin 21^\circ \\ &\doteq 0.358 \end{aligned}$$

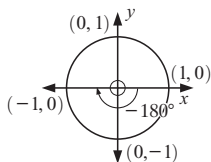
$$\begin{aligned} \text{b } & \cos 92^\circ \\ &= -\cos(180 - 92)^\circ \\ &= -\cos 88^\circ \\ &\doteq -0.035 \end{aligned}$$

$$\begin{aligned} \text{c } & \cos 75^\circ \\ &= -\cos(180 - 75)^\circ \\ &= -\cos 105^\circ \\ &\doteq 0.259 \end{aligned}$$

$$\begin{aligned} \text{d } \sin 227^\circ &= \sin(-47)^\circ \\ &= -\sin 47^\circ \\ &\doteq -0.731 \end{aligned}$$



7



$$\text{a } \cos 360^\circ = 1, \quad \sin 360^\circ = 0$$

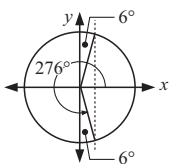
$$\text{b } \cos(-180^\circ) = -1, \quad \sin(-180^\circ) = 0$$

$$\begin{aligned} 8 \quad \text{a } & \sin 101^\circ \\ &= \sin(180 - 101)^\circ \\ &= \sin 79^\circ \\ \therefore \theta &= 79^\circ \end{aligned}$$

$$\begin{aligned} \text{b } & \sin 127^\circ \\ &= \sin(180 - 127)^\circ \\ &= \sin 53^\circ \\ \therefore \theta &= 53^\circ \end{aligned}$$

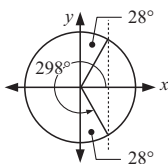
$$\begin{aligned} \text{c } & \sin 168^\circ \\ &= \sin(180 - 168)^\circ \\ &= \sin 12^\circ \\ \therefore \theta &= 12^\circ \end{aligned}$$

9 a



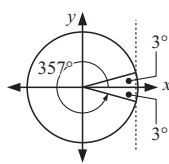
$$\therefore \theta = (90 - 6)^\circ = 84^\circ$$

b



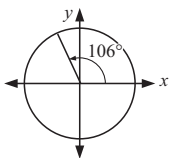
$$\therefore \theta = (90 - 28)^\circ = 62^\circ$$

c



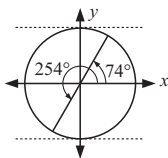
$$\therefore \theta = 3^\circ$$

10 a



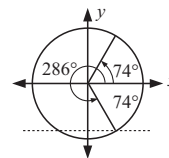
$$\begin{aligned} \sin 106^\circ &= \sin(180 - 106)^\circ \\ &= \sin 74^\circ \\ &\doteq 0.961 \end{aligned}$$

b



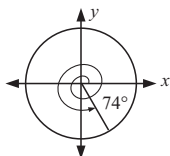
$$\begin{aligned} 254^\circ &= 74^\circ + 180^\circ \\ \therefore \sin 254^\circ &= -\sin 74^\circ \\ &\doteq -0.961 \end{aligned}$$

c



$$\begin{aligned} \sin 286^\circ &= -\sin 74^\circ \\ &= -0.961 \end{aligned}$$

d

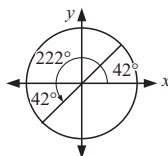


$$\begin{aligned} 646^\circ &= 360^\circ + 286^\circ \\ \therefore \sin 646^\circ &= \sin 286^\circ \\ &= -0.961 \quad \{\text{from c}\} \end{aligned}$$

11 a

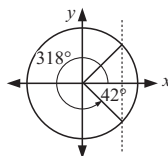
$$\begin{aligned} \cos 138^\circ &= -\cos(180 - 138)^\circ \\ &= -\cos 42^\circ \\ &\doteq -0.743 \end{aligned}$$

b



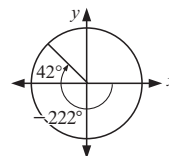
$$\begin{aligned} \cos 222^\circ &= -\cos 42^\circ \\ &= -0.743 \end{aligned}$$

c



$$\begin{aligned} \cos 318^\circ &= \cos 42^\circ \\ &= 0.743 \end{aligned}$$

d



$$\begin{aligned} \cos(-222)^\circ &= -\cos 42^\circ \\ &= -0.743 \end{aligned}$$

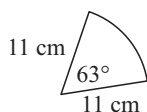
REVIEW SET 11B

1 Area = $\frac{1}{2} \times 7.3 \times 9.4 \times \sin 38^\circ$
 $\doteq 21.1 \text{ km}^2$

2 a Area = $\left(\frac{80}{360}\right) \times T \times 13^2 \doteq 118 \text{ cm}^2$

b Area = $\frac{1}{2} \times 11 \times 9 \times \sin 65^\circ \doteq 44.9 \text{ cm}^2$

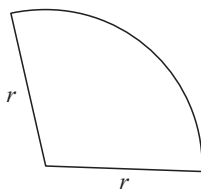
3



a Perimeter = $2 \times 11 + \left(\frac{63}{360}\right) \times 2\pi \times 11$
 $\doteq 34.1 \text{ cm}$

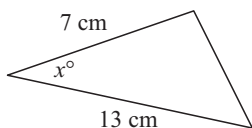
b Area = $\left(\frac{63}{360}\right) \times \pi \times 11^2$
 $\doteq 66.5 \text{ cm}^2$

4



$$\begin{aligned} \text{Perimeter} &= 2r + \left(\frac{120}{360}\right) \times 2\pi r & \text{Area} &= \left(\frac{120}{360}\right) \times \pi r^2 \\ \therefore 36 &= 2r + \frac{1}{3} \times 2\pi r & &= \frac{1}{3} \times \pi \times (8.7925)^2 \\ \therefore 36 &= r \left(2 + \frac{2\pi}{3}\right) & &\doteq 81.0 \text{ cm}^2 \\ \therefore r &= \frac{36}{2 + \frac{2\pi}{3}} \text{ cm} & & \\ \therefore r &\doteq 8.7925 \dots & & \\ \therefore r &\doteq 8.79 \text{ cm} & & \end{aligned}$$

5 Area = 42 cm^2 $\therefore \frac{1}{2} \times 7 \times 13 \times \sin x = 42$



$$\therefore \sin x = \frac{42 \times 2}{7 \times 13}$$

$$\therefore x = \sin^{-1}\left(\frac{84}{91}\right)$$

$$\therefore x \doteq 67.4 \quad \text{or} \quad 180 - 67.4$$

$$\therefore x = 67.4 \quad \text{or} \quad 112.6$$

i.e., the included angle is 67.4° or 112.6° {assuming the figure is not drawn accurately}

6 Area = 80 cm^2

$$\therefore \frac{1}{2} \times 11.3 \times 19.2 \sin x^\circ = 80$$

$$\therefore \sin x^\circ = \frac{160}{11.3 \times 19.2}$$

$$\therefore x = \sin^{-1}\left(\frac{160}{11.3 \times 19.2}\right)$$

$$\therefore x \doteq 47.5 \quad \text{or} \quad 180 - 47.5$$

$$\therefore x \doteq 47.5 \quad \text{or} \quad 132.5$$

When $x \doteq 47.5$, $AC \doteq \sqrt{19.2^2 + 11.3^2 - 2 \times 19.2 \times 11.3 \times \cos 47.5^\circ} \doteq 14.3 \text{ cm}$

When $x \doteq 132.5$, $AC \doteq \sqrt{19.2^2 + 11.3^2 - 2 \times 19.2 \times 11.3 \times \cos 132.5^\circ} \doteq 28.1 \text{ cm}$

7 Non shaded area

= area of Δ + area of sector

$$= \frac{1}{2} \times 7 \times 7 \times \sin 130^\circ + \left(\frac{230}{360}\right) \times \pi \times 7^2$$

$$\doteq 117.12 \text{ cm}^2$$

$\therefore$ shaded area

$$\doteq \pi \times 7^2 - 117.12 \text{ cm}^2$$

$$\doteq 36.8 \text{ cm}^2$$

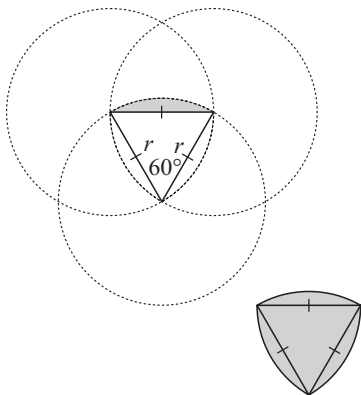
8 a $BD = \sqrt{125^2 + 120^2 - 2 \times 125 \times 120 \times \cos 75^\circ} \doteq 149.2$

$$\therefore \text{area} = \frac{1}{2} \times 120 \times 125 \sin 75^\circ + \frac{1}{2} \times 149.2 \times 90 \times \sin 30^\circ$$

$$\doteq 10\,600 \text{ m}^2$$

b $\doteq 1.06 \text{ ha}$ ($10\,000 \text{ m}^2 = 1 \text{ ha}$)

9



shaded area of sector

= area of sector – area of Δ

$$= \frac{1}{6} \pi r^2 - \frac{1}{2} \times r \times r \times \sin 60^\circ$$

$$= \frac{\pi}{6} r^2 - \frac{1}{2} r^2 \left(\frac{\sqrt{3}}{2}\right)$$

$\therefore$ shaded area of figure

$$= 3 \left[\frac{\pi}{6} r^2 - \frac{\sqrt{3}}{4} r^2 \right] + \frac{1}{2} r^2 \left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{2} r^2 - \frac{3\sqrt{3}}{4} r^2 + \frac{\sqrt{3}}{4} r^2$$

$$= \frac{\pi}{2} r^2 - \frac{1}{2} \sqrt{3} r^2$$

$$= \frac{r^2}{2} (\pi - \sqrt{3})$$

10 a 120°

$$= \left(120 \times \frac{\pi}{180}\right)^c$$

$$= \frac{2\pi}{3}^c$$

b 225°

$$= 5 \times 45^\circ$$

$$= 5 \times \frac{\pi}{4}^c$$

$$= \frac{5\pi}{4}^c$$

c 150°

$$= 5 \times 30^\circ$$

$$= 5 \times \frac{\pi}{6}^c$$

$$= \frac{5\pi}{6}^c$$

d 540°

$$= 3 \times 180^\circ$$

$$= 3\pi^c$$

11 a 71°

$$= \left(71 \times \frac{\pi}{180}\right)^c$$

$$\doteq 1.239^c$$

b 224.6°

$$= \left(224.6 \times \frac{\pi}{180}\right)^c$$

$$\doteq 2.175^c$$

c -142°

$$= \left(-142 \times \frac{\pi}{180}\right)^c$$

$$\doteq -2.478^c$$

d -25.3°

$$= \left(-25.3 \times \frac{\pi}{180}\right)^c$$

$$\doteq -0.4416^c$$

12 a	$\frac{2\pi}{5}$ $= \frac{2 \times 180^\circ}{5}$ $= 72^\circ$	b	$\frac{5\pi}{4}$ $= \frac{5 \times 180^\circ}{4}$ $= 225^\circ$	c	$\frac{7\pi}{9}$ $= \frac{7 \times 180^\circ}{9}$ $= 140^\circ$	d	$\frac{11\pi}{6}$ $= \frac{11 \times 180^\circ}{6}$ $= 330^\circ$
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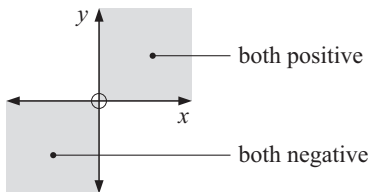
13 a	3^c $= \left(3 \times \frac{180}{\pi}\right)^\circ$ $\doteq 171.89^\circ$	b	1.46^c $= \left(1.46 \times \frac{180}{\pi}\right)^\circ$ $\doteq 83.65^\circ$	c	0.435° $= \left(0.435 \times \frac{180}{\pi}\right)^\circ$ $\doteq 24.92^\circ$	d	-5.271^c $= \left(-5.271 \times \frac{180}{\pi}\right)^\circ$ $\doteq -302.01^\circ$
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REVIEW SET 11C

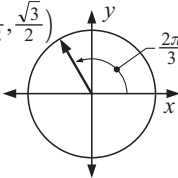
1 a The point is $(\cos 320^\circ, \sin 320^\circ)$, i.e., $(0.766, -0.643)$ approximately.

b The point is $(\cos 163^\circ, \sin 163^\circ)$, i.e., $(-0.956, 0.292)$ approximately.

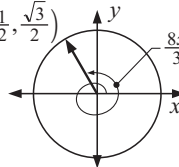
2



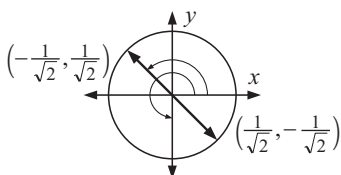
3 a $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $\therefore \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$
 $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$



b $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $\therefore \cos\left(\frac{8\pi}{3}\right) = -\frac{1}{2}$
 $\sin\left(\frac{8\pi}{3}\right) = \frac{\sqrt{3}}{2}$



4 when $\cos \theta = -\sin \theta$



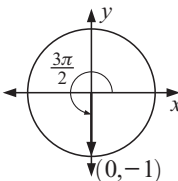
$$\frac{\sin \theta}{\cos \theta} = -1$$

and this only occurs at the two points shown.

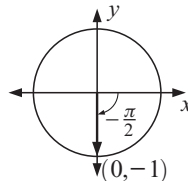
$$\therefore \tan \theta = -1$$

So, θ can be found.

5 a $\therefore \cos\left(\frac{3\pi}{2}\right) = 0$
 $\sin\left(\frac{3\pi}{2}\right) = -1$



b $\therefore \cos\left(-\frac{\pi}{2}\right) = 0$
 $\sin\left(-\frac{\pi}{2}\right) = -1$



6 $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \frac{9}{16} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{7}{16}$$

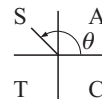
$$\therefore \sin \theta = \pm \frac{\sqrt{7}}{4}$$

7 $\cos^2 \theta + \sin^2 \theta = 1$

$$\therefore \frac{9}{16} + \sin^2 \theta = 1$$

$$\therefore \sin^2 \theta = \frac{7}{16}$$

$$\therefore \sin \theta = \pm \frac{\sqrt{7}}{4}$$

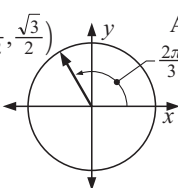


But θ is obtuse

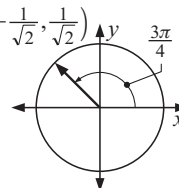
$\therefore \sin \theta$ is positive

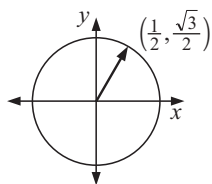
$$\therefore \sin \theta = \frac{\sqrt{7}}{4}$$

8 a $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ As $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$
 $\sin^2\left(\frac{2\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^2$
 $= \frac{3}{4}$

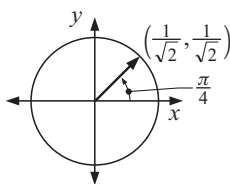


b $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $\cos\left(\frac{3\pi}{4}\right) - \sin\left(\frac{3\pi}{4}\right)$
 $= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$
 $= -\frac{2}{\sqrt{2}}$
 $= -\sqrt{2}$

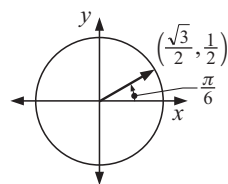


9 a


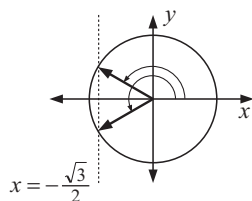
$$\begin{aligned} & 2 \sin\left(\frac{\pi}{3}\right) \cos\left(\frac{\pi}{3}\right) \\ &= 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

b


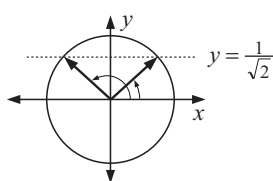
$$\begin{aligned} & \sin^2\left(\frac{\pi}{4}\right) - 1 \\ &= \left(\frac{1}{\sqrt{2}}\right)^2 - 1 \\ &= \frac{1}{2} - 1 \\ &= -\frac{1}{2} \end{aligned}$$

c


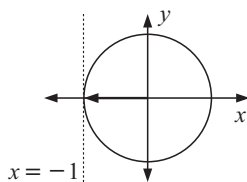
$$\begin{aligned} & \cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right) \\ &= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \\ &= \frac{3}{4} - \frac{1}{4} \\ &= \frac{2}{4} \\ &= \frac{1}{2} \end{aligned}$$

10 a


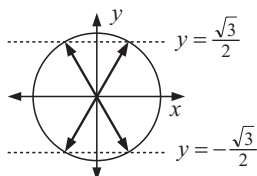
$$\therefore \theta = 150^\circ \text{ or } 210^\circ$$

b


$$\therefore \theta = 45^\circ \text{ or } 135^\circ$$

11 a


$$\begin{aligned} & x = -1 \\ \therefore \theta &= \pi + k2\pi, \quad k \in \mathbb{Z} \end{aligned}$$

b $\sin^2 \theta = \frac{3}{4} \therefore \sin \theta = \pm \frac{\sqrt{3}}{2}$


$$\therefore \theta = \left\{ \frac{\pi}{3}, \frac{2\pi}{3} \right\} + k\pi, \quad k \in \mathbb{Z}$$

Chapter 12

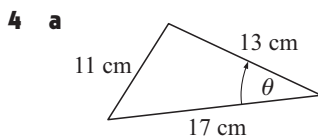
NON RIGHT ANGLED TRIANGLE TRIGONOMETRY

EXERCISE 12A

- 1 a** $BC^2 = 21^2 + 15^2 - 2 \times 21 \times 15 \times \cos 105^\circ$
 $\therefore BC = \sqrt{21^2 + 15^2 - 2 \times 21 \times 15 \times \cos 105^\circ} \doteq 28.8 \text{ cm}$
- b** $PQ^2 = 6.3^2 + 4.8^2 - 2 \times 6.3 \times 4.8 \times \cos 32^\circ$
 $\therefore PQ = \sqrt{6.3^2 + 4.8^2 - 2 \times 6.3 \times 4.8 \times \cos 32^\circ} \doteq 3.38 \text{ km}$
- c** $KM^2 = 6.2^2 + 14.8^2 - 2 \times 6.2 \times 14.8 \times \cos 72^\circ$
 $\therefore KM = \sqrt{6.2^2 + 14.8^2 - 2 \times 6.2 \times 14.8 \times \cos 72^\circ} \doteq 14.2 \text{ m}$

2 $\cos A = \frac{12^2 + 13^2 - 11^2}{2 \times 12 \times 13}$ $\cos B = \frac{13^2 + 11^2 - 12^2}{2 \times 13 \times 11}$ $C = (180 - A - B)^\circ$
 $\therefore A = \cos^{-1}\left(\frac{192}{312}\right)$ $\therefore B = \cos^{-1}\left(\frac{146}{286}\right)$ $\doteq 68.7^\circ$
 $\therefore A \doteq 52.0^\circ$ $\therefore B \doteq 59.3^\circ$

3 $\cos Q = \frac{5^2 + 7^2 - 10^2}{2 \times 5 \times 7}$
 $\therefore Q = \cos^{-1}\left(\frac{-26}{70}\right)$
 $\therefore Q \doteq 112^\circ$



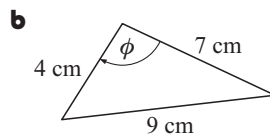
The smallest angle is opposite the shortest side.

$$\cos \theta = \frac{13^2 + 17^2 - 11^2}{2 \times 13 \times 17}$$

$$\therefore \theta = \cos^{-1}\left(\frac{337}{442}\right)$$

$$\therefore \theta \doteq 40.3$$

So, the smallest angle measures 40.3° .



The largest angle is opposite the longest side.

$$\cos \phi = \frac{4^2 + 7^2 - 9^2}{2 \times 4 \times 7}$$

$$\therefore \phi = \cos^{-1}\left(-\frac{16}{56}\right)$$

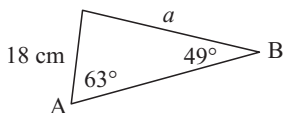
$$\therefore \phi \doteq 106.60$$

So, the largest angle measures 107° , approx.

5 a $\cos \theta = \frac{2^2 + 5^2 - 4^2}{2 \times 2 \times 5}$ **b** $x^2 = 5^2 + 3^2 - 2 \times 5 \times 3 \times \cos \theta$
 $= \frac{13}{20}$ $\therefore x = \sqrt{5^2 + 3^2 - 2 \times 5 \times 3 \times 0.65}$
 $= 0.65$ $\therefore x \doteq 3.81$

EXERCISE 12B.1

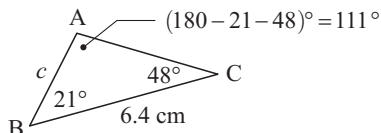
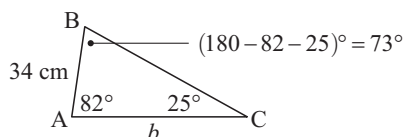
- 1 a** By the sine rule,
 $\frac{x}{\sin 48^\circ} = \frac{23}{\sin 37^\circ}$
 $\therefore x = \frac{23 \times \sin 48^\circ}{\sin 37^\circ}$
 $\therefore x \doteq 28.4$
- b** By the sine rule,
 $\frac{x}{\sin 115^\circ} = \frac{11}{\sin 48^\circ}$
 $\therefore x = \frac{11 \times \sin 115^\circ}{\sin 48^\circ}$
 $\therefore x \doteq 13.4$
- c** By the sine rule,
 $\frac{x}{\sin 51^\circ} = \frac{4.8}{\sin 80^\circ}$
 $\therefore x = \frac{4.8 \times \sin 51^\circ}{\sin 80^\circ}$
 $\therefore x \doteq 3.79$

2 a

$$\frac{a}{\sin 63^\circ} = \frac{18}{\sin 49^\circ} \quad \{\text{sine rule}\}$$

$$\therefore a = \frac{18 \times \sin 63^\circ}{\sin 49^\circ}$$

$$\therefore a \doteq 21.25 \text{ cm}$$

c**b**

$$\text{By the sine rule, } \frac{b}{\sin 73^\circ} = \frac{34}{\sin 25^\circ}$$

$$\therefore b = \frac{34 \times \sin 73^\circ}{\sin 25^\circ}$$

$$\therefore b \doteq 76.9 \text{ cm}$$

$$\text{By the sine rule, } \frac{c}{\sin 48^\circ} = \frac{6.4}{\sin 111^\circ}$$

$$\therefore c = \frac{6.4 \times \sin 48^\circ}{\sin 111^\circ}$$

$$\therefore c \doteq 5.09 \text{ cm}$$

EXERCISE 12B.2**1** By the sine rule

$$\frac{\sin C}{11} = \frac{\sin 40^\circ}{8}$$

$$\therefore \sin C = \frac{11 \times \sin 40^\circ}{8}$$

$$\therefore C = \sin^{-1} \left(\frac{11 \times \sin 40^\circ}{8} \right)$$

$$\therefore C \doteq 62.1^\circ \text{ or } (180 - 62.1)^\circ$$

$$\therefore C \doteq 62.1^\circ \text{ or } 117.9^\circ$$

$$\text{b } \frac{\sin B}{43.8} = \frac{\sin 43^\circ}{31.4}$$

$$\therefore \sin B = \frac{43.8 \times \sin 43^\circ}{31.4}$$

$$\therefore B = \sin^{-1} \left(\frac{43.8 \times \sin 43^\circ}{31.4} \right)$$

$$\therefore B \doteq 72.0^\circ \text{ or } 108^\circ$$

both of which are possible as
 $108 + 43 = 151$ which is < 180 .

$$\text{2 a } \frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\therefore \sin A = \frac{14.6 \times \sin 65^\circ}{17.4}$$

$$\therefore A = \sin^{-1} \left(\frac{14.6 \times \sin 65^\circ}{17.4} \right)$$

$$\therefore A \doteq 49.5^\circ \text{ or } 180^\circ - 49.5^\circ$$

$$\therefore A \doteq 49.5^\circ \text{ or } 130.5^\circ$$

Check: $A = 130.5^\circ$ is impossible as
 $A + B = 130.5^\circ + 65^\circ$ is already over
 180° . $\therefore A \doteq 49.5^\circ$

$$\text{c } \frac{\sin C}{4.8} = \frac{\sin 71^\circ}{6.5}$$

$$\therefore \sin C = \frac{4.8 \times \sin 71^\circ}{6.5}$$

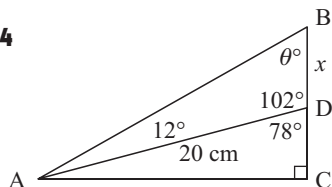
$$\therefore C = \sin^{-1} \left(\frac{4.8 \times \sin 71^\circ}{6.5} \right)$$

$$\therefore C \doteq 44.3^\circ \text{ or } 135.7^\circ$$

But $135.7 + 71$ is already > 180
 $\therefore$ this case is impossible $\therefore C \doteq 44.3^\circ$

$$\text{3 } \begin{array}{lll} \text{The third angle is} & \frac{\sin 85^\circ}{11.4} & \text{and} & \frac{\sin 27^\circ}{9.8} & \therefore \text{it is not possible as} \\ 180^\circ - 85^\circ - 68^\circ & & & & \frac{\sin 85^\circ}{11.4} \neq \frac{\sin 27^\circ}{9.8} \\ = 27^\circ & = 0.08738\dots & & = 0.04632\dots & \end{array}$$

i.e., the sine rule is violated.

4

In $\triangle ABD$,
 $\theta = (180 - 12 - 102)^\circ$
 $\therefore \theta = 66^\circ$

$$\text{Now } \frac{x}{\sin 12^\circ} = \frac{20}{\sin 66^\circ}$$

$$\therefore x = \frac{20 \times \sin 12^\circ}{\sin 66^\circ}$$

$$\therefore x \doteq 4.55$$

$$\therefore \text{BD is 4.55 cm long}$$

- 5** First we find the length of the diagonal, d m.

$$\frac{d}{\sin 118^\circ} = \frac{22}{\sin 30^\circ}$$

$$\therefore d = \frac{22 \times \sin 118^\circ}{\sin 30^\circ}$$

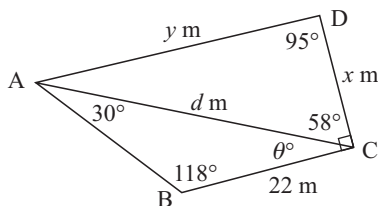
$$\therefore d \doteq 38.85$$

Using the sine rule

$$\frac{y}{\sin 58^\circ} = \frac{38.85}{\sin 95^\circ}$$

$$\therefore y = \frac{38.85 \times \sin 58^\circ}{\sin 95^\circ}$$

$$\therefore y \doteq 33.1$$



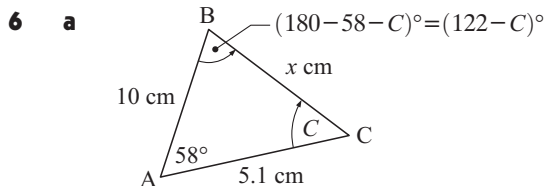
$$\theta = 180^\circ - 30^\circ - 118^\circ = 32^\circ$$

$$\therefore \angle ACD = 58^\circ$$

$$\text{and } \frac{x}{\sin(180 - 95 - 58)} = \frac{38.85}{\sin 95^\circ}$$

$$\therefore x = \frac{38.85 \times \sin 27^\circ}{\sin 95^\circ}$$

$$\therefore x \doteq 17.7$$



$$\frac{\sin C}{10} = \frac{\sin(122 - C)}{5.1}$$

$$\therefore 5.1 \sin C = 10 \sin(122 - C)$$

Using technology,

$$\angle C \doteq 88.7^\circ \text{ or } (180 - 88.7)^\circ$$

$$\text{i.e., } \angle C \doteq 88.7^\circ \text{ or } 91.3^\circ$$

b Let $BC = x$ cm $\therefore x^2 = 10^2 + 5.1^2 - 2 \times 10 \times 5.1 \cos 58^\circ$

$$\therefore x = \sqrt{10^2 + 5.1^2 - 2 \times 10 \times 5.1 \times \cos 58^\circ}$$

$$\therefore x \doteq 8.4828$$

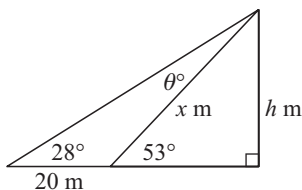
and $\cos C = \frac{5.1^2 + 8.4828^2 - 10^2}{2 \times 5.1 \times 8.4828} \doteq -0.02309$

$$\therefore \hat{C} \doteq \arccos(-0.02309) \doteq 91.3^\circ$$

- c** “When faced with using either the sine rule or the cosine rule it is better to use the *cosine rule* as it avoids the *ambiguous case*.”

EXERCISE 12C

1



$$\theta^\circ + 28^\circ = 53^\circ$$

{exterior angle of a Δ theorem}

$$\therefore \theta = 25^\circ$$

By the Sine Rule,

$$\frac{x}{\sin 28^\circ} = \frac{20}{\sin 25^\circ}$$

$$\therefore x = \frac{20 \times \sin 28^\circ}{\sin 25^\circ}$$

$$\therefore x \doteq 22.22$$

$$\text{and } \sin 53^\circ = \frac{h}{x}$$

$$\therefore h = x \sin 53^\circ$$

$$= 22.22 \times \sin 53^\circ$$

$$\doteq 17.7 \text{ m}$$

$$\therefore \text{the pole is 17.7 m high}$$

2 $PR^2 = 63^2 + 175^2 - 2 \times 63 \times 175 \times \cos 112^\circ$

$$\therefore PR = \sqrt{63^2 + 175^2 - 2 \times 63 \times 175 \times \cos 112^\circ}$$

$$\therefore PR \doteq 207 \text{ m}$$

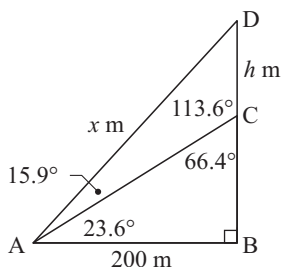
3 $\cos T = \frac{220^2 + 340^2 - 165^2}{2 \times 220 \times 340}$

$$\therefore T = \cos^{-1} \left(\frac{136775}{149600} \right)$$

$$\therefore T \doteq 23.9$$

$$\therefore \text{was } 23.9^\circ \text{ off line.}$$

4


 In $\triangle ABD$,

$$\cos(23.6 + 15.9)^\circ = \frac{200}{x}$$

$$\therefore x = \frac{200}{\cos 39.5^\circ}$$

$$\therefore x \doteq 259.2$$

 In $\triangle ACD$,

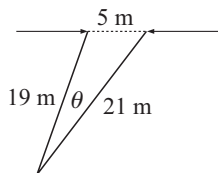
$$\frac{h}{\sin 15.9^\circ} = \frac{259.2}{\sin 113.6^\circ}$$

$$\therefore h = \frac{259.2 \times \sin 15.9^\circ}{\sin 113.6^\circ}$$

$$\therefore h \doteq 77.5$$

$$\therefore \text{is } 77.5 \text{ m high.}$$

5



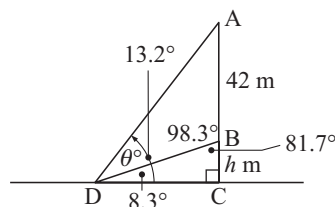
$$\cos \theta = \frac{19^2 + 21^2 - 5^2}{2 \times 19 \times 21}$$

$$\therefore \theta = \cos^{-1}\left(\frac{777}{798}\right)$$

$$\therefore \theta = 13.2$$

$$\therefore \text{angle of view is } 13.2^\circ$$

6



$$\theta = 13.2^\circ - 8.3^\circ = 4.9^\circ$$

 In $\triangle ABD$,

$$\frac{AD}{\sin 98.3^\circ} = \frac{42}{\sin 4.9^\circ}$$

$$\therefore AD = \frac{42 \times \sin 98.3^\circ}{\sin 4.9^\circ}$$

$$\therefore AD \doteq 486.56 \text{ m}$$

 In $\triangle ADC$,

$$\sin 13.2^\circ = \frac{h + 42}{486.56}$$

$$\therefore h + 42 = 486.56 \times \sin 13.2^\circ$$

$$\therefore h + 42 \doteq 111.1$$

$$\therefore h \doteq 69.1$$

$$\therefore \text{the hill is } 69.1 \text{ m high}$$

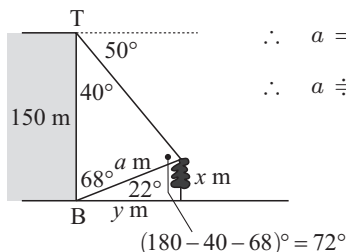
7 a

By the sine rule

$$\frac{a}{\sin 40^\circ} = \frac{150}{\sin 72^\circ}$$

$$\therefore a = \frac{150 \times \sin 40^\circ}{\sin 72^\circ}$$

$$\therefore a \doteq 101.38$$


 $\therefore$ the tree is 38.0 m high and 94.0 m from the building.

$$\text{Now } \sin 22^\circ = \frac{x}{101.38}$$

$$\therefore x = 101.38 \times \sin 22^\circ$$

$$\therefore x \doteq 38.0$$

$$\text{b and } \cos 22^\circ = \frac{y}{101.38}$$

$$\therefore y = 101.38 \times \cos 22^\circ$$

$$\therefore y \doteq 94.0$$

8 Using Pythagoras' theorem

$$RQ = \sqrt{4^2 + 7^2} = \sqrt{65} \text{ m}$$

$$PQ = \sqrt{8^2 + 7^2} = \sqrt{113} \text{ cm}$$

$$PR = \sqrt{8^2 + 4^2} = \sqrt{80} \text{ cm}$$

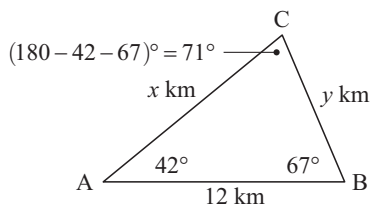
$$\text{Now } \cos Q = \frac{(\sqrt{113})^2 + (\sqrt{65})^2 - (\sqrt{80})^2}{2 \times \sqrt{113} \times \sqrt{65}}$$

$$\therefore \cos \theta \doteq \left(\frac{98}{171.4}\right)$$

$$\therefore \theta = \cos^{-1}\left(\frac{98}{171.4}\right)$$

$$\therefore \theta \doteq 55.1 \quad \text{So } \angle PQR \text{ measures } 55.1^\circ$$

9



$$\frac{x}{\sin 67^\circ} = \frac{12}{\sin 71^\circ} = \frac{y}{\sin 42^\circ}$$

$$\therefore x = \frac{12 \times \sin 67^\circ}{\sin 71^\circ} \quad \text{and} \quad y = \frac{12 \times \sin 42^\circ}{\sin 71^\circ}$$

$$\therefore x \doteq 11.7 \quad \therefore y \doteq 8.49$$

So, C is 11.7 km from A and 8.49 km from B.

10 a

$$QS = \sqrt{8^2 + 12^2 - 2 \times 8 \times 12 \times \cos 70^\circ}$$

$$\doteq 11.93$$

$$\therefore \text{area} = \frac{1}{2} \times 8 \times 12 \times \sin 70^\circ + \frac{1}{2} \times 10 \times 11.93 \times \sin 30^\circ$$

$$\doteq 74.9 \text{ km}^2$$

b

$$1 \text{ ha is } 100 \text{ m} \times 100 \text{ m}$$

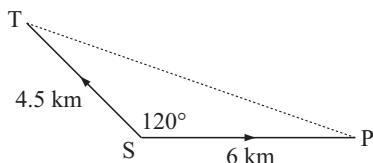
$$= 0.1 \text{ km} \times 0.1 \text{ km}$$

$$= 0.01 \text{ km}^2$$

$$\therefore 1 \text{ km}^2 = 100 \text{ ha}$$

$$\therefore \text{area} \doteq 7490 \text{ ha}$$

11



$$\text{Distance} = \text{speed} \times \text{time}$$

$$\text{So, after 45 min} = 0.75 \text{ h}$$

$$ST = 6 \times 0.75 = 4.5 \text{ km}$$

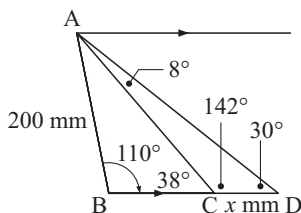
$$SP = 8 \times 0.75 = 6 \text{ km}$$

$$\text{Now } PT = \sqrt{4.5^2 + 6^2 - 2 \times 4.5 \times 6 \times \cos 120^\circ}$$

$$\therefore PT \doteq 9.12$$

So, they are 9.12 km apart.

12



$$\text{In } \triangle ABC \quad \frac{AC}{\sin 110^\circ} = \frac{200}{\sin 38^\circ}$$

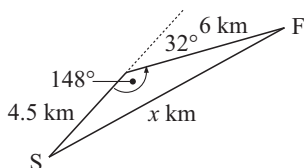
$$\therefore AC = \frac{200 \times \sin 110^\circ}{\sin 38^\circ} \doteq 305.26$$

$$\text{and in } \triangle ACD \quad \frac{x}{\sin 8^\circ} = \frac{305.26}{\sin 30^\circ}$$

$$\therefore x = \frac{305.26 \times \sin 8^\circ}{\sin 30^\circ} \doteq 84.968$$

 $\therefore$ the metal strip is 85.0 mm wide.

13

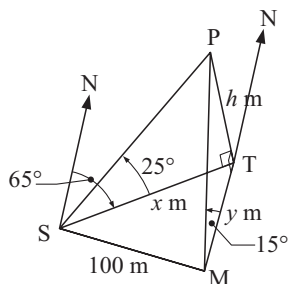


$$x = \sqrt{6^2 + (4.5)^2 - 2 \times 6 \times 4.5 \times \cos 148^\circ}$$

$$\therefore x \doteq 10.1$$

 $\therefore$ is 10.1 km from the start.

14



$$\text{In } \triangle PST, \tan 25^\circ = \frac{h}{x}$$

$$\therefore x = \frac{h}{\tan 25^\circ}$$

$$\doteq 2.145h$$

$$\text{In } \triangle PMT, \tan 15^\circ = \frac{h}{y}$$

$$\therefore y = \frac{h}{\tan 15^\circ}$$

$$\doteq 3.732h$$

 But $\angle STM = 65^\circ$ {equal alternate angles}

$$\text{and } 100^2 = x^2 + y^2 - 2xy \cos 65^\circ$$

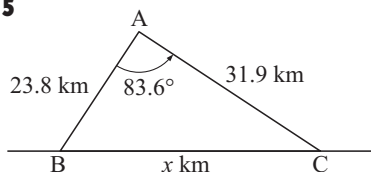
$$\therefore 10000 = (2.145h)^2 + (3.732h)^2 - 2 \times (2.145)(3.732)h^2 \cos 65^\circ$$

$$\therefore 10000 \doteq 11.763h^2$$

$$\therefore h^2 \doteq 850.15$$

$$\therefore h \doteq 29.2 \quad \text{So, the tree is 29.2 m high.}$$

15



By the cosine rule

$$x^2 = 23.8^2 + 31.9^2 - 2 \times 23.8 \times 31.9 \times \cos 83.6^\circ$$

$$\therefore x = \sqrt{23.8^2 + 31.9^2 - 2 \times 23.8 \times 31.9 \times \cos 83.6^\circ}$$

$$\therefore x \doteq 37.6$$

 $\therefore$ B and C are 37.6 km apart

Note: The helicopter's altitude of 4 km must be incorrect. It was not used, however.

REVIEW SET 12

1 a $\cos x = \frac{13^2 + 19^2 - 11^2}{2 \times 13 \times 19}$

$$\therefore \cos x = \frac{409}{494}$$

$$\therefore x = \cos^{-1}\left(\frac{409}{494}\right)$$

$$\therefore x \doteq 34.1$$

b $x = \sqrt{15^2 + 17^2 - 2 \times 15 \times 17 \times \cos 72^\circ}$

$$\therefore x \doteq 18.9$$

2 a $\cos x = \frac{11^2 + 19^2 - 13^2}{2 \times 11 \times 19}$

$$\therefore \cos x = \frac{313}{418}$$

$$\therefore x = \cos^{-1}\left(\frac{313}{418}\right)$$

$$\therefore x \doteq 41.5$$

b $x = \sqrt{14^2 + 21^2 - 2 \times 14 \times 21 \times \cos 47^\circ}$

$$\therefore x \doteq 15.4$$

3 $AC = \sqrt{11^2 + 9.8^2 - 2 \times 11 \times 9.8 \times \cos 74^\circ}$

$$\therefore AC \doteq 12.554 \dots$$

$$\therefore AC \doteq 12.6 \text{ cm}$$

Now $\frac{\sin C}{11} = \frac{\sin 74^\circ}{12.554}$

$$\therefore \sin C = \frac{11 \times \sin 74^\circ}{12.554}$$

$$\therefore C = \sin^{-1}\left(\frac{11 \times \sin 74^\circ}{12.554}\right)$$

$$\therefore C = 57.4^\circ \text{ or } 122.6^\circ$$

↑
impossible as $122.6 + 74 > 180$

$$\therefore \angle C \text{ measures } 57.4^\circ$$

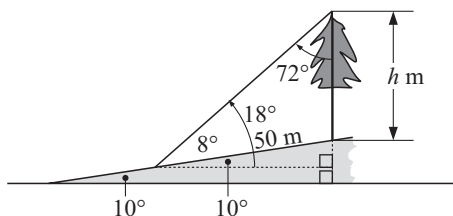
$$\therefore \angle A \text{ measures } 48.6^\circ$$

4 $DB = \sqrt{7^2 + 11^2 - 2 \times 7 \times 11 \times \cos 110^\circ}$
 $\doteq 14.922$

$$\therefore \text{total area} = \frac{1}{2} \times 7 \times 11 \times \sin 110^\circ + \frac{1}{2} \times 16 \times 14.922 \times \sin 40^\circ$$

$$\doteq 113 \text{ cm}^2$$

5



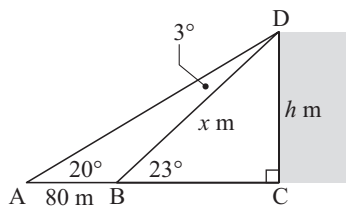
$$\frac{h}{\sin 8^\circ} = \frac{50}{\sin 72^\circ}$$

$$\therefore h = \frac{50 \times \sin 8^\circ}{\sin 72^\circ}$$

$$\therefore h \doteq 7.32$$

i.e., the tree is 7.32 m high

6



$$\text{In } \triangle ABD, \frac{x}{\sin 20^\circ} = \frac{80}{\sin 3^\circ}$$

$$\therefore x = \frac{80 \times \sin 20^\circ}{\sin 3^\circ} \div 522.8$$

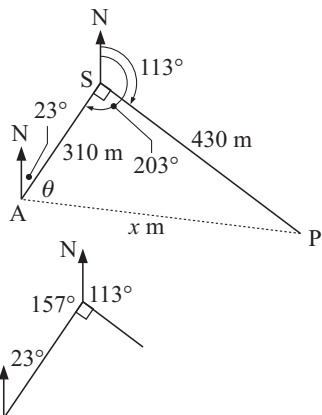
$$\text{Now } \sin 23^\circ = \frac{h}{x}$$

$$\therefore h = 522.8 \times \sin 23^\circ$$

$$\therefore h \div 204$$

So the building is 204 m tall.

7



$$\angle ASP = 203^\circ - 113^\circ = 90^\circ$$

$$\text{Now } x^2 = 310^2 + 430^2 \quad \{\text{Pythagoras}\}$$

$$\therefore x = \sqrt{310^2 + 430^2}$$

$$\therefore x \div 530$$

$\therefore$ they are 530 m apart.

$$\text{and } \tan \theta = \frac{430}{310}$$

$$\therefore \theta = \tan^{-1} \left(\frac{430}{310} \right) \div 54.2$$

$$\text{and } 23 + \theta = 77.2$$

$$\therefore \text{bearing of P from A is } 077.2^\circ$$

8

$$\text{In 45 minutes, } 140 \times \frac{3}{4} = 105 \text{ km is travelled.}$$

$$\text{In 40 minutes, } 180 \times \frac{2}{3} = 120 \text{ km is travelled.}$$

$$\text{We notice that } \theta + 43 + 32 = 180 \quad \{\text{allied angles add to } 180^\circ\}$$

$$\therefore \theta = 105$$

$$\text{and so, } x = \sqrt{120^2 + 105^2 - 2 \times 120 \times 105 \times \cos 105^\circ}$$

$$\therefore x \div 178.74$$

So, is 179 km from the start.

$$\text{Now } \frac{\sin \phi}{105} = \frac{\sin 105^\circ}{178.74}$$

$$\therefore \sin \phi = \frac{105 \times \sin 105^\circ}{178.74}$$

$$\therefore \phi \div 34.6$$

$$\therefore \theta = 180 - 105 - 34.6 - 32 = 8.4 \div 8$$

So, the bearing is 352° .

9 If the unknown is an angle, use the cosine rule to avoid the ambiguous case.

$$10 \text{ a By the cosine rule, } 7^2 = 8^2 + x^2 - 2 \times 8 \times x \times \cos 60^\circ$$

$$\therefore 49 = 64 + x^2 - 16x \left(\frac{1}{2} \right)$$

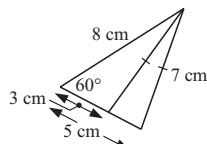
$$\therefore 49 = 64 + x^2 - 8x$$

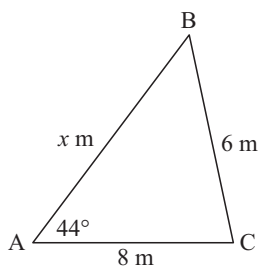
$$\therefore x^2 - 8x + 15 = 0$$

$$\therefore (x-3)(x-5) = 0$$

$$\therefore x = 3 \text{ or } 5$$

b Kady's response should be "Please supply me with additional information as there are two possibilities. Which one do you want?"



11 a

 By the cosine rule, $6^2 = x^2 + 8^2 - 2 \times x \times 8 \times \cos 44^\circ$

$$\therefore 36 = x^2 + 64 - 16x \times \cos 44^\circ$$

$$\therefore x^2 - 11.51x + 28 = 0$$

$$\therefore x = \frac{11.51 \pm \sqrt{11.51^2 - 4(1)(28)}}{2}$$

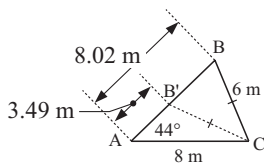
$$\therefore x = \frac{11.51 \pm 4.525}{2}$$

$$\therefore x \doteq 8.02 \quad \text{or} \quad 3.49$$

Frank needs additional information as there are two possible cases:

(1) when $AB \doteq 8.02$ m and

(2) when $AB \doteq 3.49$ m


b Volume = area $\times$ depth

$$= \frac{1}{2} \times 8 \times x \times \sin 44^\circ \times 0.1 \quad \text{and is a max. when } AB \doteq 8.02 \text{ m}$$

$$\doteq 4 \times 8.02 \times \sin 44^\circ \times 0.1$$

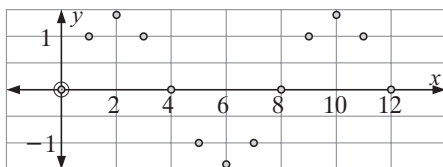
$$\doteq 2.23 \text{ m}^3$$

Chapter 13

PERIODIC PHENOMENA

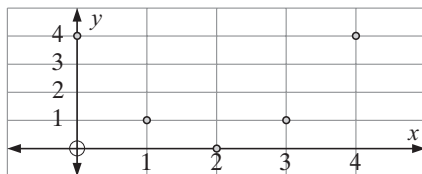
EXERCISE 13A

1 a



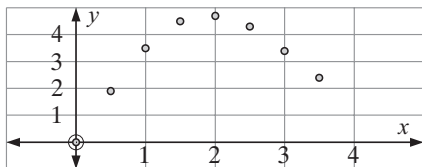
Data exhibits periodic behaviour.

b



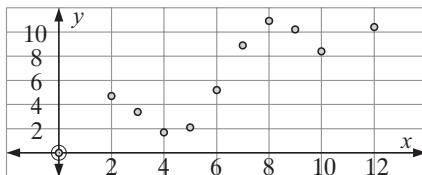
Not enough information to say data is periodic. It may in fact be quadratic.

c



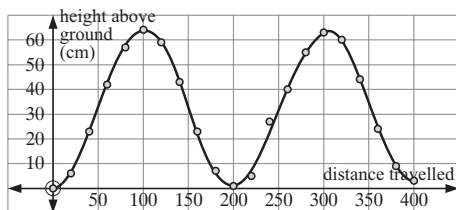
Not enough information to say data is periodic. It may in fact be quadratic.

d



Not enough information to say data is periodic.

2 a



c A curve can be fitted to the data as the distance travelled is continuous.

b The data is periodic.

i The minimum value from the table is 0 and the maximum value is 64.

So, the principle axis is $y \div \frac{0+64}{2}$,
i.e., $y \div 32$.

ii The maximum value is 64 cm (approx).

iii The period is $\div 200$ cm.

iv The amplitude is $\div 32$ cm.

3 a periodic b periodic c periodic d not periodic e periodic f periodic

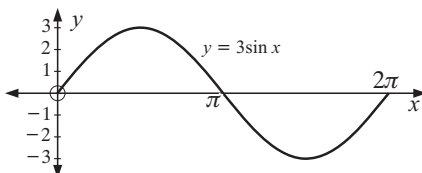
EXERCISE 13B.1

1 a

$$y = 3 \sin x$$

has amplitude 3 and period $\frac{2\pi}{1} = 2\pi$

When $x = 0$, $y = 0$.

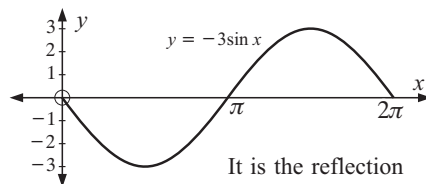


b

$$y = -3 \sin x$$

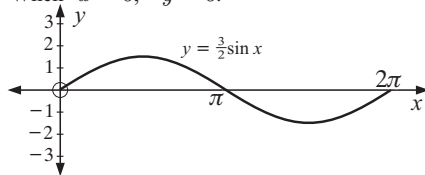
has amplitude $|-3| = 3$ and period $\frac{2\pi}{1}$

When $x = 0$, $y = 0$. $= 2\pi$.

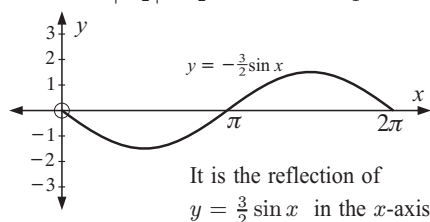


It is the reflection
of $y = 3 \sin x$ in
the x -axis.

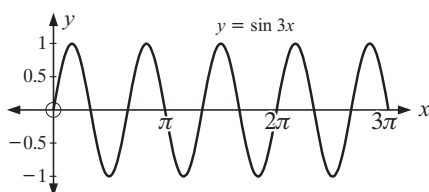
- c** $y = \frac{3}{2} \sin x$
 has amplitude $\frac{3}{2}$ and period $\frac{2\pi}{1} = 2\pi$.
 When $x = 0$, $y = 0$.



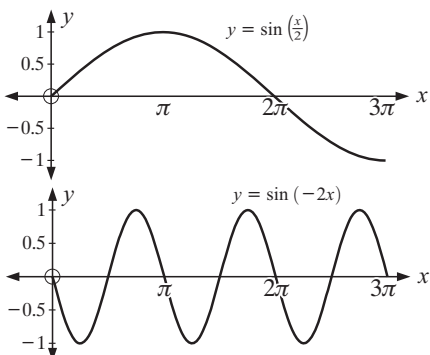
- d** $y = -\frac{3}{2} \sin x$
 has amplitude $|\frac{-3}{2}| = \frac{3}{2}$ and period $\frac{2\pi}{1} = 2\pi$.



- 2 a** $y = \sin 3x$
 has amplitude 1 and period $\frac{2\pi}{3}$.
 When $x = 0$, $y = 0$.



- c** $y = \sin(-2x)$
 has amplitude 1 and period $\frac{2\pi}{|2|} = \pi$.
 When $x = 0$, $y = 0$.



It is the reflection of $y = \sin 2x$ in the x -axis.

- 3 a** period = $\frac{2\pi}{|4|}$
 $= \frac{\pi}{2}$

- b** period = $\frac{2\pi}{|-4|}$
 $= \frac{\pi}{2}$

- c** period = $\frac{2\pi}{|\frac{1}{3}|}$
 $= 6\pi$

- d** period = $\frac{2\pi}{0.6}$
 $= \frac{20\pi}{6}$
 $= \frac{10\pi}{3}$

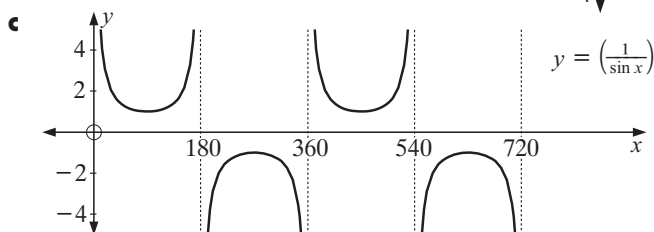
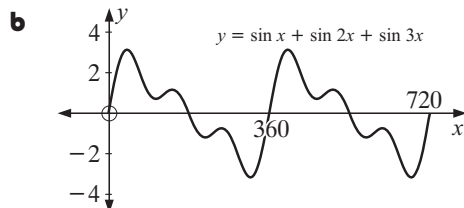
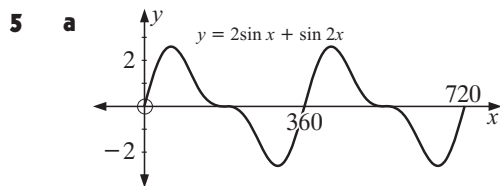
- 4 a** $\frac{2\pi}{B} = 5\pi$
 $\therefore B = \frac{2}{5}$

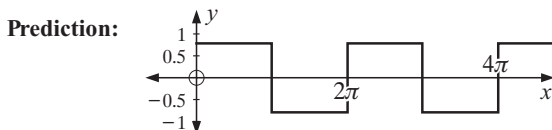
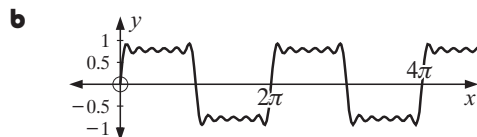
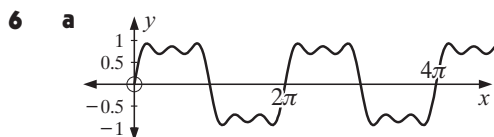
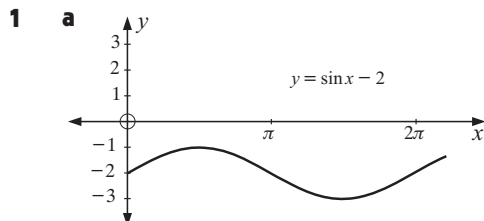
- b** $\frac{2\pi}{B} = \frac{2\pi}{3}$
 $\therefore B = 3$

- c** $\frac{2\pi}{B} = 12\pi$
 $\therefore B = \frac{1}{6}$

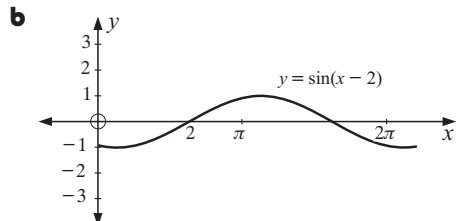
- d** $\frac{2\pi}{B} = 4$
 $\therefore B = \frac{\pi}{2}$

- e** $\frac{2\pi}{B} = 100$
 $\therefore B = \frac{2\pi}{100} = \frac{\pi}{50}$

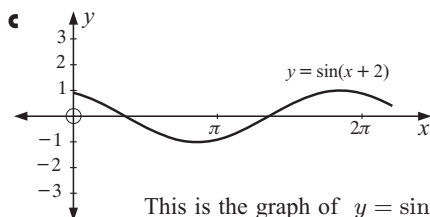


**EXERCISE 13B.2**

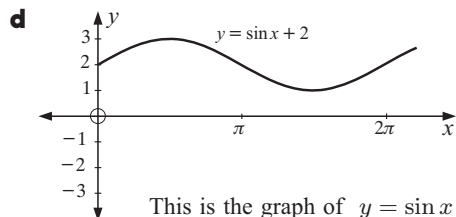
This is the graph of $y = \sin x$ translated by $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$.



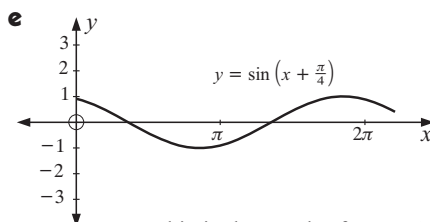
This is the graph of $y = \sin x$ translated by $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$.



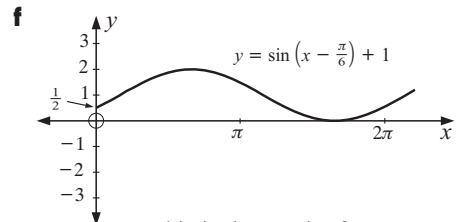
This is the graph of $y = \sin x$ translated by $\begin{bmatrix} -2 \\ 0 \end{bmatrix}$.



This is the graph of $y = \sin x$ translated by $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.



This is the graph of $y = \sin x$ translated by $\begin{bmatrix} -\frac{\pi}{4} \\ 0 \end{bmatrix}$.



This is the graph of $y = \sin x$ translated by $\begin{bmatrix} \frac{\pi}{6} \\ 1 \end{bmatrix}$.

3 a period = $\frac{2\pi}{|5|} = \frac{2\pi}{5}$

b period = $\frac{2\pi}{|\frac{1}{4}|} = 8\pi$

c period = $\frac{2\pi}{|-2|} = \pi$

4 a $\frac{2\pi}{B} = 3\pi$
 $\therefore B = \frac{2}{3}$

b $\frac{2\pi}{B} = \frac{\pi}{10}$
 $\therefore B = 20$

c $\frac{2\pi}{B} = 100\pi$
 $\therefore B = \frac{2}{100} = \frac{1}{50}$

d $\frac{2\pi}{B} = 50$
 $\therefore B = \frac{2\pi}{50} = \frac{\pi}{25}$

5 a A translation of $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$, i.e., vertically down 1 unit.

b A translation of $\begin{bmatrix} \frac{\pi}{4} \\ 0 \end{bmatrix}$, i.e., horizontally $\frac{\pi}{4}$ units right.

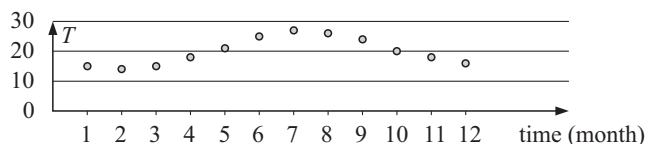
c A vertical dilation of factor 2.

d A horizontal dilation of factor $\frac{1}{4}$.

- e** A vertical dilation of factor $\frac{1}{2}$. **f** A horizontal dilation of factor $\frac{1}{4} = 4$.
g A reflection in the x -axis. **h** A translation of $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$.
i A vertical dilation of factor 2 followed by a horizontal dilation of factor $\frac{1}{3}$.
j A translation of $\begin{bmatrix} \frac{\pi}{3} \\ 2 \end{bmatrix}$.

EXERCISE 13C
1 a

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	15	14	15	18	21	25	27	26	24	20	18	16



The period is 12 months so $\frac{2\pi}{B} = 12 \quad \therefore \quad B = \frac{\pi}{6} \quad \{\text{assuming } B > 0\}$.

$$\text{Amplitude, } A \div \frac{\text{max.} - \text{min.}}{2} \div \frac{27 - 14}{2} \div 6.5$$

As the principal axis is midway between min. and max., then $D \div \frac{27 + 14}{2} \div 20.5$

When T is 20.5 (midway between min. and max.)

$$C \div \frac{2 + 7}{2} \div 4.5 \quad \{\text{average of } t \text{ values}\}$$

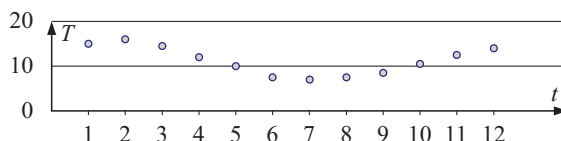
$$\therefore T \div 6.5 \sin \frac{\pi}{6}(t - 4.5) + 20.5 \quad (\text{Note: } \frac{\pi}{6} \div 0.524)$$

b Using technology, $T \div 6.14 \sin(0.575t - 2.70) + 20.4$

$$\text{i.e., } T \div 6.14 \sin 0.575(t - 4.70) + 20.4$$

2 a

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14



The period is $\frac{2\pi}{B} = 12 \quad \therefore \quad B = \frac{\pi}{6} \quad \{B > 0\}$

$$\text{Amplitude, } A \div \frac{\text{max.} - \text{min.}}{2} \div \frac{16 - 7}{2} \div 4.5$$

As the principal axis is midway between min. and max. then $D \div \frac{16 + 7}{2} \div 11.5$

$$\text{At min., } t = 7 \text{ and at max., } t = 2 + 12 = 14 \quad \therefore \quad C = \frac{7 + 14}{2} = 10.5$$

$$\text{So, } T \div 4.5 \sin \frac{\pi}{6}(t - 10.5) + 11.5$$

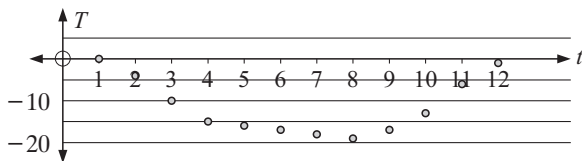
b Using tech., $T \div 4.29 \sin(0.533t + 0.769) + 11.2$ **Note:** (1) $\frac{\pi}{6} \div 0.524 \quad \checkmark$

$$\text{i.e., } T \div 4.29 \sin 0.533(t + 1.44) + 11.2$$

$$(2) \quad \frac{\pi}{6}(1.44 - (-10.5)) \div 6.25 \div 2\pi$$

3

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Temp, T	0	-4	-10	-15	-16	-17	-18	-19	-17	-13	-6	-1



The period is $\frac{2\pi}{B} = 12 \quad \therefore B = \frac{\pi}{6} \quad \{B > 0\}$

$$\text{Amplitude, } A \div \frac{\text{max.} - \text{min.}}{2} \div \frac{0 - (-19)}{2} \div 9.5$$

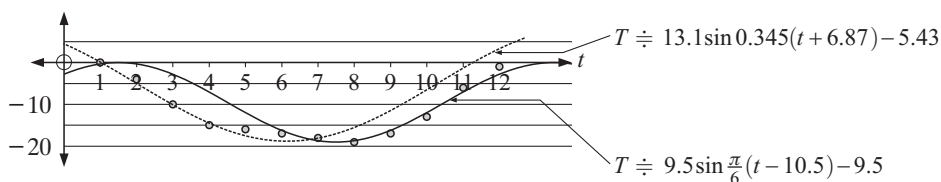
$$D \div \frac{\text{max.} + \text{min.}}{2} \div \frac{0 + (-19)}{2} \div -9.5$$

At min., $t = 8$ and at max., $t = 1 + 12 = 13 \quad \therefore C = \frac{8+13}{2} = 10.5$

$$\text{So, } T \div 9.5 \sin \frac{\pi}{6}(t - 10.5) - 9.5 \quad \dots\dots (1)$$

From technology, $T \div 13.1 \sin(0.345t + 2.37) - 5.43$

$$\text{i.e., } T \div 13.1 \sin 0.345(t + 6.87) - 5.43 \quad \dots\dots (2)$$



The model does not seem appropriate.

4 a For the model $H = A \sin B(t - C) + D$

$$\text{period} = \frac{2\pi}{B} = 12.4 \text{ hours} \quad \therefore B = \frac{2\pi}{12.4} \div 0.507$$

We let the principal axis be 0, i.e., $D = 0$

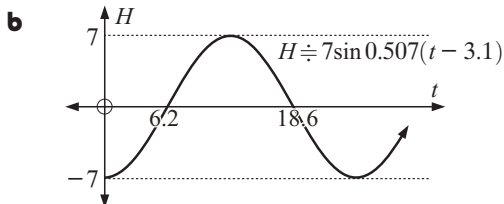
$\therefore A$, the amplitude = 7, i.e., min. is -7, max. is +7

Let $t = 0$ correspond to 'low tide' $\therefore t = 6.2$ corresponds to 'high tide'

$$\therefore C = \frac{0 + 6.2}{2} = 3.1$$

$$\text{So, } H \div 7 \sin 0.507(t - 3.1) + 0$$

$$\text{i.e., } H \div 7 \sin 0.507(t - 3.1)$$



5 Let the model be $H = A \sin B(t - C) + D$ metres

When $t = 0$, $H = 2$ and when $t = 50$, $H = 22$

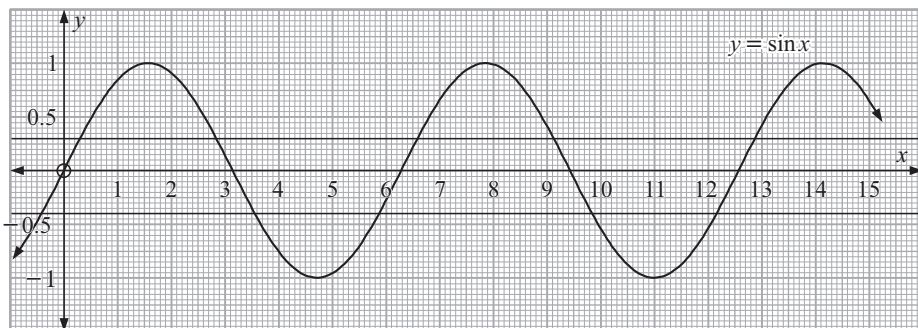
↑
min.

↑
max.

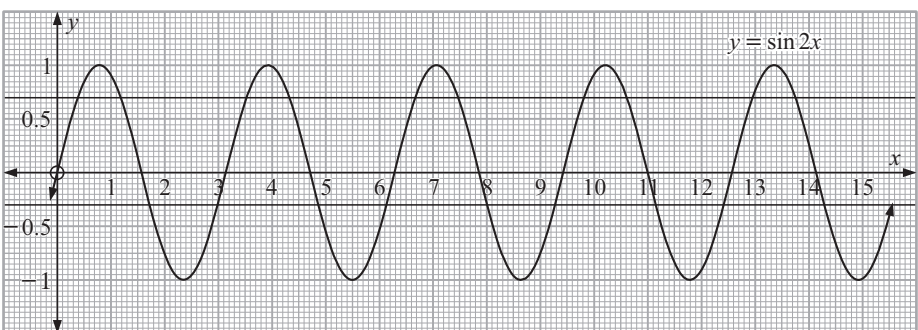
$$\text{period} = \frac{2\pi}{B} = 100 \quad \therefore B = \frac{2\pi}{100} = \frac{\pi}{50}$$

$$A = 10 \quad \{\text{from the diagram}\} \quad D = \frac{\text{max.} + \text{min.}}{2} = \frac{24}{2} = 12$$

$$C = \frac{0+50}{2} = 25 \quad \{\text{values of } t \text{ at max. and min.}\} \quad \therefore H = 10 \sin \frac{\pi}{50}(t-25) + 12$$

EXERCISE 13D.1
1


- a** When $\sin x = 0.3$, $x \doteq 0.3, 2.8, 6.6, 9.1, 12.9$
b When $\sin x = -0.4$, $x \doteq 5.9, 9.8, 12.2$

2


- a** When $\sin(2x) = 0.7$, $x \doteq 0.4, 1.2, 3.5, 4.3, 6.7, 7.5, 9.8, 10.6, 13.0, 13.8$
b When $\sin(2x) = -0.3$, $x \doteq 1.7, 3.0, 4.8, 6.1, 8.0, 9.3, 11.1, 12.4, 14.3$

EXERCISE 13D.2

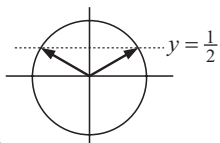
- 1 a** $\sin x = 0.414$
 $\therefore x \doteq 0.4268, 2.715, 6.710$
c $\sin x = 1.289$
 There are no solutions as all values of $\sin x$ lie between -1 and $+1$.
e $\sin\left(\frac{x}{2}\right) = -0.606$
 $\therefore x \doteq 7.585$
g $\sin(x-1.3) = 0.866$
 $\therefore x \doteq 2.347, 3.394$
i $\sin\left(\frac{2x}{3}\right) = -0.9367$ $\therefore x \doteq 6.532, 7.605$
b $\sin x = -0.673$
 $\therefore x \doteq 3.880, 5.545$
d $\sin 2x = 0.162$
 $\therefore x \doteq 0.08136, 1.489, 3.223, 4.631, 6.365, 7.773$
f $\sin(x+2) = 0.0652$
 $\therefore x \doteq 1.076, 4.348, 7.360$
h $\sin\left(x - \frac{\pi}{3}\right) = 0.7063$
 $\therefore x \doteq 1.831, 3.405$

EXERCISE 13D.3

- 1 a** $x = \frac{\pi}{6} + \frac{k12\pi}{6}$ and $0 \leq x \leq \frac{36\pi}{6}$
 $\therefore x = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}$
c $x = -\frac{\pi}{2} + \frac{k2\pi}{2}$ and $-\frac{8\pi}{2} \leq x \leq \frac{8\pi}{2}$
 $\therefore x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}, -\frac{7\pi}{2}$
b $x = -\frac{\pi}{3} + \frac{k6\pi}{3}$ and $-\frac{6\pi}{3} \leq x \leq \frac{6\pi}{3}$
 $\therefore x = -\frac{\pi}{3}, \frac{5\pi}{3}$
d $x = \frac{5\pi}{6} + \frac{k3\pi}{6}$ and $0 \leq x \leq \frac{24\pi}{6}$
 $\therefore x = \frac{2\pi}{6}, \frac{5\pi}{6}, \frac{8\pi}{6}, \frac{11\pi}{6}, \frac{14\pi}{6}, \frac{17\pi}{6}, \frac{20\pi}{6}, \frac{23\pi}{6}$
 i.e., $x = \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}, \frac{7\pi}{3}, \frac{17\pi}{6}, \frac{10\pi}{3}, \frac{23\pi}{6}$

2 a $2 \sin x = 1, \quad 0 \leq x \leq 6\pi$

$$\therefore \sin x = \frac{1}{2}$$

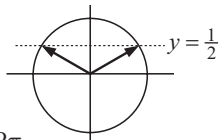


$$\therefore x = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\} + k2\pi$$

$$\therefore x = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}, \frac{5\pi}{6}, \frac{17\pi}{6}, \frac{29\pi}{6}$$

c $2 \sin x - 1 = 0, \quad -2\pi \leq x \leq 2\pi$

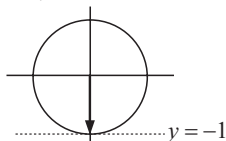
$$\therefore \sin x = \frac{1}{2}$$



$$\therefore x = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\} + k2\pi$$

$$\therefore x = \frac{\pi}{6}, -\frac{11\pi}{6}, \frac{5\pi}{6}, -\frac{7\pi}{6}$$

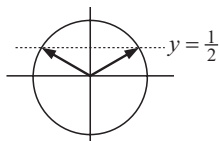
e $\sin x = -1, \quad 0 \leq x \leq 6\pi$



$$\therefore x = \frac{3\pi}{2} + k2\pi$$

$$\therefore x = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$$

g $\sin 2x = \frac{1}{2}, \quad 0 \leq x \leq 3\pi$

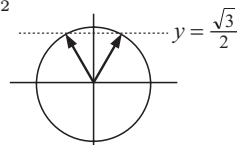


$$\therefore 2x = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\} + k2\pi \quad \therefore x = \left\{ \frac{\pi}{12}, \frac{5\pi}{12} \right\} + k\pi$$

$$\therefore x = \frac{\pi}{12}, \frac{13\pi}{12}, \frac{25\pi}{12}, \frac{5\pi}{12}, \frac{17\pi}{12}, \frac{29\pi}{12}$$

i $2 \sin 2x - \sqrt{3} = 0, \quad 0 \leq x \leq 3\pi$

$$\therefore \sin 2x = \frac{\sqrt{3}}{2}$$

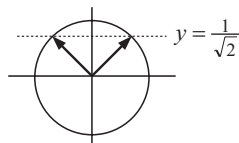


$$\therefore 2x = \left\{ \frac{\pi}{3}, \frac{2\pi}{3} \right\} + k2\pi \quad \therefore x = \left\{ \frac{\pi}{6}, \frac{\pi}{3} \right\} + k\pi$$

$$\therefore x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}$$

b $\sqrt{2} \sin x = 1, \quad 0 \leq x \leq 4\pi$

$$\therefore \sin x = \frac{1}{\sqrt{2}}$$

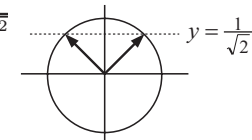


$$\therefore x = \left\{ \frac{\pi}{4}, \frac{3\pi}{4} \right\} + k2\pi$$

$$\therefore x = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{4}$$

d $\sqrt{2} \sin x - 1 = 0, \quad -4\pi \leq x \leq 0$

$$\therefore \sin x = \frac{1}{\sqrt{2}}$$

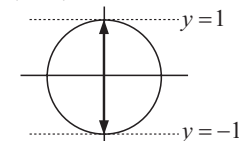


$$\therefore x = \left\{ \frac{\pi}{4}, \frac{3\pi}{4} \right\} + k2\pi$$

$$\therefore x = -\frac{7\pi}{4}, -\frac{15\pi}{4}, -\frac{5\pi}{4}, -\frac{13\pi}{4}$$

f $\sin^2 x = 1, \quad 0 \leq x \leq 4\pi$

$$\therefore \sin x = \pm 1$$

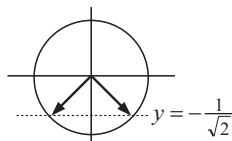


$$\therefore x = \frac{\pi}{2} + k\pi$$

$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

h $\sqrt{2} \sin 3x + 1 = 0, \quad 0 \leq x \leq 2\pi$

$$\therefore \sin 3x = -\frac{1}{\sqrt{2}}$$



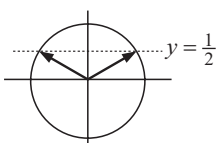
$$\therefore 3x = \left\{ \frac{5\pi}{4}, \frac{7\pi}{4} \right\} + k2\pi \quad \therefore x = \left\{ \frac{5\pi}{12}, \frac{7\pi}{12} \right\} + \frac{k2\pi}{3}$$

$$\therefore x = \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{21\pi}{12}, \frac{7\pi}{12}, \frac{15\pi}{12}, \frac{23\pi}{12}$$

$$\text{i.e., } x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$$

j $2 \sin(x + \frac{\pi}{3}) = 1, \quad -3\pi \leq x \leq 3\pi$

$$\therefore \sin(x + \frac{\pi}{3}) = \frac{1}{2}$$



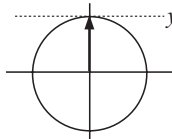
$$\therefore x + \frac{\pi}{3} = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\} + k2\pi \quad \therefore x = \left\{ -\frac{\pi}{6}, \frac{\pi}{3} \right\} + k2\pi$$

$$\therefore x = -\frac{\pi}{6}, \frac{11\pi}{6}, -\frac{13\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{3}, -\frac{3\pi}{2}$$

$$\begin{aligned} 3 \quad a \quad & \sin^2 x + \sin x - 2 = 0 \\ & \therefore (\sin x - 1)(\sin x + 2) = 0 \\ & \therefore \sin x = 1 \text{ or } -2 \end{aligned}$$

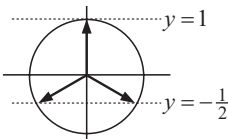
But values of sine can only lie between -1 and 1 inclusive.

$$\therefore \sin x = 1$$



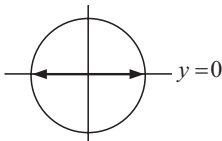
$$\therefore x = \frac{\pi}{2}, -\frac{3\pi}{2}$$

$$\begin{aligned} c \quad & 2\sin^2 x = \sin x + 1 \\ \therefore & 2\sin^2 x - \sin x - 1 = 0 \\ \therefore & (2\sin x + 1)(\sin x - 1) = 0 \\ \therefore & \sin x = -\frac{1}{2} \text{ or } 1 \end{aligned}$$



$$\therefore x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}, -\frac{\pi}{6}, -\frac{5\pi}{6}, -\frac{3\pi}{2}$$

$$4 \quad a \quad \text{The zeros of } y = \sin 2x \text{ are the solutions of } \sin 2x = 0 \quad (0 \leq x \leq \pi)$$

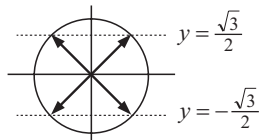


$$\therefore 2x = 0 + k\pi$$

$$\therefore x = 0 + k\frac{\pi}{2}$$

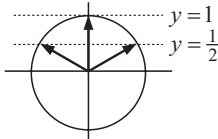
$$\therefore x = 0, \frac{\pi}{2}, \pi$$

$$\begin{aligned} b \quad & 4\sin^2 x = 3 \\ \therefore & \sin^2 x = \frac{3}{4} \\ \therefore & \sin x = \pm \frac{\sqrt{3}}{2} \end{aligned}$$



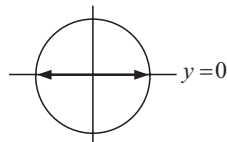
$$\therefore x = \pm \frac{\pi}{3}, \pm \frac{4\pi}{3}, \pm \frac{2\pi}{3}, \pm \frac{5\pi}{3}$$

$$\begin{aligned} d \quad & 2\sin^2 x + 1 = 3\sin x \\ \therefore & 2\sin^2 x - 3\sin x + 1 = 0 \\ \therefore & (2\sin x - 1)(\sin x - 1) = 0 \\ \therefore & \sin x = \frac{1}{2} \text{ or } 1 \end{aligned}$$



$$\therefore x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, -\frac{7\pi}{6}, -\frac{3\pi}{2}, -\frac{11\pi}{6}$$

$$b \quad \text{The zeros of } y = \sin(x - \frac{\pi}{4}) \text{ are the solutions of } \sin(x - \frac{\pi}{4}) = 0 \quad (0 \leq x \leq 3\pi)$$



$$\therefore x - \frac{\pi}{4} = 0 + k\pi$$

$$\therefore x = \frac{\pi}{4} + k\pi$$

$$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$$

EXERCISE 13D.4

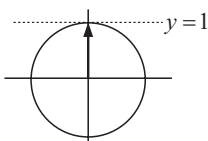
$$1 \quad a \quad P(t) = 7500 + 3000 \sin(\frac{\pi t}{8}), \quad 0 \leq t \leq 12$$

$$\begin{aligned} i \quad & P(0) = 7500 + 3000 \sin 0 \\ & = 7500 + 0 \\ & = 7500 \text{ grass-hoppers} \end{aligned}$$

$$\begin{aligned} ii \quad & P(5) = 7500 + 3000 \sin(\frac{5\pi}{8}) \\ & \div 10\,271.63 \dots \\ & \div 10\,300 \text{ grass-hoppers} \end{aligned}$$

$$b \quad \text{The greatest value of } P(t) \text{ occurs when } \sin(\frac{\pi t}{8}) = 1$$

$$\text{i.e., is } 7500 + 3000 = 10\,500 \text{ grass-hoppers when } \frac{\pi t}{8} = \frac{\pi}{2} + k2\pi$$



$$\therefore \frac{t}{8} = \frac{1}{2} + 2k$$

$$\therefore t = 4 + 16k$$

$$\therefore t = 4 \quad \{\text{as } 0 \leq t \leq 12\}$$

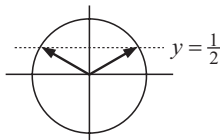
i.e., after 4 weeks

c i When $P(t) = 9000$,

$$7500 + 3000 \sin\left(\frac{\pi t}{8}\right) = 9000$$

$$\therefore 3000 \sin\left(\frac{\pi t}{8}\right) = 1500$$

$$\therefore \sin\left(\frac{\pi t}{8}\right) = \frac{1}{2}$$



$$\therefore \frac{\pi t}{8} = \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\} + k2\pi$$

$$\therefore \frac{t}{8} = \left\{ \frac{1}{6}, \frac{5}{6} \right\} + k2$$

$$\therefore t = \left\{ \frac{4}{3}, \frac{20}{3} \right\} + k16$$

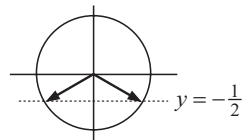
$$\therefore t = 1\frac{1}{3} \text{ or } 6\frac{2}{3}$$

i.e., at $1\frac{1}{3}$ weeks and $6\frac{2}{3}$ weeks**ii** When $P(t) = 6000$,

$$7500 + 3000 \sin\left(\frac{\pi t}{8}\right) = 6000$$

$$\therefore 3000 \sin\left(\frac{\pi t}{8}\right) = -1500$$

$$\therefore \sin\left(\frac{\pi t}{8}\right) = -\frac{1}{2}$$



$$\therefore \frac{\pi t}{8} = \left\{ \frac{7\pi}{6}, \frac{11\pi}{6} \right\} + k2\pi$$

$$\therefore \frac{t}{8} = \left\{ \frac{7}{6}, \frac{11}{6} \right\} + k2$$

$$\therefore t = \left\{ \frac{28}{3}, \frac{44}{3} \right\} + k16$$

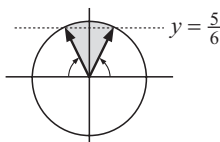
$$\therefore t = 9\frac{1}{3}$$

i.e., at $9\frac{1}{3}$ weeks**d** If $P(t) > 10\,000$, then

$$7500 + 3000 \sin\left(\frac{\pi t}{8}\right) > 10\,000$$

$$\therefore 3000 \sin\left(\frac{\pi t}{8}\right) > 2500$$

$$\therefore \sin\left(\frac{\pi t}{8}\right) > \frac{5}{6}$$

Solving $\sin\left(\frac{\pi t}{8}\right) = \frac{5}{6}$ using technology

$$t \div 2.51 \text{ or } 5.49 \quad \text{So } 2.51 \leq t \leq 5.49 \text{ weeks.}$$

2 $H(t) = 20 - 19 \sin\left(\frac{2\pi t}{3}\right)$ **a** $H(0) = 20 - 19(0)$

$$= 20 \text{ m}$$

i.e., 20 m above the ground

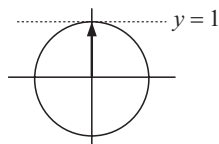
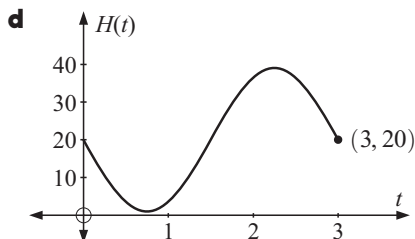
b H is smallest when $\sin\left(\frac{2\pi t}{3}\right) = 1$

$$\therefore \frac{2\pi t}{3} = \frac{\pi}{2} + k2\pi$$

$$\therefore \frac{2t}{3} = \frac{1}{2} + k2$$

$$\therefore t = \frac{3}{4} + k3$$

$$\therefore t = \frac{3}{4} \text{ min } \{\text{as } k = 0\}$$

**c** period = $\frac{2\pi}{\frac{2\pi}{3}} = 3 \text{ min}$ $\therefore$ one revolution takes 3 min

3 $P(t) = 400 + 250 \sin\left(\frac{\pi t}{2}\right)$ years

a $P(0) = 400 + 250(0)$
 $= 400$ water buffalo

b i $P\left(\frac{1}{2}\right) = 400 + 250 \sin\left(\frac{\pi(\frac{1}{2})}{2}\right)$
 $= 400 + 250 \sin\left(\frac{\pi}{4}\right)$
 $= 400 + 250 \times \frac{1}{\sqrt{2}}$
 $\div 577$ water buffalo

c $P(1) = 400 + 250 \sin\left(\frac{\pi}{2}\right)$
 $= 400 + 250 \times 1$
 $= 650$ water buffalo

ii $P(2) = 400 + 250 \sin \pi$
 $= 400 + 250(0)$
 $= 400$ water buffalo

This is the maximum herd size.

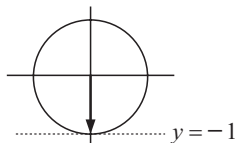
d $P(t)$ is smallest when $\sin\left(\frac{\pi t}{2}\right) = -1$ and is $400 - 250 = 150$ water buffalo.

It occurs when

$$\frac{\pi t}{2} = \frac{3\pi}{2} + k2\pi$$

$$\therefore \frac{t}{2} = \frac{3}{2} + k2$$

$$\therefore t = 3 + 4k \quad \text{So, the first time is after 3 years.}$$



e If $P(t) > 500$ then

$$400 + 250 \sin\left(\frac{\pi t}{2}\right) > 500$$

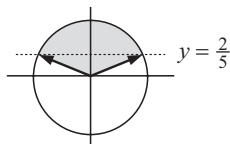
$$\therefore 250 \sin\left(\frac{\pi t}{2}\right) > 100$$

$$\therefore \sin\left(\frac{\pi t}{2}\right) > \frac{2}{5}$$

$$\sin\left(\frac{\pi t}{2}\right) = \frac{2}{5} \quad \text{when} \quad \frac{\pi t}{2} = 0.4115 \quad \text{or} \quad \pi - 0.4115$$

$$\therefore t \div 0.262 \quad \text{or} \quad 1.74$$

$$\text{So, for } \sin\left(\frac{\pi t}{2}\right) > \frac{2}{5}, \quad 0.26 < t < 1.74$$



4 $C(t) = 9.2 \sin \frac{\pi}{7}(t - 4) + 107.8$ cents/L

a i 107.8 is the median value. Values are between $107.8 - 9.2$ and $107.8 + 9.2$
 i.e., 98.6 cents/L and 117.0 cents/L
 $\uparrow$ $\uparrow$
 min. max.
 $\therefore$ the statement is true.

ii period $= \frac{2\pi}{\frac{\pi}{7}} = 14$ days $\therefore$ true

b $C(7) = 9.2 \sin \frac{\pi}{7}(3) + 107.8 \div 116.8$ cents/L

c When $C(t) = \$1.10/\text{L}$ then $9.2 \sin \frac{\pi}{7}(t - 4) + 107.8 = 110$

$$\therefore \sin \frac{\pi}{7}(t - 4) = \frac{2.2}{9.2} \div 0.23913$$

$$\therefore \frac{\pi}{7}(t - 4) \div 0.23913 \quad \text{or} \quad \pi - 0.23913$$

$$\therefore t - 4 \div 0.533 \quad \text{or} \quad 6.467$$

$$\therefore t \div 4.53 \quad \text{or} \quad 10.47$$

i.e., on the 5th and 11th days

d The min. cost/litre is $-9.2 + 107.8 = 98.6$

cents/L when $\sin \frac{\pi}{7}(t-4) = -1$

$$\text{i.e., } \frac{\pi}{7}(t-4) = \frac{3\pi}{2}$$

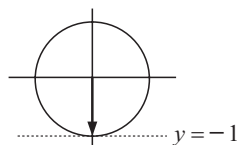
$$\therefore \frac{t-4}{7} = \frac{3}{2}$$

$$\therefore 2t - 8 = 21$$

$$\therefore 2t = 29$$

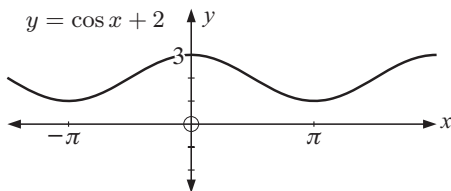
$$\therefore t = 14.5$$

i.e., on the 15th day



EXERCISE 13E

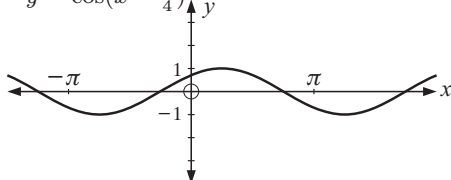
1 a $y = \cos x + 2$



i.e., a vertical translation of

$y = \cos x$ through $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

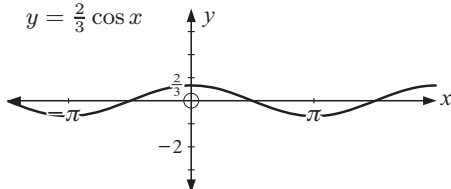
c $y = \cos(x - \frac{\pi}{4})$



i.e., a horizontal translation of

$y = \cos x$ through $\begin{bmatrix} \frac{\pi}{4} \\ 0 \end{bmatrix}$.

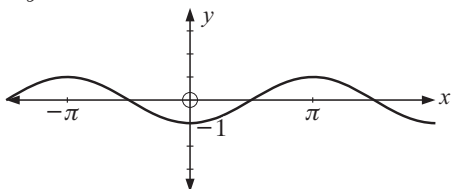
e $y = \frac{2}{3} \cos x$



i.e., a vertical dilation of

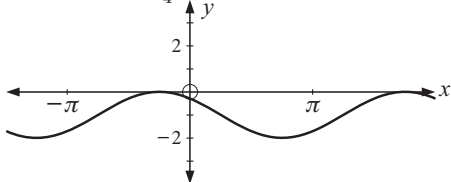
$y = \cos x$ with factor $\frac{2}{3}$.

g $y = -\cos x$



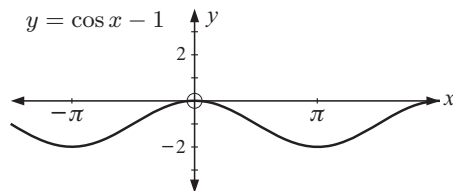
i.e., a reflection of $y = \cos x$
in the x -axis.

i $y = \cos(x + \frac{\pi}{4}) - 1$



i.e., a translation of $\begin{bmatrix} -\frac{\pi}{4} \\ -1 \end{bmatrix}$.

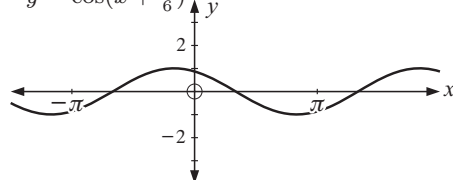
b $y = \cos x - 1$



i.e., a vertical translation of

$y = \cos x$ through $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

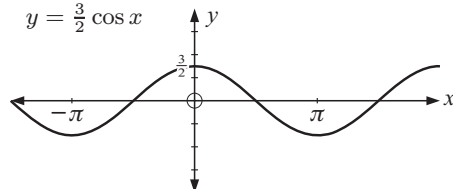
d $y = \cos(x + \frac{\pi}{6})$



i.e., a horizontal translation of

$y = \cos x$ through $\begin{bmatrix} -\frac{\pi}{6} \\ 0 \end{bmatrix}$.

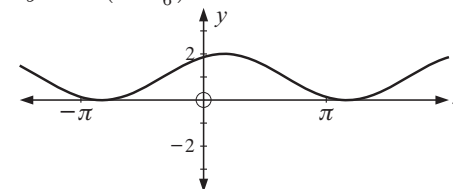
f $y = \frac{3}{2} \cos x$



i.e., a vertical dilation of

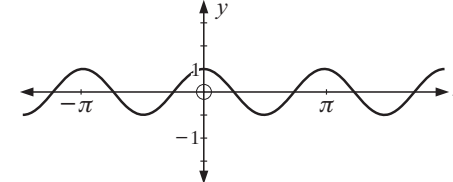
$y = \cos x$ with factor $\frac{3}{2}$.

h $y = \cos(x - \frac{\pi}{6}) + 1$

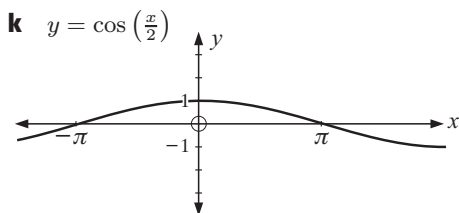


i.e., a translation of $\begin{bmatrix} \frac{\pi}{6} \\ 1 \end{bmatrix}$.

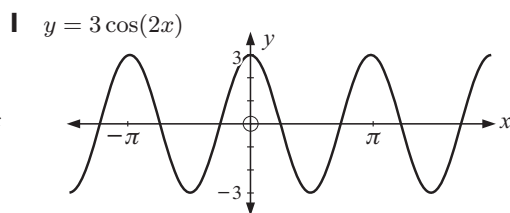
j $y = \cos(2x)$



i.e., a horizontal dilation of factor $\frac{1}{2}$.



i.e., a horizontal dilation of factor 2.



i.e., a horizontal dilation of factor $\frac{1}{2}$
followed by a vertical dilation of factor 3.

2 a period = $\frac{2\pi}{3}$ **b** period = $\frac{2\pi}{\frac{1}{3}} = 6\pi$ **c** period = $\frac{2\pi}{\frac{\pi}{50}} = 100$

3 A controls the amplitude B controls the period, period = $\frac{2\pi}{|B|}$
 C controls the horizontal translation D controls the vertical translation

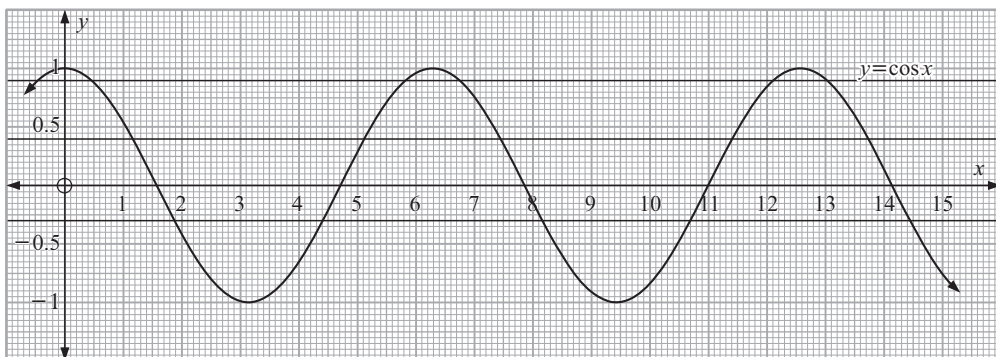
4 a If $y = A\cos B(x - C) + D$, then $A = 2$, $\pi = \frac{2\pi}{B} \therefore B = 2$
 C and D are 0 as there is no horizontal or vertical shift. $\therefore y = 2\cos(2x)$

b If $y = A\cos B(x - C) + D$, then $A = 1$, $4\pi = \frac{2\pi}{B} \therefore B = \frac{1}{2}$
A vertical shift of 2 units, no horizontal shift $\therefore D = 2$, $C = 0$.
So, $y = \cos\frac{1}{2}(x) + 2$ i.e., $y = \cos\left(\frac{x}{2}\right) + 2$.

c If $y = A\cos B(x - C) + D$, then $A = -5$, $6 = \frac{2\pi}{B} \therefore B = \frac{\pi}{3}$
 $C = D = 0$ {as no translation} $\therefore y = -5\cos\left(\frac{\pi}{3}x\right)$

EXERCISE 13F

1



a $x \div 1.2, 5.1, 7.4$

b $x \div 4.4, 8.2, 10.7$

2 a $\cos x = 0.561$, $0 \leq x \leq 10$
 $\therefore x \div 0.975, 5.308, 7.258$

b $\cos 2x = 0.782$, $0 \leq x \leq 6$
 $\therefore x \div 0.336, 2.805, 3.478, 5.947$

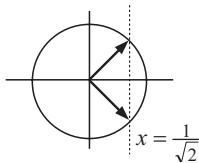
c $\cos(x - 1.3) = -0.609$, $0 \leq x \leq 12$
 $\therefore x \div 3.526, 5.358, 9.809, 11.641$

d $4\cos 3x + 1 = 0$, $0 \leq x \leq 5$
 $\therefore \cos 3x = -0.25$, $0 \leq x \leq 5$
 $\therefore x \div 0.608, 1.487, 2.702,$
 $3.581, 4.797$

3 a $\cos x = \frac{1}{\sqrt{2}}, \quad 0 \leq x \leq 4\pi$

$$\therefore x = \left\{ \frac{\pi}{4}, \frac{7\pi}{4} \right\} + k2\pi$$

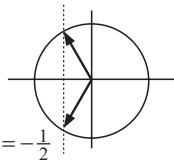
$$\therefore x = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{7\pi}{4}, \frac{15\pi}{4}$$



b $\cos x = -\frac{1}{2}, \quad 0 \leq x \leq 5\pi$

$$\therefore x = \left\{ \frac{2\pi}{3}, \frac{4\pi}{3} \right\} + k2\pi$$

$$\therefore x = \frac{2\pi}{3}, \frac{8\pi}{3}, \frac{14\pi}{3}, \frac{4\pi}{3}, \frac{10\pi}{3}, x = -\frac{1}{2}$$

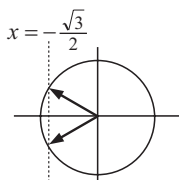


c $2 \cos x + \sqrt{3} = 0, \quad 0 \leq x \leq 3\pi$

$$\therefore \cos x = -\frac{\sqrt{3}}{2}$$

$$\therefore x = \left\{ \frac{5\pi}{6}, \frac{7\pi}{6} \right\} + k2\pi$$

$$\therefore x = \frac{5\pi}{6}, \frac{17\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}$$

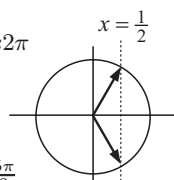


d $\cos(x - \frac{2\pi}{3}) = \frac{1}{2}, \quad -2\pi \leq x \leq 2\pi$

$$\therefore x - \frac{2\pi}{3} = \left\{ \frac{\pi}{3}, \frac{5\pi}{3} \right\} + k2\pi$$

$$\therefore x = \left\{ \frac{\pi}{3}, \frac{7\pi}{3} \right\} + k2\pi$$

$$\therefore x = \pi, -\pi, \frac{\pi}{3}, -\frac{5\pi}{3}$$



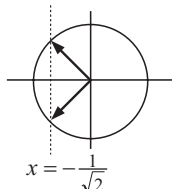
e $\sqrt{2} \cos(x - \frac{\pi}{4}) + 1 = 0, \quad 0 \leq x \leq 3\pi$

$$\therefore \cos(x - \frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$$

$$\therefore x - \frac{\pi}{4} = \left\{ \frac{3\pi}{4}, \frac{5\pi}{4} \right\} + k2\pi$$

$$\therefore x = \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\} + k2\pi$$

$$\therefore x = \pi, 3\pi, \frac{3\pi}{2}, \frac{9\pi}{2}$$



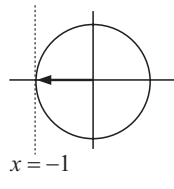
f $\cos 2x + 1 = 0, \quad 0 \leq x \leq 2\pi$

$$\therefore \cos 2x = -1$$

$$\therefore 2x = \pi + k2\pi$$

$$\therefore x = \frac{\pi}{2} + k\pi$$

$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$$



4 a The period is 4 seconds.

$$\therefore \frac{2\pi}{B} = 4$$

$$\therefore B = \frac{\pi}{2}$$

Amplitude is 3

$$\therefore A = 3$$

$$D = 1 + 3 = 4$$

$$C = 0$$

$$\therefore H(t) = 3 \cos \frac{\pi}{2}(t - 0) + 4 \text{ metres}$$

$$\text{i.e., } H(t) = 3 \cos(\frac{\pi}{2}t) + 4 \text{ metres}$$

Check: When $t = 0$, $H(0) = 3 \cos 0 + 4 = 7$ ✓

b X enters the water when $H(t) = 2$

$$\therefore 3 \cos\left(\frac{\pi t}{2}\right) + 4 = 2 \quad \therefore \cos\left(\frac{\pi t}{2}\right) = -\frac{2}{3}$$

Using technology, $t \doteq 1.46$ sec

EXERCISE 13G.1

1 a $\sin \theta + \sin \theta$
 $= 2 \sin \theta$

d $3 \sin \theta - 2 \sin \theta$
 $= \sin \theta$

2 a $3 \sin^2 \theta + 3 \cos^2 \theta$
 $= 3(\sin^2 \theta + \cos^2 \theta)$
 $= 3(1)$
 $= 3$

d $3 - 3 \sin^2 \theta$
 $= 3(1 - \sin^2 \theta)$
 $= 3 \cos^2 \theta$

b $2 \cos \theta + \cos \theta$
 $= 3 \cos \theta$

e $\cos \theta - 3 \cos \theta$
 $= -2 \cos \theta$

b $-2 \sin^2 \theta - 2 \cos^2 \theta$
 $= -2(\sin^2 \theta + \cos^2 \theta)$
 $= -2(1)$
 $= -2$

e $4 - 4 \cos^2 \theta$
 $= 4(1 - \cos^2 \theta)$
 $= 4 \sin^2 \theta$

c $3 \sin \theta - \sin \theta$
 $= 2 \sin \theta$

f $2 \cos \theta - 5 \cos \theta$
 $= -3 \cos \theta$

c $-\cos^2 \theta - \sin^2 \theta$
 $= -(\cos^2 \theta + \sin^2 \theta)$
 $= -(1)$
 $= -1$

f $\sin^3 \theta + \sin \theta \cos^2 \theta$
 $= \sin \theta (\sin^2 \theta + \cos^2 \theta)$
 $= \sin \theta (1)$
 $= \sin \theta$

$$\begin{aligned} \mathbf{g} \quad & \cos^2 \theta - 1 \\ &= 1 - \sin^2 \theta - 1 \\ &= -\sin^2 \theta \end{aligned}$$

$$\begin{aligned} \mathbf{j} \quad & \frac{1 - \sin^2 \theta}{\cos^2 \theta} \\ &= \frac{\cos^2 \theta}{\cos^2 \theta} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \sin^2 \theta - 1 \\ &= 1 - \cos^2 \theta - 1 \\ &= -\cos^2 \theta \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad & \frac{1 - \cos^2 \theta}{\sin \theta} \\ &= \frac{\sin^2 \theta}{\sin \theta} \\ &= \sin \theta \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & 2 \cos^2 \theta - 2 \\ &= -2(1 - \cos^2 \theta) \\ &= -2 \sin^2 \theta \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad & \frac{\cos^2 \theta - 1}{-\sin \theta} \\ &= \frac{1 - \sin^2 \theta - 1}{-\sin \theta} \\ &= \frac{-\sin^2 \theta}{-\sin \theta} \\ &= \sin \theta \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & (1 + \sin \theta)^2 \\ &= 1 + 2 \sin \theta + \sin^2 \theta \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & (\cos \alpha - 1)^2 \\ &= \cos^2 \alpha - 2 \cos \alpha + 1 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & (\sin \beta - \cos \beta)^2 \\ &= \sin^2 \beta - 2 \sin \beta \cos \beta + \cos^2 \beta \\ &= 1 - 2 \sin \beta \cos \beta \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & (\sin \alpha - 2)^2 \\ &= \sin^2 \alpha - 4 \sin \alpha + 4 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & (\sin \alpha + \cos \alpha)^2 \\ &= \sin^2 \alpha + 2 \sin \alpha \cos \alpha + \cos^2 \alpha \\ &= 1 + 2 \sin \alpha \cos \alpha \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & -(2 - \cos \alpha)^2 \\ &= -[4 - 4 \cos \alpha + \cos^2 \alpha] \\ &= -4 + 4 \cos \alpha - \cos^2 \alpha \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad & 1 - \sin^2 \theta \\ &= (1 + \sin \theta)(1 - \sin \theta) \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \sin^2 \alpha - \cos^2 \alpha \\ &= (\sin \alpha + \cos \alpha)(\sin \alpha - \cos \alpha) \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \cos^2 \alpha - 1 \\ &= (\cos \alpha + 1)(\cos \alpha - 1) \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & 2 \sin^2 \beta - \sin \beta \\ &= \sin \beta(2 \sin \beta - 1) \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & 2 \cos \phi + 3 \cos^2 \phi \\ &= \cos \phi(2 + 3 \cos \phi) \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & 3 \sin^2 \theta - 6 \sin \theta \\ &= 3 \sin \theta(\sin \theta - 2) \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \sin^2 \theta + 5 \sin \theta + 6 \\ &= (\sin \theta + 2)(\sin \theta + 3) \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & 2 \cos^2 \theta + 7 \cos \theta + 3 \\ &= (2 \cos \theta + 1)(\cos \theta + 3) \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & 6 \cos^2 \alpha - \cos \alpha - 1 \\ &= (3 \cos \alpha + 1)(2 \cos \alpha - 1) \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad & \frac{1 - \sin^2 \alpha}{1 - \sin \alpha} \\ &= \frac{(1 + \sin \alpha)(1 - \sin \alpha)}{1 - \sin \alpha}^1 \\ &= 1 + \sin \alpha \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \frac{\cos^2 \beta - 1}{\cos \beta + 1} \\ &= \frac{(\cos \beta - 1)(\cos \beta + 1)}{\cos \beta + 1}^1 \\ &= \cos \beta - 1 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \frac{\cos^2 \phi - \sin^2 \phi}{\cos \phi + \sin \phi} \\ &= \frac{(\cos \phi - \sin \phi)(\cos \phi + \sin \phi)}{\cos \phi + \sin \phi}^1 \\ &= \cos \phi - \sin \phi \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \frac{\cos^2 \phi - \sin^2 \phi}{\cos \phi - \sin \phi} \\ &= \frac{(\cos \phi + \sin \phi)(\cos \phi - \sin \phi)}{\cos \phi - \sin \phi}^1 \\ &= \cos \phi + \sin \phi \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \frac{\sin \alpha + \cos \alpha}{\sin^2 \alpha - \cos^2 \alpha} \\ &= \frac{\sin \alpha + \cos \alpha}{(\sin \alpha + \cos \alpha)(\sin \alpha - \cos \alpha)}^1 \\ &= \frac{1}{\sin \alpha - \cos \alpha} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \frac{3 - 3 \sin^2 \theta}{6 \cos \theta} \\ &= \frac{3(1 - \sin^2 \theta)}{6 \cos \theta} \\ &= \frac{3 \cos^2 \theta}{6 \cos \theta} \\ &= \frac{\cos \theta}{2} \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad & (\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta)^2 \\
 &= \cos^2 \theta + \cancel{2 \cos \theta \sin \theta} + \sin^2 \theta \\
 &\quad + \cos^2 \theta - \cancel{2 \cos \theta \sin \theta} + \sin^2 \theta \\
 &= 2 \cos^2 \theta + 2 \sin^2 \theta \\
 &= 2(\cos^2 \theta + \sin^2 \theta) \\
 &= 2(1) \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & (1 - \cos \theta) \left(1 + \frac{1}{\cos \theta} \right) \\
 &= 1 + \frac{1}{\cos \theta} - \cos \theta - 1 \\
 &= \frac{1}{\cos \theta} - \cos \theta \\
 &= \frac{1}{\cos \theta} - \cos \theta \left(\frac{\cos \theta}{\cos \theta} \right) \\
 &= \frac{1 - \cos^2 \theta}{\cos \theta} \\
 &= \frac{\sin^2 \theta}{\cos \theta}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & (2 \sin \theta + 3 \cos \theta)^2 + (3 \sin \theta - 2 \cos \theta)^2 \\
 &= 4 \sin^2 \theta + \cancel{12 \sin \theta \cos \theta} + 9 \cos^2 \theta \\
 &\quad + 9 \sin^2 \theta - \cancel{12 \sin \theta \cos \theta} + 4 \cos^2 \theta \\
 &= 13 \sin^2 \theta + 13 \cos^2 \theta \\
 &= 13(\sin^2 \theta + \cos^2 \theta) \\
 &= 13(1) \\
 &= 13
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \left(1 + \frac{1}{\sin \theta} \right) (\sin \theta - \sin^2 \theta) \\
 &= \sin \theta - \sin^2 \theta + 1 - \sin \theta \\
 &= 1 - \sin^2 \theta \\
 &= \cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \frac{\sin \theta}{1 + \cos \theta} + \frac{1 + \cos \theta}{\sin \theta} \\
 &= \frac{\sin^2 \theta + (1 + \cos \theta)(1 + \cos \theta)}{\sin \theta(1 + \cos \theta)} \\
 &= \frac{\sin^2 \theta + 1 + 2 \cos \theta + \cos^2 \theta}{\sin \theta(1 + \cos \theta)} \\
 &= \frac{1 + 1 + 2 \cos \theta}{\sin \theta(1 + \cos \theta)} \\
 &= \frac{\cancel{2(1 + \cos \theta)}^1}{\sin \theta \cancel{(1 + \cos \theta)}_1} \\
 &= \frac{2}{\sin \theta}
 \end{aligned}$$

EXERCISE 13G.2

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \sin \theta + \sin(-\theta) \\
 &= \sin \theta - \sin \theta \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & 3 \sin \theta - \sin(-\theta) \\
 &= 3 \sin \theta - (-\sin \theta) \\
 &= 3 \sin \theta + \sin \theta \\
 &= 4 \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \cos(-\alpha) \cos \alpha - \sin(-\alpha) \sin \alpha \\
 &= \cos \alpha \cos \alpha - (-\sin \alpha \sin \alpha) \\
 &= \cos^2 \alpha + \sin^2 \alpha \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \sin(-\theta) - \sin \theta \\
 &= -\sin \theta - \sin \theta \\
 &= -2 \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \cos^2(-\alpha) \\
 &= \cos(-\alpha) \times \cos(-\alpha) \\
 &= \cos \alpha \times \cos \alpha \\
 &= \cos^2 \alpha
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 2 \cos \theta + \cos(-\theta) \\
 &= 2 \cos \theta + \cos \theta \\
 &= 3 \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \sin^2(-\alpha) \\
 &= \sin(-\alpha) \times \sin(-\alpha) \\
 &= -\sin \alpha \times -\sin \alpha \\
 &= \sin^2 \alpha
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & 2 \sin \theta - \cos(90^\circ - \theta) \\
 &= 2 \sin \theta - \sin \theta \\
 &= \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \sin(-\theta) - \cos(90^\circ - \theta) \\
 &= -\sin \theta - \sin \theta \\
 &= -2 \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \sin(90^\circ - \theta) - \cos \theta \\
 &= \cos \theta - \cos \theta \\
 &= 0
 \end{aligned}$$

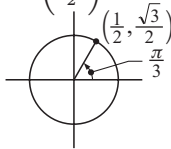
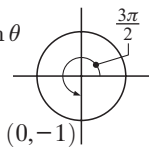
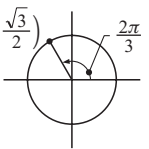
$$\begin{aligned}
 \mathbf{d} \quad & 3 \cos(-\theta) - 4 \sin\left(\frac{\pi}{2} - \theta\right) \\
 &= 3 \cos \theta - 4 \cos \theta \\
 &= -\cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & 3 \cos \theta + \sin\left(\frac{\pi}{2} - \theta\right) \\
 &= 3 \cos \theta + \cos \theta \\
 &= 4 \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \cos\left(\frac{\pi}{2} - \theta\right) + 4 \sin \theta \\
 &= \sin \theta + 4 \sin \theta \\
 &= 5 \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad & \sin(\theta - \phi) & \cos(\theta - \phi) \\
 &= \sin(-[\phi - \theta]) &= \cos(-[\phi - \theta]) \\
 &= -\sin(\phi - \theta) &= \cos(\phi - \theta)
 \end{aligned}$$

EXERCISE 13H

- 1**
- a** $\sin(M + N) = \sin M \cos N + \cos M \sin N$
- c** $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$
- e** $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
- g** $\sin(\alpha - 2\beta) = \sin \alpha \cos 2\beta - \cos \alpha \sin 2\beta$
- i** $\cos(B - 2C) = \cos B \cos 2C + \sin B \sin 2C$
- 2**
- a** $\sin(90^\circ + \theta)$
 $= \sin 90^\circ \cos \theta + \cos 90^\circ \sin \theta$
 $= (1) \cos \theta + (0) \sin \theta$
 $= \cos \theta$
- c** $\sin(180^\circ - \alpha)$
 $= \sin 180^\circ \cos \alpha - \cos 180^\circ \sin \alpha$
 $= (0) \cos \alpha - (-1) \sin \alpha$
 $= \sin \alpha$
- e** $\sin(2\pi - A)$
 $= \sin 2\pi \cos A - \cos 2\pi \sin A$
 $= (0) \cos A - (1) \sin A$
 $= -\sin A$
- 3**
- a** $\sin\left(\theta + \frac{\pi}{3}\right)$
 $= \sin \theta \cos\left(\frac{\pi}{3}\right) + \cos \theta \sin\left(\frac{\pi}{3}\right)$
 $= \sin \theta \times \left(\frac{1}{2}\right) + \cos \theta \times \left(\frac{\sqrt{3}}{2}\right)$
 $= \frac{1}{2} \sin \theta + \frac{\sqrt{3}}{2} \cos \theta$
- 
- c** $\cos\left(\theta + \frac{\pi}{4}\right)$
 $= \cos \theta \cos\left(\frac{\pi}{4}\right) - \sin \theta \sin\left(\frac{\pi}{4}\right)$
 $= \frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta$
 $= -\frac{1}{\sqrt{2}} \sin \theta + \frac{1}{\sqrt{2}} \cos \theta$
- 4**
- a** $\cos 2\theta \cos \theta + \sin 2\theta \sin \theta$
 $= \cos(2\theta - \theta)$
 $= \cos \theta$
- c** $\cos A \sin B - \sin A \cos B$
 $= \sin B \cos A - \cos B \sin A$
 $= \sin(B - A)$
- e** $\sin \phi \sin \theta - \cos \phi \cos \theta$
 $= -[\cos \phi \cos \theta - \sin \phi \sin \theta]$
 $= -\cos(\phi + \theta)$
- 5**
- a** $\cos(\alpha + \beta) \cos(\alpha - \beta) - \sin(\alpha + \beta) \sin(\alpha - \beta)$
 $= \cos[(\alpha + \beta) + (\alpha - \beta)]$
 $= \cos(2\alpha)$
- b** $\sin(\theta - 2\phi) \cos(\theta + \phi) - \cos(\theta - 2\phi) \sin(\theta + \phi)$
 $= \sin[(\theta - 2\phi) - (\theta + \phi)]$
 $= \sin(-3\phi)$
 $= -\sin(3\phi)$
- b** $\cos(T - S) = \cos T \cos S + \sin T \sin S$
- d** $\sin(\phi + \theta) = \sin \phi \cos \theta + \cos \phi \sin \theta$
- f** $\cos(2\theta - \alpha) = \cos 2\theta \cos \alpha + \sin 2\theta \sin \alpha$
- h** $\cos(3A + B) = \cos 3A \cos B - \sin 3A \sin B$
- b** $\cos(90^\circ + \theta)$
 $= \cos 90^\circ \cos \theta - \sin 90^\circ \sin \theta$
 $= (0) \cos \theta - (1) \sin \theta$
 $= -\sin \theta$
- d** $\cos(\pi + \alpha)$
 $= \cos \pi \cos \alpha - \sin \pi \sin \alpha$
 $= (-1) \cos \alpha - (0) \sin \alpha$
 $= -\cos \alpha$
- f** $\cos\left(\frac{3\pi}{2} - \theta\right)$
 $= \cos\left(\frac{3\pi}{2}\right) \cos \theta + \sin\left(\frac{3\pi}{2}\right) \sin \theta$
 $= (0) \cos \theta + (-1) \sin \theta$
 $= -\sin \theta$
- 
- b** $\cos\left(\frac{2\pi}{3} - \theta\right)$
 $= \cos\left(\frac{2\pi}{3}\right) \cos \theta + \sin\left(\frac{2\pi}{3}\right) \sin \theta$
 $= \left(-\frac{1}{2}\right) \cos \theta + \left(\frac{\sqrt{3}}{2}\right) \sin \theta$
 $= -\frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta$
 $= \frac{\sqrt{3}}{2} \sin \theta - \frac{1}{2} \cos \theta$
- 
- d** $\sin\left(\frac{\pi}{6} - \theta\right)$
 $= \sin\left(\frac{\pi}{6}\right) \cos \theta - \cos\left(\frac{\pi}{6}\right) \sin \theta$
 $= \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta$
 $= -\frac{\sqrt{3}}{2} \sin \theta + \frac{1}{2} \cos \theta$
- b** $\sin 2A \cos A + \cos 2A \sin A$
 $= \sin(2A + A)$
 $= \sin 3A$
- d** $\sin \alpha \sin \beta + \cos \alpha \cos \beta$
 $= \cos \alpha \cos \beta + \sin \alpha \sin \beta$
 $= \cos(\alpha - \beta)$
- f** $2 \sin \alpha \cos \beta - 2 \cos \alpha \sin \beta$
 $= 2 [\sin \alpha \cos \beta - \cos \alpha \sin \beta]$
 $= 2 \sin(\alpha - \beta)$
- c** $\cos \alpha \cos(\beta - \alpha) - \sin \alpha \sin(\beta - \alpha)$
 $= \cos[\alpha + (\beta - \alpha)]$
 $= \cos \beta$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad & \cos 75^\circ \\
 &= \cos(45^\circ + 30^\circ) \\
 &= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\
 &= \left(\frac{1}{\sqrt{2}}\right) \left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right) \left(\frac{1}{2}\right) \\
 &= \frac{\sqrt{3}-1}{2\sqrt{2}} \\
 &= \left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) \frac{\sqrt{2}}{\sqrt{2}} \\
 &= \frac{\sqrt{6}-\sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \cos\left(\frac{13\pi}{12}\right) \\
 &= \cos\left(\frac{13 \times 180^\circ}{12}\right) \\
 &= \cos 195^\circ \\
 &= \cos(150^\circ + 45^\circ) \\
 &= \cos 150^\circ \cos 45^\circ - \sin 150^\circ \sin 45^\circ \\
 &= \left(-\frac{\sqrt{3}}{2}\right) \left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2} \left(\frac{1}{\sqrt{2}}\right) \\
 &= \left(\frac{-\sqrt{3}-1}{2\sqrt{2}}\right) \frac{\sqrt{2}}{\sqrt{2}} \\
 &= \frac{-\sqrt{6}-\sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \sin 105^\circ \\
 &= \sin(60^\circ + 45^\circ) \\
 &= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ \\
 &= \left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\right) \left(\frac{1}{\sqrt{2}}\right) \\
 &= \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) \frac{\sqrt{2}}{\sqrt{2}} \\
 &= \frac{\sqrt{6}+\sqrt{2}}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{7} \quad \mathbf{a} \quad & \sqrt{2} \cos\left(\theta + \frac{\pi}{4}\right) \\
 &= \sqrt{2} \left[\cos \theta \cos\left(\frac{\pi}{4}\right) - \sin \theta \sin\left(\frac{\pi}{4}\right)\right] \\
 &= \sqrt{2} \left[\frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta\right] \\
 &= \cos \theta - \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 2 \cos\left(\theta - \frac{\pi}{3}\right) \\
 &= 2 \left[\cos \theta \cos\left(\frac{\pi}{3}\right) + \sin \theta \sin\left(\frac{\pi}{3}\right)\right] \\
 &= 2 \left[\frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta\right] \\
 &= \cos \theta + \sqrt{3} \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \cos(\alpha + \beta) - \cos(\alpha - \beta) \\
 &= \cos \alpha \cos \beta - \sin \alpha \sin \beta - [\cos \alpha \cos \beta + \sin \alpha \sin \beta] \\
 &= \cancel{\cos \alpha \cos \beta} - \sin \alpha \sin \beta - \cancel{\cos \alpha \cos \beta} - \sin \alpha \sin \beta \\
 &= -2 \sin \alpha \sin \beta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \cos(\alpha + \beta) \cos(\alpha - \beta) \\
 &= [\cos \alpha \cos \beta - \sin \alpha \sin \beta] [\cos \alpha \cos \beta + \sin \alpha \sin \beta] \\
 &= \cos^2 \alpha \cos^2 \beta - \sin^2 \alpha \sin^2 \beta \\
 &= \cos^2 \alpha [1 - \sin^2 \beta] - [1 - \cos^2 \alpha] \sin^2 \beta \\
 &= \cos^2 \alpha - \cancel{\cos^2 \alpha \sin^2 \beta} - \sin^2 \beta + \cancel{\cos^2 \alpha \sin^2 \beta} \\
 &= \cos^2 \alpha - \sin^2 \beta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad & \sin(A + B) + \sin(A - B) \\
 &= \sin A \cos B + \cancel{\cos A \sin B} + \sin A \cos B - \cancel{\cos A \sin B} \\
 &= 2 \sin A \cos B
 \end{aligned}$$

$$\mathbf{b} \quad \therefore \sin A \cos B = \frac{1}{2} \sin(A + B) + \frac{1}{2} \sin(A - B)$$

$$\begin{aligned}
 \mathbf{i} \quad & \sin 3\theta \cos \theta \\
 &= \frac{1}{2} \sin(3\theta + \theta) + \frac{1}{2} \sin(3\theta - \theta) \\
 &= \frac{1}{2} \sin 4\theta + \frac{1}{2} \sin 2\theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{iii} \quad & 2 \sin 5\beta \cos \beta \\
 &= 2 \left[\frac{1}{2} \sin(5\beta + \beta) + \frac{1}{2} \sin(5\beta - \beta)\right] \\
 &= \sin 6\beta + \sin 4\beta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{v} \quad & 6 \cos 4\alpha \sin 3\alpha \\
 &= 6 \sin 3\alpha \cos 4\alpha \\
 &= 6 \left[\frac{1}{2} \sin 7\alpha + \frac{1}{2} \sin(-\alpha)\right] \\
 &= 3 \sin 7\alpha + 3 \sin(-\alpha) \\
 &= 3 \sin 7\alpha - 3 \sin \alpha
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad & \sin 6\alpha \cos \alpha \\
 &= \frac{1}{2} \sin(6\alpha + \alpha) + \frac{1}{2} \sin(6\alpha - \alpha) \\
 &= \frac{1}{2} \sin 7\alpha + \frac{1}{2} \sin 5\alpha
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{iv} \quad & 4 \cos \theta \sin 4\theta \\
 &= 4 [\sin 4\theta \cos \theta] \\
 &= 4 \left[\frac{1}{2} \sin 5\theta + \frac{1}{2} \sin 3\theta\right] \\
 &= 2 \sin 5\theta + 2 \sin 3\theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{vi} \quad & \frac{1}{3} \cos 5A \sin 3A \\
 &= \frac{1}{3} \sin 3A \cos 5A \\
 &= \frac{1}{3} \left[\frac{1}{2} \sin 8A + \frac{1}{2} \sin(-2A)\right] \\
 &= \frac{1}{6} \sin 8A - \frac{1}{6} \sin 2A
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{9} \quad \mathbf{a} \quad & \cos(A+B) + \cos(A-B) \\
 &= \cos A \cos B - \cancel{\sin A \sin B} + \cos A \cos B + \cancel{\sin A \sin B} \\
 &= 2 \cos A \cos B
 \end{aligned}$$

$$\mathbf{b} \quad \therefore \cos A \cos B = \frac{1}{2} \cos(A+B) + \frac{1}{2} \cos(A-B)$$

$$\begin{aligned}
 \mathbf{i} \quad & \cos 4\theta \cos \theta \\
 &= \frac{1}{2} \cos 5\theta + \frac{1}{2} \cos 3\theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad & \cos 7\alpha \cos \alpha \\
 &= \frac{1}{2} \cos 8\alpha + \frac{1}{2} \cos 6\alpha
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{iii} \quad & 2 \cos 3\beta \cos \beta \\
 &= 2 \left[\frac{1}{2} \cos 4\beta + \frac{1}{2} \cos 2\beta \right] \\
 &= \cos 4\beta + \cos 2\beta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{iv} \quad & 6 \cos x \cos 7x \\
 &= 6 \cos 7x \cos x \\
 &= 6 \left[\frac{1}{2} \cos 8x + \frac{1}{2} \cos 6x \right] \\
 &= 3 \cos 8x + 3 \cos 6x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{v} \quad & 3 \cos P \cos 4P \\
 &= 3 \cos 4P \cos P \\
 &= 3 \left[\frac{1}{2} \cos 5P + \frac{1}{2} \cos 3P \right] \\
 &= \frac{3}{2} \cos 5P + \frac{3}{2} \cos 3P
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{vi} \quad & \frac{1}{4} \cos 4x \cos 2x \\
 &= \frac{1}{4} \left[\frac{1}{2} \cos 6x + \frac{1}{2} \cos 2x \right] \\
 &= \frac{1}{8} \cos 6x + \frac{1}{8} \cos 2x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{10} \quad \mathbf{a} \quad & \cos(A-B) - \cos(A+B) \\
 &= \cancel{\cos A \cos B} + \sin A \sin B - [\cancel{\cos A \cos B} - \sin A \sin B] \\
 &= \sin A \sin B + \sin A \sin B \\
 &= 2 \sin A \sin B
 \end{aligned}$$

$$\mathbf{b} \quad \therefore \sin A \sin B = \frac{1}{2} \cos(A-B) - \frac{1}{2} \cos(A+B)$$

$$\begin{aligned}
 \mathbf{i} \quad & \sin 3\theta \sin \theta \\
 &= \frac{1}{2} \cos 2\theta - \frac{1}{2} \cos 4\theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{ii} \quad & \sin 6\alpha \sin \alpha \\
 &= \frac{1}{2} \cos 5\alpha - \frac{1}{2} \cos 7\alpha
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{iii} \quad & 2 \sin 5\beta \sin \beta \\
 &= 2 \left[\frac{1}{2} \cos 4\beta - \frac{1}{2} \cos 6\beta \right] \\
 &= \cos 4\beta - \cos 6\beta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{iv} \quad & 4 \sin \theta \sin 4\theta \\
 &= 4 \sin 4\theta \sin \theta \\
 &= 4 \left[\frac{1}{2} \cos 3\theta - \frac{1}{2} \cos 5\theta \right] \\
 &= 2 \cos 3\theta - 2 \cos 5\theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{v} \quad & 10 \sin 2A \sin 8A \\
 &= 10 \sin 8A \sin 2A \\
 &= 10 \left[\frac{1}{2} \cos 6A - \frac{1}{2} \cos 10A \right] \\
 &= 5 \cos 6A - 5 \cos 10A
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{vi} \quad & \frac{1}{5} \sin 3M \sin 7M \\
 &= \frac{1}{5} \sin 7M \sin 3M \\
 &= \frac{1}{5} \left[\frac{1}{2} \cos 4M - \frac{1}{2} \cos 10M \right] \\
 &= \frac{1}{10} \cos 4M - \frac{1}{10} \cos 10M
 \end{aligned}$$

$$\mathbf{11} \quad (1) \quad \text{becomes} \quad \sin A \cos B = \frac{1}{2} \sin(2A)$$

$$(2) \quad \text{becomes} \quad \cos^2 A = \frac{1}{2} \cos(2A) + \frac{1}{2} \cos 0, \quad \text{i.e.,} \quad \cos^2 A = \frac{1}{2} \cos(2A) + \frac{1}{2}$$

$$(3) \quad \text{becomes} \quad \sin^2 A = \frac{1}{2} \cos 0 - \frac{1}{2} \cos(2A), \quad \text{i.e.,} \quad \sin^2 A = \frac{1}{2} - \frac{1}{2} \cos(2A)$$

$$\mathbf{12} \quad \mathbf{a} \quad A+B=S \quad A-B=D \quad \therefore 2A=S+D \quad \text{i.e.,} \quad A=\frac{S+D}{2}$$

$$\text{and} \quad B=S-A=S-\left(\frac{S+D}{2}\right)=\frac{2S}{2}-\left(\frac{S+D}{2}\right)=\frac{2S-S-D}{2}=\frac{S-D}{2}$$

$$\mathbf{b} \quad \sin A \cos B = \frac{1}{2} \sin(A+B) + \frac{1}{2} \sin(A-B) \quad \text{becomes}$$

$$\sin\left(\frac{S+D}{2}\right) \cos\left(\frac{S-D}{2}\right) = \frac{1}{2} \sin S + \frac{1}{2} \sin D$$

$$\text{i.e.,} \quad \sin S + \sin D = 2 \sin\left(\frac{S+D}{2}\right) \cos\left(\frac{S-D}{2}\right) \quad \dots\dots (4)$$

$$\mathbf{c} \quad \text{Replacing } D \text{ by } (-D) \text{ in (4) gives}$$

$$\sin S + \sin(-D) = 2 \sin\left(\frac{S-D}{2}\right) \cos\left(\frac{S+D}{2}\right) \quad \dots\dots (4)$$

$$\text{i.e.,} \quad \sin S - \sin D = 2 \cos\left(\frac{S+D}{2}\right) \sin\left(\frac{S-D}{2}\right)$$

$$\mathbf{d} \quad \cos A \cos B = \frac{1}{2} \cos(A+B) + \frac{1}{2} \cos(A-B) \quad \text{becomes}$$

$$\cos\left(\frac{S+D}{2}\right) \cos\left(\frac{S-D}{2}\right) = \frac{1}{2} \cos S + \frac{1}{2} \cos D$$

$$\text{i.e.,} \quad \cos S + \cos D = 2 \cos\left(\frac{S+D}{2}\right) \cos\left(\frac{S-D}{2}\right)$$

e $\sin A \sin B = \frac{1}{2} \cos(A - B) - \frac{1}{2} \cos(A + B)$ becomes

$$\sin\left(\frac{S+D}{2}\right) \sin\left(\frac{S-D}{2}\right) = \frac{1}{2} \cos D - \frac{1}{2} \cos S$$

i.e., $\cos D - \cos S = 2 \sin\left(\frac{S+D}{2}\right) \sin\left(\frac{S-D}{2}\right)$

or $\cos S - \cos D = -2 \sin\left(\frac{S+D}{2}\right) \sin\left(\frac{S-D}{2}\right)$

13 a $\sin 5x + \sin x$
 $= 2 \sin\left(\frac{5x+x}{2}\right) \cos\left(\frac{5x-x}{2}\right)$
 $= 2 \sin 3x \cos 2x$

b $\cos 8A + \cos 2A$
 $= 2 \cos\left(\frac{8A+2A}{2}\right) \cos\left(\frac{8A-2A}{2}\right)$
 $= 2 \cos 5A \cos 3A$

c $\cos 3\alpha - \cos \alpha$
 $= -2 \sin\left(\frac{3\alpha+\alpha}{2}\right) \sin\left(\frac{3\alpha-\alpha}{2}\right)$
 $= -2 \sin 2\alpha \sin \alpha$

d $\sin 5\theta - \sin 3\theta$
 $= 2 \cos\left(\frac{5\theta+3\theta}{2}\right) \sin\left(\frac{5\theta-3\theta}{2}\right)$
 $= 2 \cos 4\theta \sin \theta$

e $\cos 7\alpha - \cos \alpha$
 $= -2 \sin\left(\frac{7\alpha+\alpha}{2}\right) \sin\left(\frac{7\alpha-\alpha}{2}\right)$
 $= -2 \sin 4\alpha \sin 3\alpha$

f $\sin 3\alpha + \sin 7\alpha$
 $= \sin 7\alpha + \sin 3\alpha$
 $= 2 \sin\left(\frac{7\alpha+3\alpha}{2}\right) \cos\left(\frac{7\alpha-3\alpha}{2}\right)$
 $= 2 \sin 5\alpha \cos 2\alpha$

g $\cos 2B - \cos 4B$
 $= -[\cos 4B - \cos 2B]$
 $= -[-2 \sin\left(\frac{4B+2B}{2}\right) \sin\left(\frac{4B-2B}{2}\right)]$
 $= 2 \sin 3B \sin B$

h $\sin(x+h) - \sin x$
 $= 2 \cos\left(\frac{x+h+x}{2}\right) \sin\left(\frac{x+h-x}{2}\right)$
 $= 2 \cos\left(\frac{2x+h}{2}\right) \sin\left(\frac{h}{2}\right)$
 $= 2 \cos\left(x + \frac{h}{2}\right) \sin\left(\frac{h}{2}\right)$

i $\cos(x+h) - \cos x$
 $= -2 \sin\left(\frac{x+h+x}{2}\right) \sin\left(\frac{x+h-x}{2}\right)$
 $= -2 \sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{h}{2}\right)$
 $= -2 \sin\left(x + \frac{h}{2}\right) \sin\left(\frac{h}{2}\right)$

EXERCISE 13I

1 a $\sin 2A = 2 \sin A \cos A$
 $= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$
 $= \frac{24}{25}$

b $\cos 2A = \cos^2 A - \sin^2 A$
 $= \frac{9}{25} - \frac{16}{25}$
 $= -\frac{7}{25}$

2 $\cos 2A = 2 \cos^2 A - 1$
 $= 2\left(\frac{1}{3}\right)^2 - 1$
 $= 2 \times \frac{1}{9} - 1$
 $= \frac{2}{9} - 1$
 $= -\frac{7}{9}$

3 $\cos 2\phi = 1 - 2 \sin^2 \phi$
 $= 1 - 2\left(-\frac{2}{3}\right)^2$
 $= 1 - 2\left(\frac{4}{9}\right)$
 $= 1 - \frac{8}{9}$
 $= \frac{1}{9}$

4 a $\sin \alpha = -\frac{2}{3}$
 α is in Q3
 $\therefore \cos \alpha < 0$



$$\begin{aligned} \cos^2 \alpha + \sin^2 \alpha &= 1 \\ \therefore \cos^2 \alpha + \frac{4}{9} &= 1 \\ \therefore \cos^2 \alpha &= \frac{5}{9} \\ \therefore \cos \alpha &= -\frac{\sqrt{5}}{3} \end{aligned}$$

$$\begin{aligned} \sin 2\alpha &= 2 \sin \alpha \cos \alpha \\ &= 2\left(-\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right) \\ &= \frac{4\sqrt{5}}{9} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \cos \beta &= \frac{2}{5} \\ \beta &\text{ is in Q4} \\ \therefore \sin \beta &< 0 \end{aligned}$$



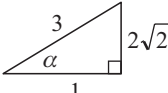
$$\begin{aligned} \cos^2 \beta + \sin^2 \beta &= 1 \\ \therefore \frac{4}{25} + \sin^2 \beta &= 1 \\ \therefore \sin^2 \beta &= \frac{21}{25} \\ \therefore \sin \beta &= -\frac{\sqrt{21}}{5} \end{aligned}$$

$$\begin{aligned} \sin 2\beta &= 2 \sin \beta \cos \beta \\ &= 2 \left(-\frac{\sqrt{21}}{5} \right) \left(\frac{2}{5} \right) \\ &= -\frac{4\sqrt{21}}{25} \end{aligned}$$

5 α is acute $\therefore \cos \alpha$ and $\sin \alpha$ are positive

$$\begin{aligned} \mathbf{a} \quad \cos 2\alpha &= 2 \cos^2 \alpha - 1 \\ \therefore -\frac{7}{9} &= 2 \cos^2 \alpha - 1 \\ \therefore 2 \cos^2 \alpha &= \frac{2}{9} \\ \therefore \cos^2 \alpha &= \frac{1}{9} \\ \therefore \cos \alpha &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \cos 2\alpha &= 1 - 2 \sin^2 \alpha \\ \therefore -\frac{7}{9} &= 1 - 2 \sin^2 \alpha \\ \therefore 2 \sin^2 \alpha &= 1\frac{7}{9} = \frac{16}{9} \\ \therefore \sin^2 \alpha &= \frac{8}{9} \\ \therefore \sin \alpha &= \frac{2\sqrt{2}}{3} \end{aligned}$$

or 

$$\sin \alpha = \frac{2\sqrt{2}}{3}$$

6 a $2 \sin \alpha \cos \alpha$
 $= \sin 2\alpha$

b $4 \cos \alpha \sin \alpha$
 $= 2(2 \sin \alpha \cos \alpha)$
 $= 2 \sin 2\alpha$

c $\sin \alpha \cos \alpha$
 $= \frac{1}{2}(2 \sin \alpha \cos \alpha)$
 $= \frac{1}{2} \sin 2\alpha$

d $2 \cos^2 \beta - 1$
 $= \cos 2\beta$

e $1 - 2 \cos^2 \phi$
 $= -(2 \cos^2 \phi - 1)$
 $= -\cos 2\phi$

f $1 - 2 \sin^2 N$
 $= \cos 2N$

g $2 \sin^2 M - 1$
 $= -(1 - 2 \sin^2 M)$
 $= -\cos 2M$

h $\cos^2 \alpha - \sin^2 \alpha$
 $= \cos 2\alpha$

i $\sin^2 \alpha - \cos^2 \alpha$
 $= -(\cos^2 \alpha - \sin^2 \alpha)$
 $= -\cos 2\alpha$

j $2 \sin 2A \cos 2A$
 $= \sin 2(2A)$
 $= \sin 4A$

k $2 \cos 3\alpha \sin 3\alpha$
 $= \sin 2(3\alpha)$
 $= \sin 6\alpha$

l $2 \cos^2 4\theta - 1$
 $= \cos 2(4\theta)$
 $= \cos 8\theta$

m $1 - 2 \cos^2 3\beta$
 $= -(2 \cos^2 3\beta - 1)$
 $= -\cos 2(3\beta)$
 $= -\cos 6\beta$

n $1 - 2 \sin^2 5\alpha$
 $= \cos 2(5\alpha)$
 $= \cos 10\alpha$

o $2 \sin^2 3D - 1$
 $= -(1 - 2 \sin^2 3D)$
 $= -\cos 2(3D)$
 $= -\cos 6D$

p $\cos^2 2A - \sin^2 2A$
 $= \cos 2(2A)$
 $= \cos 4A$

q $\cos^2(\frac{\alpha}{2}) - \sin^2(\frac{\alpha}{2})$
 $= \cos 2(\frac{\alpha}{2})$
 $= \cos \alpha$

r $2 \sin^2 3P - 2 \cos^2 3P$
 $= -2[\cos^2 3P - \sin^2 3P]$
 $= -2 \cos 2(3P)$
 $= -2 \cos 6P$

7 a $(\sin \theta + \cos \theta)^2$
 $= \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta$
 $= \underbrace{\sin^2 \theta + \cos^2 \theta}_{=1} + 2 \sin \theta \cos \theta$
 $= 1 + \sin 2\theta$

b $\cos^4 \theta - \sin^4 \theta$
 $= (\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta)$
 $= 1 \times \cos 2\theta$
 $= \cos 2\theta$

EXERCISE 13J.1

1 a $\tan 0^\circ$
 $= 0$

b $\tan 15^\circ$
 $\div 0.27$

c $\tan 20^\circ$
 $\div 0.36$

d $\tan 25^\circ$
 $\div 0.47$

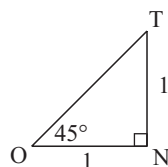
e $\tan 35^\circ$
 $\div 0.70$

f $\tan 45^\circ$
 $= 1.00$

g $\tan 50^\circ$
 $\div 1.19$

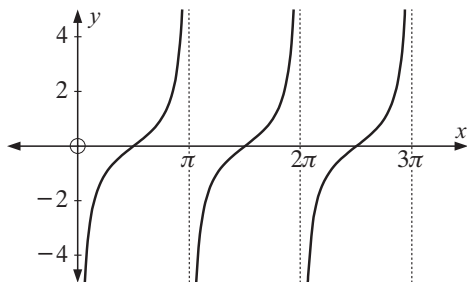
h $\tan 55^\circ$
 $\div 1.43$

2 In $\triangle TON$, $ON = NT = 1$ ($\triangle$ is isosceles) $\tan 45^\circ = \frac{NT}{ON} = \frac{1}{1} = 1$

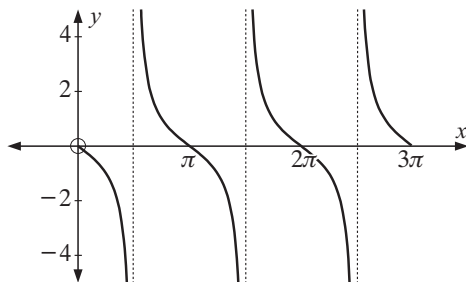


EXERCISE 13J.2

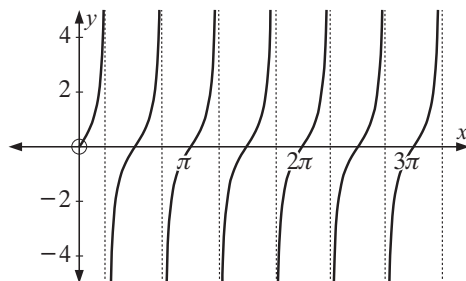
- 1 a i** $y = \tan(x - \frac{\pi}{2})$ is $y = \tan x$ translated $\begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}$.



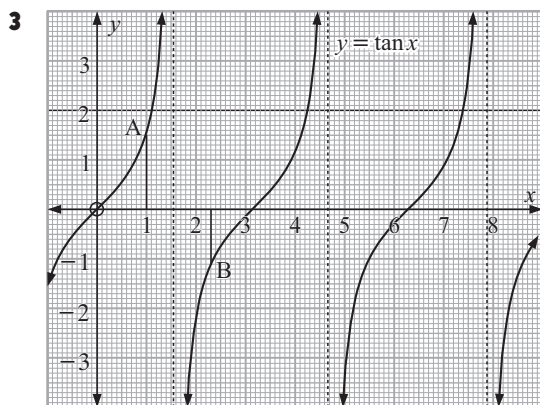
- ii** $y = -\tan x$ is $y = \tan x$ reflected in the x -axis.



- iii** $y = \tan 2x$ comes from $y = \tan x$ under a horizontal dilation of factor $\frac{1}{2}$.



- 2 a** translation through $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **b** reflection in x -axis **c** horizontal dilation, factor $k = 2$



- a i** $\tan 1 \div 1.6$ {point A}
ii $\tan 2.3 \div -1.1$ {point B}
b i $\tan 1 \div 1.557$
ii $\tan 2.3 \div -1.119$
c i When $\tan x = 2$, $x \div 1.1, 4.2, 7.4$
ii When $\tan x = -1.4$, $x \div 2.2, 5.3$

- 4 a** period $= \frac{\pi}{1} = \pi$ **b** period $= \frac{\pi}{2}$ **c** period $= \frac{\pi}{n}$

EXERCISE 13K.1

- 1** $X = \tan^{-1} 2 \therefore X \div 1.107 + k\pi$

a If $\tan 2x = 2$

then $2x = 1.107 + k\pi$

$\therefore x = 0.554 + k(\frac{\pi}{2})$

b $\tan(\frac{x}{3}) = 2$

$\therefore \frac{x}{3} = 1.107 + k\pi$

$\therefore x = 3.32 + k3\pi$

c $\tan(x + 1.2) = 2$

$\therefore x + 1.2 = 1.107 + k\pi$

$\therefore x = -0.0929 + k\pi$

$$2 \quad X = \tan^{-1}(-3) \div -1.249 + k\pi$$

$$\begin{aligned} \mathbf{a} \quad \tan(x-2) &= -3 \\ \therefore x-2 &\div -1.249 + k\pi \\ \therefore x &\div 0.751 + k\pi \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \tan(3x) &= -3 \\ \therefore 3x &= -1.249 + k\pi \\ \therefore x &\div -0.416 + \frac{k\pi}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \tan\left(\frac{x}{2}\right) &= -3 \\ \therefore \frac{x}{2} &= -1.249 + k\pi \\ \therefore x &= -2.50 + k2\pi \end{aligned}$$

$$3 \quad X = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} + k\pi$$

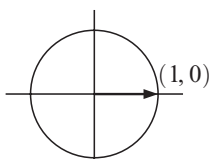
$$\begin{aligned} \mathbf{a} \quad \tan\left(x - \frac{\pi}{6}\right) &= \sqrt{3} \\ \therefore x - \frac{\pi}{6} &= \frac{\pi}{3} + k\pi \\ \therefore x &= \frac{\pi}{2} + k\pi \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \tan 4x &= \sqrt{3} \\ \therefore 4x &= \frac{\pi}{3} + k\pi \\ \therefore x &= \frac{\pi}{12} + \frac{k\pi}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \tan^2 x &= 3 \\ \therefore \tan x &= \pm\sqrt{3} \\ \therefore x &= \left\{ \frac{\pi}{3}, -\frac{\pi}{3} \right\} + k\pi \end{aligned}$$

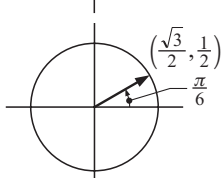
EXERCISE 13K.2

1 **a**



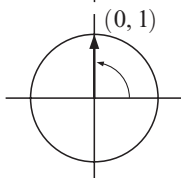
$$\begin{aligned} \tan 0 &= \frac{0}{1} \\ &= 0 \end{aligned}$$

c



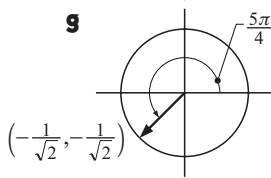
$$\begin{aligned} \tan\left(\frac{\pi}{6}\right) &= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

e



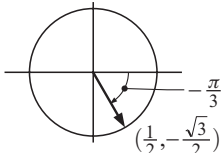
$$\begin{aligned} \tan\left(\frac{\pi}{2}\right) &= \frac{1}{0} \\ \text{i.e., undefined} \end{aligned}$$

g



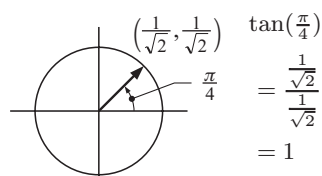
$$\begin{aligned} \tan\left(\frac{5\pi}{4}\right) &= -\frac{\sqrt{3}}{2} \div \frac{1}{2} \\ &= -\sqrt{3} \end{aligned}$$

i



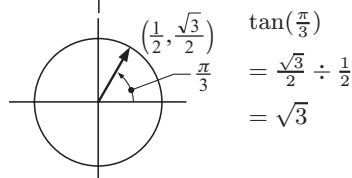
$$\begin{aligned} \tan\left(-\frac{\pi}{3}\right) &= -\frac{\sqrt{3}}{2} \div \frac{1}{2} \\ &= -\sqrt{3} \end{aligned}$$

b



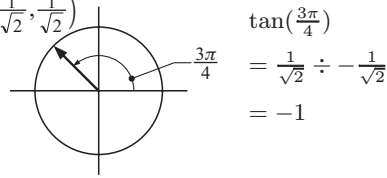
$$\begin{aligned} \tan\left(\frac{\pi}{4}\right) &= \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \\ &= 1 \end{aligned}$$

d



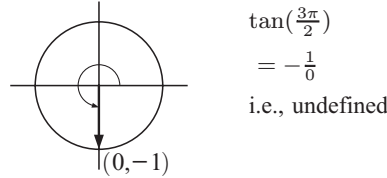
$$\begin{aligned} \tan\left(\frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2} \div \frac{1}{2} \\ &= \sqrt{3} \end{aligned}$$

f



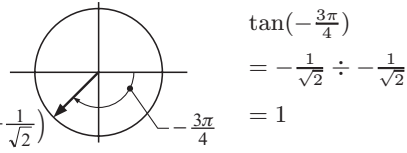
$$\begin{aligned} \tan\left(\frac{3\pi}{4}\right) &= \frac{1}{\sqrt{2}} \div -\frac{1}{\sqrt{2}} \\ &= -1 \end{aligned}$$

h



$$\begin{aligned} \tan\left(\frac{3\pi}{2}\right) &= -\frac{1}{0} \\ \text{i.e., undefined} \end{aligned}$$

j



$$\begin{aligned} \tan\left(-\frac{3\pi}{4}\right) &= -\frac{1}{\sqrt{2}} \div -\frac{1}{\sqrt{2}} \\ &= 1 \end{aligned}$$

2 **a**

$$\begin{aligned} \frac{\pi}{4}, \frac{\pi}{4} + \pi \\ \text{i.e., } \frac{\pi}{4}, \frac{5\pi}{4} \end{aligned}$$

b

$$\begin{aligned} \frac{3\pi}{4}, \frac{3\pi}{4} + \pi \\ \text{i.e., } \frac{3\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

c

$$\begin{aligned} \frac{\pi}{3}, \frac{\pi}{3} + \pi \\ \text{i.e., } \frac{\pi}{3}, \frac{4\pi}{3} \end{aligned}$$

d

$$\begin{aligned} 0, 0 + \pi, 0 + 2\pi \\ \text{i.e., } 0, \pi, 2\pi \end{aligned}$$

e

$$\begin{aligned} \frac{\pi}{6}, \frac{\pi}{6} + \pi \\ \text{i.e., } \frac{\pi}{6}, \frac{7\pi}{6} \end{aligned}$$

f

$$\begin{aligned} \frac{5\pi}{3}, \frac{5\pi}{3} - \pi \\ \text{i.e., } \frac{5\pi}{3}, \frac{2\pi}{3} \end{aligned}$$

3 a $3 \tan x - \tan x$
 $= 2 \tan x$

b $\tan x - 4 \tan x$
 $= -3 \tan x$

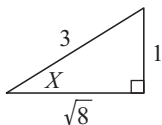
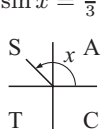
c $\tan x \cos x$
 $= \frac{\sin x}{\cos x} \times \cos x$
 $= \sin x$

d $\frac{\sin x}{\tan x}$
 $= \sin x \div \frac{\sin x}{\cos x}$
 $= \sin x \times \frac{\cos x}{\sin x}$
 $= \cos x$

e $3 \sin x + 2 \cos x \tan x$
 $= 3 \sin x + 2 \cos x \frac{\sin x}{\cos x}$
 $= 3 \sin x + 2 \sin x$
 $= 5 \sin x$

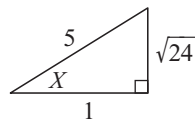
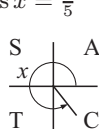
f $\frac{2 \tan x}{\sin x}$
 $= 2 \left(\frac{\sin x}{\cos x} \right) \div \frac{\sin x}{1}$
 $= \frac{2 \sin x}{\cos x} \times \frac{1}{\sin x}$
 $= \frac{2}{\cos x}$

4 a $\sin x = \frac{1}{3}$



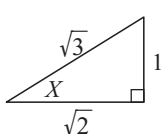
$\therefore \tan x = -\frac{1}{\sqrt{8}} = -\frac{1}{2\sqrt{2}}$

b $\cos x = \frac{1}{5}$



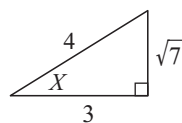
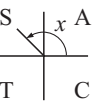
$\therefore \tan x = -\frac{\sqrt{24}}{1} = -2\sqrt{6}$

c $\sin x = -\frac{1}{\sqrt{3}}$



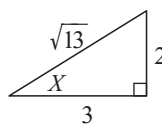
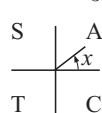
$\therefore \tan x = \frac{1}{\sqrt{2}}$

d $\cos x = -\frac{3}{4}$



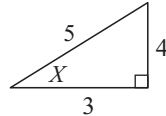
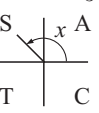
$\therefore \tan x = -\frac{\sqrt{7}}{3}$

5 a $\tan x = \frac{2}{3}$



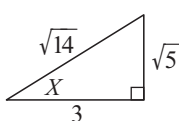
$\therefore \sin x = \frac{2}{\sqrt{13}}, \cos x = \frac{3}{\sqrt{13}}$

b $\tan x = -\frac{4}{3}$



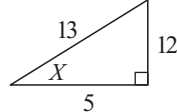
$\therefore \sin x = \frac{4}{5}, \cos x = -\frac{3}{5}$

c $\tan x = \frac{\sqrt{5}}{3}$



$\therefore \sin x = \frac{\sqrt{5}}{\sqrt{14}}, \cos x = \frac{3}{\sqrt{14}}$

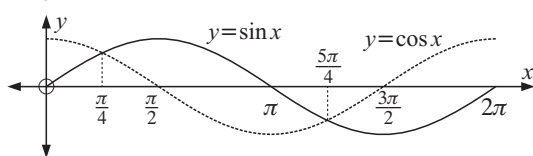
d $\tan x = -\frac{12}{5}$



$\therefore \sin x = \frac{12}{13}, \cos x = -\frac{5}{13}$

EXERCISE 13L

1 a



b $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

c If $\sin x = \cos x$ then $\frac{\sin x}{\cos x} = 1$
 $\therefore \tan x = 1$
 $\therefore x = \frac{\pi}{4} + k\pi$
 $\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \sin x = -\cos x \\
 & \therefore \frac{\sin x}{\cos x} = \frac{-\cos x}{\cos x} \\
 & \therefore \tan x = -1 \\
 & \therefore x = \frac{3\pi}{4} + k\pi \\
 & \therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}
 \end{aligned}$$

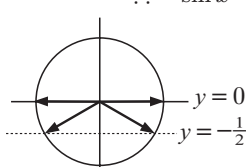
$$\begin{aligned}
 \mathbf{c} \quad & \sin(2x) = \sqrt{3} \cos(2x) \\
 & \therefore \frac{\sin(2x)}{\cos(2x)} = \sqrt{3} \\
 & \therefore \tan(2x) = \sqrt{3} \\
 & \therefore 2x = \frac{\pi}{3} + k\pi \\
 & \therefore x = \frac{\pi}{6} + \frac{k\pi}{2} \\
 & \therefore x = \frac{\pi}{6}, \frac{4\pi}{6}, \frac{7\pi}{6}, \frac{10\pi}{6} \\
 & \therefore x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3},
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \sin(3x) = \cos(3x) \\
 & \therefore \frac{\sin(3x)}{\cos(3x)} = 1 \\
 & \therefore \tan(3x) = 1 \\
 & \therefore 3x = \frac{\pi}{4} + k\pi \\
 & \therefore x = \frac{\pi}{12} + \frac{k\pi}{3} \\
 & \therefore x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12} \\
 & \therefore x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}
 \end{aligned}$$

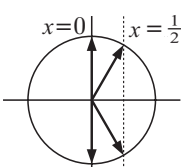
$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & \sin x = 5 \cos x \\
 & \therefore \tan x = 5 \\
 & \therefore x = \tan^{-1}(5) \\
 & \therefore x = 1.373 + k\pi \\
 & \therefore x \doteq 1.37, 4.51, 7.66
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & 4 \sin x + 3 \cos x = 0 \\
 & \therefore 4 \sin x = -3 \cos x \\
 & \therefore \frac{\sin x}{\cos x} = -\frac{3}{4} \\
 & \therefore \tan x = -\frac{3}{4} \\
 & \therefore x = \tan^{-1}\left(-\frac{3}{4}\right) \\
 & \therefore x \doteq -1.081 + k\pi \\
 & \therefore x \doteq 2.50, 5.64, 8.78
 \end{aligned}$$

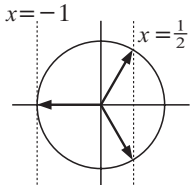
EXERCISE 13M

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & 2 \sin^2 x + \sin x = 0 \\
 & \therefore \sin x(2 \sin x + 1) = 0 \\
 & \therefore \sin x = 0 \text{ or } -\frac{1}{2}
 \end{aligned}$$


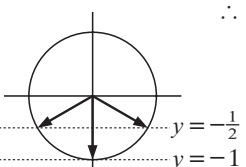
$$\therefore x = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$$

$$\begin{aligned}
 \mathbf{b} \quad & 2 \cos^2 x = \cos x \\
 & \therefore 2 \cos^2 x - \cos x = 0 \\
 & \therefore \cos x(2 \cos x - 1) = 0 \\
 & \therefore \cos x = 0 \text{ or } \frac{1}{2}
 \end{aligned}$$


$$\therefore x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{3}$$

$$\begin{aligned}
 \mathbf{c} \quad & 2 \cos^2 x + \cos x - 1 = 0 \\
 & \therefore (2 \cos x - 1)(\cos x + 1) = 0 \\
 & \therefore \cos x = \frac{1}{2} \text{ or } -1
 \end{aligned}$$


$$\therefore x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

$$\begin{aligned}
 \mathbf{d} \quad & 2 \sin^2 x + 3 \sin x + 1 = 0 \\
 & \therefore (2 \sin x + 1)(\sin x + 1) = 0 \\
 & \therefore \sin x = -\frac{1}{2} \text{ or } -1
 \end{aligned}$$


$$\therefore x = \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$$

$$\begin{aligned}
 \mathbf{e} \quad & \sin^2 x = 2 - \cos x \\
 & \therefore 1 - \cos^2 x = 2 - \cos x \\
 & \therefore \cos^2 x - \cos x + 1 = 0 \\
 & \text{where } \Delta = (-1)^2 - 4(1)(1) \\
 & \quad = 1 - 4 \\
 & \quad = -3 \\
 & \therefore \text{no real solutions exist}
 \end{aligned}$$

$$\text{f} \quad 2 \cos^2 x = \sin x$$

$$\therefore 2(1 - \sin^2 x) - \sin x = 0$$

$$\therefore 2 - 2 \sin^2 x - \sin x = 0$$

$$\therefore 2 \sin^2 x + \sin x - 2 = 0$$

$$\therefore \sin x = \frac{-1 \pm \sqrt{1 - 4(2)(-2)}}{4}$$

$$\therefore \sin x = \frac{-1 \pm \sqrt{17}}{4}$$

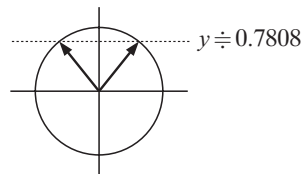
$$\div 0.7808 \text{ or } 1.281$$

$$\therefore \sin x \div 0.7808 \text{ as}$$

$$-1 \leq \sin x \leq 1$$

$$\therefore x \div \arcsin(0.7808) \text{ or } \pi - \arcsin(0.7808)$$

$$\therefore x \div 0.896 \text{ or } 2.246$$

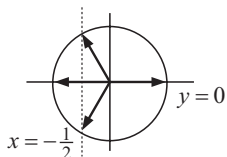


$$\text{2 a} \quad \sin 2x + \sin x = 0$$

$$\therefore 2 \sin x \cos x + \sin x = 0$$

$$\therefore \sin x(2 \cos x + 1) = 0$$

$$\therefore \sin x = 0 \text{ or } \cos x = -\frac{1}{2}$$



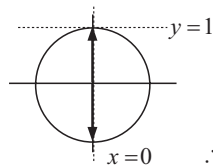
$$\therefore x = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$$

$$\text{b} \quad \sin 2x - 2 \cos x = 0$$

$$\therefore 2 \sin x \cos x - 2 \cos x = 0$$

$$\therefore 2 \cos x(\sin x - 1) = 0$$

$$\therefore \cos x = 0 \text{ or } \sin x = 1$$



$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$$

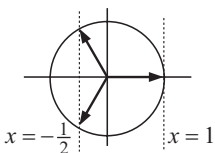
$$\text{c} \quad \cos 2x - \cos x = 0$$

$$\therefore 2 \cos^2 x - 1 - \cos x = 0$$

$$\therefore 2 \cos^2 x - \cos x - 1 = 0$$

$$\therefore (2 \cos x + 1)(\cos x - 1) = 0$$

$$\therefore \cos x = -\frac{1}{2} \text{ or } 1$$



$$\therefore x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$$

$$\text{d} \quad \cos 2x + 3 \cos x = 1$$

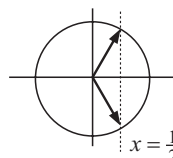
$$\therefore 2 \cos^2 x - 1 + 3 \cos x - 1 = 0$$

$$\therefore 2 \cos^2 x + 3 \cos x - 2 = 0$$

$$\therefore (2 \cos x - 1)(\cos x + 2) = 0$$

$$\therefore \cos x = \frac{1}{2} \text{ or } -2$$

$$\therefore \cos x = \frac{1}{2} \text{ \{as } -1 \leq \cos x \leq 1\}}$$



$$\therefore x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\text{e} \quad \cos 2x + 5 \sin x = 0$$

$$\therefore 1 - 2 \sin^2 x + 5 \sin x = 0$$

$$\therefore 2 \sin^2 x - 5 \sin x - 1 = 0$$

$$\therefore \sin x = \frac{5 \pm \sqrt{25 - 4(2)(-1)}}{4}$$

$$= \frac{5 \pm \sqrt{33}}{4}$$

$$\div 2.6861 \text{ or } -0.1861$$

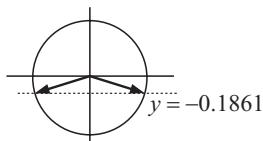
$$\therefore \sin x = -0.1861$$

$$\{\text{as } -1 \leq \sin x \leq 1\}$$

$$\therefore x \div \pi + 0.1872$$

$$\text{or } 2\pi - 0.1872$$

$$\therefore x \div 3.33 \text{ or } 6.10$$



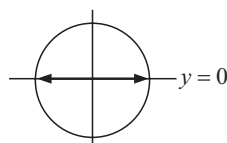
$$\text{f} \quad \sin 2x + 3 \sin x = 0$$

$$\therefore 2 \sin x \cos x + 3 \sin x = 0$$

$$\therefore \sin x(2 \cos x + 3) = 0$$

$$\therefore \sin x = 0 \text{ or } \cos x = -\frac{3}{2}$$

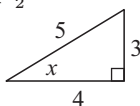
↑
impossible



$$\therefore x = 0, \pi, 2\pi$$

EXERCISE 13N

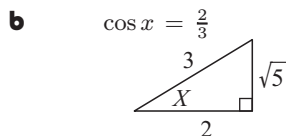
1 a $\sin x = \frac{3}{5}, \quad 0 \leq x \leq \frac{\pi}{2}$



$$\therefore \csc x = \frac{1}{\sin x} = \frac{5}{3}$$

$$\sec x = \frac{1}{\cos x} = \frac{5}{4}$$

$$\cot x = \frac{1}{\tan x} = \frac{4}{3}$$

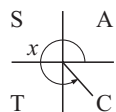


$$\therefore \sin x = -\frac{\sqrt{5}}{3} \quad \text{and} \quad \tan x = -\frac{\sqrt{5}}{2}$$

$$\therefore \csc x = -\frac{3}{\sqrt{5}}$$

$$\sec x = \frac{3}{2}$$

$$\cot x = -\frac{2}{\sqrt{5}}$$



2 a $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

$$\therefore \csc(\frac{\pi}{3}) = \frac{2}{\sqrt{3}}$$

b $\tan(\frac{2\pi}{3}) = -\sqrt{3}$

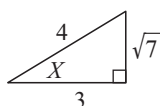
$$\therefore \cot(\frac{2\pi}{3}) = -\frac{1}{\sqrt{3}}$$

c $\cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$

$$\therefore \sec(\frac{5\pi}{6}) = -\frac{2}{\sqrt{3}}$$

d $\tan(\pi) = 0$
 $\therefore \cot(\pi)$ is undefined.

3 a $\cos x = \frac{3}{4}$



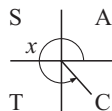
$$\therefore \sin x = -\frac{\sqrt{7}}{4}$$

$$\tan x = -\frac{\sqrt{7}}{3}$$

$$\csc x = -\frac{4}{\sqrt{7}}$$

$$\sec x = \frac{4}{3}$$

$$\cot x = -\frac{3}{\sqrt{7}}$$



b $\sin x = -\frac{2}{3}$

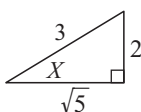
$$\therefore \cos x = -\frac{\sqrt{5}}{3}$$

$$\tan x = \frac{2}{\sqrt{5}}$$

$$\csc x = -\frac{3}{2}$$

$$\sec x = -\frac{3}{\sqrt{5}}$$

$$\cot x = \frac{\sqrt{5}}{2}$$



c $\sec x = \frac{5}{2}$

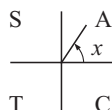
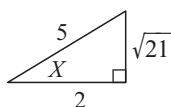
$$\therefore \cos x = \frac{2}{5}$$

$$\therefore \sin x = \frac{\sqrt{21}}{5}$$

$$\tan x = \frac{\sqrt{21}}{2}$$

$$\csc x = \frac{5}{\sqrt{21}}$$

$$\cot x = \frac{2}{\sqrt{21}}$$



d $\csc x = 2$

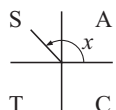
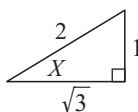
$$\therefore \sin x = \frac{1}{2}$$

$$\therefore \cos x = -\frac{\sqrt{3}}{2}$$

$$\tan x = -\frac{1}{\sqrt{3}}$$

$$\sec x = -\frac{2}{\sqrt{3}}$$

$$\cot x = -\sqrt{3}$$



e $\tan x = \frac{1}{2}$

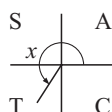
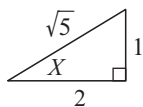
$$\therefore \cot x = 2$$

$$\sin x = -\frac{1}{\sqrt{5}}$$

$$\cos x = -\frac{2}{\sqrt{5}}$$

$$\csc x = -\sqrt{5}$$

$$\sec x = -\frac{\sqrt{5}}{2}$$



f $\cot x = \frac{4}{3}$

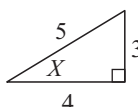
$$\therefore \tan x = \frac{3}{4}$$

$$\therefore \sin x = -\frac{3}{5}$$

$$\cos x = -\frac{4}{5}$$

$$\csc x = -\frac{5}{3}$$

$$\sec x = -\frac{5}{4}$$



$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & \tan x \cot x \\
 &= \frac{\sin x}{\cos x} \times \frac{\cos x}{\sin x} \\
 &= 1
 \end{aligned}$$

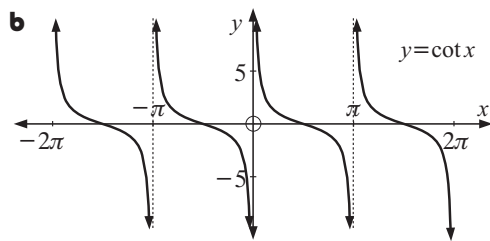
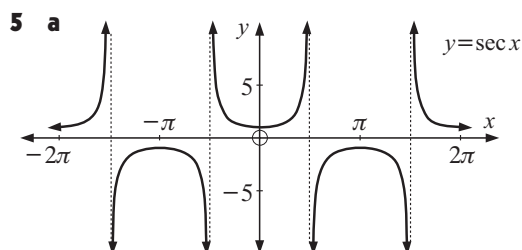
$$\begin{aligned}
 \mathbf{b} \quad & \sin x \csc x \\
 &= \sin x \times \frac{1}{\sin x} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \csc x \cot x \\
 &= \frac{1}{\sin x} \times \frac{\cos x}{\sin x} \\
 &= \frac{\cos x}{\sin^2 x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \sin x \cot x \\
 &= \sin x \times \frac{\cos x}{\sin x} \\
 &= \cos x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \frac{\cot x}{\csc x} \\
 &= \frac{\cos x}{\sin x} \div \frac{1}{\sin x} \\
 &= \frac{\cos x}{\sin x} \times \frac{\sin x}{1} \\
 &= \cos x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \frac{2 \sin x \cot x + 3 \cos x}{\cot x} \\
 &= \frac{2 \sin x \times \frac{\cos x}{\sin x} + 3 \cos x}{\frac{\cos x}{\sin x}} \\
 &= \frac{2 \cos x + 3 \cos x}{\frac{\cos x}{\sin x}} \times \frac{\sin x}{\cos x} \\
 &= 5 \cos x \times \frac{\sin x}{\cos x} \\
 &= 5 \sin x
 \end{aligned}$$



6 a

$$\begin{aligned}
 \sec x &= 2 \\
 \therefore \cos x &= \frac{1}{2} \\
 \therefore x &= \frac{\pi}{3}, \frac{5\pi}{3}
 \end{aligned}$$

b

$$\begin{aligned}
 \csc x &= -\sqrt{2} \\
 \therefore \sin x &= -\frac{1}{\sqrt{2}} \\
 \therefore x &= \frac{5\pi}{4}, \frac{7\pi}{4}
 \end{aligned}$$

c

$$\begin{aligned}
 \cot x &= 4 \\
 \therefore \tan x &= \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \therefore x &= \arctan\left(\frac{1}{4}\right) + k\pi \\
 \therefore x &= 0.245 \text{ or } 3.387
 \end{aligned}$$

d

$$\begin{aligned}
 \sec 2x &= \frac{1}{3} \\
 \therefore \cos 2x &= 3 \text{ which is impossible} \\
 &\text{as all values of cosine lie between} \\
 &-1 \text{ and } 1, \therefore \text{no solution exists.}
 \end{aligned}$$

e

$$\begin{aligned}
 \csc x &= -\frac{2}{3} \\
 \therefore \sin x &= -\frac{3}{2} \text{ which is} \\
 &\text{impossible as } -1 \leq \sin x \leq 1 \\
 \therefore &\text{no solutions exist.}
 \end{aligned}$$

f

$$\begin{aligned}
 \cot\left(2x - \frac{\pi}{4}\right) + 3 &= 0 \\
 \therefore \cot\left(2x - \frac{\pi}{4}\right) &= -3 \\
 \therefore \tan\left(2x - \frac{\pi}{4}\right) &= -\frac{1}{3} \\
 \therefore 2x - \frac{\pi}{4} &= \arctan\left(-\frac{1}{3}\right) + k\pi \\
 \therefore 2x - \frac{\pi}{4} &\doteq -0.3218 + k\pi \\
 \therefore 2x &\doteq 0.4636 + k\pi \\
 \therefore x &\doteq 0.232 + k\frac{\pi}{2} \\
 \therefore x &= 0.232, 1.803, 3.373, 4.944
 \end{aligned}$$

7 a

$$\begin{aligned}
 \sin^2 x + \cos^2 x &= 1 \\
 \therefore \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\
 \therefore \tan^2 x + 1 &= \sec^2 x
 \end{aligned}$$

b

$$\begin{aligned}
 \sin^2 x + \cos^2 x &= 1 \\
 \therefore \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} &= \frac{1}{\sin^2 x} \\
 \therefore 1 + \cot^2 x &= \csc^2 x
 \end{aligned}$$

$$\begin{aligned}
 \text{8 a } \sin^2 x + \cot^2 x \sin^2 x &= \sin^2 x + \frac{\cos^2 x}{\sin^2 x} \sin^2 x \\
 &= \sin^2 x + \cos^2 x \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \\
 &= \frac{\sin x}{\cos x} \left(\frac{\sin x}{\sin x} \right) + \frac{\cos x}{\sin x} \left(\frac{\cos x}{\cos x} \right) \\
 &= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} \\
 &= \frac{1}{\cos x \sin x} \\
 &= \frac{1}{\sin x} \times \frac{1}{\cos x} \\
 &= \csc x \sec x
 \end{aligned}$$

$$\begin{aligned}
 \text{c } \sec x - \tan x \sin x &= \frac{1}{\cos x} - \left(\frac{\sin x}{\cos x} \right) \sin x \\
 &= \frac{1 - \sin^2 x}{\cos x} \\
 &= \frac{\cos^2 x}{\cos x} \\
 &= \cos x
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \frac{\sin \theta}{1 - \cos \theta} - \frac{\sin \theta}{1 + \cos \theta} &= \frac{\sin \theta}{1 - \cos \theta} \left(\frac{1 + \cos \theta}{1 + \cos \theta} \right) - \frac{\sin \theta}{1 + \cos \theta} \left(\frac{1 - \cos \theta}{1 - \cos \theta} \right) \\
 &= \frac{\cancel{\sin \theta} + \sin \theta \cos \theta - \cancel{\sin \theta} + \sin \theta \cos \theta}{(1 - \cos \theta)(1 + \cos \theta)} \\
 &= \frac{2 \sin \theta \cos \theta}{1 - \cos^2 \theta} \\
 &= \frac{2 \sin \theta \cos \theta}{\sin^2 \theta} \\
 &= \frac{2 \cos \theta}{\sin \theta} \\
 &= 2 \cot \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \frac{1}{1 - \sin \theta} + \frac{1}{1 + \sin \theta} &= \frac{1}{1 - \sin \theta} \left(\frac{1 + \sin \theta}{1 + \sin \theta} \right) + \frac{1}{1 + \sin \theta} \left(\frac{1 - \sin \theta}{1 - \sin \theta} \right) \\
 &= \frac{1 + \sin \theta + 1 - \sin \theta}{(1 - \sin \theta)(1 + \sin \theta)} \\
 &= \frac{2}{1 - \sin^2 \theta} \\
 &= \frac{2}{\cos^2 \theta} \\
 &= 2 \sec^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{f } \frac{(1 - \cot \theta)^2}{\csc^2 \theta} + \sin 2\theta &= \frac{1 - 2 \cot \theta + \cot^2 \theta}{\frac{1}{\sin^2 \theta}} + \sin 2\theta \\
 &= \left(1 - \frac{2 \cos \theta}{\sin \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} \right) \sin^2 \theta + 2 \sin \theta \cos \theta \\
 &= \sin^2 \theta - \cancel{2 \sin \theta \cos \theta} + \cos^2 \theta + \cancel{2 \sin \theta \cos \theta} \\
 &= 1
 \end{aligned}$$

EXERCISE 130

$$\begin{aligned}
 \text{1 a } 1 + \sin x + \sin^2 x + \sin^3 x + \dots + \sin^{n-1} x &\text{ is a geometric series with} \\
 u_1 = 1, \quad r = \sin x
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{ sum} &= \frac{u_1(1 - r^n)}{1 - r} \\
 &= \frac{1(1 - \sin^n x)}{1 - \sin x} \\
 &= \frac{1 - \sin^n x}{1 - \sin x}
 \end{aligned}$$

$$\begin{aligned}
 \text{b } S_\infty &= \frac{u_1}{1 - r} = \frac{1}{1 - \sin x} \\
 \text{as } -1 \leq \sin x \leq 1 &\Rightarrow -1 \leq r \leq 1
 \end{aligned}$$

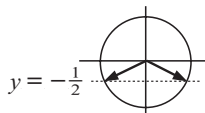
$$\text{c } \text{If } S_\infty = \frac{2}{3}, \quad \frac{1}{1 - \sin x} = \frac{2}{3}$$

$$\therefore 3 = 2 - 2 \sin x$$

$$\therefore 2 \sin x = -1$$

$$\therefore \sin x = -\frac{1}{2}$$

$$\therefore x = \frac{7\pi}{6} \quad \text{or} \quad \frac{11\pi}{6}$$



$$\begin{array}{ll}
 \mathbf{2} \quad \mathbf{a} \quad \mathbf{i} & 2 \sin x(\cos x + \cos 3x) \\
 & = 2 \sin x \cos x + 2 \sin x \cos 3x \\
 & = \sin 2x + \sin 4x + \sin(-2x) \\
 & = \cancel{\sin 2x} + \sin 4x - \cancel{\sin 2x} \\
 & = \sin 4x \\
 \mathbf{ii} & 2 \sin x(\cos x + \cos 3x + \cos 5x) \\
 & = 2 \sin x(\cos x + \cos 3x) + 2 \sin x \cos 5x \\
 & = \sin 4x + \sin 6x + \sin(-4x) \quad \{\text{from i}\} \\
 & = \cancel{\sin 4x} + \sin 6x - \cancel{\sin 4x} \\
 & = \sin 6x
 \end{array}$$

{ $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$ was used}

$$\mathbf{b} \quad \mathbf{i} \quad 2 \sin x(\cos x + \cos 3x + \cos 5x + \dots + \cos 7x) = \sin 8x$$

$$\mathbf{ii} \quad 2 \sin x(\cos x + \cos 3x + \cos 5x + \dots + \cos 19x) = \sin 20x$$

$$\therefore \cos x + \cos 3x + \cos 5x + \dots + \cos 19x = \frac{\sin 20x}{2 \sin x}$$

$$\mathbf{c} \quad \text{In general, } \cos x + \cos 3x + \cos 5x + \dots + \cos(2n-1)x = \frac{\sin 2nx}{2 \sin x}$$

$$\begin{array}{ll}
 \mathbf{3} \quad \mathbf{a} \quad \mathbf{i} & \sin x \cos x \cos 2x \\
 & = \frac{1}{2}(2 \sin x \cos x) \cos 2x \\
 & = \frac{1}{2} \sin 2x \cos 2x \\
 & = \frac{1}{4} 2 \sin 2x \cos 2x \\
 & = \frac{1}{4} \sin 4x \quad \dots (1) \\
 & = \frac{\sin(2^2 x)}{2^2} \\
 \mathbf{ii} & (\sin x \cos x \cos 2x) \cos 4x \\
 & = \frac{1}{4} \sin 4x \cos 4x \quad \{\text{from (1)}\} \\
 & = \frac{1}{8} (2 \sin 4x \cos 4x) \\
 & = \frac{\sin 8x}{8} \\
 & = \frac{\sin(2^3 x)}{2^3}
 \end{array}$$

$$\mathbf{b} \quad \mathbf{i} \quad \frac{\sin(2^4 x)}{2^4} \quad \mathbf{ii} \quad \frac{\sin(2^6 x)}{2^6}$$

$$\mathbf{c} \quad \sin x \cos x \cos 2x \cos 4x \dots \cos(2^n x) = \frac{\sin(2^{n+1} x)}{2^{n+1}}$$

$$\text{or } \cos x \cos 2x \cos 4x \dots \cos(2^n x) = \frac{\sin(2^{n+1} x)}{2^{n+1} \sin x}$$

$$\mathbf{4} \quad \mathbf{a} \quad P_n \text{ is " } \cos \theta + \cos 3\theta + \cos 5\theta + \dots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2 \sin \theta}, \quad n \in \mathbb{Z}^+ \text{"}$$

Proof: (By the Principle of Mathematical Induction)

$$(1) \quad \text{If } n = 1, \text{ LHS} = \cos \theta \text{ and RHS} = \frac{\sin 2\theta}{2 \sin \theta} = \frac{2 \sin \theta \cos \theta}{2 \sin \theta} = \cos \theta \quad \therefore P_1 \text{ is true.}$$

(2) If P_k is true, then

$$\cos \theta + \cos 3\theta + \cos 5\theta + \dots + \cos(2k-1)\theta = \frac{\sin 2k\theta}{2 \sin \theta}$$

$$\therefore \cos \theta + \cos 3\theta + \cos 5\theta + \dots + \cos(2k-1)\theta + \cos(2k+1)\theta$$

$$= \frac{\sin 2k\theta}{2 \sin \theta} + \cos(2k+1)\theta$$

$$= \frac{\sin 2k\theta + 2 \sin \theta \cos(2k+1)\theta}{2 \sin \theta}$$

$$= \frac{\sin 2k\theta + \sin[\theta + (2k+1)\theta] + \sin[\theta - (2k+1)\theta]}{2 \sin \theta}$$

$$= \frac{\sin 2k\theta + \sin(\theta + 2k\theta + \theta) + \sin(\theta - 2k\theta - \theta)}{2 \sin \theta}$$

$$= \frac{\sin 2k\theta + \sin(2k\theta + 2\theta) + \sin(-2k\theta)}{2 \sin \theta}$$

$$= \frac{\cancel{\sin 2k\theta} + \sin 2(k+1)\theta - \cancel{\sin 2k\theta}}{2 \sin \theta}$$

$$= \frac{\sin 2(k+1)\theta}{2 \sin \theta}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore P_n$ is true {Principle of Mathematical Induction}

$$\text{b } \cos \theta + \cos 3\theta + \cos 5\theta + \dots + \cos 31\theta = \frac{\sin 32\theta}{2 \sin \theta} \quad \{n = 16\}$$

$$5 \quad P_n \text{ is } \sin \theta + \sin 3\theta + \sin 5\theta + \dots + \sin(2n-1)\theta = \frac{1 - \cos 2n\theta}{2 \sin \theta}, \quad n \in \mathbb{Z}^+$$

Proof: (By the Principle of Mathematical Induction)

$$(1) \quad \text{If } n = 1, \text{ LHS} = \sin \theta \text{ and } \text{RHS} = \frac{1 - \cos 2\theta}{2 \sin \theta} = \frac{1 - (1 - 2 \sin^2 \theta)}{2 \sin \theta} = \frac{2 \sin^2 \theta}{2 \sin \theta} = \sin \theta$$

$\therefore P_1$ is true.

(2) If P_k is true, then

$$\sin \theta + \sin 3\theta + \sin 5\theta + \dots + \sin(2k-1)\theta = \frac{1 - \cos 2k\theta}{2 \sin \theta}$$

$$\therefore \sin \theta + \sin 3\theta + \sin 5\theta + \dots + \sin(2k-1)\theta + \sin(2k+1)\theta$$

$$= \frac{1 - \cos 2k\theta}{2 \sin \theta} + \sin(2k+1)\theta$$

$$= \frac{1 - \cos 2k\theta + 2 \sin(2k+1)\theta \sin \theta}{2 \sin \theta}$$

$$= \frac{1 - \cos 2k\theta + \cos[(2k+1)\theta - \theta] - \cos[(2k+1)\theta + \theta]}{2 \sin \theta}$$

$$= \frac{1 - \cancel{\cos 2k\theta} + \cancel{\cos 2k\theta} - \cos[(2k+2)\theta]}{2 \sin \theta}$$

$$= \frac{1 - \cos 2(k+1)\theta}{2 \sin \theta}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore P_n$ is true {Principle of Mathematical Induction}

$$\text{Thus } \sin \frac{\pi}{7} + \sin \frac{3\pi}{7} + \sin \frac{5\pi}{7} + \dots + \sin \frac{13\pi}{7} \text{ has } 2n-1 = 13 \text{ and } \theta = \frac{\pi}{7}$$

i.e., $n = 7$ and $\theta = \frac{\pi}{7}$

$$\therefore \text{the sum is } \frac{1 - \cos(2 \times 7 \times \frac{\pi}{7})}{2 \sin \frac{\pi}{7}} = \frac{1 - \cos 2\pi}{2 \sin \frac{\pi}{7}} = \frac{1 - 1}{2 \sin \frac{\pi}{7}} = 0$$

$$6 \quad P_n \text{ is } \cos x \times \cos 2x \times \cos 4x \times \cos 8x \times \dots \times \cos(2^{n-1}x) = \frac{\sin(2^n x)}{2^n \times \sin x}, \quad n \in \mathbb{Z}^+$$

Proof: (By the Principle of Mathematical Induction)

$$(1) \quad \text{If } n = 1, \text{ LHS} = \cos x, \text{ RHS} = \frac{\sin 2x}{2 \sin x} = \frac{2 \sin x \cos x}{2 \sin x} = \cos x \quad \therefore P_1 \text{ is true.}$$

(2) If P_k is true, then

$$\cos x \times \cos 2x \times \cos 4x \times \dots \times \cos(2^{k-1}x) = \frac{\sin(2^k x)}{2^k \sin x}$$

$$\therefore \cos x \times \cos 2x \times \cos 4x \times \dots \times \cos(2^{k-1}x) \times \cos(2^k x)$$

$$= \frac{\sin(2^k x)}{2^k \sin x} \times \cos(2^k x)$$

$$= \frac{2 \sin(2^k x) \cos(2^k x)}{2 \times 2^k \sin x}$$

$$= \frac{\sin(2 \times 2^k x)}{2^{k+1} \sin x} \quad \{2 \sin \theta \cos \theta = \sin 2\theta\}$$

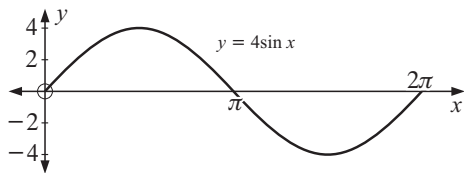
$$= \frac{\sin(2^{k+1} x)}{2^{k+1} \sin x}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore P_n$ is true {Principle of Mathematical Induction}

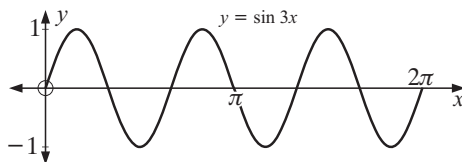
REVIEW SET 13A

- 1** This is the graph of $y = \sin x$ under a vertical stretch of factor 4. The amplitude is 4.



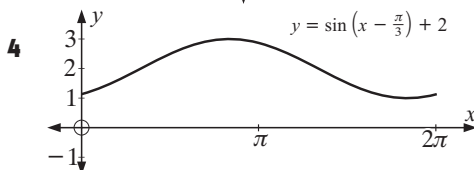
- 2** This is the graph of $y = \sin x$ under a horizontal stretch of factor $\frac{1}{3}$.

The period is $\frac{2\pi}{2}$.



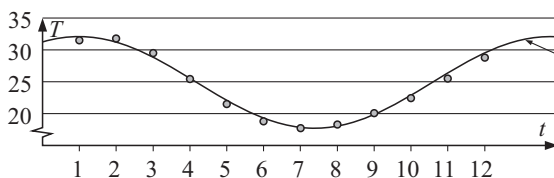
- 3 a** period = $\frac{2\pi}{\frac{1}{3}} = 6\pi$

b period = $\frac{2\pi}{4} = \frac{\pi}{2}$



5

Month	1	2	3	4	5	6	7	8	9	10	11	12
Temp	31.5	31.8	29.5	25.4	21.5	18.8	17.7	18.3	20.1	22.4	25.5	28.8



$$T = A \sin B(t - C) + D \quad \text{period} = \frac{2\pi}{B} = 12, \quad \therefore B = \frac{2\pi}{12} = \frac{\pi}{6}$$

$$\text{max.} = 31.8 \quad \therefore A = \frac{\text{max.} - \text{min.}}{2} \div \frac{31.8 - 17.7}{2} \div 7.05$$

$$\text{min.} = 17.7$$

$$D = \frac{\text{max.} + \text{min.}}{2} \div \frac{31.8 + 17.7}{2} \div 24.75$$

$$C = \frac{2 + 7}{2} = 4.5 \quad \{\text{values of } t \text{ at max. and min.}\}$$

$$\text{So, } T \div 7.05 \sin \frac{\pi}{6}(t - 10.5) + 24.75$$

$$\text{From technology, } T \div 7.21 \sin(0.488t + 1.082) + 24.75$$

$$\div 7.21 \sin 0.488(t + 2.22) + 24.75 \quad \text{Note: } 0.488(2.22 - -10.5) \div 6.21 \div 2\pi$$

- 6 a** $\sin x = 0.382$

$$\therefore x \div 0.392, 2.750, 6.675$$

- b** $\sin\left(\frac{x}{2}\right) = -0.458$

$$\therefore x \div 7.235$$

- 7 a** $\sin(x - 2.4) = 0.754$

$$\therefore x \div 3.254, 4.687$$

- b** $\sin\left(x + \frac{\pi}{3}\right) = 0.6049$

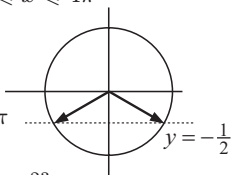
$$\therefore x \div 1.445, 5.89, 7.73$$

- 8 a** $2 \sin x = -1, \quad 0 \leq x \leq 4\pi$

$$\therefore \sin x = -\frac{1}{2}$$

$$\therefore x = \left\{ \frac{7\pi}{6}, \frac{11\pi}{6} \right\} + k2\pi$$

$$\therefore x = \frac{7\pi}{6}, \frac{19\pi}{6}, \frac{11\pi}{6}, \frac{23\pi}{6}$$

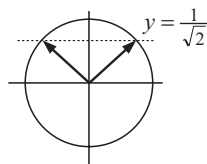


- b** $\sqrt{2} \sin x - 1 = 0, \quad -2\pi \leq x \leq 2\pi$

$$\therefore \sin x = \frac{1}{\sqrt{2}}$$

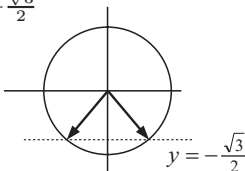
$$\therefore x = \left\{ \frac{\pi}{4}, \frac{3\pi}{4} \right\} + k2\pi$$

$$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{7\pi}{4}, -\frac{5\pi}{4}$$



9 a $2 \sin 3x + \sqrt{3} = 0, \quad 0 \leq x \leq 2\pi$

$$\therefore \sin 3x = -\frac{\sqrt{3}}{2}$$



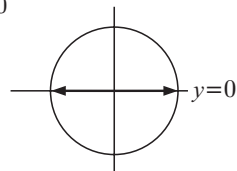
$$\therefore 3x = \left\{ \begin{array}{l} \frac{4\pi}{3} \\ \frac{5\pi}{3} \end{array} \right\} + k2\pi$$

$$\therefore x = \left\{ \begin{array}{l} \frac{4\pi}{9} \\ \frac{5\pi}{9} \end{array} \right\} + k \frac{2\pi}{3}$$

$$\therefore x = \frac{4\pi}{9}, \frac{10\pi}{9}, \frac{16\pi}{9}, \frac{5\pi}{9}, \frac{11\pi}{9}, \frac{17\pi}{9}$$

b $\sqrt{2} \sin(x + \frac{\pi}{4}) = 0, \quad 0 \leq x \leq 3\pi$

$$\therefore \sin(x + \frac{\pi}{4}) = 0$$



$$\therefore x + \frac{\pi}{4} = 0 + k\pi$$

$$\therefore x = -\frac{\pi}{4} + k\pi$$

$$\therefore x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$$

10 $P(t) = 5 + 2 \sin\left(\frac{\pi t}{3}\right), \quad 0 \leq t \leq 8$

a $P(0) = 5 + 2 \sin 0$
 $= 5$

i.e., 5000 water beetles

b Smallest $P = 5 + 2(-1) = 3$

Largest $P = 5 + 2(1) = 7$

$\therefore$ smallest is 3000 water beetles

largest is 7000 water beetles

c If population is > 6000 ,

then $P(t) > 6$

$$\therefore 5 + 2 \sin\left(\frac{\pi t}{3}\right) > 6$$

$$\therefore 2 \sin\left(\frac{\pi t}{3}\right) > 1$$

$$\therefore \sin\left(\frac{\pi t}{3}\right) > \frac{1}{2}$$

Using technology,

$$0.5 < t < 2.5 \quad \text{and} \quad 6.5 < t < 8$$

REVIEW SET 13B

1 a $\sin^2 x - \sin x - 2 = 0$

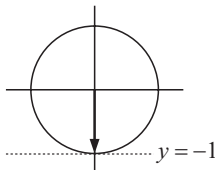
$$\therefore (\sin x - 2)(\sin x + 1) = 0$$

$$\therefore \sin x = 2 \text{ or } -1$$

But $\sin x$ values lie
 between -1 and
 1 inclusive

$$\therefore \sin x = -1$$

$$\therefore x = \frac{3\pi}{2} + k2\pi$$

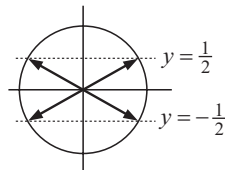


b $4 \sin^2 x = 1$

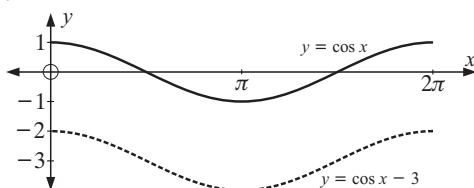
$$\therefore \sin^2 x = \frac{1}{4}$$

$$\therefore \sin x = \pm \frac{1}{2}$$

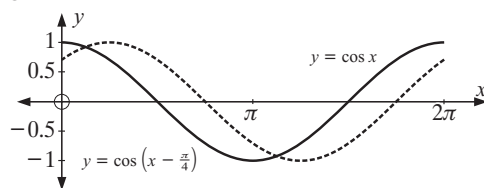
$$\therefore x = \left\{ \begin{array}{l} \frac{\pi}{6} \\ \frac{5\pi}{6} \end{array} \right\} + k\pi$$



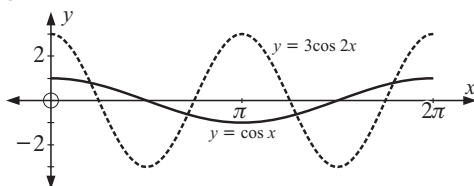
2 a



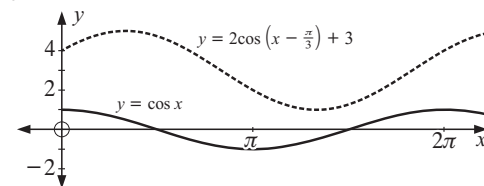
b



c



d



3 $P(t) = 40 + 12 \sin \frac{2\pi}{7} \left(t - \frac{37}{12} \right) \text{ m}^3$

a $P(t)$ is a minimum of $40 + 12(-1) = 28 \text{ mg/m}^3$

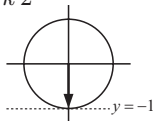
b when $\sin \frac{2\pi}{7} \left(t - \frac{37}{12} \right) = -1$
 $\therefore \frac{2\pi}{7} \left(t - \frac{37}{12} \right) = \frac{3\pi}{2} + k2\pi$
 $\therefore \frac{2}{7} \left(t - \frac{37}{12} \right) = \frac{3}{2} + k2$

So, $t - \frac{37}{12} = \frac{21}{4} + k7$

$\therefore t = 8\frac{1}{3} + k7$

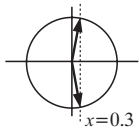
$\therefore t = 1\frac{1}{3}, 8\frac{1}{3}, 15\frac{1}{3}, \text{ etc.}$

$\therefore$ on Mondays at 8.00 am
 $\{1\frac{1}{3} \text{ days after midnight Sat.}\}$



5 a $\cos x = 0.4379, \quad 0 \leq x \leq 10$
 $\therefore x \div 1.12, 5.17, 7.40$

6 a $\cos 4x = 0.3, \quad \text{for all } x$
 $\therefore 4x = \cos^{-1}(0.3)$



$\therefore 4x = \left. \begin{matrix} 1.2661 \\ 2\pi - 1.2661 \end{matrix} \right\} + k2\pi$

$\therefore x = \left. \begin{matrix} 0.317 \\ 1.254 \end{matrix} \right\} + k\left(\frac{\pi}{2}\right)$

4 a If $y = A \cos B(t - C) + D$

then $A = -4, \quad \frac{2\pi}{B} = \pi$
 $\therefore B = 2$

$C = D = 0$

$\therefore y = -4 \cos 2x$

b If $y = A \cos B(x - C) + D$

then $A = 1, \quad \frac{2\pi}{B} = 8 \quad \therefore B = \frac{\pi}{4}$

$D = \frac{\text{max.} + \text{min.}}{2} = \frac{3 + 1}{2} = 2$

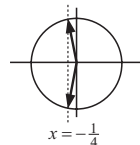
$C = 0$

So, $y = \cos\left(\frac{\pi}{4}x\right) + 2$

b $\cos(x - 2.4) = -0.6014, \quad 0 \leq x \leq 6$
 $\therefore x \div 0.184, 4.62$

b $4 \cos 2x + 1 = 0, \quad 0 \leq x \leq 5$
 $\therefore \cos 2x = -\frac{1}{4}$

$\therefore 2x = \left. \begin{matrix} \pi - 1.318 \\ \pi + 1.318 \end{matrix} \right\} + k2\pi$

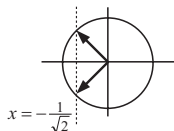


$\therefore x = \left. \begin{matrix} 0.912 \\ 2.230 \end{matrix} \right\} + k\pi$

$\therefore x = 0.912, 4.05, 2.23$

7 a $\cos x = -\frac{1}{\sqrt{2}}, \quad 0 \leq x \leq 4\pi$

$x = \left. \begin{matrix} \frac{3\pi}{4} \\ \frac{5\pi}{4} \end{matrix} \right\} + k2\pi$



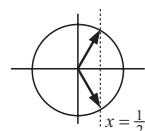
$\therefore x = \frac{3\pi}{4}, \frac{11\pi}{4}, \frac{5\pi}{4}, \frac{13\pi}{4}$

b $\cos\left(x + \frac{2\pi}{3}\right) = \frac{1}{2}, \quad -2\pi \leq x \leq 2\pi$

$x + \frac{2\pi}{3} = \left. \begin{matrix} \frac{\pi}{3} \\ \frac{5\pi}{3} \end{matrix} \right\} + k2\pi$

$\therefore x = \left. \begin{matrix} -\frac{\pi}{3} \\ \frac{\pi}{3} \end{matrix} \right\} + k2\pi$

$\therefore x = -\frac{\pi}{3}, \frac{5\pi}{3}, \pi, -\pi$

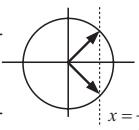


8 a $\sqrt{2} \cos\left(x + \frac{\pi}{4}\right) - 1 = 0, \quad 0 \leq x \leq 4\pi$
 $\therefore \cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$\therefore x + \frac{\pi}{4} = \left. \begin{matrix} \frac{\pi}{4} \\ \frac{7\pi}{4} \end{matrix} \right\} + k2\pi$

$\therefore x = \left. \begin{matrix} 0 \\ \frac{3\pi}{2} \end{matrix} \right\} + k2\pi$

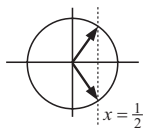
$\therefore x = 0, 2\pi, 4\pi, \frac{3\pi}{2}, \frac{7\pi}{2}$



b $2 \cos 2x - 1 = 0, \quad \text{for all } x$
 $\therefore \cos 2x = \frac{1}{2}$

$\therefore 2x = \left. \begin{matrix} \frac{\pi}{3} \\ \frac{5\pi}{3} \end{matrix} \right\} + k2\pi$

$\therefore x = \left. \begin{matrix} \frac{\pi}{6} \\ \frac{5\pi}{6} \end{matrix} \right\} + k\pi$



$$\begin{aligned}
 \mathbf{9} \quad \mathbf{a} \quad & \cos^3 \theta + \sin^2 \theta \cos \theta \\
 &= \cos \theta (\cos^2 \theta + \sin^2 \theta) \\
 &= \cos \theta (1) \\
 &= \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{\cos^2 \theta - 1}{\sin \theta} \\
 &= \frac{-(1 - \cos^2 \theta)}{\sin \theta} \\
 &= -\frac{\sin^2 \theta}{\sin \theta} \\
 &= -\sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & 3 \cos \theta - \cos \theta \\
 &= 2 \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & 5 - 5 \sin^2 \theta \\
 &= 5(1 - \sin^2 \theta) \\
 &= 5 \cos^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \frac{\sin^2 \theta - 1}{\cos \theta} \\
 &= -\frac{(1 - \sin^2 \theta)}{\cos \theta} \\
 &= -\frac{\cos^2 \theta}{\cos \theta} \\
 &= -\cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{10} \quad \mathbf{a} \quad & (2 \sin \alpha - 1)^2 \\
 &= 4 \sin^2 \alpha - 4 \sin \alpha + 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & (\cos \alpha - \sin \alpha)^2 \\
 &= \cos^2 \alpha - 2 \sin \alpha \cos \alpha + \sin^2 \alpha \\
 &= \cos^2 \alpha + \sin^2 \alpha - 2 \sin \alpha \cos \alpha \\
 &= 1 - \sin 2\alpha
 \end{aligned}$$

REVIEW SET 13C

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \frac{1 - \cos^2 \theta}{1 + \cos \theta} \\
 &= \frac{(1 + \cos \theta)(1 - \cos \theta)}{(1 + \cos \theta) \cdot 1} \\
 &= 1 - \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{\sin \alpha - \cos \alpha}{\sin^2 \alpha - \cos^2 \alpha} \\
 &= \frac{\sin \alpha - \cos \alpha}{(\sin \alpha + \cos \alpha)(\sin \alpha - \cos \alpha)} \cdot 1 \\
 &= \frac{1}{\sin \alpha + \cos \alpha}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{4 \sin^2 \alpha - 4}{8 \cos \alpha} \\
 &= \frac{-4(1 - \sin^2 \alpha)}{8 \cos \alpha} \\
 &= \frac{-4 \cos^2 \alpha}{8 \cos \alpha} \\
 &= -\frac{1}{2} \cos \alpha
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \frac{\cos \theta}{1 + \sin \theta} + \frac{1 + \sin \theta}{\cos \theta} \\
 &= \frac{\cos^2 \theta + (1 + \sin \theta)^2}{(1 + \sin \theta) \cos \theta} \\
 &= \frac{\cos^2 \theta + 1 + 2 \sin \theta + \sin^2 \theta}{(1 + \sin \theta) \cos \theta} \\
 &= \frac{2 + 2 \sin \theta}{(1 + \sin \theta) \cos \theta} \quad \{\cos^2 \theta + \sin^2 \theta = 1\} \\
 &= \frac{2(1 + \sin \theta)}{(1 + \sin \theta) \cos \theta} \cdot 1 \\
 &= \frac{2}{\cos \theta} \quad \text{or} \quad 2 \sec \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \left(1 + \frac{1}{\cos \theta}\right) (\cos \theta - \cos^2 \theta) \\
 &= \cancel{\cos \theta} - \cos^2 \theta + 1 - \cancel{\cos \theta} \\
 &= 1 - \cos^2 \theta \\
 &= \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & \sin 2A = 2 \sin A \cos A \\
 &= 2\left(\frac{5}{13}\right)\left(\frac{12}{13}\right) \\
 &= \frac{120}{169}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \cos 2A = \cos^2 A - \sin^2 A \\
 &= \left(\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 \\
 &= \frac{144 - 25}{169} \\
 &= \frac{119}{169}
 \end{aligned}$$

4

$$\cos^2 \alpha + \sin^2 \alpha = 1$$

$$\therefore \cos^2 \alpha + \frac{9}{16} = 1$$

$$\therefore \cos^2 \alpha = \frac{7}{16}$$

$$\therefore \cos \alpha = \pm \frac{\sqrt{7}}{4}$$

But in Q3, $\cos \alpha < 0$

$$\therefore \cos \alpha = -\frac{\sqrt{7}}{4}$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$= 2\left(-\frac{3}{4}\right)\left(-\frac{\sqrt{7}}{4}\right)$$

$$= \frac{3\sqrt{7}}{8}$$

5

$$\cos 2A = 1 - 2 \sin^2 A$$

$$\therefore \cos x = 1 - 2 \sin^2 \left(\frac{x}{2}\right) \quad \{\text{letting } 2A = x, A = \frac{x}{2}\}$$

$$\therefore -\frac{3}{4} = 1 - 2 \sin^2 \left(\frac{x}{2}\right)$$

$$\therefore 2 \sin^2 \left(\frac{x}{2}\right) = \frac{7}{4}$$

$$\therefore \sin^2 \left(\frac{x}{2}\right) = \frac{7}{8}$$

$$\therefore \sin \left(\frac{x}{2}\right) = \pm \frac{\sqrt{7}}{2\sqrt{2}}$$

$$\text{But } \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \quad (\text{in Q2}) \quad \therefore \sin \left(\frac{x}{2}\right) = \frac{\sqrt{7}}{2\sqrt{2}}$$

6

a i $\tan x = 4$

$$\therefore x \div 1.326 + k\pi$$

$$\text{i.e., } x \div 1.33 + k\pi$$

ii $\tan \left(\frac{x}{4}\right) = 4$

$$\therefore \frac{x}{4} \div 1.326 + k\pi$$

$$\therefore x \div 5.30 + k4\pi$$

iii $\tan(x - 1.5) = 4$

$$\therefore x - 1.5 \div 1.326 + k2\pi$$

$$\therefore x \div 2.83 + k\pi$$

b i $\tan\left(x + \frac{\pi}{6}\right) = -\sqrt{3}$

$$\therefore x + \frac{\pi}{6} = \frac{2\pi}{3} + k\pi$$

$$\therefore x = \frac{\pi}{2} + k\pi$$

ii $\tan 2x = -\sqrt{3}$

$$\therefore 2x = \frac{2\pi}{3} + k\pi$$

$$\therefore x = \frac{\pi}{3} + \frac{k\pi}{2}$$

iii $\tan^2 x - 3 = 0$

$$\therefore \tan x = \pm\sqrt{3}$$

$$\therefore x = \left\{ \frac{\pi}{3}, \frac{2\pi}{3} \right\} + k\pi$$

c $3 \tan(x - 1.2) = -2$

$$\therefore \tan(x - 1.2) = -\frac{2}{3}$$

$$\therefore x - 1.2 \div -0.588 + k\pi$$

$$\therefore x \div 0.612 + k\pi$$

7

$$\tan \theta = -\frac{2}{3}, \quad \frac{\pi}{2} < \theta < \pi$$

$$\therefore \frac{\sin \theta}{\cos \theta} = -\frac{2}{3}$$

$$\therefore \sin \theta = -2k, \quad \cos \theta = 3k$$

$$\text{but } \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore 9k^2 + 4k^2 = 1$$

$$\therefore 13k^2 = 1$$

$$\therefore k = \pm \frac{1}{\sqrt{13}}$$



But in Q2,

$$\sin \theta > 0, \quad \cos \theta < 0$$

$$\therefore k = -\frac{1}{\sqrt{13}}$$

$$\therefore \sin \theta = \frac{2}{\sqrt{13}}, \quad \cos \theta = -\frac{3}{\sqrt{13}}$$

8

$$\frac{\sin 2\alpha - \sin \alpha}{\cos 2\alpha - \cos \alpha + 1} = \frac{2 \sin \alpha \cos \alpha - \sin \alpha}{2 \cos^2 \alpha - 1 - \cos \alpha + 1}$$

$$= \frac{\sin \alpha (2 \cos \alpha - 1)}{\cos \alpha (2 \cos \alpha - 1)}$$

$$= \frac{\sin \alpha}{\cos \alpha}$$

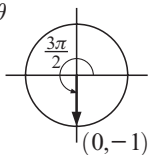
$$= \tan \alpha$$

$$9 \quad a \quad \cos\left(\frac{3\pi}{2} - \theta\right)$$

$$= \cos\left(\frac{3\pi}{2}\right) \cos \theta + \sin \frac{3\pi}{2} \sin \theta$$

$$= (0) \cos \theta + (-1) \sin \theta$$

$$= -\sin \theta$$

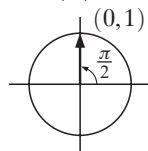


$$b \quad \sin\left(\theta + \frac{\pi}{2}\right)$$

$$= \sin \theta \cos\left(\frac{\pi}{2}\right) + \cos \theta \sin\left(\frac{\pi}{2}\right)$$

$$= \sin \theta(0) + \cos \theta(1)$$

$$= \cos \theta$$



$$10 \quad a \quad \tan(\alpha + \beta)$$

$$= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}$$

$$= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}$$

$$= \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta} + \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}}$$

$$= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$b \quad \text{In } a \text{ let } \beta = \alpha$$

$$\therefore \tan(\alpha + \alpha)$$

$$= \frac{\tan \alpha + \tan \alpha}{1 - \tan \alpha \tan \alpha}$$

$$\text{i.e., } \tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

$$c \quad \text{If } \tan 2\alpha = 3, \text{ then}$$

$$\frac{2 \tan \alpha}{1 - \tan^2 \alpha} = 3$$

$$\therefore 2 \tan \alpha = 3 - 3 \tan^2 \alpha$$

$$\therefore 3 \tan^2 \alpha + 2 \tan \alpha - 3 = 0$$

$$\therefore \tan \alpha = \frac{-2 \pm \sqrt{40}}{6}$$

$$\therefore \tan \alpha = \frac{-1 \pm \sqrt{10}}{3}$$

REVIEW SET 13D

$$1 \quad a \quad \sqrt{2} \cos\left(\theta + \frac{\pi}{4}\right)$$

$$= \sqrt{2} \left[\cos \theta \cos\left(\frac{\pi}{4}\right) - \sin \theta \sin\left(\frac{\pi}{4}\right) \right]$$

$$= \sqrt{2} \left[\cos \theta \times \frac{1}{\sqrt{2}} - \sin \theta \times \frac{1}{\sqrt{2}} \right]$$

$$= \cos \theta - \sin \theta$$

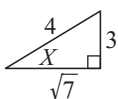
$$b \quad \cos \alpha \cos(\beta - \alpha) - \sin \alpha \sin(\beta - \alpha)$$

$$= \cos[\alpha + (\beta - \alpha)]$$

$$= \cos[\alpha + \beta - \alpha]$$

$$= \cos \beta$$

$$2 \quad a \quad \sin x = \frac{3}{4}$$



$$\therefore \cos x = -\frac{\sqrt{7}}{4}$$

$$b \quad \sin 2x$$

$$= 2 \sin x \cos x$$

$$= 2 \left(\frac{3}{4}\right) \left(-\frac{\sqrt{7}}{4}\right)$$

$$= -\frac{3\sqrt{7}}{8}$$

$$c \quad \cos 2x$$

$$= 1 - 2 \sin^2 x$$

$$= 1 - 2 \left(\frac{9}{16}\right)$$

$$= 1 - \frac{9}{8}$$

$$= -\frac{1}{8}$$

$$d \quad \tan 2x$$

$$= \frac{\sin 2x}{\cos 2x}$$

$$= -\frac{3\sqrt{7}}{8} \div -\frac{1}{8}$$

$$= 3\sqrt{7}$$

$$3 \quad \cos 2\theta = 1 - 2 \sin^2 \theta$$

$$\therefore \cos\left(\frac{\pi}{4}\right) = 1 - 2 \sin^2\left(\frac{\pi}{8}\right) \quad \{\text{letting } \theta = \frac{\pi}{8}\}$$

$$\therefore \frac{1}{\sqrt{2}} = 1 - 2 \sin^2\left(\frac{\pi}{8}\right)$$

$$\therefore 2 \sin^2\left(\frac{\pi}{8}\right) = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$$\therefore \sin^2\left(\frac{\pi}{8}\right) = \frac{\sqrt{2}-1}{2\sqrt{2}}$$

$$\therefore \sin^2\left(\frac{\pi}{8}\right) = \left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right) \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$\therefore \sin^2\left(\frac{\pi}{8}\right) = \frac{2-\sqrt{2}}{4}$$

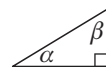
$$\therefore \sin^2\left(\frac{\pi}{8}\right) = \pm \frac{\sqrt{2}-\sqrt{2}}{2}$$

But $\sin\left(\frac{\pi}{8}\right)$ is positive as $\frac{\pi}{8}$ is in quad 1.

$$\therefore \sin\left(\frac{\pi}{8}\right) = \frac{1}{2} \sqrt{2-\sqrt{2}}$$

$$4 \quad \alpha + \beta = \frac{\pi}{2} \quad \{\text{angles of a } \Delta\}$$

$$\therefore \beta = \frac{\pi}{2} - \alpha$$



$$\text{So, } \sin 2\beta = \sin(\pi - 2\alpha)$$

$$= \sin \pi \cos 2\alpha - \cos \pi \sin 2\alpha$$

$$= (0) \cos 2\alpha - (-1) \sin 2\alpha$$

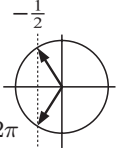
$$= \sin 2\alpha$$

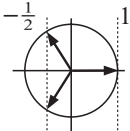
$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad & (\sin \theta + \cos \theta)^2 \\
 &= \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta \\
 &= [\sin^2 \theta + \cos^2 \theta] + 2 \sin \theta \cos \theta \\
 &= 1 + \sin 2\theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \csc(2x) + \cot(2x) = \frac{1}{\sin 2x} + \frac{\cos 2x}{\sin 2x} \\
 &= \frac{1 + \cos 2x}{\sin 2x} \\
 &= \frac{\cancel{1} + 2 \cos^2 x \cancel{1}}{2 \sin x \cos x} \\
 &= \frac{\cancel{2} \cos x \cancel{\cos x}}{\cancel{2} \sin x \cancel{\cos x}} \\
 &= \cot x
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad & \sin 3\theta \\
 &= \sin(2\theta + \theta) \\
 &= \sin 2\theta \cos \theta + \cos 2\theta \sin \theta \\
 &= 2 \sin \theta \cos \theta \times \cos \theta + [1 - 2 \sin^2 \theta] \sin \theta \\
 &= 2 \sin \theta \cos^2 \theta + \sin \theta - 2 \sin^3 \theta \\
 &= 2 \sin \theta (1 - \sin^2 \theta) + \sin \theta - 2 \sin^3 \theta \\
 &= 2 \sin \theta - 2 \sin^3 \theta + \sin \theta - 2 \sin^3 \theta \\
 &= 3 \sin \theta - 4 \sin^3 \theta
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \cos 3\theta \\
 &= \cos(2\theta + \theta) \\
 &= \cos 2\theta \cos \theta - \sin 2\theta \sin \theta \\
 &= [2 \cos^2 \theta - 1] \cos \theta - 2 \sin \theta \cos \theta \sin \theta \\
 &= [2 \cos^2 \theta - 1] \cos \theta - 2 \cos \theta \sin^2 \theta \\
 &= 2 \cos^3 \theta - \cos \theta - 2 \cos \theta [1 - \cos^2 \theta] \\
 &= 2 \cos^3 \theta - \cos \theta - 2 \cos \theta + 2 \cos^3 \theta \\
 &= 4 \cos^3 \theta - 3 \cos \theta \\
 \therefore \quad & a = 4 \quad \text{and} \quad b = -3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{7} \quad \mathbf{a} \quad & 2 \cos(2x) + 1 = 0 \\
 \therefore \quad & \cos(2x) = -\frac{1}{2} \\
 \therefore \quad & 2x = \left. \begin{array}{l} \frac{2\pi}{3} \\ \frac{4\pi}{3} \end{array} \right\} + k2\pi \\
 \therefore \quad & x = \left. \begin{array}{l} \frac{\pi}{3} \\ \frac{2\pi}{3} \end{array} \right\} + k\pi \\
 \therefore \quad & x = \frac{\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}
 \end{aligned}$$


$$\begin{aligned}
 \mathbf{b} \quad & \cos 2x = \cos x \\
 \therefore \quad & 2 \cos^2 x - 1 = \cos x \\
 \therefore \quad & 2 \cos^2 x - \cos x - 1 = 0 \\
 \therefore \quad & (2 \cos x + 1)(\cos x - 1) = 0 \\
 \therefore \quad & \cos x = -\frac{1}{2} \text{ or } 1 \\
 \therefore \quad & x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi
 \end{aligned}$$


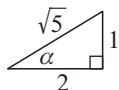
$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad & \sin 2\theta = 2 \sin \theta \cos \theta \\
 &= 2 \left(\frac{b}{c} \right) \left(\frac{a}{c} \right) \\
 &= \frac{2ab}{c^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \cos 2\theta = \cos^2 \theta - \sin^2 \theta \\
 &= \left(\frac{a}{c} \right)^2 - \left(\frac{b}{c} \right)^2 \\
 &= \frac{a^2 - b^2}{c^2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{9} \quad & \tan 2\alpha = \frac{4}{3}, \quad 0 < \alpha < \frac{\pi}{2} \\
 \therefore \quad & \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{4}{3} \\
 \therefore \quad & 6 \tan \alpha = 4 - 4 \tan^2 \alpha \\
 \therefore \quad & 4 \tan^2 \alpha + 6 \tan \alpha - 4 = 0 \\
 \therefore \quad & 2 \tan^2 \alpha + 3 \tan \alpha - 2 = 0 \\
 \therefore \quad & (2 \tan \alpha - 1)(\tan \alpha + 2) = 0 \\
 \therefore \quad & \tan \alpha = \frac{1}{2} \text{ or } -2
 \end{aligned}$$

But α is in quad 1.

$$\therefore \tan \alpha = \frac{1}{2}$$



$$\therefore \sin \alpha = \frac{1}{\sqrt{5}}$$

$$\begin{aligned}
 \mathbf{10} \quad \mathbf{a} \quad & \text{By the sine rule} \quad \frac{\sin 2\alpha}{5} = \frac{\sin \alpha}{3} \\
 \therefore \quad & \frac{2 \sin \alpha \cos \alpha}{\sin \alpha} = \frac{5}{3} \\
 \therefore \quad & 2 \cos \alpha = \frac{5}{3} \quad \text{p.v. } \sin \alpha \neq 0 \\
 & \quad \quad \quad \text{and this is so.} \\
 \therefore \quad & \cos \alpha = \frac{5}{6}
 \end{aligned}$$

b Using the cosine rule

$$3^2 = x^2 + 5^2 - 2 \times x \times 5 \times \cos \alpha$$

$$\therefore 9 = x^2 + 25 - 10x \left(\frac{5}{6} \right)$$

$$\therefore x^2 - \frac{25}{3}x + 16 = 0$$

$$\therefore 3x^2 - 25x + 48 = 0$$

$$\begin{aligned}
 \mathbf{c} \quad & (3x - 16)(x - 3) = 0 \\
 \therefore \quad & x = \frac{16}{3} \text{ or } 3
 \end{aligned}$$

Chapter 14

MATRICES

EXERCISE 14A

- 1 a** 1 row and 4 columns $\therefore 1 \times 4$
c 2 rows and 2 columns $\therefore 2 \times 2$

- b** 2 rows and 1 column $\therefore 2 \times 1$
d 3 rows and 3 columns $\therefore 3 \times 3$

2 $\begin{bmatrix} 2 & 1 & 6 & 1 \end{bmatrix}$

b $\begin{bmatrix} 1.95 \\ 2.35 \\ 0.15 \\ 0.95 \end{bmatrix}$

- c** $(2 \times 1.95) + (1 \times 2.35) + (6 \times 0.15) + (1 \times 0.95)$
 represents the total cost of the groceries.

3

200 g	300 g	500 g
1000	1500	1250
1500	1000	1000
800	2300	1300
1200	1200	1200

 $\begin{bmatrix} 1000 & 1500 & 1250 \\ 1500 & 1000 & 1000 \\ 800 & 2300 & 1300 \\ 1200 & 1200 & 1200 \end{bmatrix}$
 week 1
 week 2
 week 3
 week 4

4

pies	pasties	rolls	buns
40	50	55	40
25	65	44	30
35	40	40	35
35	40	35	50

 $\begin{bmatrix} 40 & 50 & 55 & 40 \\ 25 & 65 & 44 & 30 \\ 35 & 40 & 40 & 35 \\ 35 & 40 & 35 & 50 \end{bmatrix}$
 Friday
 Saturday
 Sunday
 Monday

EXERCISE 14B

1 a $A + B$
 $= \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$
 $= \begin{bmatrix} 9 & 1 \\ 3 & 3 \end{bmatrix}$

b $A + B + C$
 $= \begin{bmatrix} 9 & 1 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ -4 & -2 \end{bmatrix}$
 $= \begin{bmatrix} 6 & 8 \\ -1 & 1 \end{bmatrix}$

c $B + C$
 $= \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ -4 & -2 \end{bmatrix}$
 $= \begin{bmatrix} 3 & 4 \\ -6 & -1 \end{bmatrix}$

d $C + B - A$
 $= \begin{bmatrix} -3 & 7 \\ -4 & -2 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$
 $= \begin{bmatrix} 0 & 0 \\ -11 & -3 \end{bmatrix}$

2 a $P + Q$
 $= \begin{bmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{bmatrix} + \begin{bmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{bmatrix}$
 $= \begin{bmatrix} 20 & 1 & -8 \\ 8 & 10 & -2 \\ 1 & -5 & 18 \end{bmatrix}$

b $P - Q$
 $= \begin{bmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{bmatrix} - \begin{bmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{bmatrix}$
 $= \begin{bmatrix} -14 & 9 & -14 \\ 12 & -6 & 14 \\ -5 & 3 & -4 \end{bmatrix}$

c $Q - P = \begin{bmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{bmatrix} - \begin{bmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 14 & -9 & 14 \\ -12 & 6 & -14 \\ 5 & -3 & 4 \end{bmatrix}$

3 a

Friday	Saturday
$\begin{bmatrix} 85 \\ 92 \\ 52 \end{bmatrix}$	$\begin{bmatrix} 102 \\ 137 \\ 49 \end{bmatrix}$

b Total for Friday and Saturday $= \begin{bmatrix} 85 \\ 92 \\ 52 \end{bmatrix} + \begin{bmatrix} 102 \\ 137 \\ 49 \end{bmatrix} = \begin{bmatrix} 187 \\ 229 \\ 101 \end{bmatrix}$

4 a i Cost price **ii** Selling price **b** In order to find David's profit/loss matrix we subtract the cost price matrix from the selling price matrix.

$$\begin{bmatrix} 1.72 \\ 27.85 \\ 0.92 \\ 2.53 \\ 3.56 \end{bmatrix} \quad \begin{bmatrix} 1.79 \\ 28.75 \\ 1.33 \\ 2.25 \\ 3.51 \end{bmatrix}$$

c Profit/Loss matrix $= \begin{bmatrix} 1.79 \\ 28.75 \\ 1.33 \\ 2.25 \\ 3.51 \end{bmatrix} - \begin{bmatrix} 1.72 \\ 27.85 \\ 0.92 \\ 2.53 \\ 3.56 \end{bmatrix} = \begin{bmatrix} 0.07 \\ 0.90 \\ 0.41 \\ -0.28 \\ -0.05 \end{bmatrix}$

5 a Lou Rose **b** Lou Rose **c** Total sales for November and December

$$\begin{bmatrix} 23 & 19 \\ 17 & 29 \\ 31 & 24 \end{bmatrix} \quad \begin{bmatrix} 18 & 25 \\ 7 & 13 \\ 36 & 19 \end{bmatrix} = \begin{bmatrix} 23 & 19 \\ 17 & 29 \\ 31 & 24 \end{bmatrix} + \begin{bmatrix} 18 & 25 \\ 7 & 13 \\ 36 & 19 \end{bmatrix} = \begin{bmatrix} 41 & 44 \\ 24 & 42 \\ 67 & 43 \end{bmatrix}$$

6 a $\begin{bmatrix} x & x^2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} y & 4 \\ 3 & y+1 \end{bmatrix}$ **b** $\begin{bmatrix} x & y \\ y & x \end{bmatrix} = \begin{bmatrix} -y & x \\ x & -y \end{bmatrix}$

Equating corresponding elements:

$$\begin{aligned} x &= y, \quad x^2 = 4 \quad \text{and} \quad -1 = y + 1 \\ \therefore y &= -2 \quad \text{and} \quad x = \pm 2 \\ \text{But } x &= y \quad \therefore x = y = -2 \end{aligned}$$

Equating corresponding elements:

$$\left. \begin{aligned} x &= -y \\ y &= x \end{aligned} \right\} \therefore y = 0, x = 0$$

7 a $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$ $\mathbf{B} + \mathbf{A} = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$

$$\begin{aligned} &= \begin{bmatrix} 2+(-1) & 1+2 \\ 3+2 & -1+3 \end{bmatrix} &= \begin{bmatrix} -1+2 & 2+1 \\ 2+3 & 3+(-1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} &= \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} \end{aligned}$$

b $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ because addition of numbers is commutative.

8 a $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \left(\begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix} \right) + \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$

$$\begin{aligned} &= \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 3 \\ -1 & 6 \end{bmatrix} \end{aligned}$$

$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} + \left(\begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix} \right)$

$$\begin{aligned} &= \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 7 & 3 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 3 \\ -1 & 6 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \text{Let } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\
 & \therefore (\mathbf{A} + \mathbf{B}) + \mathbf{C} \qquad \qquad \qquad \mathbf{A} + (\mathbf{B} + \mathbf{C}) \\
 & = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} p & q \\ r & s \end{bmatrix} \right) + \begin{bmatrix} w & x \\ y & z \end{bmatrix} \qquad = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \left(\begin{bmatrix} p & q \\ r & s \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} \right) \\
 & = \begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix} + \begin{bmatrix} w & x \\ y & z \end{bmatrix} \qquad = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} p+w & q+x \\ r+y & s+z \end{bmatrix} \\
 & = \begin{bmatrix} a+p+w & b+q+x \\ c+r+y & d+s+z \end{bmatrix} \qquad = \begin{bmatrix} a+p+w & b+q+x \\ c+r+y & d+s+z \end{bmatrix} \\
 & \qquad \qquad \qquad = (\mathbf{A} + \mathbf{B}) + \mathbf{C}
 \end{aligned}$$

EXERCISE 14C

$$\begin{array}{llll}
 \mathbf{1} \quad \mathbf{a} \quad 2\mathbf{B} & \mathbf{b} \quad \frac{1}{3}\mathbf{B} & \mathbf{c} \quad \frac{1}{12}\mathbf{B} & \mathbf{d} \quad -\frac{1}{2}\mathbf{B} \\
 = 2 \begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix} & = \frac{1}{3} \begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix} & = \frac{1}{12} \begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix} & = -\frac{1}{2} \begin{bmatrix} 6 & 12 \\ 24 & 6 \end{bmatrix} \\
 = \begin{bmatrix} 12 & 24 \\ 48 & 12 \end{bmatrix} & = \begin{bmatrix} 2 & 4 \\ 8 & 2 \end{bmatrix} & = \begin{bmatrix} \frac{1}{2} & 1 \\ 2 & \frac{1}{2} \end{bmatrix} & = \begin{bmatrix} -3 & -6 \\ -12 & -3 \end{bmatrix}
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{2} \quad \mathbf{a} \quad \mathbf{A} + \mathbf{B} & \mathbf{b} \quad \mathbf{A} - \mathbf{B} \\
 = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} & = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\
 = \begin{bmatrix} 2+1 & 3+2 & 5+1 \\ 1+1 & 6+2 & 4+3 \end{bmatrix} & = \begin{bmatrix} 2-1 & 3-2 & 5-1 \\ 1-1 & 6-2 & 4-3 \end{bmatrix} \\
 = \begin{bmatrix} 3 & 5 & 6 \\ 2 & 8 & 7 \end{bmatrix} & = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 4 & 1 \end{bmatrix} \\
 \mathbf{c} \quad 2\mathbf{A} + \mathbf{B} & \mathbf{d} \quad 3\mathbf{A} - \mathbf{B} \\
 = 2 \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} & = 3 \begin{bmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\
 = \begin{bmatrix} 4 & 6 & 10 \\ 2 & 12 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} & = \begin{bmatrix} 6 & 9 & 15 \\ 3 & 18 & 12 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\
 = \begin{bmatrix} 4+1 & 6+2 & 10+1 \\ 2+1 & 12+2 & 8+3 \end{bmatrix} & = \begin{bmatrix} 6-1 & 9-2 & 15-1 \\ 3-1 & 18-2 & 12-3 \end{bmatrix} \\
 = \begin{bmatrix} 5 & 8 & 11 \\ 3 & 14 & 11 \end{bmatrix} & = \begin{bmatrix} 5 & 7 & 14 \\ 2 & 16 & 9 \end{bmatrix}
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{3} \quad \mathbf{a} & \begin{bmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{bmatrix} = \begin{bmatrix} 35 & 46 & 46 & 69 \\ 58 & 46 & 35 & 86 \\ 46 & 46 & 58 & 58 \\ 12 & 23 & 23 & 17 \end{bmatrix} \quad \begin{array}{l} \text{rounded to the} \\ \text{nearest whole} \\ \text{number.} \end{array} \\
 \mathbf{b} & \begin{bmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{bmatrix} = \begin{bmatrix} 26 & 34 & 34 & 51 \\ 43 & 34 & 26 & 64 \\ 34 & 34 & 43 & 43 \\ 9 & 17 & 17 & 13 \end{bmatrix} \quad \begin{array}{l} \text{rounded to the} \\ \text{nearest whole} \\ \text{number.} \end{array}
 \end{array}$$

4 a Weekdays Weekends

$$\begin{bmatrix} 75 \\ 27 \\ 102 \end{bmatrix} \quad \begin{bmatrix} 136 \\ 43 \\ 129 \end{bmatrix} \quad \begin{array}{l} \text{VHS} \\ \text{DVD} \\ \text{games} \end{array}$$

c The sum matrix of **b** represents total weekly average hirings.

b

$$\begin{bmatrix} 75 \\ 27 \\ 102 \end{bmatrix} + \begin{bmatrix} 136 \\ 43 \\ 129 \end{bmatrix} = \begin{bmatrix} 211 \\ 70 \\ 231 \end{bmatrix}$$

5 The matrix is $12\mathbf{F} = 12 \begin{bmatrix} 1 \\ 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 48 \\ 24 \\ 12 \end{bmatrix}$

EXERCISE 14D

1 a $\mathbf{A} + 2\mathbf{A} = 3\mathbf{A}$ **b** $3\mathbf{B} - 3\mathbf{B} = \mathbf{O}$ **c** $\mathbf{C} - 2\mathbf{C} = -\mathbf{C}$
d $-\mathbf{B} + \mathbf{B} = \mathbf{O}$ **e** $2(\mathbf{A} + \mathbf{B}) = 2\mathbf{A} + 2\mathbf{B}$ **f** $-(\mathbf{A} + \mathbf{B}) = -\mathbf{A} - \mathbf{B}$
g $-(2\mathbf{A} - \mathbf{C}) = -2\mathbf{A} + \mathbf{C}$ **h** $3\mathbf{A} - (\mathbf{B} - \mathbf{A}) = 3\mathbf{A} - \mathbf{B} + \mathbf{A} = 4\mathbf{A} - \mathbf{B}$ **i** $\mathbf{A} + 2\mathbf{B} - (\mathbf{A} - \mathbf{B}) = \mathbf{A} + 2\mathbf{B} - \mathbf{A} + \mathbf{B} = 3\mathbf{B}$

2 a if $\mathbf{X} + \mathbf{B} = \mathbf{A}$
 then $\mathbf{X} + \mathbf{B} + (-\mathbf{B}) = \mathbf{A} + (-\mathbf{B})$
 $\therefore \mathbf{X} + \mathbf{O} = \mathbf{A} - \mathbf{B}$
 $\therefore \mathbf{X} = \mathbf{A} - \mathbf{B}$

c if $4\mathbf{B} + \mathbf{X} = 2\mathbf{C}$
 then $4\mathbf{B} + \mathbf{X} + (-4\mathbf{B}) = 2\mathbf{C} + (-4\mathbf{B})$
 $\therefore \mathbf{O} + \mathbf{X} = 2\mathbf{C} - 4\mathbf{B}$
 $\therefore \mathbf{X} = 2\mathbf{C} - 4\mathbf{B}$

e if $3\mathbf{X} = \mathbf{B}$
 then $\frac{1}{3}(3\mathbf{X}) = \frac{1}{3}\mathbf{B}$
 $\therefore 1\mathbf{X} = \frac{1}{3}\mathbf{B}$ i.e., $\mathbf{X} = \frac{1}{3}\mathbf{B}$

g if $\frac{1}{2}\mathbf{X} = \mathbf{C}$
 then $2(\frac{1}{2}\mathbf{X}) = 2\mathbf{C}$
 $\therefore 1\mathbf{X} = 2\mathbf{C}$ i.e., $\mathbf{X} = 2\mathbf{C}$

i if $\mathbf{A} - 4\mathbf{X} = \mathbf{C}$
 then $\mathbf{A} - 4\mathbf{X} + 4\mathbf{X} = \mathbf{C} + 4\mathbf{X}$
 $\therefore \mathbf{A} + \mathbf{O} = \mathbf{C} + 4\mathbf{X}$
 $\therefore \mathbf{A} = \mathbf{C} + 4\mathbf{X}$
 and $\mathbf{A} - \mathbf{C} = 4\mathbf{X}$
 $\therefore \frac{1}{4}(\mathbf{A} - \mathbf{C}) = \frac{1}{4}(4\mathbf{X})$
 $\therefore \mathbf{X} = \frac{1}{4}(\mathbf{A} - \mathbf{C})$

b if $\mathbf{B} + \mathbf{X} = \mathbf{C}$
 then $\mathbf{B} + \mathbf{X} + (-\mathbf{B}) = \mathbf{C} + (-\mathbf{B})$
 $\therefore \mathbf{O} + \mathbf{X} = \mathbf{C} - \mathbf{B}$
 $\therefore \mathbf{X} = \mathbf{C} - \mathbf{B}$

d if $2\mathbf{X} = \mathbf{A}$
 then $\frac{1}{2}(2\mathbf{X}) = \frac{1}{2}\mathbf{A}$
 $\therefore 1\mathbf{X} = \frac{1}{2}\mathbf{A}$ i.e., $\mathbf{X} = \frac{1}{2}\mathbf{A}$

f if $\mathbf{A} - \mathbf{X} = \mathbf{B}$
 then $\mathbf{A} - \mathbf{X} + \mathbf{X} = \mathbf{B} + \mathbf{X}$
 $\therefore \mathbf{A} + \mathbf{O} = \mathbf{B} + \mathbf{X}$
 $\therefore \mathbf{A} = \mathbf{B} + \mathbf{X}$
 and $\mathbf{A} + (-\mathbf{B}) = \mathbf{B} + \mathbf{X} + (-\mathbf{B})$
 $\therefore \mathbf{A} - \mathbf{B} = \mathbf{X} + \mathbf{O}$
 $\therefore \mathbf{A} - \mathbf{B} = \mathbf{X}$
 $\therefore \mathbf{X} = \mathbf{A} - \mathbf{B}$

h if $2(\mathbf{X} + \mathbf{A}) = \mathbf{B}$
 then $\frac{1}{2}[2(\mathbf{X} + \mathbf{A})] = \frac{1}{2}\mathbf{B}$
 $\therefore 1(\mathbf{X} + \mathbf{A}) = \frac{1}{2}\mathbf{B}$
 $\therefore \mathbf{X} + \mathbf{A} = \frac{1}{2}\mathbf{B}$
 $\therefore \mathbf{X} + \mathbf{A} + (-\mathbf{A}) = \frac{1}{2}\mathbf{B} + (-\mathbf{A})$
 $\therefore \mathbf{X} + \mathbf{O} = \frac{1}{2}\mathbf{B} - \mathbf{A}$
 $\therefore \mathbf{X} = \frac{1}{2}\mathbf{B} - \mathbf{A}$

3 a if $\frac{1}{3}\mathbf{X} = \mathbf{M}$
 then $3(\frac{1}{3}\mathbf{X}) = 3\mathbf{M}$
 $\therefore \mathbf{X} = 3\mathbf{M} = 3 \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$
 $= \begin{bmatrix} 3 & 6 \\ 9 & 18 \end{bmatrix}$

b if $4\mathbf{X} = \mathbf{N}$ then $\frac{1}{4}(4\mathbf{X}) = \frac{1}{4}\mathbf{N}$
 $\therefore \mathbf{X} = \frac{1}{4}\mathbf{N}$
 $\therefore \mathbf{X} = \frac{1}{4} \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{3}{4} & \frac{5}{4} \end{bmatrix}$

$$\text{c} \quad \text{if } \mathbf{A} - 2\mathbf{X} = 3\mathbf{B}$$

$$\text{then } \mathbf{A} - 2\mathbf{X} + 2\mathbf{X} = 3\mathbf{B} + 2\mathbf{X}$$

$$\therefore \mathbf{A} = 3\mathbf{B} - 2\mathbf{X}$$

$$\therefore \mathbf{A} + (-3\mathbf{B}) = 3\mathbf{B} - 2\mathbf{X} + (-3\mathbf{B})$$

$$\therefore \mathbf{A} - 3\mathbf{B} = 2\mathbf{X}$$

$$\therefore \frac{1}{2}(\mathbf{A} - 3\mathbf{B}) = \frac{1}{2}(2\mathbf{X})$$

$$\therefore \frac{1}{2}(\mathbf{A} - 3\mathbf{B}) = \mathbf{X}$$

$$\therefore \mathbf{X} = \frac{1}{2}(\mathbf{A} - 3\mathbf{B})$$

$$= \frac{1}{2} \left(\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{2} \begin{bmatrix} -2 & -12 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -6 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

EXERCISE 14E.1

1 a

$$\begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$= [3 \times 5 + (-1) \times 4]$$

$$= [15 - 4]$$

$$= [11]$$

b

$$\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$$

$$= [1 \times 5 + 3 \times 1 + 2 \times 7]$$

$$= [5 + 3 + 14]$$

$$= [22]$$

c

$$\begin{bmatrix} 6 & -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 4 \end{bmatrix} = [6 \times 1 + (-1) \times 0 + 2 \times (-1) + 3 \times 4]$$

$$= [6 + 0 - 2 + 12]$$

$$= [16]$$

2

$$\begin{bmatrix} w & x & y & z \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = [w + x + y + z]$$

$\therefore \frac{1}{4}(w + x + y + z)$, which is the average of w , x , y and z , can be represented as

$$\begin{bmatrix} w & x & y & z \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

3

$$\text{a} \quad \mathbf{Q} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 27 & 35 & 39 \end{bmatrix}$$

$$\text{b} \quad \text{total cost} = \mathbf{PQ} = \begin{bmatrix} 27 & 35 & 39 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$= [27 \times 4 + 35 \times 3 + 39 \times 2]$$

$$= [291] \quad \therefore \text{total cost is \$291}$$

4

$$\text{a} \quad \mathbf{P} = \begin{bmatrix} 10 & 6 & 3 & 1 \end{bmatrix}$$

$$\mathbf{N} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 2 \end{bmatrix}$$

b

$$\text{total points} = \mathbf{PN} = \begin{bmatrix} 10 & 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 4 \\ 2 \end{bmatrix}$$

$$= [10 \times 3 + 6 \times 2 + 3 \times 4 + 1 \times 2]$$

$$= [30 + 12 + 12 + 2]$$

$$= [56] \quad \text{So, the number of points awarded is 56.}$$

EXERCISE 14E.2

1

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 1 \end{bmatrix} \quad \text{which is 1 row} \times 3 \text{ columns}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{which is 2 rows} \times 3 \text{ columns}$$

$\mathbf{AB}$ cannot be found because the number of columns in $\mathbf{A}$ does not equal the number of rows in $\mathbf{B}$.

6 a me friend

$$\mathbf{R} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{array}{l} \text{hammers} \\ \text{screwdrivers} \\ \text{cans of paint} \end{array}$$

b $\mathbf{P} = \begin{bmatrix} 7 & 3 & 19 \\ 6 & 2 & 22 \end{bmatrix}$ store A
store B

c $\mathbf{P}$ is 2×3 and $\mathbf{R}$ is 3×2 , $\therefore \mathbf{PR}$ is 2×2

$$\mathbf{PR} = \begin{bmatrix} 7 & 3 & 19 \\ 6 & 2 & 22 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 7 \times 1 + 3 \times 1 + 19 \times 2 & 7 \times 1 + 3 \times 2 + 19 \times 3 \\ 6 \times 1 + 2 \times 1 + 22 \times 2 & 6 \times 1 + 2 \times 2 + 22 \times 3 \end{bmatrix}$$

$$\therefore \mathbf{PR} = \begin{bmatrix} 7 + 3 + 38 & 7 + 6 + 57 \\ 6 + 2 + 44 & 6 + 4 + 66 \end{bmatrix} = \begin{bmatrix} 48 & 70 \\ 52 & 76 \end{bmatrix}$$

d My costs at Store A are \$48; my friend's costs at Store B are \$76.

e My costs at Store B are \$52. Therefore I should shop at Store A, which is cheaper.

EXERCISE 14F

1 a $\begin{bmatrix} 16 & 18 & 15 \\ 13 & 21 & 16 \\ 10 & 22 & 24 \end{bmatrix}$

b $\begin{bmatrix} 10 & 6 & -7 \\ 9 & 3 & 0 \\ 4 & -4 & -10 \end{bmatrix}$

c $\begin{bmatrix} 22 & 0 & 132 & 176 & 198 \\ 44 & 154 & 88 & 110 & 0 \\ 176 & 44 & 88 & 88 & 132 \end{bmatrix}$

d $\begin{bmatrix} 115 \\ 136 \\ 46 \\ 106 \end{bmatrix}$

2 a 
Numbers matrix $\mathbf{N} = \begin{bmatrix} 3 & 3 & 2 \end{bmatrix}$

b $\mathbf{P} = \begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix}$ Bay View Terrace Staunton Star
room
breakfast
dinner

c Total prices for each venue = numbers matrix $\times$ prices matrix = $\mathbf{NP}$

$\mathbf{N}$ is 1×3 and $\mathbf{P}$ is 3×3 , $\therefore \mathbf{NP}$ is 1×3

$$\mathbf{NP} = \begin{bmatrix} 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} = \begin{bmatrix} 657 & 730 & 670 \end{bmatrix} \quad \{\text{using technology}\}$$

$\therefore$ \$657 for Bay View, \$730 for Terrace, \$670 for Staunton Star.

d Total prices = $\begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} = \begin{bmatrix} 369 & 420 & 385 \end{bmatrix}$, {using technology}

i.e., \$369 for Bay View, \$420 for Terrace, \$385 for Staunton Star.

e To include both scenarios we calculate

$$\begin{bmatrix} 3 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 125 & 150 & 140 \\ 44 & 40 & 40 \\ 75 & 80 & 65 \end{bmatrix} = \begin{bmatrix} 657 & 730 & 670 \\ 369 & 420 & 385 \end{bmatrix}, \text{ using technology}$$

3

Prices matrix = $\begin{bmatrix} 125 \\ 315 \\ 405 \\ 375 \end{bmatrix}$ To find total income we calculate:
 prices matrix $\times$ numbers matrix.

$$= \begin{bmatrix} 125 \\ 315 \\ 405 \\ 375 \end{bmatrix} \begin{bmatrix} 50 & 42 & 18 & 65 \\ 65 & 37 & 25 & 82 \\ 120 & 29 & 23 & 75 \\ 42 & 36 & 19 & 72 \end{bmatrix} = \begin{bmatrix} 51\,145 \\ 60\,655 \\ 61\,575 \\ 51\,285 \end{bmatrix} \text{ using tech.}$$

$$\therefore \text{total income} = \$51\,145 + \$60\,655 + \$61\,575 + \$51\,285 = \$224\,660$$

4 a

$$\text{Income matrix } \mathbf{I} = \begin{bmatrix} 125 & 195 & 225 \end{bmatrix}$$

$$\text{Cost matrix } \mathbf{C} = \begin{bmatrix} 85 & 120 & 130 \end{bmatrix}$$

$$\text{Numbers (bookings) matrix } \mathbf{N} = \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix}$$

$$\begin{aligned} \text{profit per day} &= (\text{income from room}) \times (\text{bookings per day}) \\ &\quad - (\text{maintenance cost per room}) \times (\text{bookings per day}) \end{aligned}$$

$$= \mathbf{IN} - \mathbf{CN}$$

$$= \begin{bmatrix} 125 & 195 & 225 \end{bmatrix} \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} 85 & 120 & 130 \end{bmatrix} \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1185 & 800 & 1350 & 970 & 845 & 1130 & 845 \end{bmatrix}, \text{ using technology}$$

$$\begin{aligned} \therefore \text{profit for the week} &= \$1185 + \$800 + \$1350 + \$970 + \$845 + \$1130 + \$845 \\ &= \$7125 \end{aligned}$$

b If the hotel maintained every room every day we would need to calculate

$$(\text{income from room}) \times (\text{bookings per day}) - (\text{maintenance costs per room}) \times (\text{number of rooms})$$

$$= \begin{bmatrix} 125 & 195 & 225 \end{bmatrix} \times \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} 85 & 120 & 130 \end{bmatrix} \times \begin{bmatrix} 20 & 20 & 20 & 20 & 20 & 20 & 20 \\ 15 & 15 & 15 & 15 & 15 & 15 & 15 \\ 5 & 5 & 5 & 5 & 5 & 5 & 5 \end{bmatrix}$$

$$\therefore \text{using technology, profit per day} = \begin{bmatrix} -820 & -1840 & -455 & -1485 & -1725 & -920 & -1785 \end{bmatrix}$$

$$\therefore \text{the profit per week would be } (-\$820) + \dots + (-\$1785) = -\$9030, \text{ i.e., a loss of } \$9030.$$

c Profit per room matrix = income per room matrix – cost per room matrix

$$= \mathbf{I} - \mathbf{C}$$

$$= \begin{bmatrix} 125 & 195 & 225 \end{bmatrix} - \begin{bmatrix} 85 & 120 & 130 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & 75 & 95 \end{bmatrix}$$

 $\therefore$ the result in **a** can be obtained if we calculate:

$$\begin{bmatrix} 40 & 75 & 95 \end{bmatrix} \begin{bmatrix} 15 & 12 & 13 & 11 & 14 & 16 & 8 \\ 4 & 3 & 6 & 2 & 0 & 4 & 7 \\ 3 & 1 & 4 & 4 & 3 & 2 & 0 \end{bmatrix} \quad \begin{array}{l} \text{This checks} \\ \text{using technology.} \end{array}$$

EXERCISE 14G

$$1 \quad \mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1+0 & 1+0 \\ -1+0 & 1+6 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 7 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 0+2 \\ 0+3 & 0+6 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 3 & 6 \end{bmatrix}$$

$\mathbf{AB} \neq \mathbf{BA}$ $\therefore$ in the general case $\mathbf{AB}$ does not necessarily equal $\mathbf{BA}$.

$$2 \quad \mathbf{AO} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$$

$$\mathbf{OA} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O} \quad \therefore \mathbf{AO} = \mathbf{OA} = \mathbf{O}$$

$$3 \quad \text{a} \quad \text{Let } \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}, \text{ say}$$

$$\begin{aligned} \mathbf{A(B+C)} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 3+4 & 4+4 \\ 9+8 & 12+8 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 8 \\ 17 & 20 \end{bmatrix} \end{aligned} \quad \begin{aligned} \mathbf{AB+AC} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+2 & 1+2 \\ 3+4 & 3+4 \end{bmatrix} + \begin{bmatrix} 2+2 & 3+2 \\ 6+4 & 9+4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 10 & 13 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 8 \\ 17 & 20 \end{bmatrix} \\ &= \mathbf{A(B+C)} \end{aligned}$$

$$\text{b} \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \mathbf{B} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$\mathbf{A(B+C)} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p+w & q+x \\ r+y & s+z \end{bmatrix} = \begin{bmatrix} ap+aw+br+by & aq+ax+bs+bz \\ cp+cw+dr+dy & cq+cx+ds+dz \end{bmatrix}$$

$$\begin{aligned} \mathbf{AB+AC} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ &= \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix} + \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} \\ &= \begin{bmatrix} ap+aw+br+by & aq+ax+bs+bz \\ cp+cw+dr+dy & cq+cx+ds+dz \end{bmatrix} = \mathbf{A(B+C)} \end{aligned}$$

c Using the matrices in a,

$$\begin{aligned} (\mathbf{AB})\mathbf{C} &= \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6+3 & 9+3 \\ 14+7 & 21+7 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 12 \\ 21 & 28 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{A(BC)} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3+6 & 4+8 \\ 9+12 & 12+16 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 12 \\ 21 & 28 \end{bmatrix} = (\mathbf{AB})\mathbf{C} \end{aligned}$$

d Using the matrices in **b**,

$$\begin{aligned}
 (\mathbf{AB})\mathbf{C} &= \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\
 &= \begin{bmatrix} apw+brw+aqy+bsy & apx+brx+aqz+bsz \\ cpw+drw+cqy+dsy & cpx+drx+cqz+dsz \end{bmatrix} \\
 \mathbf{A}(\mathbf{BC}) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left[\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \right] \\
 &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} pw+qy & px+qz \\ rw+sy & rx+sz \end{bmatrix} \\
 &= \begin{bmatrix} apw+brw+aqy+bsy & apx+brx+aqz+bsz \\ cpw+drw+cqy+dsy & cpx+drx+cqz+dsz \end{bmatrix} = (\mathbf{AB})\mathbf{C}
 \end{aligned}$$

4 a If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
 $\therefore$ equating coefficients of corresponding elements: $w = 1, y = 0, x = 0, z = 1$
 which checks with the coefficients in the second line. $\therefore w = z = 1, \text{ i.e., } \mathbf{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
 $x = y = 0$

b In **a** we showed that $\mathbf{AX} = \mathbf{A}$ where $\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore \mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all
 2×2 matrices $\mathbf{A}$ where $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5 a $\mathbf{A}^2 = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 4+3 & 2+(-2) \\ 6+(-6) & 3+4 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

b $\mathbf{A}^3 = \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$
 $= \begin{bmatrix} 25+(-2) & -5+(-4) \\ 10+8 & -2+16 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$
 $= \begin{bmatrix} 23 & -9 \\ 18 & 14 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$
 $= \begin{bmatrix} 115+(-18) & -23+(-36) \\ 90+28 & -18+56 \end{bmatrix}$
 $= \begin{bmatrix} 97 & -59 \\ 118 & 38 \end{bmatrix}$

6 a $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $\therefore \mathbf{A}^2$ is 3×2 by 3×2

$2 \neq 3$ so $\mathbf{A}^2$ does not exist.

b We can square a matrix when the number of columns equals the number of rows,
 i.e., if it is a square matrix.

7 $\mathbf{I}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I} \therefore \mathbf{I}^3 = \mathbf{I}^2 = \mathbf{I} = \mathbf{I}^2 = \mathbf{I}$

8 a $\mathbf{A}(\mathbf{A} + \mathbf{I})$
 $= \mathbf{A}^2 + \mathbf{AI}$
 $= \mathbf{A}^2 + \mathbf{A}$

b $(\mathbf{B} + 2\mathbf{I})\mathbf{B}$
 $= \mathbf{B}^2 + 2\mathbf{IB}$
 $= \mathbf{B}^2 + 2\mathbf{B}$

c $\mathbf{A}(\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I})$
 $= \mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{AI}$
 $= \mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A}$

$$\begin{aligned} \mathbf{d} \quad & A(A^2 + A - 2I) \\ &= A^3 + A^2 - 2AI \\ &= A^3 + A^2 - 2A \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & (A + B)(C + D) \\ &= (A + B)C + (A + B)D \\ &= AC + BC + AD + BD \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & (A + B)^2 \\ &= (A + B)(A + B) \\ &= (A + B)A + (A + B)B \\ &= A^2 + BA + AB + B^2 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & (A + B)(A - B) \\ &= (A + B)A - (A + B)B \\ &= A^2 + BA - AB - B^2 \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & (A + I)^2 \\ &= (A + I)(A + I) \\ &= (A + I)A + (A + I)I \\ &= A^2 + IA + AI + I^2 \\ &= A^2 + A + A + I \\ &= A^2 + 2A + I \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & (3I - B)^2 \\ &= (3I - B)(3I - B) \\ &= (3I - B)3I - (3I - B)B \\ &= 9I^2 - 3BI - 3IB + B^2 \\ &= 9I - 3B - 3B + B^2 \\ &= 9I - 6B + B^2 \end{aligned}$$

$$\begin{aligned} \mathbf{9} \quad \mathbf{a} \quad & A^2 = 2A - I \quad \therefore A^3 = A \times A^2 \quad \text{and} \quad A^4 = A \times A^3 \\ &= A(2A - I) \quad = A(3A - 2I) \\ &= 2A^2 - AI \quad = 3A^2 - 2AI \\ &= 2(2A - I) - A \quad = 3(2A - I) - 2A \\ &= 4A - 2I - A \quad = 6A - 3I - 2A \\ &= 3A - 2I \quad = 4A - 3I \end{aligned}$$

$$\mathbf{b} \quad B^2 = 2I - B$$

$$\begin{aligned} \therefore B^3 &= B \times B^2 \quad \text{and} \quad B^4 = B \times B^3 \quad \text{and} \quad B^5 = B \times B^4 \\ &= B(2I - B) \quad = B(3B - 2I) \quad = B(6I - 5B) \\ &= 2BI - B^2 \quad = 3B^2 - 2BI \quad = 6BI - 5B^2 \\ &= 2B - (2I - B) \quad = 3(2I - B) - 2B \quad = 6B - 5(2I - B) \\ &= 2B - 2I + B \quad = 6I - 3B - 2B \quad = 6B - 10I + 5B \\ &= 3B - 2I \quad = 6I - 5B \quad = 11B - 10I \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & C^2 = 4C - 3I \quad C^3 = C \times C^2 \quad C^5 = C^2 \times C^3 \\ &= C(4C - 3I) \quad = (4C - 3I)(13C - 12I) \\ &= 4C^2 - 3CI \quad = (4C - 3I)13C - (4C - 3I)12I \\ &= 4(4C - 3I) - 3C \quad = 52C^2 - 39IC - 48CI + 36I^2 \\ &= 16C - 12I - 3C \quad = 52(4C - 3I) - 39C - 48C + 36I \\ &= 13C - 12I \quad = 208C - 156I - 87C + 36I \\ & \quad = 121C - 120I \end{aligned}$$

$$\mathbf{10} \quad \mathbf{a} \quad \text{If } A^2 = I:$$

$$\begin{aligned} \mathbf{i} \quad & A(A + 2I) \\ &= A^2 + 2AI \\ &= I + 2A \end{aligned}$$

$$\begin{aligned} \mathbf{ii} \quad & (A - I)^2 \\ &= (A - I)(A - I) \\ &= (A - I)A - (A - I)I \\ &= A^2 - IA - AI + I^2 \\ &= I - A - A + I \\ &= 2I - 2A \end{aligned}$$

$$\begin{aligned} \mathbf{iii} \quad & A(A + 3I)^2 \\ &= A(A + 3I)(A + 3I) \\ &= A[(A + 3I)A + (A + 3I)3I] \\ &= A[A^2 + 3IA + 3AI + 9I^2] \\ &= A[I + 3A + 3A + 9I] \\ &= A[10I + 6A] \\ &= 10AI + 6A^2 \\ &= 10A + 6I \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{If } \mathbf{A}^3 = \mathbf{I}, \quad \mathbf{A}^2(\mathbf{A} + \mathbf{I})^2 &= \mathbf{A}^2(\mathbf{A}^2 + 2\mathbf{A} + \mathbf{I}) \\
 &= \mathbf{A}^4 + 2\mathbf{A}^3 + \mathbf{A}^2\mathbf{I} \\
 &= \mathbf{A}(\mathbf{A}^3) + 2\mathbf{A}^3 + \mathbf{A}^2\mathbf{I} \\
 &= \mathbf{AI} + 2\mathbf{I} + \mathbf{A}^2 \\
 \mathbf{c} \quad \text{If } \mathbf{A}^2 = \mathbf{O}: &= \mathbf{A}^2 + \mathbf{A} + 2\mathbf{I}
 \end{aligned}$$

$$\begin{array}{lll}
 \mathbf{i} \quad \mathbf{A}(2\mathbf{A} - 3\mathbf{I}) & \mathbf{ii} \quad \mathbf{A}(\mathbf{A} + 2\mathbf{I})(\mathbf{A} - \mathbf{I}) & \mathbf{iii} \quad \mathbf{A}(\mathbf{A} + \mathbf{I})^3 \\
 = 2\mathbf{A}^2 - 3\mathbf{AI} & = \mathbf{A}[(\mathbf{A} + 2\mathbf{I})\mathbf{A} - (\mathbf{A} + 2\mathbf{I})\mathbf{I}] & = \mathbf{A}(\mathbf{A} + \mathbf{I})(\mathbf{A} + \mathbf{I})^2 \\
 = 2\mathbf{O} - 3\mathbf{AI} & = \mathbf{A}(\mathbf{A}^2 + 2\mathbf{IA} - \mathbf{AI} - 2\mathbf{I}^2) & = \mathbf{A}(\mathbf{A} + \mathbf{I})(\mathbf{A}^2 + 2\mathbf{A} + \mathbf{I}) \\
 = -3\mathbf{AI} & = \mathbf{A}(\mathbf{O} + \mathbf{A} - 2\mathbf{I}) & = (\mathbf{A}^2 + \mathbf{AI})(\mathbf{O} + 2\mathbf{A} + \mathbf{I}) \\
 & = \mathbf{A}^2 - 2\mathbf{AI} & = (\mathbf{O} + \mathbf{A})(2\mathbf{A} + \mathbf{I}) \\
 & = \mathbf{O} - 2\mathbf{A} & = 2\mathbf{A}^2 + \mathbf{AI} \\
 & = -2\mathbf{A} & = 2\mathbf{O} + \mathbf{A} \\
 & & = \mathbf{A}
 \end{array}$$

$$\mathbf{11} \quad \mathbf{a} \quad \mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}$$

$$\mathbf{b} \quad \mathbf{A}^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} + \frac{1}{4} & \frac{1}{4} + \frac{1}{4} \\ \frac{1}{4} + \frac{1}{4} & \frac{1}{4} + \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \mathbf{A}$$

$$\begin{aligned}
 \mathbf{c} \quad \mathbf{A}^2 &= \mathbf{A} \\
 \therefore \mathbf{A}^2 - \mathbf{A} &= \mathbf{O} \\
 \therefore \mathbf{A}(\mathbf{A} - \mathbf{I}) &= \mathbf{O} \\
 \therefore \mathbf{A} &= \mathbf{O} \text{ or } \mathbf{A} - \mathbf{I} = \mathbf{O} \\
 \therefore \mathbf{A} &= \mathbf{O} \text{ or } \mathbf{I}
 \end{aligned}$$

The argument contains a false step. As the example in **a** illustrates, $\mathbf{AB} = \mathbf{O}$ does not imply that $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$.

This is a property of real numbers that does not hold for matrices. Therefore it is false to say that if $\mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{O}$, then $\mathbf{A} = \mathbf{O}$ or $\mathbf{A} - \mathbf{I}$ equals $\mathbf{O}$.

$$\begin{aligned}
 \mathbf{d} \quad \text{Let } \mathbf{A} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad \text{If } \mathbf{A}^2 = \mathbf{A}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
 &\quad \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
 \end{aligned}$$

Equating corresponding elements:

$$a^2 + bc = a \quad \therefore \quad bc = a(1 - a) \quad \dots (1)$$

$$ab + bd = b \quad \therefore \quad b(a + d - 1) = 0 \quad \dots (2)$$

$$ac + cd = c \quad \therefore \quad c(a + d - 1) = 0 \quad \dots (3)$$

$$bc + d^2 = d \quad \therefore \quad bc = d(1 - d) \quad \dots (4)$$

If $a + d - 1 \neq 0$ then from (2) and (3), $b = c = 0$.

$\therefore$ from (1) and (4), $a = 0$ or 1 and $d = 0$ or 1

$\therefore a = 0, d = 0$ or $a = 0, d = 1$ or $a = 1, d = 0$ or $a = 1, d = 1$

where the last two cases are not possible as $a + d \neq 1$.

$$\text{So, if } a = 0, d = 0 \text{ then } \mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and if } a = 1, d = 1 \text{ then } \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If $a + d - 1 = 0$ then $d = 1 - a$ and $c = \frac{a - a^2}{b}$

$$\text{So } \mathbf{A} \text{ is } \begin{bmatrix} a & b \\ \frac{a - a^2}{b} & 1 - a \end{bmatrix} \text{ provided } b \neq 0.$$

12 Choose $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$,

$$\text{then } \mathbf{A}^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+(-1) & -1+1 \\ 1+(-1) & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \mathbf{A}^2 = \mathbf{O}$, but $\mathbf{A} \neq \mathbf{O}$ so “if $\mathbf{A}^2 = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$ ” is a false statement.

13 a Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$, $\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = a \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1+(-2) & 2+4 \\ -1+(-2) & -2+4 \end{bmatrix} = \begin{bmatrix} a & 2a \\ -a & 2a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

$$\therefore \begin{bmatrix} -1 & 6 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} a+b & 2a \\ -a & 2a+b \end{bmatrix}$$

$$\therefore a+b = -1 \quad \text{and} \quad 2a = 6$$

$$\text{i.e., } a = 3 \quad \text{and} \quad b = -4$$

Checking for consistency: $-a = -3$, $2a + b = 6 + (-4) = 2 \quad \checkmark$

$$\therefore \mathbf{A}^2 = 3\mathbf{A} - 4\mathbf{I}$$

b Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$, $\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} = a \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} + b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 9+2 & 3+(-2) \\ 6+(-4) & 2+4 \end{bmatrix} = \begin{bmatrix} 3a & a \\ 2a & -2a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$$

$$\therefore \begin{bmatrix} 11 & 1 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 3a+b & a \\ 2a & -2a+b \end{bmatrix}$$

$$\therefore 3a+b = 11 \quad \text{and} \quad a = 1$$

$$\therefore a = 1 \quad \text{and} \quad b = 8$$

Checking for consistency: $2a = 2(1) = 2$, $-2a + b = -2(1) + 8 = 6 \quad \checkmark$

$$\therefore \mathbf{A}^2 = \mathbf{A} + 8\mathbf{I}$$

14 If $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$

$$\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} = p \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} + q \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+(-2) & 2+(-6) \\ -1+3 & -2+9 \end{bmatrix} = \begin{bmatrix} p & 2p \\ -p & -3p \end{bmatrix} + \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$$

$$\begin{bmatrix} -1 & -4 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} p+q & 2p \\ -p & -3p+q \end{bmatrix}$$

$$\therefore p+q = -1 \quad \text{and} \quad 2p = -4$$

$$\therefore p = -2 \quad \text{and} \quad q = 1$$

Checking for consistency: $-p = -(-2) = 2$, $-3p + q = -3(-2) + 1 = 7 \quad \checkmark$

$$\therefore \mathbf{A}^2 = -2\mathbf{A} + \mathbf{I}$$

a $\mathbf{A}^3 = \mathbf{A} \times \mathbf{A}^2$

$$= \mathbf{A}(-2\mathbf{A} + \mathbf{I})$$

$$= -2\mathbf{A}^2 + \mathbf{A}\mathbf{I}$$

$$= -2(-2\mathbf{A} + \mathbf{I}) + \mathbf{A}$$

$$= 4\mathbf{A} - 2\mathbf{I} + \mathbf{A}$$

$$= 5\mathbf{A} - 2\mathbf{I}$$

b $\mathbf{A}^4 = \mathbf{A} \times \mathbf{A}^3$

$$= \mathbf{A}(5\mathbf{A} - 2\mathbf{I})$$

$$= 5\mathbf{A}^2 - 2\mathbf{A}\mathbf{I}$$

$$= 5(-2\mathbf{A} + \mathbf{I}) - 2\mathbf{A}$$

$$= -10\mathbf{A} + 5\mathbf{I} - 2\mathbf{A}$$

$$= -12\mathbf{A} + 5\mathbf{I}$$

EXERCISE 14H

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\
 &= 3\mathbf{I} \\
 \therefore & \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} \times \frac{1}{3} \begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix} = \mathbf{I} \\
 \therefore & \begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & -2 \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \\
 &= 10\mathbf{I} \\
 \therefore & \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} \times \frac{1}{10} \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} = \mathbf{I} \\
 \therefore & \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{10} \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 0.2 & 0.4 \\ -0.1 & 0.3 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 2\mathbf{I} \\
 \therefore & \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{bmatrix} = \mathbf{I} \\
 \text{and so} \quad & \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{11}{2} & \frac{9}{2} & \frac{15}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & -3 & -5 \end{bmatrix}
 \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad |\mathbf{A}| = 12 - 14 = -2 \quad \mathbf{b} \quad |\mathbf{A}| = 2 - 3 = -1 \quad \mathbf{c} \quad |\mathbf{A}| = 0 - 0 = 0 \quad \mathbf{d} \quad |\mathbf{A}| = 1 - 0 = 1$$

$$\mathbf{3} \quad \mathbf{a} \quad \det \mathbf{B} = 12 - -14 = 26 \quad \mathbf{b} \quad \det \mathbf{B} = 6 - 0 = 6 \quad \mathbf{c} \quad \det \mathbf{B} = 0 - 1 = -1 \quad \mathbf{d} \quad \det \mathbf{B} = a^2 - -a = a^2 + a$$

$$\mathbf{4} \quad \mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\mathbf{a} \quad |\mathbf{A}| = 2(-1) - (-1)(-1) = -3$$

$$\begin{aligned}
 \mathbf{b} \quad \mathbf{A}^2 &= \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} 4+1 & -2+1 \\ -2+1 & 1+1 \end{bmatrix} \\
 &= \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad 2\mathbf{A} &= 2 \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & -2 \\ -2 & -2 \end{bmatrix}
 \end{aligned}$$

$$\therefore |2\mathbf{A}| = 4(-2) - (-2)(-2) = -12$$

$$\therefore |\mathbf{A}^2| = 5(2) - (-1)(-1) = 9$$

$$\begin{aligned}
 \mathbf{5} \quad \text{Let } \mathbf{A} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then } k\mathbf{A} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \quad \text{and } |k\mathbf{A}| = ka(kd) - kb(kc) \\
 &= k^2(ad - bc) \\
 &= k^2|\mathbf{A}|
 \end{aligned}$$

$$6 \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$a \quad |\mathbf{A}| = ad - bc \\ \text{and } |\mathbf{B}| = wz - xy$$

$$b \quad \mathbf{AB} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \\ = \begin{bmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{bmatrix} \\ \therefore |\mathbf{AB}| = (aw + by)(cx + dz) - (ax + bz)(cw + dy)$$

c Expanding brackets,

$$|\mathbf{AB}| = awcx + awdz + bycx + bydz - axcw - axdy - bzcw - bzdy \\ = wz(ad - bc) - xy(ad - bc) \\ = (ad - bc)(wz - xy) \\ = |\mathbf{A}||\mathbf{B}|$$

$$7 \quad \mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$a \quad i \quad |\mathbf{A}| = 1(4) - 2(3) \quad ii \quad |2\mathbf{A}| = 2^2 |\mathbf{A}| \quad iii \quad |-\mathbf{A}| = (-1)^2 |\mathbf{A}| \\ = 4 - 6 \quad = 4(-2) \quad = 1(-2) \\ = -2 \quad = -8 \quad = -2$$

$$iv \quad |\mathbf{B}| = (-1)(1) - 2(0) \quad \therefore |-3\mathbf{B}| = (-3)^2 |\mathbf{B}| \quad v \quad |\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| \\ = -1 \quad = 9(-1) \quad = (-2)(-1) \\ = -9 \quad = 2$$

b Checking:

$$ii \quad 2\mathbf{A} = 2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \quad iii \quad |-\mathbf{A}| = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix} \\ |2\mathbf{A}| = 2(8) - 4(6) = -8 \quad \checkmark \quad = (-1)(-4) - (-2)(-3) = -2 \quad \checkmark$$

$$iv \quad -3\mathbf{B} = -3 \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \quad v \quad \mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 3 & -6 \\ 0 & -3 \end{bmatrix} \quad = \begin{bmatrix} -1 + 0 & 2 + 2 \\ -3 + 0 & 6 + 4 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -3 & 10 \end{bmatrix} \\ |-3\mathbf{B}| = (3)(-3) - (-6)(0) \quad |\mathbf{AB}| = (-1)(10) - 4(-3) = 2 \quad \checkmark \\ = -9 \quad \checkmark$$

$$8 \quad a \quad \begin{bmatrix} 2 & 4 \\ -1 & 5 \end{bmatrix}^{-1} = \frac{1}{2(5) - 4(-1)} \begin{bmatrix} 5 & -4 \\ -(-1) & 2 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 5 & -4 \\ 1 & 2 \end{bmatrix}$$

$$b \quad \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{1(-1) - 0(1)} \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = - \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$c \quad \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}^{-1} \text{ does not exist, since } ad - bc = 2(2) - 4(1) = 0$$

$$d \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{1(1) - 0(0)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e \quad \begin{bmatrix} 3 & 5 \\ -6 & -10 \end{bmatrix}^{-1} \text{ does not exist, since } ad - bc = 3(-10) - 5(-6) = 0$$

$$\mathbf{f} \quad \begin{bmatrix} -1 & 2 \\ 4 & 7 \end{bmatrix}^{-1} = \frac{1}{(-1)(7) - 2(4)} \begin{bmatrix} 7 & -2 \\ -4 & -1 \end{bmatrix} = -\frac{1}{15} \begin{bmatrix} 7 & -2 \\ -4 & -1 \end{bmatrix}$$

$$\mathbf{g} \quad \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}^{-1} = \frac{1}{3(2) - (-1)(4)} \begin{bmatrix} 2 & -4 \\ -(-1) & 3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 2 & -4 \\ 1 & 3 \end{bmatrix}$$

$$\mathbf{h} \quad \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}^{-1} = \frac{1}{(-1)(3) - (-1)(2)} \begin{bmatrix} 3 & -(-1) \\ -2 & -1 \end{bmatrix} = - \begin{bmatrix} 3 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$$

EXERCISE 14I

$$\mathbf{1} \quad \mathbf{a} \quad \left. \begin{array}{l} 3x - y = 8 \\ 2x + 3y = 6 \end{array} \right\} \text{ can be written as } \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$\mathbf{b} \quad \left. \begin{array}{l} 4x - 3y = 11 \\ 3x + 2y = -5 \end{array} \right\} \text{ can be written as } \begin{bmatrix} 4 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$$

$$\mathbf{c} \quad \left. \begin{array}{l} 3a - b = 6 \\ 2a + 7b = -4 \end{array} \right\} \text{ can be written as } \begin{bmatrix} 3 & -1 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

$$\mathbf{2} \quad \mathbf{a} \quad \begin{array}{l} 2x - y = 6 \\ x + 3y = 14 \end{array} \quad \text{In matrix form, the system is: } \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 14 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 18 + 14 \\ -6 + 28 \end{bmatrix} = \begin{bmatrix} \frac{32}{7} \\ \frac{22}{7} \end{bmatrix} \quad \text{and so } x = \frac{32}{7}, \quad y = \frac{22}{7}$$

$$\mathbf{b} \quad \begin{array}{l} 5x - 4y = 5 \\ 2x + 3y = -13 \end{array} \quad \text{In matrix form, the system is: } \begin{bmatrix} 5 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -13 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ -13 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ -13 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{23} \begin{bmatrix} 15 + (-52) \\ -10 + (-65) \end{bmatrix} = \begin{bmatrix} -\frac{37}{23} \\ -\frac{75}{23} \end{bmatrix} \quad \text{and so } x = -\frac{37}{23}, \quad y = -\frac{75}{23}$$

$$\mathbf{c} \quad \begin{array}{l} x - 2y = 7 \\ 5x + 3y = -2 \end{array} \quad \text{In matrix form, the system is: } \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ -2 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 21 + (-4) \\ -35 + (-2) \end{bmatrix} = \begin{bmatrix} \frac{17}{13} \\ -\frac{37}{13} \end{bmatrix} \quad \text{and so } x = \frac{17}{13}, \quad y = -\frac{37}{13}$$

$$\mathbf{d} \quad \begin{array}{l} 3x + 5y = 4 \\ 2x - y = 11 \end{array} \quad \text{In matrix form, the system is: } \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 11 \end{bmatrix} = \frac{1}{-13} \begin{bmatrix} -1 & -5 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 11 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{13} \begin{bmatrix} -4 + (-55) \\ -8 + 33 \end{bmatrix} = \begin{bmatrix} \frac{59}{13} \\ -\frac{25}{13} \end{bmatrix} \quad \text{and so } x = \frac{59}{13}, \quad y = -\frac{25}{13}$$

e $4x - 7y = 8$ In matrix form, the system is: $\begin{bmatrix} 4 & -7 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ 3 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 0 \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} -5 & 7 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -40 + 0 \\ -24 + 0 \end{bmatrix} \quad \text{and so } x = -40, \quad y = -24$$

f $7x + 11y = 18$ In matrix form, the system is: $\begin{bmatrix} 7 & 11 \\ 11 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18 \\ -11 \end{bmatrix}$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 11 & -7 \end{bmatrix}^{-1} \begin{bmatrix} 18 \\ -11 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-170} \begin{bmatrix} -7 & -11 \\ -11 & 7 \end{bmatrix} \begin{bmatrix} 18 \\ -11 \end{bmatrix} = -\frac{1}{170} \begin{bmatrix} -126 + 121 \\ -198 - 77 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{170} \begin{bmatrix} -5 \\ -275 \end{bmatrix} = \begin{bmatrix} \frac{1}{34} \\ \frac{55}{34} \end{bmatrix} \quad \text{and so } x = \frac{1}{34}, \quad y = \frac{55}{34}$$

3 a If $\mathbf{AX} = \mathbf{B}$

then $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$ {pre-mult by $\mathbf{A}^{-1}$ }

$$\therefore \mathbf{IX} = \mathbf{A}^{-1}\mathbf{B}$$

$$\therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

If $\mathbf{XA} = \mathbf{B}$

then $\mathbf{XAA}^{-1} = \mathbf{BA}^{-1}$ {post-mult by $\mathbf{A}^{-1}$ }

$$\therefore \mathbf{XI} = \mathbf{BA}^{-1}$$

$$\therefore \mathbf{X} = \mathbf{BA}^{-1}$$

b i $\mathbf{X} \begin{bmatrix} 1 & 2 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 14 & -5 \\ 22 & 0 \end{bmatrix}$

$$\begin{aligned} \therefore \mathbf{X} &= \begin{bmatrix} 14 & -5 \\ 22 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & -1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 14 & -5 \\ 22 & 0 \end{bmatrix} \frac{1}{-11} \begin{bmatrix} -1 & -2 \\ -5 & 1 \end{bmatrix} \\ &= -\frac{1}{11} \begin{bmatrix} -14 + 25 & -28 - 5 \\ -22 + 0 & -44 + 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 \\ 2 & 4 \end{bmatrix} \end{aligned}$$

ii $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}$

$$\begin{aligned} \therefore \mathbf{X} &= \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} \\ &= \frac{1}{-7} \begin{bmatrix} -1 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} \\ &= -\frac{1}{7} \begin{bmatrix} -1 + (-12) & 3 + (-6) \\ -2 + 4 & 6 + 2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{13}{7} & \frac{3}{7} \\ -\frac{2}{7} & -\frac{8}{7} \end{bmatrix} \end{aligned}$$

4 a i $|\mathbf{A}| = 0$ when $2k - (-6) = 0$ i.e., $k = -3$ i.e., singular when $k = -3$

ii $\mathbf{A} = \begin{bmatrix} k & 1 \\ -6 & 2 \end{bmatrix}$ $\therefore \mathbf{A}^{-1} = \frac{1}{2k+6} \begin{bmatrix} 2 & -1 \\ 6 & k \end{bmatrix}$, provided that $k \neq -3$

b i $|\mathbf{A}| = 0$ when $3k - 0 = 0$ i.e., $k = 0$ i.e., singular when $k = 0$

ii $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 0 & k \end{bmatrix}$ $\therefore \mathbf{A}^{-1} = \frac{1}{3k} \begin{bmatrix} k & 1 \\ 0 & 3 \end{bmatrix}$, provided that $k \neq 0$

c i $|\mathbf{A}| = 0$ when $k(k+1) - 2 = 0$
 i.e., $k^2 + k - 2 = 0$
 $(k-1)(k+2) = 0$
 $\therefore k = 1$ or -2 i.e., singular when $k = 1$ or -2

$$\begin{aligned} \text{ii} \quad \mathbf{A} &= \begin{bmatrix} k+1 & 2 \\ 1 & k \end{bmatrix} \quad \therefore \mathbf{A}^{-1} = \frac{1}{k(k+1)-2} \begin{bmatrix} k & -2 \\ -1 & k+1 \end{bmatrix}, \\ &= \frac{1}{(k+2)(k-1)} \begin{bmatrix} k & -2 \\ -1 & k+1 \end{bmatrix}, \quad \text{provided that } k \neq -2 \text{ or } 1 \end{aligned}$$

5 a $\mathbf{A}$ is 2×3 and $\mathbf{B}$ is 3×2 $\therefore \mathbf{AB}$ is 2×2

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -4 & 6 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+0+2 & 2+0+(-2) \\ 1+(-4)+3 & -2+6+(-3) \end{bmatrix} \\ \therefore \mathbf{AB} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I} \end{aligned}$$

b $\mathbf{B}$ is 3×2 and $\mathbf{A}$ is 2×3 $\therefore \mathbf{BA}$ is 3×3 whereas $\mathbf{AB}$ is 2×2 . Hence $\mathbf{BA} \neq \mathbf{AB}$, and $\mathbf{A}$ and $\mathbf{B}$ are not inverses. The inverse of a matrix $\mathbf{A}$ satisfies $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1}$. This requires that $\mathbf{A}$ has the same number of rows as columns, i.e., that $\mathbf{A}$ is a square matrix.

6 $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$

Since $\mathbf{AXB} = \mathbf{C}$,

$$\begin{aligned} \text{then } \mathbf{A}^{-1}\mathbf{AXB} &= \mathbf{A}^{-1}\mathbf{C} \quad \{\text{premult. by } \mathbf{A}^{-1}\} & \mathbf{XBB}^{-1} &= \mathbf{A}^{-1}\mathbf{CB}^{-1} \quad \{\text{postmult. by } \mathbf{B}^{-1}\} \\ \therefore \mathbf{IXB} &= \mathbf{A}^{-1}\mathbf{C} & \therefore \mathbf{XI} &= \mathbf{A}^{-1}\mathbf{CB}^{-1} \\ \therefore \mathbf{XB} &= \mathbf{A}^{-1}\mathbf{C} & \therefore \mathbf{X} &= \mathbf{A}^{-1}\mathbf{CB}^{-1} \end{aligned}$$

$$\therefore \mathbf{X} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0+(-1) & 3+(-2) \\ 0+2 & 0+4 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \mathbf{X} = \frac{1}{4} \begin{bmatrix} -1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0+1 & 2+1 \\ 0+4 & -4+4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ 1 & 0 \end{bmatrix}$$

7 a i $\begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$
 and $|\mathbf{A}| = -2 - (-12)$
 $= -2 + 12$
 $= 10$

ii As $|\mathbf{A}| \neq 0$, the system has a unique solution

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 11 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} -1 & 3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 11 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 25 \\ -10 \end{bmatrix} \\ &= \begin{bmatrix} \frac{5}{2} \\ -1 \end{bmatrix} \\ \therefore x &= \frac{5}{2}, \quad y = -1 \end{aligned}$$

b i $\begin{bmatrix} 2 & k \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$
 and $|\mathbf{A}| = -2 - 4k$

ii The system has a unique solution if
 $-2 - 4k \neq 0$ i.e., $k \neq -\frac{1}{2}$

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 & k \\ 4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 11 \end{bmatrix} \\ &= \frac{1}{-2-4k} \begin{bmatrix} -1 & -k \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 11 \end{bmatrix} \\ &= \frac{1}{-2-4k} \begin{bmatrix} -8-11k \\ -10 \end{bmatrix} \\ \therefore x &= \frac{8+11k}{2+4k}, \quad y = \frac{5}{1+2k}, \quad k \neq -\frac{1}{2} \end{aligned}$$

is the unique solution

iii When $k = -\frac{1}{2}$, the equations are

$$\begin{cases} 2x - \frac{1}{2}y = 8 \\ 4x - y = 11 \end{cases} \quad \text{i.e.,} \quad \begin{cases} 4x - y = 16 \\ 4x - y = 11 \end{cases}$$

So, we have no solutions (as the lines are parallel and so do not meet).

8 a If $\mathbf{A} = \mathbf{A}^{-1}$, then $\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

b If $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is its own inverse, then $\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} a^2 + b^2 & 2ab \\ 2ab & b^2 + a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{lll} \therefore a^2 + b^2 = 1 & \text{then } a = 0 & \text{or } b = 0 \\ \text{and } 2ab = 0 & \text{and } b^2 = 1 & \text{and } a^2 = 1 \\ \text{If } 2ab = 0, & \therefore a = 0 & \text{and } b = \pm 1 \quad \therefore a = \pm 1 \text{ and } b = 0 \end{array}$$

This gives four possible combinations: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

9 a $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad \therefore \mathbf{A}^{-1} = \frac{1}{2} \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$$\therefore (\mathbf{A}^{-1})^{-1} = \frac{1}{\frac{1}{2}} \begin{bmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \mathbf{A}$$

b If $\mathbf{A}^{-1} = \mathbf{B}$ then $(\mathbf{A}^{-1})^{-1}(\mathbf{A}^{-1}) = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$ and $(\mathbf{A}^{-1})(\mathbf{A}^{-1})^{-1} = \mathbf{B}\mathbf{B}^{-1} = \mathbf{I}$

c We can deduce from **b** that $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$ and $(\mathbf{A}^{-1})^{-1}$ is the inverse of $\mathbf{A}^{-1}$.

10 a $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix}$

i $\mathbf{A}^{-1} = \frac{1}{-3} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

ii $\mathbf{B}^{-1} = \frac{1}{-2} \begin{bmatrix} -3 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$

iii $(\mathbf{AB})^{-1}$

$$= \left(\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 0+2 & 1+(-3) \\ 0+(-2) & 2+3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}^{-1}$$

$$= \frac{1}{6} \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \text{ i.e., } = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

iv $(\mathbf{BA})^{-1}$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 0+2 & 0+(-1) \\ 2+(-6) & 2+3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}^{-1}$$

$$= \frac{1}{6} \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix} \text{ i.e., } = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

v $\mathbf{A}^{-1}\mathbf{B}^{-1}$

$$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{6} + \frac{1}{3} & \frac{1}{6} + 0 \\ \frac{6}{6} + (-\frac{1}{3}) & \frac{2}{6} + 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

vi $\mathbf{B}^{-1}\mathbf{A}^{-1}$

$$= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{6} + \frac{2}{6} & \frac{3}{6} + (-\frac{1}{6}) \\ \frac{1}{3} + 0 & \frac{1}{3} + 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

b Choose appropriate vectors and repeat question **a**.

c The results of **a** and **b** suggest that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and $(\mathbf{BA})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$.

$$\begin{aligned} \mathbf{d} \quad & (\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) \quad \text{and} \quad (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) \\ & = (\mathbf{AB})(\mathbf{AB})^{-1} \quad \{\text{from c}\} \quad = (\mathbf{AB})^{-1}(\mathbf{AB}) \quad \{\text{from c}\} \\ & = \mathbf{I} \quad \quad \quad = \mathbf{I} \end{aligned}$$

$\therefore (\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}$ i.e., $\mathbf{AB}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$ are inverses.

$$11 \quad (k\mathbf{A})\left(\frac{1}{k}\mathbf{A}^{-1}\right) = k \times \frac{1}{k}(\mathbf{AA}^{-1}) = \mathbf{I} \quad \text{also} \quad \left(\frac{1}{k}\mathbf{A}^{-1}\right)(k\mathbf{A}) = \frac{1}{k} \times k(\mathbf{A}^{-1}\mathbf{A}) = \mathbf{I}$$

$\therefore (k\mathbf{A})\left(\frac{1}{k}\mathbf{A}^{-1}\right) = \left(\frac{1}{k}\mathbf{A}^{-1}\right)(k\mathbf{A}) = \mathbf{I}$ i.e., $k\mathbf{A}$ and $\frac{1}{k}\mathbf{A}^{-1}$ are inverses.

$$12 \quad \mathbf{X} = \mathbf{AY} \quad \text{and} \quad \mathbf{Y} = \mathbf{BZ}$$

$$\mathbf{a} \quad \mathbf{X} = \mathbf{AY} = \mathbf{A}(\mathbf{BZ}) = \mathbf{ABZ}$$

$$\mathbf{b} \quad (\mathbf{AB})^{-1}\mathbf{X} = (\mathbf{AB})^{-1}\mathbf{ABZ} \quad \{\text{premult. by } (\mathbf{AB})^{-1}\}$$

$$(\mathbf{AB})^{-1}\mathbf{X} = \mathbf{IZ}$$

$$\therefore \mathbf{Z} = \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{X} \quad \{\text{as } (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}\}$$

13 In each example we premultiply by $\mathbf{A}^{-1}$.

$$\mathbf{a} \quad \mathbf{A}^2 = 4\mathbf{A} - \mathbf{I}$$

$$\therefore \mathbf{A}^{-1}\mathbf{A}^2 = \mathbf{A}^{-1}(4\mathbf{A} - \mathbf{I})$$

$$\therefore \mathbf{A}^{-1}\mathbf{AA} = 4\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}\mathbf{I}$$

$$\therefore \mathbf{IA} = 4\mathbf{I} - \mathbf{A}^{-1}$$

$$\therefore \mathbf{A} - 4\mathbf{I} = -\mathbf{A}^{-1}$$

$$\therefore \mathbf{A}^{-1} = 4\mathbf{I} - \mathbf{A}$$

$$\mathbf{b} \quad 5\mathbf{A} = \mathbf{I} - \mathbf{A}^2$$

$$\therefore \mathbf{A}^{-1}5\mathbf{A} = \mathbf{A}^{-1}(\mathbf{I} - \mathbf{A}^2)$$

$$\therefore 5\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}^{-1}\mathbf{I} - \mathbf{A}^{-1}\mathbf{AA}$$

$$\therefore 5\mathbf{I} = \mathbf{A}^{-1} - \mathbf{IA}$$

$$\therefore 5\mathbf{I} = \mathbf{A}^{-1} - \mathbf{A}$$

$$\therefore \mathbf{A}^{-1} = 5\mathbf{I} + \mathbf{A}$$

$$\mathbf{c} \quad 2\mathbf{I} = 3\mathbf{A}^2 - 4\mathbf{A}$$

$$\therefore \mathbf{A}^{-1}2\mathbf{I} = \mathbf{A}^{-1}3\mathbf{A}^2 - \mathbf{A}^{-1}4\mathbf{A}$$

$$\therefore 2\mathbf{A}^{-1} = 3\mathbf{A}^{-1}\mathbf{AA} - 4\mathbf{A}^{-1}\mathbf{A}$$

$$\therefore 2\mathbf{A}^{-1} = 3\mathbf{IA} - 4\mathbf{I}$$

$$\therefore 2\mathbf{A}^{-1} = 3\mathbf{A} - 4\mathbf{I}$$

$$\therefore \mathbf{A}^{-1} = \frac{3}{2}\mathbf{A} - 2\mathbf{I}$$

$$14 \quad \text{If } \mathbf{A} = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}, \text{ let } \mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$$

$$\therefore \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} = p \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} + q \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 9 + (-4) & 6 + (-2) \\ -6 + 2 & -4 + 1 \end{bmatrix} = \begin{bmatrix} 3p & 2p \\ -2p & -p \end{bmatrix} + \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$$

$$\therefore \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 3p + q & 2p \\ -2p & -p + q \end{bmatrix}$$

$$\therefore 5 = 3p + q \quad \text{and} \quad 4 = 2p$$

$$\therefore p = 2 \quad \text{and} \quad q = -1 \quad \text{and} \quad \mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$$

Checking for consistency:

$$-2p = -2(2) = -4 \quad \checkmark$$

$$-p + q = -2 + (-1) = -3 \quad \checkmark$$

$$\text{Now } \mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$$

$$\therefore \mathbf{A}^{-1}\mathbf{A}^2 = \mathbf{A}^{-1}2\mathbf{A} - \mathbf{A}^{-1}\mathbf{I} \quad \{\text{premultiplying by } \mathbf{A}^{-1}\}$$

$$\therefore \mathbf{A}^{-1}\mathbf{A}\mathbf{A} = 2\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}$$

$$\therefore \mathbf{I}\mathbf{A} = 2\mathbf{I} - \mathbf{A}^{-1}$$

$$\therefore \mathbf{A} = 2\mathbf{I} - \mathbf{A}^{-1}$$

$$\therefore \mathbf{A}^{-1} = 2\mathbf{I} - \mathbf{A}$$

15 If $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$,

$$\text{then } \mathbf{A}^2 = \mathbf{AA}$$

$$= (\mathbf{AB})\mathbf{A}$$

$$= \mathbf{A}(\mathbf{BA}) \quad \{\text{associative rule}\}$$

$$= \mathbf{AB}$$

$$\therefore \mathbf{A}^2 = \mathbf{A}$$

$ab = ac$ implies that $b = c$ for real numbers, but this property does not hold for matrices.

Thus from $\mathbf{AB} = \mathbf{AI} = \mathbf{A}$ it does not follow that $\mathbf{B} = \mathbf{I}$.

16 If $\mathbf{AB} = \mathbf{AC}$

$$\text{then } \mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{AC} \quad \{\text{premultiplying by } \mathbf{A}^{-1}\}$$

$$\therefore \mathbf{IB} = \mathbf{IC}$$

$$\therefore \mathbf{B} = \mathbf{C}$$

i.e., if $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$ if $\mathbf{A}^{-1}$ exists, i.e., if $ad - bc \neq 0$.

17 If $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{A}^3 = \mathbf{I}$

$$\text{then } \mathbf{X}^3 = (\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP})$$

$$= (\mathbf{P}^{-1}\mathbf{A})(\mathbf{PP}^{-1})\mathbf{A}(\mathbf{PP}^{-1})\mathbf{AP} \quad \{\text{associative rule}\}$$

$$= \mathbf{P}^{-1}\mathbf{AIAIAP}$$

$$= \mathbf{P}^{-1}\mathbf{AAAP}$$

$$= \mathbf{P}^{-1}\mathbf{A}^3\mathbf{P}$$

$$= \mathbf{P}^{-1}\mathbf{IP}$$

$$= \mathbf{P}^{-1}\mathbf{P} \quad \text{and so } \mathbf{X}^3 = \mathbf{I}$$

18 If $a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{O}$ and $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$

$$\text{then } a\mathbf{X}^2 + b\mathbf{X} + c\mathbf{I} = a(\mathbf{P}^{-1}\mathbf{AP})(\mathbf{P}^{-1}\mathbf{AP}) + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I}$$

$$= a\mathbf{P}^{-1}\mathbf{A}(\mathbf{PP}^{-1})\mathbf{AP} + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I}$$

$$= a\mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} + b\mathbf{P}^{-1}\mathbf{AP} + c\mathbf{I}$$

$$= \mathbf{P}^{-1}(a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I})\mathbf{P}$$

$$= \mathbf{P}^{-1}\mathbf{OP}$$

$$= \mathbf{O}$$

EXERCISE 14J

1 a

$$\begin{vmatrix} 2 & 3 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 5 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 \\ 0 & 5 \end{vmatrix} + 3 \begin{vmatrix} 1 & -1 \\ 5 & 2 \end{vmatrix} + 0 \begin{vmatrix} -1 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= 2(10 - 0) + 3(2 - 5) + 0$$

$$= 41$$

b

$$\begin{vmatrix} -1 & 2 & -3 \\ 1 & 0 & 0 \\ -1 & 2 & 1 \end{vmatrix} = -1 \begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} + (-3) \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}$$

$$= -1(0 - 0) + 2(0 - 1) - 3(2 - 0)$$

$$= -8$$

$$\begin{aligned} \mathbf{c} \quad \begin{vmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix} &= 2 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} + 3 \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} \\ &= 2(3-2) + 1(4-3) + 3(-1-2) \\ &= 0 \end{aligned}$$

$$\mathbf{d} \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1 \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + 0 \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} + 0 \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 1(6-0) = 6$$

$$\mathbf{e} \quad \begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = 0 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} = 2(0-3) = -6$$

$$\begin{aligned} \mathbf{f} \quad \begin{vmatrix} 4 & 1 & 3 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{vmatrix} &= 4 \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + 3 \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} \\ &= 4(0-2) + 1(-2-1) + 3(-1-0) \\ &= -12 \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad \begin{vmatrix} x & 2 & 9 \\ 3 & 1 & 2 \\ -1 & 0 & x \end{vmatrix} &= x \begin{vmatrix} 1 & 2 \\ 0 & x \end{vmatrix} + 2 \begin{vmatrix} 2 & 3 \\ x & -1 \end{vmatrix} + 9 \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} \\ &= x(x) + 2(-2-3x) + 9(1) \\ &= x^2 - 4 - 6x + 9 \\ &= x^2 - 6x + 5 \\ &= (x-5)(x-1) \end{aligned}$$

The matrix is singular if its determinant is 0
i.e., when $x = 5$ or $x = 1$

b This means that the matrix has an inverse for all x in R , $x \neq 1$ or 5 .

$$\mathbf{3} \quad \mathbf{a} \quad \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = a \begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} + 0 \begin{vmatrix} 0 & 0 \\ c & 0 \end{vmatrix} + 0 \begin{vmatrix} 0 & b \\ 0 & 0 \end{vmatrix} = a(bc-0) = abc$$

$$\begin{aligned} \mathbf{b} \quad \begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix} &= 0 \begin{vmatrix} 0 & z \\ -z & 0 \end{vmatrix} + x \begin{vmatrix} z & -x \\ 0 & -y \end{vmatrix} + y \begin{vmatrix} -x & 0 \\ -y & -z \end{vmatrix} \\ &= x(-zy) + y(xz) \\ &= -xyz + xyz \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} &= a \begin{vmatrix} c & a \\ a & b \end{vmatrix} + b \begin{vmatrix} a & b \\ b & c \end{vmatrix} + c \begin{vmatrix} b & c \\ c & a \end{vmatrix} \\ &= a(cb-a^2) + b(ac-b^2) + c(ba-c^2) \\ &= abc - a^3 + abc - b^3 + abc - c^3 \\ &= 3abc - a^3 - b^3 - c^3 \end{aligned}$$

$$\mathbf{4} \quad \begin{cases} x+2y-3z=5 \\ 2x-y-z=8 \\ kx+y+2z=14 \end{cases} \quad \text{has matrix equation} \quad \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ k & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 14 \end{bmatrix}$$

$\mathbf{A} \qquad \qquad \mathbf{X} \qquad \qquad \mathbf{B}$

$\therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$ has a unique solution if $|\mathbf{A}| \neq 0$.

$$\begin{aligned} \text{Now} \quad \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -1 \\ k & 1 & 2 \end{vmatrix} &= 1 \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} -1 & 2 \\ 2 & k \end{vmatrix} + (-3) \begin{vmatrix} 2 & -1 \\ k & 1 \end{vmatrix} \\ &= 1(-2-1) + 2(-k-4) - 3(2-k) \\ &= -1-2k-8-6-3k \\ &= -15-5k \quad \text{i.e., a unique solution for all } k \text{ provided } k \neq -3. \end{aligned}$$

$$5 \quad \begin{cases} 2x - y - 4z = 8 \\ 3x - ky + z = 1 \\ 5x - y + kz = -2 \end{cases} \quad \text{has matrix equation} \quad \begin{bmatrix} 2 & -1 & -4 \\ 3 & -k & 1 \\ 5 & -1 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ -2 \end{bmatrix}$$

A **X** **B**

This has a unique solution if $|\mathbf{A}| \neq 0$.

$$\begin{aligned} \text{Now } \begin{vmatrix} 2 & -1 & -4 \\ 3 & -k & 1 \\ 5 & -1 & k \end{vmatrix} &= 2 \begin{vmatrix} -k & 1 \\ -1 & k \end{vmatrix} + (-1) \begin{vmatrix} 1 & 3 \\ k & 5 \end{vmatrix} + (-4) \begin{vmatrix} 3 & -k \\ 5 & -1 \end{vmatrix} \\ &= 2(-k^2 - 1) - 1(5 - 3k) - 4(-3 - 5k) \\ &= -2k^2 + 2 - 5 + 3k + 12 - 20k \\ &= -2k^2 - 17k + 9 \\ &= -(2k^2 + 17k - 9) \\ &= -(2k - 1)(k + 9) \end{aligned}$$

$\therefore$ there is a unique solution for all k provided $k \neq \frac{1}{2}$ and $k \neq -9$.

$$6 \quad \text{a} \quad \begin{vmatrix} 1 & k & 3 \\ k & 1 & -1 \\ 3 & 4 & 2 \end{vmatrix} = 7 \quad \therefore \quad 1 \begin{vmatrix} 1 & -1 \\ 4 & 2 \end{vmatrix} + k \begin{vmatrix} -1 & k \\ 2 & 3 \end{vmatrix} + 3 \begin{vmatrix} k & 1 \\ 3 & 4 \end{vmatrix} = 7$$

$$\therefore 1(2 - 4) + k(-3 - 2k) + 3(4k - 3) = 7$$

$$\therefore 6 - 3k - 2k^2 + 12k - 9 = 7$$

$$\therefore 2k^2 - 9k + 10 = 0$$

$$\therefore (2k - 5)(k - 2) = 0$$

and so $k = \frac{5}{2}$ or 2

$$\text{b} \quad \begin{vmatrix} k & 2 & 1 \\ 2 & k & 2 \\ 1 & 2 & k \end{vmatrix} = 0$$

$$\therefore k \begin{vmatrix} 2 & 2 \\ k & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 2 \\ k & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & k \\ 1 & 2 \end{vmatrix} = 0$$

$$\therefore k(k^2 - 4) + 2(2 - 2k) + (4 - k) = 0$$

$$\therefore k^3 - 4k + 4 - 4k + 4 - k = 0$$

$$\therefore k^3 - 9k + 8 = 0$$

Using technology there is one rational zero, $k = 1$

$$\therefore (k - 1)(k^2 + k - 8) = 0$$

$$\therefore k = 1 \quad \text{or} \quad k = \frac{-1 \pm \sqrt{1 - 4(1)(-8)}}{2} = \frac{-1 \pm \sqrt{33}}{2}$$

7 Using technology,

$$\text{a} \quad \begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 5 \end{vmatrix} = 16$$

$$\text{b} \quad \begin{vmatrix} 1 & 2 & 3 & 4 & 6 \\ 2 & 3 & 4 & 5 & 0 \\ 1 & 2 & 0 & 1 & 4 \\ 2 & 1 & 0 & 1 & 5 \\ 3 & 0 & 1 & 2 & 1 \end{vmatrix} = -34$$

$$\text{inverse} = \begin{bmatrix} -\frac{21}{16} & -\frac{17}{16} & \frac{5}{4} & \frac{11}{16} \\ -\frac{17}{16} & -\frac{29}{16} & \frac{5}{4} & \frac{15}{16} \\ \frac{5}{4} & \frac{5}{4} & -1 & -\frac{3}{4} \\ \frac{11}{16} & \frac{15}{16} & -\frac{3}{4} & -\frac{5}{16} \end{bmatrix}$$

$$\text{inverse} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -1 & \frac{3}{2} & -\frac{1}{2} \\ -\frac{15}{34} & \frac{1}{2} & -\frac{4}{17} & \frac{29}{34} & -\frac{23}{34} \\ -\frac{29}{34} & \frac{3}{2} & -\frac{61}{17} & \frac{149}{34} & -\frac{83}{34} \\ \frac{39}{34} & -\frac{3}{2} & \frac{58}{17} & -\frac{157}{34} & \frac{87}{34} \\ \frac{1}{17} & 0 & -\frac{4}{17} & \frac{6}{17} & -\frac{3}{17} \end{bmatrix}$$

- 8 a** Let o, a, p, c, l represent (respectively) the cost (in dollars) of each orange, apple, pear, cabbage and lettuce. The system, in matrix form, becomes:

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 3 & 5 & 2 & 2 \end{bmatrix} \begin{bmatrix} o \\ a \\ p \\ c \\ l \end{bmatrix} = \begin{bmatrix} 6.3 \\ 6.7 \\ 7.7 \\ 9.8 \\ 10.9 \end{bmatrix}$$

A **X** **B**

b $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

Using technology, $|\mathbf{A}| = 0$,
 $\therefore \mathbf{A}^{-1}$ does not exist and $\mathbf{X}$
 cannot be found using this
 information.

- c** If the last line is amended, the matrix $\mathbf{A}$ becomes

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \det \mathbf{A} = 6$$

$$\therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 2 & 1 & 1 & 3 \\ 3 & 1 & 2 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6.3 \\ 6.7 \\ 7.7 \\ 9.8 \\ 9.2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.8 \\ 0.7 \\ 2 \\ 1.5 \end{bmatrix} \quad \{\text{using technology}\}$$

$\therefore$ oranges cost 50 cents, apples cost 80 cents, pears cost 70 cents, cabbages cost \$2.00, and
 lettuces cost \$1.50

EXERCISE 14K

1 a $\mathbf{A}^{-1} = \begin{bmatrix} \frac{5}{4} & \frac{3}{4} & -\frac{7}{4} \\ -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{4} & \frac{5}{4} \end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{11}{2} & \frac{9}{2} & \frac{15}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & -3 & -5 \end{bmatrix}$

2 a $\mathbf{B}^{-1} = \begin{bmatrix} 0.050\,23 & -0.011\,48 & -0.066\,34 \\ 4.212 \times 10^{-4} & 0.013\,53 & 0.027\,75 \\ -0.029\,90 & 0.039\,33 & 0.030\,06 \end{bmatrix} \div \begin{bmatrix} 0.050 & -0.011 & -0.066 \\ 0.000 & 0.014 & 0.028 \\ -0.030 & 0.039 & 0.030 \end{bmatrix}$

b $\mathbf{B}^{-1} = \begin{bmatrix} 1.596 & -0.9964 & -0.1686 \\ -3.224 & 1.925 & 0.6291 \\ 2.000 & -1.086 & -0.3958 \end{bmatrix} \div \begin{bmatrix} 1.596 & -0.996 & -0.169 \\ -3.224 & 1.925 & 0.629 \\ 2.000 & -1.086 & -0.396 \end{bmatrix}$

EXERCISE 14L

1 a $\left. \begin{array}{l} x - y - z = 2 \\ x + y + 3z = 7 \\ 9x - y - 3z = -1 \end{array} \right\}$ has matrix equation $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 9 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$

b $\left. \begin{array}{l} 2x + y - z = 3 \\ y + 2z = 6 \\ x - y + z = 13 \end{array} \right\}$ has matrix equation $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 13 \end{bmatrix}$

c $\left. \begin{array}{l} a + b - c = 7 \\ a - b + c = 6 \\ 2a + b - 3c = -2 \end{array} \right\}$ has matrix equation $\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ -2 \end{bmatrix}$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{AB} &= \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{bmatrix} \\
 &= \begin{bmatrix} 8-1-6 & 14-2-12 & -6+1+5 \\ -4-2+6 & -7-4+12 & 3+2-5 \\ 0-6+6 & 0-12+12 & 0+6-5 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \therefore \mathbf{AB} = \mathbf{I} \quad \text{and so} \quad \mathbf{A} = \mathbf{B}^{-1}
 \end{aligned}$$

$$\left. \begin{aligned} 4a + 7b - 3c &= -8 \\ -a - 2b + c &= 3 \\ 6a + 12b - 5c &= -15 \end{aligned} \right\} \text{ has matrix equation } \begin{bmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ -15 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{bmatrix}^{-1} \begin{bmatrix} -8 \\ 3 \\ -15 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 6 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ 3 \\ -15 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -16+3+15 \\ 8+6-15 \\ 0+18-15 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad \therefore a = 2, \quad b = -1, \quad c = 3$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{MN} &= \begin{bmatrix} 5 & 3 & -7 \\ -1 & -3 & 3 \\ -3 & -1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 15+3-14 & 10-3-7 & 15+6-21 \\ -3-3+6 & -2+3+3 & -3-6+9 \\ -9-1+10 & -6+1+5 & -9-2+15 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad \therefore \mathbf{MN} = 4\mathbf{I} \quad \therefore \left(\frac{1}{4}\mathbf{M}\right)\mathbf{N} = \mathbf{I} \quad \text{and so} \quad \frac{1}{4}\mathbf{M} = \mathbf{N}^{-1}
 \end{aligned}$$

$$\text{Now } \left. \begin{aligned} 3u + 2v + 3w &= 18 \\ u - v + 2w &= 6 \\ 2u + v + 3w &= 16 \end{aligned} \right\} \text{ has matrix equation } \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \\ 16 \end{bmatrix}$$

$$\therefore \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 18 \\ 6 \\ 16 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 5 & 3 & -7 \\ -1 & -3 & 3 \\ -3 & -1 & 5 \end{bmatrix} \begin{bmatrix} 18 \\ 6 \\ 16 \end{bmatrix}$$

$$\therefore \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 90+18-112 \\ -18-18+48 \\ -54-6+80 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4 \\ 12 \\ 20 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \quad \therefore u = -1, \quad v = 3, \quad w = 5$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 14 \\ -8 \\ 13 \end{bmatrix} \\
 \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 3 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 14 \\ -8 \\ 13 \end{bmatrix}
 \end{aligned}$$

Using technology, $x = \frac{23}{10}$, $y = \frac{13}{10}$, $z = -\frac{9}{2}$

$$\mathbf{b} \quad \begin{bmatrix} 1 & -1 & -2 \\ 5 & 1 & 2 \\ 3 & -4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 17 \end{bmatrix}$$

Using technology,

$$x = -\frac{1}{3},$$

$$y = -\frac{95}{21},$$

$$z = \frac{2}{21}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 5 & 1 & 2 \\ 3 & -4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -6 \\ 17 \end{bmatrix}$$

$$\mathbf{c} \quad \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 7 \\ 0 \end{bmatrix}$$

Using technology,

$$x = 2,$$

$$y = 4,$$

$$z = -1$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 15 \\ 7 \\ 0 \end{bmatrix}$$

- 5** **a** $x = 2, \quad y = -1, \quad z = 5$ **b** $x = 4, \quad y = -2, \quad z = 1$
 c $x = 4, \quad y = -3, \quad z = 2$ **d** $x = 4, \quad y = 6, \quad z = -7$
 e $x = 3, \quad y = 11, \quad z = -7$ **f** $x = 0.326, \quad y = 7.652, \quad z = 4.156$

- 6** **a** Let x be the cost of a football in dollars,
 y be the cost of baseball in dollars, and
 z be the cost of a basketball in dollars.

Using technology, $x = 14, \quad y = 11, \quad z = 17$.

- b** Cost of 4 footballs and 5 baseballs is $4x + 5y = 4(14) + 5(11) = 111$, i.e., \$111
 $\therefore$ amount left for basketballs is $\$315 - \$111 = \$204$

$$\text{Number of basketballs bought} = \frac{204}{17} = 12 \quad \text{i.e., 12 basketballs are bought.}$$

- 7** **a** System of equations is: $2x + 3y + 8z = 352$
 $x + 5y + 4z = 274$
 $x + 2y + 11z = 351$

- b** Using technology, $x = 42, \quad y = 28, \quad z = 23$,
 i.e., the salaries are: manager \$42 000, clerk \$28 000 and labourer \$23 000.

c Salary bill is $3x + 8y + 37z$
 $= 3(42) + 8(28) + 37(23)$
 $= 1201 \quad (\text{thousands of dollars}) \quad \text{i.e., } \$1\,201\,000$

- 8** Let x be the cost in dollars of 1 kg of cashews,
 y be the cost in dollars of 1 kg of macadamias, and
 z be the cost in dollars of 1 kg of Brazil nuts.

The cost of 1 kg of mix A is $0.5x + 0.3y + 0.2z = 12.5$,

the cost of 1 kg of mix B is $0.2x + 0.4y + 0.4z = 12.4$,

the cost of 1 kg of mix C is $0.6x + 0.1y + 0.3z = 11.7$.

Using technology, $x = 12, \quad y = 15 \quad \text{and} \quad z = 10$

i.e., the cost of 1 kg of cashews is \$12, the cost of 1 kg of macadamias is \$15 and the cost of 1 kg of Brazil nuts is \$10.

Cost per kg of 400 g cashews, 200 g macadamias and 400 g Brazil nuts
 $= 0.4 \times 12 + 0.2 \times 15 + 0.4 \times 10 \quad \text{dollars}$
 $= \$11.80$

9 a Number of students who study Chemistry is $\frac{1}{3}p + \frac{1}{3}q + \frac{2}{5}r = 27$ (1)

number of students who study Maths is $\frac{1}{2}p + \frac{2}{3}q + \frac{1}{5}r = 35$ (2)

number of students who study Geography is $\frac{1}{4}p + \frac{1}{3}q + \frac{3}{5}r = 30$ (3)

The required system of equations is $5p + 5q + 6r = 405$ {(1) $\times$ 15}

$15p + 20q + 6r = 1050$ {(2) $\times$ 30}

$15p + 20q + 36r = 1800$ {(3) $\times$ 60}

b Using technology, $p = 24$, $q = 27$, $r = 25$.

10 a As t is the number of years after 2000, then

profit in year 2000 is $P(0) = b + \frac{c}{4} = 160\,000$

profit in year 2001 is $P(1) = a + b + \frac{c}{5} = 198\,000$

profit in year 2002 is $P(2) = 2a + b + \frac{c}{6} = 240\,000$

Using technology, $a = 50\,000$, $b = 100\,000$ and $c = 240\,000$.

b Using the model given, the profit in 1999 would be

$P(-1) = -a + b + \frac{c}{3} = 50\,000 + 100\,000 + 80\,000 = 130\,000$

i.e., the profit would be \$130 000, which fits the model.

c Predicted profit in 2003 is $P(3) = 3a + b + \frac{c}{7} = 3(50\,000) + 100\,000 + \frac{240\,000}{7}$
 $\div \$284\,000$

Predicted profit in 2005 is $P(5) = 5a + b + \frac{c}{9} = 5(50\,000) + 100\,000 + \frac{240\,000}{9}$
 $\div \$377\,000$

EXERCISE 14M.1

1 a In augmented matrix form, the system is:

$$\begin{bmatrix} 1 & -2 & 8 \\ 4 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 8 \\ 0 & 9 & -27 \end{bmatrix} \leftarrow \text{Replace } R_2 \text{ with } R_2 - 4R_1$$

From R_2 , $9y = -27$

i.e., $y = -3$

Now $x - 2y = 8$

$\therefore x - 2(-3) = 8$

$\therefore x = 2$

So the solution is $x = 2$, $y = -3$.

b In augmented matrix form, the system is:

$$\begin{bmatrix} 4 & 5 & 21 \\ 5 & -3 & -20 \end{bmatrix} \sim \begin{bmatrix} 4 & 5 & 21 \\ 0 & -37 & -185 \end{bmatrix} \leftarrow \text{Replace } R_2 \text{ with } 4R_2 - 5R_1$$

From R_2 , $-37y = -185$

i.e., $y = 5$

Now $4x + 5y = 21$

$\therefore 4x + 25 = 21$

$\therefore 4x = -4$

$\therefore x = -1$

So the solution is $x = -1$, $y = 5$.

c In augmented matrix form, the system is:

$$\begin{bmatrix} 3 & 1 & -10 \\ 2 & 5 & -24 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & -10 \\ 0 & 13 & -52 \end{bmatrix} \leftarrow \text{Replace } R_2 \text{ with } 3R_2 - 2R_1 \quad \begin{array}{ccc} 6 & 15 & -72 \\ -6 & -2 & 20 \\ \hline 0 & 13 & -52 \end{array}$$

From R_2 , $13y = -52$

i.e., $y = -4$

Now $3x + y = -10$

$\therefore 3x + (-4) = -10$

$\therefore 3x = -6$

$\therefore x = -2$

So the solution is $x = -2$, $y = -4$.

2 a One equation is not a multiple of the other and their gradients are not the same, so the lines are intersecting.

b $x + y = 7$ can be written as $3x + 3y = 21$ and the other line is $3x + 3y = 1$,
 $\therefore$ the lines are parallel.

c The lines intersect at $(2\frac{1}{2}, 2)$.

d $x - 2y = 4$ can be written as $2x - 4y = 8$, so the lines are coincident.

e The lines are intersecting.

f $3x - 4y = 5$ can be written as $-3x + 4y = -5$ and the other line is $-3x + 4y = 2$,
 $\therefore$ the lines are parallel.

3 a $x + 2y = 3$ can be written as $2x + 4y = 6$, $\therefore$ the equations represent coincident lines,
i.e., there are an infinite number of solutions (all the points on the line).

b As the second equation is an exact multiple of the first, it will give the same solutions as the first so it can be ignored.

c i If $x = t$, $t + 2y = 3$

$$\therefore 2y = 3 - t$$

$$\therefore y = \frac{3-t}{2}, \text{ i.e., the solutions are } x = t, y = \frac{3-t}{2}, t \in \mathcal{R}.$$

ii If $y = s$, $x + 2s = 3$

$$\therefore x = 3 - 2s, \text{ i.e., the solutions are } x = 3 - 2s, y = s, s \in \mathcal{R}.$$

4 a In augmented matrix form, the system is:

$$\begin{bmatrix} 2 & 3 & 5 \\ 2 & 3 & 11 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix} \leftarrow \text{Replace } R_2 \text{ with } R_2 - R_1 \quad \begin{array}{ccc} 2 & 3 & 11 \\ -2 & -3 & -5 \\ \hline 0 & 0 & 6 \end{array}$$

R_2 shows $0x + 0y = 6$ i.e., there are no solutions {as the lines are parallel}

b In augmented matrix form, the system is:

$$\begin{bmatrix} 2 & 3 & 5 \\ 4 & 6 & 10 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{Replace } R_2 \text{ with } R_2 - 2R_1 \quad \begin{array}{ccc} 4 & 6 & 10 \\ -4 & -6 & -10 \\ \hline 0 & 0 & 0 \end{array}$$

R_2 shows $0x + 0y = 0$, which is true for all x and y

i.e., there are infinitely many solutions {as the lines are coincident}

- 5 a** In augmented matrix form, the system is:

$$\begin{bmatrix} 3 & -1 & 2 \\ 6 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{Replace } R_2 \text{ by } R_2 - 2R_1$$

$$\begin{array}{ccc} 6 & -2 & 4 \\ -6 & 2 & -4 \\ \hline 0 & 0 & 0 \end{array}$$

R_2 shows $0x + 0y = 0$, which is true for all x and y

i.e., there are infinitely many solutions {as the lines are coincident}

Substitute $x = t$ in the first equation $3x - y = 2$

$$\therefore 3t - y = 2$$

$$y = 3t - 2$$

i.e., the solutions have form $x = t$, $y = 3t - 2$, t in $\mathcal{R}$.

- b** $3x - y = 2$ (1)
 $6x - 2y = k$ (2)

If $k = 4$ then $6x - 2y = 4$, which is an exact multiple ($\times 2$) of equation (1), $\therefore$ the lines are coincident and there are an infinite number of solutions of the form $x = t$, $y = 3t - 2$.

If $k \neq 4$ then the equations represent parallel lines $\therefore$ there are no solutions.

- 6 a** In augmented matrix form, the system is:

$$\begin{bmatrix} 3 & -1 & 8 \\ 6 & -2 & k \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & 8 \\ 0 & 0 & k - 16 \end{bmatrix} \leftarrow \text{Replace } R_2 \text{ by } R_2 - 2R_1$$

$$\begin{array}{ccc} 6 & -2 & k \\ -6 & 2 & -16 \\ \hline 0 & 0 & k - 16 \end{array}$$

- b** If $k - 16 = 0$, i.e., $k = 16$, there are infinitely many solutions.

- c** Substitute $x = t$ in $3x - y = 8$, then $3t - y = 8$ i.e., $y = 3t - 8$.
 The solutions are $x = t$, $y = 3t - 8$, t in $\mathcal{R}$.

- d** The system has no solutions when $k - 16 \neq 0$, i.e., $k \neq 16$.

- 7 a** In augmented matrix form, the system is:

$$\begin{bmatrix} 4 & 8 & 1 \\ 2 & -a & 11 \end{bmatrix} \sim \begin{bmatrix} 4 & 8 & 1 \\ 0 & -2a - 8 & 21 \end{bmatrix} \leftarrow \text{Replace } R_2 \text{ by } 2R_2 - R_1$$

$$\begin{array}{ccc} 4 & -2a & 22 \\ -4 & -8 & -1 \\ \hline 0 & -2a - 8 & 21 \end{array}$$

- b** A unique solution exists provided $-2a - 8 \neq 0$ i.e., provided $a \neq -4$.

- c** From R_2 , $(-2a - 8)y = 21$
- $$\therefore y = \frac{-21}{2a + 8}$$
- $$\therefore 4x(2a + 8) - 168 = 2a + 8$$
- and $4x + 8y = 1$
- $$\therefore 2x(2a + 8) - 84 = a + 4$$
- $$\therefore 4x + 8\left(\frac{-21}{2a + 8}\right) = 1$$
- $$\therefore 2x(2a + 8) = a + 88$$
- $$\therefore x = \frac{a + 88}{4a + 16}$$

The solution is $x = \frac{a + 88}{4a + 16}$, $y = \frac{-21}{2a + 8}$, $a \neq -4$.

- d** When $a = -4$ there are no solutions as the lines are parallel.

8 In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} m & 2 & 6 \\ 2 & m & 6 \end{array} \right] \sim \left[\begin{array}{cc|c} m & 2 & 6 \\ 0 & m^2 - 4 & 6m - 12 \end{array} \right] \leftarrow \text{Replace } R_2 \text{ by } mR_2 - 2R_1$$

$$\begin{array}{ccc} 2m & m^2 & 6m \\ -2m & -4 & -12 \\ \hline 0 & m^2 - 4 & 6m - 12 \end{array}$$

A unique solution exists provided $m^2 - 4 \neq 0$

i.e., there is a unique solution for all m except $m = \pm 2$.

a In R_2 , $(m^2 - 4)y = 6m - 12$

Substituting in $mx + 2y = 6$

$$\therefore y = \frac{6(m-2)}{(m-2)(m+2)}$$

$$\text{gives } mx + 2\left(\frac{6}{m+2}\right) = 6$$

$$\therefore y = \frac{6}{m+2} \text{ provided } m \neq \pm 2 \quad \therefore m(m+2)x + 12 = 6(m+2)$$

$$\therefore m(m+2)x = 6m + 12 - 12$$

$$\therefore m(m+2)x = 6m$$

$$\therefore x = \frac{6}{m+2}$$

i.e., the unique solution is $x = \frac{6}{m+2}$, $y = \frac{6}{m+2}$ when $x \neq \pm 2$.

b When $m = 2$, the equations are $2x + 2y = 6$ and $2x + 2y = 6$, i.e., the lines are coincident so there are an infinite number of solutions of the form $x = t$, $y = \frac{6-2t}{2} = 3-t$ for all t in $\mathcal{R}$.

When $m = -2$, the equations are $-2x + 2y = 6$ and $2x - 2y = 6$

i.e., $-2x + 2y = -6$

$\therefore$ the lines are parallel and there are no solutions.

EXERCISE 14M.2

1 a In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & 5 \\ 2 & 3 & 1 & 6 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & -7 \\ 0 & 1 & -1 & -6 \end{array} \right] \begin{array}{l} \leftarrow R_2 \rightarrow R_2 - 2R_1 \\ \leftarrow R_3 \rightarrow R_3 - 2R_1 \end{array}$$

$$\begin{array}{ccc} 2 & 4 & 1 & 5 \\ -2 & -2 & -2 & -12 \\ \hline 0 & 2 & -1 & -7 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & -1 & -5 \end{array} \right] \leftarrow R_3 \rightarrow 2R_3 - R_2$$

$$\begin{array}{ccc} 2 & 3 & 1 & 6 \\ -2 & -2 & -2 & -12 \\ \hline 0 & 1 & -1 & -6 \end{array}$$

$$\begin{array}{ccc} 0 & 2 & -2 & -12 \\ 0 & -2 & 1 & 7 \\ \hline 0 & 0 & -1 & -5 \end{array}$$

The last row gives $-z = -5$ i.e., $z = 5$

$\therefore$ in row 2, $2y - z = -7$

and in row 1, $x + y + z = 6$

$$\therefore 2y - 5 = -7$$

$$\therefore x - 1 + 5 = 6$$

$$\therefore 2y = -2$$

$$\therefore x + 4 = 6$$

$$\therefore y = -1$$

$$\therefore x = 2$$

Thus we have a unique solution $x = 2$, $y = -1$, $z = 5$.

b In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 1 & 6 & 17 & 9 \\ 1 & 4 & 8 & 4 \end{array} \right] \quad \begin{array}{cccc} 1 & 6 & 17 & 9 \\ -1 & -4 & -11 & -7 \\ \hline 0 & 2 & 6 & 2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 0 & 2 & 6 & 2 \\ 0 & 0 & -3 & -3 \end{array} \right] \begin{array}{l} \leftarrow R_2 \rightarrow R_2 - R_1 \\ \leftarrow R_3 \rightarrow R_3 - R_1 \end{array} \quad \begin{array}{cccc} 1 & 4 & 8 & 4 \\ -1 & -4 & -11 & -7 \\ \hline 0 & 0 & -3 & -3 \end{array}$$

The last row gives $-3z = -3 \quad \therefore z = 1$

$\therefore$ in row 2, $2y + 6z = 2$

and in row 1, $x + 4y + 11z = 7$

$\therefore 2y + 6 = 2$

$\therefore x + 4(-2) + 11(1) = 7$

$\therefore y = -2$

$\therefore x + 3 = 7$

$\therefore x = 4$

Thus we have a unique solution $x = 4, y = -2, z = 1$

c In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 2 & -2 & -5 & 4 \\ 3 & 2 & 2 & 10 \end{array} \right] \quad \begin{array}{cccc} 2 & -2 & -5 & 4 \\ -2 & 1 & -3 & -17 \\ \hline 0 & -1 & -8 & -13 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & -1 & -8 & -13 \\ 0 & 7 & -5 & -31 \end{array} \right] \begin{array}{l} \leftarrow R_2 \rightarrow R_2 - R_1 \\ \leftarrow R_3 \rightarrow 2R_3 - 3R_1 \end{array} \quad \begin{array}{cccc} 6 & 4 & 4 & 20 \\ -6 & 3 & -9 & -51 \\ \hline 0 & 7 & -5 & -31 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 3 & 17 \\ 0 & -1 & -8 & -13 \\ 0 & 0 & -61 & -122 \end{array} \right] \leftarrow R_3 \rightarrow R_3 + 7R_2 \quad \begin{array}{cccc} 0 & 7 & -5 & -31 \\ 0 & -7 & -56 & -91 \\ \hline 0 & 0 & -61 & -122 \end{array}$$

The last row gives $-61z = -122$

$\therefore z = 2$

$\therefore$ in row 2, $-y - 8z = -13$

and in row 1, $2x - y + 3z = 17$

$\therefore -y - 16 = -13$

$\therefore 2x + 3 + 6 = 17$

$\therefore y = -3$

$\therefore 2x = 8$

$\therefore x = 4$

Thus we have a unique solution $x = 4, y = -3, z = 2$.

EXERCISE 14M.3

1 a In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 2 & -4 & 8 & 2 \\ -3 & 6 & 7 & -3 \end{array} \right] \quad \begin{array}{cccc} 2 & -4 & 8 & 2 \\ -2 & 4 & -10 & -2 \\ \hline 0 & 0 & -2 & 0 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 5 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 22 & 0 \end{array} \right] \begin{array}{l} \leftarrow R_2 \rightarrow R_2 - 2R_1 \\ \leftarrow R_3 \rightarrow R_3 + 3R_1 \end{array} \quad \begin{array}{cccc} -3 & 6 & 7 & -3 \\ 3 & -6 & 15 & 3 \\ \hline 0 & 0 & 22 & 0 \end{array}$$

Rows 2 and 3 show $-2z = 0$ and $22z = 0$ i.e., $z = 0$

Row 1 becomes $x - 2y + 5(0) = 1$

let $y = t$, then $x - 2t = 1$

i.e., $x = 1 + 2t$

$\therefore$ there are infinitely many solutions of the form $x = 1 + 2t, y = t, z = 0$ (t is in $\mathcal{R}$)

b In augmented matrix form, the system is:

$$\begin{aligned}
 & \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 3 & 2 & 1 & 7 \\ 5 & 2 & 3 & 11 \end{array} \right] & \begin{array}{cccc} 3 & 2 & 1 & 7 \\ -3 & -6 & 3 & -12 \\ \hline 0 & -4 & 4 & -5 \end{array} \\
 & \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -4 & 4 & -5 \\ 0 & -8 & 8 & -9 \end{array} \right] \begin{array}{l} \leftarrow R_2 \rightarrow R_2 - 3R_1 \\ \leftarrow R_3 \rightarrow R_3 - 5R_1 \end{array} & \begin{array}{cccc} 5 & 2 & 3 & 11 \\ -5 & -10 & 5 & -20 \\ \hline 0 & -8 & 8 & -9 \end{array} \\
 & \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -4 & 4 & -5 \\ 0 & 0 & 0 & 1 \end{array} \right] \leftarrow R_3 \rightarrow R_3 - 2R_2 & \begin{array}{cccc} 0 & -8 & 8 & -9 \\ 0 & 8 & -8 & 10 \\ \hline 0 & 0 & 0 & 1 \end{array}
 \end{aligned}$$

In row 3, $0z = 1$ which has no solution. $\therefore$ the system has no real solutions.

c In augmented matrix form, the system is:

$$\begin{aligned}
 & \left[\begin{array}{ccc|c} 2 & 4 & 1 & 1 \\ 3 & 5 & 0 & 1 \\ 5 & 13 & 7 & 4 \end{array} \right] & \begin{array}{cccc} 6 & 10 & 0 & 2 \\ -6 & -12 & -3 & -3 \\ \hline 0 & -2 & -3 & -1 \end{array} \\
 & \sim \left[\begin{array}{ccc|c} 2 & 4 & 1 & 1 \\ 0 & -2 & -3 & -1 \\ 0 & 6 & 9 & 3 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 - 3R_1 \\ R_3 \rightarrow 2R_3 - 5R_1 \end{array} & \begin{array}{cccc} 10 & 26 & 14 & 8 \\ -10 & -20 & -5 & -5 \\ \hline 0 & 6 & 9 & 3 \end{array} \\
 & \sim \left[\begin{array}{ccc|c} 2 & 4 & 1 & 1 \\ 0 & -2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] R_3 \rightarrow R_3 + 3R_2 & \begin{array}{cccc} 0 & 6 & 9 & 3 \\ 0 & -6 & -9 & -3 \\ \hline 0 & 0 & 0 & 0 \end{array}
 \end{aligned}$$

The row of zeros indicates infinitely many solutions.

If we let $z = t$ in row 2, $-2y - 3t = -1$

$$\therefore 2y = 1 - 3t$$

$$\therefore y = \frac{1 - 3t}{2}$$

Thus in equation 1, $2x + 4\left(\frac{1 - 3t}{2}\right) + t = 1$

$$\therefore 2x + 2(1 - 3t) + t = 1$$

$$\therefore 2x + 2 - 6t + t = 1$$

$$\therefore 2x = 5t - 1 \quad \text{and so} \quad x = \frac{5t - 1}{2}$$

$\therefore$ the solutions have form $x = \frac{5t - 1}{2}$, $y = \frac{1 - 3t}{2}$, $z = t$, for all t in $\mathcal{R}$.

d In augmented matrix form, the system is:

$$\begin{aligned}
 & \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 5 & 6 & 7 & 2 \\ 8 & 9 & 10 & 4 \end{array} \right] & \begin{array}{cccc} 10 & 12 & 14 & 4 \\ -10 & -15 & -20 & -5 \\ \hline 0 & -3 & -6 & -1 \end{array} \\
 & \sim \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & -3 & -6 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 - 5R_1 \\ R_3 \rightarrow R_3 - 4R_1 \end{array} & \begin{array}{cccc} 8 & 9 & 10 & 4 \\ -8 & -12 & -16 & -4 \\ \hline 0 & -3 & -6 & 0 \end{array} \\
 & \sim \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & 0 & 0 & 1 \end{array} \right] R_3 \rightarrow R_3 - R_2 & &
 \end{aligned}$$

In row 3, $0z = 1$ which has no solution. $\therefore$ the system has no real solutions.

2 In augmented matrix form, the system is $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & 4 & 1 \\ 1 & 7 & -1 & k \end{array} \right]$

$$\begin{aligned} \mathbf{a} \quad & \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & -1 & 4 & 1 \\ 1 & 7 & -1 & k \end{array} \right] & \begin{array}{cccc} 2 & -1 & 4 & 1 \\ -2 & -4 & -2 & -6 \\ \hline 0 & -5 & 2 & -5 \end{array} \\ \sim & \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & 2 & -5 \\ 0 & 5 & -2 & k-3 \end{array} \right] & \begin{array}{cccc} & & & \\ R_2 \rightarrow R_2 - 2R_1 & & & \\ R_3 \rightarrow R_3 - R_1 & & & \\ \hline 1 & 7 & -1 & k \\ -1 & -2 & -1 & -3 \\ \hline 0 & 5 & -2 & k-3 \end{array} \\ \sim & \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 0 & k-8 \end{array} \right] & \begin{array}{cccc} & & & \\ & & & \\ R_3 \rightarrow R_3 + R_2 & & & \end{array} \end{aligned}$$

- b** If $k - 8 \neq 0$, we have $0z \neq 0$, so the system is inconsistent and there are no solutions.
 If $k - 8 = 0$, we have $0z = 0$, which is true for all z , so we have infinitely many solutions.

Let $z = t$ in row 2, then $-5y + 2t = -5 \quad \therefore 5y = 2t + 5$
 $\therefore y = \frac{2t + 5}{5}$

and substituting in row 1, $x + 2\left(\frac{2t + 5}{5}\right) + t = 3$

$$\therefore x + \frac{4}{5}t + 2 + t = 3$$

$$\therefore x = 1 - \frac{9}{5}t = \frac{5 - 9t}{5}$$

$\therefore$ the solutions have form $x = \frac{5 - 9t}{5}$, $y = \frac{2t + 5}{5}$, $z = t$, t in $\mathcal{R}$.

- c** There is no unique solution because the system reduces to 2 equations in 3 unknowns when $k = 8$.

3 a In augmented matrix form, the system is:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 1 & -1 & 3 & -1 \\ 1 & -7 & k & -k \end{array} \right] & \begin{array}{cccc} 1 & -1 & 3 & -1 \\ -1 & -2 & 2 & -5 \\ \hline 0 & -3 & 5 & -6 \end{array} \\ \sim & \left[\begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 0 & -3 & 5 & -6 \\ 0 & -9 & k+2 & -k-5 \end{array} \right] & \begin{array}{cccc} & & & \\ R_2 \rightarrow R_2 - R_1 & & & \\ R_3 \rightarrow R_3 - R_1 & & & \\ \hline 1 & -7 & k & -k \\ -1 & -2 & 2 & -5 \\ \hline 0 & -9 & k+2 & -k-5 \end{array} \\ \sim & \left[\begin{array}{ccc|c} 1 & 2 & -2 & 5 \\ 0 & -3 & 5 & -6 \\ 0 & 0 & k-13 & 13-k \end{array} \right] & \begin{array}{cccc} & & & \\ & & & \\ R_3 \rightarrow R_3 + 3R_2 & & & \\ \hline 0 & -9 & k+2 & -k-5 \\ 0 & 9 & -15 & 18 \\ \hline 0 & 0 & k-13 & 13-k \end{array} \end{aligned}$$

- b** If $k = 13$, row 3 is a row of zeros, so there are infinitely many solutions.

Let $z = t$ in row 2, then $-3y + 5t = -6 \quad \therefore 3y = 6 + 5t$
 $\therefore y = \frac{6 + 5t}{3}$

and substituting in row 1 gives $x + 2\left(\frac{6 + 5t}{3}\right) - 2t = 5$

$$\therefore x + 4 + \frac{10}{3}t - 2t = 5$$

$$\therefore x = 1 - \frac{4}{3}t = \frac{3 - 4t}{3}$$

i.e., there are infinitely many solutions of the form $x = \frac{3 - 4t}{3}$, $y = \frac{6 + 5t}{3}$, $z = t$, t in $\mathcal{R}$.

c If $k \neq 13$, then $(k-13)z = 13-k \quad \therefore z = \frac{13-k}{k-13} = -1$

From row 2, $-3y + 5(-1) = -6 \quad \therefore -3y - 5 = -6 \quad \therefore y = \frac{1}{3}$

and from row 1, $x + 2y - 2z = x + \frac{2}{3} + 2 = 5 \quad \therefore x = 2\frac{1}{3}$

i.e., the unique solution is $x = \frac{7}{3}$, $y = \frac{1}{3}$, $z = -1$.

4 a In augmented matrix form, the system is:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 2 & -1 & 1 & 7 \\ 3 & -5 & a & 16 \end{array} \right] & \begin{array}{cccc} 2 & -1 & 1 & 7 \\ -2 & -6 & -6 & -2a+2 \\ \hline 0 & -7 & -5 & 9-2a \end{array} \\ \sim & \left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & -14 & a-9 & 19-3a \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} & \begin{array}{cccc} 3 & -5 & a & 16 \\ -3 & -9 & -9 & -3a+3 \\ \hline 0 & -14 & a-9 & 19-3a \end{array} \\ \sim & \left[\begin{array}{ccc|c} 1 & 3 & 3 & a-1 \\ 0 & -7 & -5 & 9-2a \\ 0 & 0 & a+1 & a+1 \end{array} \right] \begin{array}{l} \\ \\ R_3 \rightarrow R_3 - 2R_2 \end{array} & \begin{array}{cccc} 0 & -14 & a-9 & 19-3a \\ 0 & 14 & 10 & 4a-18 \\ \hline 0 & 0 & a+1 & a+1 \end{array} \end{aligned}$$

b If $a = -1$, row 3 is a row of zeros, so there are infinitely many solutions, as we have 2 equations in 3 unknowns.

Let $z = t$ in row 2, then $-7y - 5t = 9 - 2(-1) \quad \therefore -7y - 5t = 11$
 $\therefore y = \frac{-5t-11}{7}$

and substituting in row 1 gives $x + 3\left(\frac{-5t-11}{7}\right) + 3t = (-1) - 1$

$$\therefore x - \frac{15t}{7} - \frac{33}{7} + 3t = -2 \text{ and so, } x = \frac{19}{7} - \frac{6t}{7}$$

i.e., there are infinitely many solutions of form $x = \frac{19-6t}{7}$, $y = \frac{-5t-11}{7}$, $z = t$, t in $\mathcal{R}$.

c If $a \neq -1$, then $(a+1)z = a+1 \quad \therefore z = 1$

From row 2, $-7y - 5(1) = 9 - 2a \quad \therefore -7y = -2a + 14$

$$y = \frac{2a-14}{7}$$

and substituting in row 1 gives $x + 3\left(\frac{2a-14}{7}\right) + 3 = a - 1$

$$\therefore x + \frac{6a}{7} - 6 + 3 = a - 1 \quad \therefore x = \frac{a}{7} + 2 = \frac{a+14}{7}$$

i.e., the unique solution is $x = \frac{a+14}{7}$, $y = \frac{2a-14}{7}$, $z = 1$.

5 In augmented matrix form, the system is:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ m & -2 & 1 & 1 \\ 1 & 2 & m & -1 \end{array} \right] & \begin{array}{cccc} 2m & -4 & 2 & 2 \\ -2m & -m & m & -3m \\ \hline 0 & -m-4 & m+2 & -3m+2 \end{array} \\ \sim & \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & -m-4 & m+2 & -3m+2 \\ 0 & 3 & 2m+1 & -5 \end{array} \right] \begin{array}{l} \\ R_2 \rightarrow 2R_2 - mR_1 \\ R_3 \rightarrow 2R_3 - R_1 \end{array} & \begin{array}{cccc} 2 & 4 & 2m & -2 \\ -2 & -1 & 1 & -3 \\ \hline 0 & 3 & 2m+1 & -5 \end{array} \\ \sim & \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & -m-4 & m+2 & -3m+2 \\ 0 & 0 & 2(m+1)(m+5) & -14(m+1) \end{array} \right] & R_3 \rightarrow (m+4)R_3 + 3R_2 \end{aligned}$$

$$\begin{array}{rcll}
 0 & 3m+12 & (m+4)(2m+1) & -5(m+4) \\
 0 & -3m-12 & 3(m+2) & -9m+6 \\
 \hline
 0 & 0 & 2m^2+12m+10 & -14m-14 \\
 & & = 2(m^2+6m+5) & = -14(m+1) \\
 & & = 2(m+1)(m+5) &
 \end{array}$$

- a** If $m = -5$, row 3 becomes $0x + 0y + 0z = -56$, i.e., the system is inconsistent and there are no solutions.
- b** If $m = -1$, row 3 is a row of zeros, so we have 2 equations in 3 unknowns and $\therefore$ there are infinitely many solutions.
- c** If $m \neq -1$ and $m \neq -5$, then row 3 becomes

$$2(m+5)(m+1)z = -14(m+1)$$

$$\therefore z = \frac{-7}{m+5} \quad \text{and substituting in row 2 gives}$$

$$-(m+4)y + (m+2)\left(\frac{-7}{m+5}\right) = -3m+2$$

$$\therefore -(m+4)(m+5)y - 7(m+2) = (-3m+2)(m+5)$$

$$\therefore -(m+4)(m+5)y = -3m^2 - 13m + 10 + 7m + 14$$

$$\therefore -(m+4)(m+5)y = -3m^2 - 6m + 24$$

$$\therefore -(m+4)(m+5)y = -3(m^2 + 2m - 8)$$

$$\therefore (m+4)(m+5)y = 3(m-2)(m+4)$$

$$\therefore y = \frac{3(m-2)}{m+5}$$

$$\text{and substituting in row 1 gives} \quad 2x + \frac{3(m-2)}{m+5} - \frac{-7}{m+5} = 3$$

$$\therefore 2x(m+5) + 3(m-2) + 7 = 3(m+5)$$

$$\therefore 2x(m+5) + 3m - 6 + 7 = 3m + 15$$

$$\therefore 2x(m+5) = 14$$

$$\therefore x = \frac{7}{m+5}$$

$\therefore$ the system has a unique solution for all m except $m = -5$ and $m = -1$, and the solution is $x = \frac{7}{m+5}$, $y = \frac{3(m-2)}{m+5}$, $z = \frac{-7}{m+5}$.

- 6 a** In augmented matrix form, the system is:

$$\begin{array}{l}
 \left[\begin{array}{ccc|c} 1 & 3 & k & 2 \\ k & -2 & 3 & k \\ 4 & -3 & 10 & 5 \end{array} \right] \quad \begin{array}{ccc|c} k & -2 & 3 & k \\ -k & -3k & -k^2 & -2k \\ \hline 0 & -2-3k & 3-k^2 & -k \end{array} \\
 \sim \left[\begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -2-3k & 3-k^2 & -k \\ 0 & -15 & 10-4k & -3 \end{array} \right] \quad \begin{array}{l} R_2 \rightarrow R_2 - kR_1 \\ R_3 \rightarrow R_3 - 4R_1 \end{array} \quad \begin{array}{ccc|c} 4 & -3 & 10 & 5 \\ -4 & -12 & -4k & -8 \\ \hline 0 & -15 & 10-4k & -3 \end{array} \\
 \sim \left[\begin{array}{ccc|c} 1 & 3 & k & 2 \\ 0 & -2-3k & 3-k^2 & -k \\ 0 & 0 & (3k+25)(k-1) & 6(k-1) \end{array} \right] \quad R_3 \rightarrow (3k+2)R_3 - 15R_2
 \end{array}$$

$$\begin{array}{cccc}
 0 & -15(3k+2) & (3k+2)(10-4k) & -3(3k+2) \\
 0 & 30+45k & 15k^2-45 & 15k \\
 \hline
 0 & 0 & -12k^2+22k+20+15k^2-45 & 6k-6 \\
 & & = 3k^2+22k-25 & = 6(k-1) \\
 & & = (3k+25)(k-1) &
 \end{array}$$

- b** If $k = 1$, row 3 is a row of zeros, so there are infinitely many solutions.

The system becomes $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 2 \\ 0 & -5 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$. Let $z = t$ in row 2, then

$$\begin{aligned}
 -5y + 2t &= -1 \\
 \therefore -5y &= -1 - 2t \\
 \therefore y &= \frac{1+2t}{5}
 \end{aligned}$$

and substituting in row 1 gives $x + 3\left(\frac{1+2t}{5}\right) + t = 2$

$$\begin{aligned}
 \therefore x + \frac{3}{5} + \frac{6}{5}t + t &= 2 \\
 \therefore x &= \frac{7}{5} - \frac{11}{5}t = \frac{7-11t}{5}
 \end{aligned}$$

$\therefore$ the solutions are of the form $x = \frac{7-11t}{5}$, $y = \frac{1+2t}{5}$, $z = t$, for all t in $\mathcal{R}$.

- c** If $k = -\frac{25}{3}$ the system is inconsistent ($0z = -56$) $\therefore$ there are no real solutions.

- d** The system has a unique solution for all $k \neq -1$ or $-\frac{25}{3}$.

EXERCISE 14M.4

- 1 a** In augmented matrix form, the system is:

$$\begin{aligned}
 &\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & -1 & 1 & 3 \end{array} \right] \\
 &\sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -3 & 1 & 1 \end{array} \right] \quad R_2 \rightarrow 2R_2 - R_1 \quad \begin{array}{cccc} 2 & -2 & 2 & 6 \\ -2 & -1 & -1 & -5 \\ \hline 0 & -3 & 1 & 1 \end{array}
 \end{aligned}$$

Let $z = t$ in row 2, then $-3y + t = 1 \quad \therefore -3y = 1 - t \quad \therefore y = \frac{t-1}{3}$

and substituting in equation 1 gives $2x + \left(\frac{t-1}{3}\right) + t = 5$

$$\begin{aligned}
 \therefore 2x + \frac{1}{3}t - \frac{1}{3} + t &= 5 \\
 \therefore 2x &= \frac{16}{3} - \frac{4}{3}t \\
 \therefore x &= \frac{8}{3} - \frac{2}{3}t = \frac{8-2t}{3}
 \end{aligned}$$

$\therefore$ the solutions are $x = \frac{8-2t}{3}$, $y = \frac{t-1}{3}$, $z = t$, t in $\mathcal{R}$.

- b** In augmented matrix form, the system is:

$$\begin{aligned}
 &\left[\begin{array}{ccc|c} 3 & 1 & 2 & 10 \\ 1 & -2 & 1 & -4 \end{array} \right] \\
 &\sim \left[\begin{array}{ccc|c} 3 & 1 & 2 & 10 \\ 0 & -7 & 1 & -22 \end{array} \right] \quad R_2 \rightarrow 3R_2 - R_1 \quad \begin{array}{cccc} 3 & -6 & 3 & -12 \\ -3 & -1 & -2 & -10 \\ \hline 0 & -7 & 1 & -22 \end{array}
 \end{aligned}$$

Let $z = t$ in row 2, then $-7y + t = -22 \quad \therefore -7y = -t - 22$

$$\therefore y = \frac{t+22}{7}$$

and substituting in equation 1 gives $3x + \left(\frac{t+22}{7}\right) + 2t = 10$

$$\therefore 3x + \frac{1}{7}t + \frac{22}{7} + 2t = 10$$

$$\therefore 3x = \frac{48}{7} - \frac{15}{7}t$$

$$\therefore x = \frac{16}{7} - \frac{5}{7}t = \frac{16-5t}{7}$$

$\therefore$ the solutions are $x = \frac{16-5t}{7}$, $y = \frac{t+22}{7}$, $z = t$, t in $\mathcal{R}$.

c In augmented matrix form, the system is:

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 2 & 4 & 2 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 0 & 0 & 6 \end{bmatrix} \quad R_2 \rightarrow R_2 - 2R_1 \quad \begin{array}{cccc} 2 & 4 & 2 & 16 \\ -2 & -4 & -2 & -10 \\ \hline 0 & 0 & 0 & 6 \end{array}$$

Row 2 shows $0z = 6$ which has no solution

$\therefore$ the system is inconsistent and has no real solutions.

2 In augmented matrix form, the system is:

$$\begin{bmatrix} 1 & -3 & 1 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 7 & -4 & 0 \end{bmatrix} \quad R_2 \rightarrow R_2 - 2R_1 \quad \begin{array}{cccc} 2 & 1 & -2 & 0 \\ -2 & 6 & -2 & 0 \\ \hline 0 & 7 & -4 & 0 \end{array}$$

Let $z = t$ in row 2, then $7y - 4t = 0 \quad \therefore y = \frac{4}{7}t$

and substituting in equation 1 gives $x - 3\left(\frac{4}{7}t\right) + t = 0 \quad \therefore x = -t + \frac{12}{7}t = \frac{5}{7}t$

$\therefore$ the solution is $x = \frac{5}{7}t$, $y = \frac{4}{7}t$, $z = t$ for t in $\mathcal{R}$

(or $x = 5s$, $y = 4s$, $z = 7s$ for s in $\mathcal{R}$)

To solve the new system, substitute the solution of the first two equations into the third equation.

$$\therefore 3\left(\frac{5}{7}t\right) - \left(\frac{4}{7}t\right) + t = 18 \quad \therefore 15t - 4t + 7t = 126$$

$$\therefore 18t = 126$$

$$\therefore t = 7$$

$\therefore$ the solution is $x = 5$, $y = 4$, $z = 7$

3 In augmented matrix form, the system is:

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & -5 & 3 & 0 \end{bmatrix} \quad R_2 \rightarrow 2R_2 - R_1 \quad \begin{array}{cccc} 2 & -2 & 4 & 0 \\ -2 & -3 & -1 & 0 \\ \hline 0 & -5 & 3 & 0 \end{array}$$

Let $z = t$ in row 2, then $-5y + 3t = 0$

$$\therefore y = \frac{3}{5}t$$

and substituting in equation 1 gives $2x + 3\left(\frac{3}{5}t\right) + t = 0 \quad \therefore 2x = -\frac{14}{5}t$

$$\therefore x = -\frac{7}{5}t$$

$\therefore$ the solution is $x = -\frac{7}{5}t$, $y = \frac{3}{5}t$, $z = t$ for t in $\mathcal{R}$

(or $x = -7s$, $y = 3s$, $z = 5s$ for s in $\mathcal{R}$)

To solve the new system, substitute the solution of the first two equations into the third equation.

$$\begin{aligned}\therefore a\left(-\frac{7}{5}t\right) + \left(\frac{3}{5}t\right) - t &= 0 & \therefore -7at + 3t - 5t &= 0 \\ & & -7at - 2t &= 0 \\ & & \therefore t(7a + 2) &= 0 \\ & & \therefore t = 0 &\text{ or } a = -\frac{2}{7}\end{aligned}$$

If $t = 0$, then the solution is $x = 0$, $y = 0$, $z = 0$.

If $a = -\frac{2}{7}$, then the solution is $x = -\frac{7}{5}t$, $y = \frac{3}{5}t$, $z = t$ for $t \in \mathcal{R}$
(or $x = -7s$, $y = 3s$, $z = 5s$ for $s \in \mathcal{R}$)

- 4 a** $P(x) = ax^2 + bx + c$ in thousands of dollars where x is in thousands.

Profit is $8 \times \$1000$ when 1×1000 items are produced.

$$\therefore P(1) = a + b + c = 8$$

and profit is $17 \times \$1000$ when 4×1000 items are produced

$$\therefore P(4) = 16a + 4b + c = 17$$

$$\text{i.e., } a + b + c = 8 \quad \text{and} \quad 16a + 4b + c = 17$$

- b** If $a = t$, $b = 3 - 5t$, $c = 5 + 4t$,

$$\text{then } a + b + c = t + 3 - 5t + 5 + 4t = 8$$

$$\text{and } 16a + 4b + c = 16t + 4(3 - 5t) + (5 + 4t) = 16t + 12 - 20t + 5 + 4t = 17$$

$\therefore a = t$, $b = 3 - 5t$, $c = 5 + 4t$ are the possible solutions for the system.

- c** Now $P(x) = tx^2 + (3 - 5t)x + (5 + 4t)$

$$\begin{aligned}\text{and } P(2.5) &= t(2.5)^2 + (3 - 5t)(2.5) + 5 + 4t \\ &= 6.25t + 7.5 - 12.5t + 5 + 4t \\ &= -2.25t + 12.5\end{aligned}$$

$$\text{But } P(2.5) = 19.75 \quad \therefore -2.25t + 12.5 = 19.75$$

$$\therefore -2.25t = 7.25$$

$$\therefore t = -\frac{7.25}{2.25} = -\frac{29}{9}$$

$$\therefore a = -\frac{29}{9}, \quad b = 3 - 5\left(-\frac{29}{9}\right) = \frac{172}{9} \quad \text{and} \quad c = 5 + 4\left(-\frac{29}{9}\right) = -\frac{71}{9}$$

$$\therefore P(x) = -\frac{29}{9}x^2 + \frac{172}{9}x - \frac{71}{9}$$

- d** Now $P'(x) = -\frac{58}{9}x + \frac{172}{9} = \frac{1}{9}(-58x + 172)$

Sign diagram for $P'(x)$ is:

$\therefore P(x)$ is a maximum when $x \div 2.966$. Now $P(2.966) \div 20.448$

$\therefore$ maximum profit is approximately \$20 448 when 2966 items are produced.

EXERCISE 14N

$$\begin{aligned}\mathbf{1} \quad \mathbf{a} \quad \mathbf{M} &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} & \mathbf{M}^3 &= \mathbf{M}^2 \mathbf{M} & \mathbf{M}^4 &= \mathbf{M}^3 \mathbf{M} \\ \therefore \mathbf{M}^2 &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} & &= \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} & &= \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} & &= \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} & &= \begin{bmatrix} 1 & 8 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

b P_n is: “If $\mathbf{M} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ then $\mathbf{M}^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$ for $n \in \mathbb{Z}^+$.”

c Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $\mathbf{M}^1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \mathbf{M} \therefore P_1$ is true.

(2) If P_k is true, then $\mathbf{M}^k = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix}$

$$\therefore \mathbf{M}^{k+1} = \mathbf{M}^k \mathbf{M}$$

$$= \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2+2k \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2(k+1) \\ 0 & 1 \end{bmatrix}$$

Thus, P_{k+1} is true whenever P_k is true and P_1 is true.

$\Rightarrow P_n$ is true {Principle of Math. Induction}

2 a $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $\mathbf{A}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ $\mathbf{A}^3 = \mathbf{A}^2 \mathbf{A}$

$$= \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$$

$$\mathbf{A}^4 = \mathbf{A}^3 \mathbf{A}$$

$$= \begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 80 \\ 0 & 81 \end{bmatrix}$$

$$\mathbf{A}^5 = \mathbf{A}^4 \mathbf{A}$$

$$= \begin{bmatrix} 1 & 80 \\ 0 & 81 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 242 \\ 0 & 243 \end{bmatrix}$$

b P_n is “If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ then $\mathbf{A}^n = \begin{bmatrix} 1 & 3^n - 1 \\ 0 & 3^n \end{bmatrix}$, $n \in \mathbb{Z}^+$.”

c Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $\mathbf{A}^1 = \begin{bmatrix} 1 & 3-1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \therefore P_1$ is true.

(2) If P_k is true, then $\mathbf{A}^k = \begin{bmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{bmatrix}$

$$\therefore \mathbf{A}^{k+1} = \mathbf{A}^k \mathbf{A} = \begin{bmatrix} 1 & 3^k - 1 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2+3(3^k-1) \\ 0 & 3^k \times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2+3^{k+1}-3 \\ 0 & 3^{k+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3^{k+1}-1 \\ 0 & 3^{k+1} \end{bmatrix}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\Rightarrow P_n$ is true {Princ. of Math. Induct.}

$$\mathbf{d} \quad \text{If } n = -1, \quad \mathbf{A}^{-1} = \begin{bmatrix} 1 & 3^{-1} - 1 \\ 0 & 3^{-1} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$\text{But } \mathbf{A}^{-1} = \frac{1}{3-0} \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$$

So, the result is true when $n = -1$.

$$\mathbf{3} \quad \mathbf{a} \quad \mathbf{P} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{P}^2 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{P}^3 = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} \quad = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$$

$$\mathbf{P}^4 = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix}$$

$$\mathbf{b} \quad P_n \text{ is "If } \mathbf{P} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \text{ then } \mathbf{P}^n = \begin{bmatrix} n+1 & n \\ -n & 1-n \end{bmatrix}, \quad n \in \mathbb{Z}^+."$$

c Proof: (By the Principle of Mathematical Induction)

$$(1) \quad \text{If } n = 1, \quad \mathbf{P}^1 = \mathbf{P} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \quad \therefore P_1 \text{ is true.}$$

$$(2) \quad \text{If } P_k \text{ is true then } \mathbf{P}^k = \begin{bmatrix} k+1 & k \\ -k & 1-k \end{bmatrix}$$

$$\therefore \mathbf{P}^{k+1} = \mathbf{P}^k \mathbf{P}$$

$$= \begin{bmatrix} k+1 & k \\ -k & 1-k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2k+2-k & k+1+0 \\ -2k-1+k & -k+0 \end{bmatrix}$$

$$= \begin{bmatrix} k+2 & k+1 \\ -k-1 & -k \end{bmatrix}$$

$$= \begin{bmatrix} (k+1)+1 & (k+1) \\ -(k+1) & 1-(k+1) \end{bmatrix}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\therefore P_n$ is true {Princ. of Math. Induc.}

$$\mathbf{4} \quad P_n \text{ is "If } u_1 = u_2 = 1 \text{ and } u_{n+2} = u_n + u_{n+1} \text{ and } \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ then}$$

$$\mathbf{A}^{n+1} = \begin{bmatrix} u_{n+2} & u_{n+1} \\ u_{n+1} & u_n \end{bmatrix} \text{ for } n \in \mathbb{Z}^+."$$

Proof: (By the Principle of Mathematical Induction)

$$(1) \quad \text{If } n = 1, \quad \mathbf{A}^2 = \begin{bmatrix} u_3 & u_2 \\ u_2 & u_1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \therefore P_1 \text{ is true.}$$

$$(2) \quad \text{If } P_k \text{ is true, then } \mathbf{A}^{k+1} = \begin{bmatrix} u_{k+2} & u_{k+1} \\ u_{k+1} & u_k \end{bmatrix}$$

$$\therefore \mathbf{A}^{k+2} = \mathbf{A}^{k+1} \mathbf{A} = \begin{bmatrix} u_{k+2} & u_{k+1} \\ u_{k+1} & u_k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} u_{k+2} + u_{k+1} & u_{k+2} + 0 \\ u_{k+1} + u_k & u_{k+1} + 0 \end{bmatrix}$$

$$= \begin{bmatrix} u_{k+3} & u_{k+2} \\ u_{k+2} & u_{k+1} \end{bmatrix} \quad \text{Thus } P_{k+1} \text{ is true whenever } P_k \text{ is true and } P_1 \text{ is true.}$$

$\therefore P_n \text{ is true } \{\text{Principle of Mathematical Induction}\}$

REVIEW SET 14A

1 a $A + B$

$$= \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}$$

b $3A$

$$= 3 \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 \\ 0 & -3 \end{bmatrix}$$

c $-2B$

$$= -2 \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 4 & -8 \end{bmatrix}$$

d $A - B$

$$= \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & -5 \end{bmatrix}$$

e $B - 2A$

$$= \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 4 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -4 \\ -2 & 6 \end{bmatrix}$$

f $3A - 2B$

$$= 3 \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 6 \\ 4 & -11 \end{bmatrix}$$

g AB

$$= \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 8 \\ 2 & -4 \end{bmatrix}$$

h BA

$$= \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ -6 & -8 \end{bmatrix}$$

i A^{-1}

$$= \frac{1}{-3} \begin{bmatrix} -1 & -2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & -1 \end{bmatrix}$$

j A^2

$$= \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 4 \\ 0 & 1 \end{bmatrix}$$

k ABA

$$= (AB)A$$

$$= \begin{bmatrix} -1 & 8 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -10 \\ 6 & 8 \end{bmatrix}$$

l $(AB)^{-1}$

$$= \begin{bmatrix} -1 & 8 \\ 2 & -4 \end{bmatrix}^{-1}$$

$$= \frac{1}{4 - 16} \begin{bmatrix} -4 & -8 \\ -2 & -1 \end{bmatrix}$$

$$= \frac{1}{-12} \begin{bmatrix} -4 & -8 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{12} \end{bmatrix}$$

2 a Equating corresponding elements,

$$a = -a$$

$$b - 2 = 3$$

$$c = 2 - c \quad \therefore a = 0, \quad b = 5$$

$$d = -4 \quad c = 1, \quad d = -4$$

b Equating corresponding elements,

$$3 + b = a \quad \therefore a = 2$$

$$2a - a = 2 \quad b = -1$$

$$b + c = 2 \quad c = 3$$

$$-2 + d = 6 \quad d = 8$$

3 a $B - Y = A$

$$\therefore -Y = A - B$$

$$\therefore Y = -(A - B)$$

$$\therefore Y = B - A$$

b $2Y + C = D$

$$\therefore 2Y = D - C$$

$$\therefore Y = \frac{1}{2}(D - C)$$

c $AY = B$

$$\therefore A^{-1}AY = A^{-1}B$$

$$\therefore IY = A^{-1}B$$

$$\therefore Y = A^{-1}B$$

d $YB = C$

$$\therefore YBB^{-1} = CB^{-1}$$

$$\therefore YI = CB^{-1}$$

$$\therefore Y = CB^{-1}$$

e $C - AY = B$

$$\therefore -AY = B - C$$

$$\therefore AY = C - B$$

$$\therefore A^{-1}AY = A^{-1}(C - B)$$

$$\therefore Y = A^{-1}(C - B)$$

f $AY^{-1} = B$

$$\therefore A^{-1}AY^{-1} = A^{-1}B$$

$$\therefore Y^{-1} = A^{-1}B$$

$$\therefore (Y^{-1})^{-1} = (A^{-1}B)^{-1}$$

$$\therefore Y = B^{-1}(A^{-1})^{-1}$$

$$\therefore Y = B^{-1}A$$

$$\mathbf{4} \quad \mathbf{a} \quad \begin{aligned} 3x - 4y &= 2 \\ 5x + 2y &= -1 \end{aligned}$$

$$\begin{aligned} \therefore \begin{bmatrix} 3 & -4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ \therefore \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 3 & -4 \\ 5 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \frac{1}{26} \begin{bmatrix} 2 & 4 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &= \frac{1}{26} \begin{bmatrix} 0 \\ -13 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \\ \therefore x &= 0, \quad y = -\frac{1}{2} \end{aligned}$$

$$\mathbf{b} \quad \begin{aligned} 4x - y &= 5 \\ 2x + 3y &= 9 \end{aligned}$$

$$\begin{aligned} \therefore \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 5 \\ 9 \end{bmatrix} \\ \therefore \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 9 \end{bmatrix} \\ &= \frac{1}{14} \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} \\ &= \frac{1}{14} \begin{bmatrix} 24 \\ 26 \end{bmatrix} \\ &= \begin{bmatrix} \frac{12}{7} \\ \frac{13}{7} \end{bmatrix} \\ \therefore x &= \frac{12}{7}, \quad y = \frac{13}{7} \end{aligned}$$

$$\mathbf{c} \quad \mathbf{X} \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix}$$

$$\begin{aligned} \therefore \mathbf{X} &= \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}^{-1} \\ \therefore \mathbf{X} &= \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix} \frac{1}{-1} \begin{bmatrix} 1 & -4 \\ -1 & 3 \end{bmatrix} \\ \therefore \mathbf{X} &= \begin{bmatrix} 5 & 4 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} \\ \therefore \mathbf{X} &= \begin{bmatrix} -1 & 8 \\ -2 & 6 \end{bmatrix} \end{aligned}$$

$$\mathbf{d} \quad \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \mathbf{X} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \therefore \mathbf{X} &= \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ \therefore \mathbf{X} &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\ \therefore \mathbf{X} &= \frac{1}{2} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ \therefore \mathbf{X} &= \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} \end{aligned}$$

$$\mathbf{e} \quad \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{aligned} \therefore \mathbf{X} &= \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\ \therefore \mathbf{X} &= \frac{1}{-3} \begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\ \therefore \mathbf{X} &= \frac{1}{-3} \begin{bmatrix} -14 \\ -1 \end{bmatrix} \\ \therefore \mathbf{X} &= \begin{bmatrix} \frac{14}{3} \\ \frac{1}{3} \end{bmatrix} \end{aligned}$$

$$\mathbf{f} \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{X} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$$

$$\begin{aligned} \therefore \mathbf{X} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\ \therefore \mathbf{X} &= \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix} \frac{1}{-3} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= -\frac{1}{6} \begin{bmatrix} 5 & -2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} \\ &= -\frac{1}{6} \begin{bmatrix} -3 & -9 \\ -9 & 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \end{aligned}$$

5 a $2\mathbf{B}$ **b** $\frac{1}{2}\mathbf{B}$ **c** $\mathbf{AB}$ **d** $\mathbf{B}$ is 3×2 and $\mathbf{A}$ is 1×3 not equal $\therefore \mathbf{BA}$ does not exist.

$$= 2 \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 0 & 2 \\ 6 & 4 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 12 \end{bmatrix}$$

6 a $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{Q} - \mathbf{P}$ **c** $\frac{3}{2}\mathbf{P} - \mathbf{Q}$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 3 \\ \frac{3}{2} & 0 \\ 3 & \frac{9}{2} \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 4 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & 4 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & 3 \\ \frac{1}{2} & -4 \\ 2 & \frac{7}{2} \end{bmatrix}$$

7 In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 1 & 4 & 2 \\ k & 3 & -6 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 4 & 2 \\ 0 & 3-4k & -6-2k \end{array} \right] \quad R_2 \rightarrow R_2 - kR_1$$

$$\begin{array}{ccc} k & 3 & -6 \\ -k & -4k & -2k \\ \hline 0 & 3-4k & -6-2k \end{array}$$

Row 2 gives $(3-4k)y = -6-2k$

$$\text{i.e., } y = \frac{-2(3+k)}{3-4k} \text{ which is not defined if } k = \frac{3}{4}$$

So the system has a unique solution for all $k \neq \frac{3}{4}$, k in $\mathcal{R}$.

$$\text{If } k = \frac{3}{4}, \text{ equation 2 becomes } \frac{3}{4}x + 3y = -6$$

$$\therefore 3x + 12y = -24$$

$$\text{i.e., } x + 4y = -8$$

which is the equation of a line parallel to $x + 4y = 2$ $\therefore$ there are no solutions.

8 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 2 & 3 & -1 & -3 \\ 1 & -2 & 3 & 2 \end{array} \right] \sim \left[\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 0 & 11 & -7 & -15 \\ 0 & -5 & 7 & 3 \end{array} \right] \quad \begin{array}{l} R_2 \rightarrow 3R_2 - 2R_1 \\ R_3 \rightarrow 3R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & -1 & 2 & 3 \\ 0 & 11 & -7 & -15 \\ 0 & 0 & 42 & -42 \end{array} \right] \quad R_3 \rightarrow 11R_3 + 5R_2$$

$$\begin{array}{cccc} 6 & 9 & -3 & -9 \\ -6 & 2 & -4 & -6 \\ \hline 0 & 11 & -7 & -15 \\ 3 & -6 & 9 & 6 \\ -3 & 1 & -2 & -3 \\ \hline 0 & -5 & 7 & 3 \\ 0 & -55 & 77 & 33 \\ 0 & 55 & -35 & -75 \\ \hline 0 & 0 & 42 & -42 \end{array}$$

From row 3, $42z = -42$ $\therefore z = -1$

Substituting in row 2 gives $11y - 7(-1) = -15$ and from row 1, $3x - (-2) + 2(-1) = 3$

$$\therefore 11y + 7 = -15$$

$$\therefore 3x + 2 - 2 = 3$$

$$\therefore 11y = -22$$

$$\therefore x = 1$$

$$\therefore y = -2$$

i.e., $x = 1$, $y = -2$, $z = -1$

9 $x^2 + y^2 + ax + by + c = 0$

- a** $(-2, 4)$ lies on the circle, $\therefore 4 + 16 - 2a + 4b + c = 0$, i.e., $2a - 4b - c = 20$
 $(1, 3)$ lies on the circle, $\therefore 1 + 9 + a + 3b + c = 0$, i.e., $a + 3b + c = -10$

In augmented matrix form, the system is:

$$\begin{bmatrix} 2 & -4 & -1 & 20 \\ 1 & 3 & 1 & -10 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & -1 & 20 \\ 0 & 10 & 3 & -40 \end{bmatrix} \quad R_2 \rightarrow 2R_2 - R_1$$

$$\begin{array}{cccc} 2 & -4 & -1 & 20 \\ -2 & 4 & 1 & -20 \\ \hline 0 & 10 & 3 & -40 \end{array}$$

Let $c = t$ in row 3, then $10b + 3t = -40$ $10b = -3t - 40$ $b = \frac{-3t - 40}{10}$

and substituting in row 1 gives $2a - 4\left(\frac{-3t - 40}{10}\right) - t = 20$

$$\therefore 20a + 12t + 160 - 10t = 200$$

$$\therefore 20a = 40 - 2t \quad \therefore a = \frac{40 - 2t}{20}$$

i.e., $a = 2 - \frac{t}{10}$, $b = -4 - \frac{3t}{10}$, $c = t$ for all t in $\mathcal{R}$.

- b** There are infinitely many solutions as we have 2 equations in 3 unknowns.

- c** If $(2, 2)$ is on the circle then $(2, 2)$ satisfies the equation of the circle.

$$\therefore 2^2 + 2^2 + \left(\frac{20 - t}{10}\right)(2) + \left(\frac{-3t - 40}{10}\right)(2) + t = 0$$

$$\therefore 40 + 40 + 40 - 2t - 6t - 80 + 10t = 0 \quad \therefore 2t = -40$$

$$\therefore t = -20$$

and when $t = -20$, $a = 2 - \left(\frac{-20}{10}\right) = 4$, $b = -4 - \left(\frac{-60}{10}\right) = 2$, $c = -20$

i.e., the equation of the circle is $x^2 + y^2 + 4x + 2y - 20 = 0$

- 10** In augmented matrix form, the system is:

$$\begin{bmatrix} 2 & 3 & -4 & 13 \\ 1 & -1 & 3 & -1 \\ 3 & 7 & -11 & k \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & -1 \\ 2 & 3 & -4 & 13 \\ 3 & 7 & -11 & k \end{bmatrix} \quad \text{interchanging } R_1 \text{ and } R_2$$

$$\begin{array}{cccc} 2 & 3 & -4 & 13 \\ -2 & 2 & -6 & 2 \\ \hline 0 & 5 & -10 & 15 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & -1 \\ 0 & 5 & -10 & 15 \\ 0 & 10 & -20 & k + 3 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$\begin{array}{cccc} 3 & 7 & -11 & k \\ -3 & 3 & -9 & 3 \\ \hline 0 & 10 & -20 & k + 3 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & -1 \\ 0 & 5 & -10 & 15 \\ 0 & 0 & 0 & k - 27 \end{bmatrix} \quad R_3 \rightarrow R_3 - 2R_2$$

$$\begin{array}{cccc} 0 & 10 & -20 & k + 3 \\ 0 & -10 & 20 & -30 \\ \hline 0 & 0 & 0 & k - 27 \end{array}$$

If $k - 27 = 0$, i.e., $k = 27$ we have a row of zeros.

$\therefore$ we have 2 equations in 3 unknowns which gives an infinite number of solutions.

Let $z = t$ in row 2. then $5y - 10t = 15$ and $x - (3 + 2t) + 3t = -1$
 $\therefore y = 3 + 2t$ $\therefore x - 3 - 2t + 3t = -1$
 $\therefore x = 2 - t$

i.e., if $k = 27$, we have solutions of the form $x = 2 - t$, $y = 3 + 2t$, $z = t$ where t is in $\mathcal{R}$.

If $k \neq 27$, the system is inconsistent and there are no solutions.

11 In augmented matrix form, the system is:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 1 & 1 & 2 & 0 \end{array} \right] \\ & \sim \left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & 2 & 7 & 0 \end{array} \right] \quad R_2 \rightarrow 3R_2 - R_1 \end{aligned} \quad \begin{array}{ccc|c} 3 & 3 & 6 & 0 \\ -3 & -1 & 1 & 0 \\ \hline 0 & 2 & 7 & 0 \end{array}$$

Let $z = t$ in row 2.

then $2y + 7t = 0$ i.e., $y = -\frac{7}{2}t$ and $3x + (-\frac{7}{2}t) - t = 0$

$$\therefore 3x = \frac{9}{2}t$$

$$\therefore x = \frac{3}{2}t$$

$\therefore$ we have infinitely many solutions of the form $x = \frac{3}{2}t$, $y = -\frac{7}{2}t$, $z = t$ where t is in $\mathcal{R}$
(or $x = 3t$, $y = -7t$, $z = 2t$ for t in $\mathcal{R}$).

REVIEW SET 14B

1 $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \therefore a = 1, b = 0$ i.e., $a = 1, b = 0, c = 0, d = 1$
 $\therefore \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad a+c=1, \quad b+d=1 \therefore$ matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$

2 a AB

$$\begin{aligned} &= \begin{bmatrix} 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \\ &= [10] \end{aligned}$$

b BA

$$\begin{aligned} &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

c AC

$$\begin{aligned} &= \begin{bmatrix} 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 15 & 18 & 21 \end{bmatrix} \end{aligned}$$

d CA does not exist as **C** is 3×3 and **A** is 1×3

e CB $= \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix}$

3 a $\mathbf{A}^{-1} = \frac{1}{42-40} \begin{bmatrix} 7 & -8 \\ -5 & 6 \end{bmatrix}$
 $= \begin{bmatrix} \frac{7}{2} & -4 \\ -\frac{5}{2} & 3 \end{bmatrix}$

b $\mathbf{A}^{-1}$ does not exist as $|\mathbf{A}| = -24 - -24 = 0$

c $\mathbf{A}^{-1} = \frac{1}{-3} \begin{bmatrix} -3 & -5 \\ 6 & 11 \end{bmatrix}$
 $= \begin{bmatrix} 1 & \frac{5}{3} \\ -2 & -\frac{11}{3} \end{bmatrix}$

4 $\mathbf{A} = 2\mathbf{A}^{-1}$

a $\mathbf{A}^2 = \mathbf{A} \times \mathbf{A}$
 $= \mathbf{A}(2\mathbf{A}^{-1})$
 $= 2\mathbf{A}\mathbf{A}^{-1}$
 $= 2\mathbf{I}$

b $(\mathbf{A} - \mathbf{I})(\mathbf{A} + 3\mathbf{I}) = (\mathbf{A} - \mathbf{I})\mathbf{A} + (\mathbf{A} - \mathbf{I})3\mathbf{I}$
 $= \mathbf{A}^2 - \mathbf{I}\mathbf{A} + 3\mathbf{A}\mathbf{I} - 3\mathbf{I}^2$
 $= 2\mathbf{I} - \mathbf{A} + 3\mathbf{A} - 3\mathbf{I}$
 $= 2\mathbf{A} - \mathbf{I}$

5 Sales matrix is $\begin{bmatrix} 42-27 & 54-31 \\ 36-28 & 27-15 \\ 34-28 & 30-22 \end{bmatrix} = \begin{bmatrix} 15 & 23 \\ 8 & 12 \\ 6 & 8 \end{bmatrix}$ Totals matrix is $\begin{bmatrix} 38 \\ 20 \\ 14 \end{bmatrix}$

$\therefore$ total profit $= \begin{bmatrix} 0.75 & 0.55 & 1.20 \end{bmatrix} \begin{bmatrix} 38 \\ 20 \\ 14 \end{bmatrix} = [56.3]$ dollars i.e., \$56.30

6

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 + (-8) + 6 & -2 + 2 + 0 & -1 - 2 + 3 \\ 6 - 20 + 14 & -4 + 5 + 0 & -2 - 5 + 7 \\ -6 + 16 - 10 & 4 - 4 + 0 & 2 + 4 - 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} = \begin{bmatrix} 3 - 4 + 2 & 6 - 10 + 4 & 9 - 14 + 5 \\ -4 + 2 + 2 & -8 + 5 + 4 & -12 + 7 + 5 \\ 2 + 0 - 2 & 4 + 0 - 4 & 6 + 0 - 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{i.e., } \mathbf{AB} = \mathbf{BA} = \mathbf{I} \quad \therefore \mathbf{A}^{-1} = \mathbf{B}$$

- 7 Using technology, the solution of $\begin{aligned} 2x + y + z &= 8 \\ 4x - 7y + 3z &= 10 \\ 3x - 2y + z &= 1 \end{aligned}$ is $x = 2, y = 1, z = 3$

- 8 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 6 & 3 & 2 & 4 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 3 \\ 0 & -9 & 20 & -14 \end{array} \right] \quad R_2 \rightarrow R_2 - 6R_1$$

$$\begin{array}{cccc} 6 & 3 & 2 & 4 \\ -6 & -12 & 18 & -18 \\ \hline 0 & -9 & 20 & -14 \end{array}$$

Let $z = t$ in row 2, then

$$\begin{aligned} -9y + 20t &= -14 & \text{and in row 1, } x + 2\left(\frac{14 + 20t}{9}\right) - 3t &= 3 \\ \therefore -9y &= -14 - 20t & \therefore 9x + 28 + 40t - 27t &= 27 \\ \text{i.e., } y &= \frac{14 + 20t}{9} & \therefore 9x &= -1 - 13t \\ & & \therefore x &= \frac{-1 - 13t}{9} \end{aligned}$$

$\therefore$ there are infinitely many solutions of the form $x = \frac{-1 - 13t}{9}, y = \frac{14 + 20t}{9}, z = t$ where t is in $\mathcal{R}$.

- 9 a In augmented matrix form, the system is:

$$\left[\begin{array}{cc|c} 2 & -3 & 9 \\ m & -7 & n \end{array} \right] \sim \left[\begin{array}{cc|c} 2 & -3 & 9 \\ 3m - 14 & 0 & 3n - 63 \end{array} \right] \quad R_2 \rightarrow 3R_2 - 7R_1$$

$$\begin{array}{ccc} 3m & -21 & 3n \\ -14 & 21 & -63 \\ \hline 3m - 14 & 0 & 3n - 63 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 2 & -3 & 9 \\ 14 - 3m & 0 & 63 - 3n \end{array} \right] \quad R_2 \rightarrow -R_2$$

- b When $14 - 3m \neq 0$, i.e., when $m \neq \frac{14}{3}$, the system has a unique solution.
 {If $m = \frac{14}{3}$, the second row becomes all zeros so there are infinitely many solutions.}

REVIEW SET 14C

1 In matrix form $\begin{bmatrix} k & 3 \\ 1 & k+2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$ has a unique solution if $\begin{vmatrix} k & 3 \\ 1 & k+2 \end{vmatrix} \neq 0$
 i.e., $k^2 + 2k - 3 \neq 0$
 $(k-1)(k+3) \neq 0$
 $k \neq 1 \text{ or } -3$

and if $k \neq 1 \text{ or } -3$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k & 3 \\ 1 & k+2 \end{bmatrix}^{-1} \begin{bmatrix} -6 \\ 2 \end{bmatrix}$

$$= \frac{1}{(k-1)(k+3)} \begin{bmatrix} k+2 & -3 \\ -1 & k \end{bmatrix} \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$= \frac{1}{(k-1)(k+3)} \begin{bmatrix} -6k-18 \\ 6+2k \end{bmatrix}$$

$$= \frac{1}{(k-1)(k+3)} (k+3) \begin{bmatrix} -6 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-6}{k-1} \\ \frac{2}{k-1} \end{bmatrix}$$
 So the unique solution is $x = \frac{-6}{k-1}, y = \frac{2}{k-1}$

If $k = 1$, the equations are: $\begin{cases} x+3y = -6 \\ x+3y = 2 \end{cases}$ parallel lines $\therefore$ no solutions exist

If $k = -3$, the equations are: $\begin{cases} -3x+3y = -6 \\ x-y = 2 \end{cases}$ coincident lines $\therefore$ infinitely many solutions

2 $\begin{vmatrix} x & 2 & 0 \\ 2 & x+1 & -2 \\ 0 & -2 & x+2 \end{vmatrix} = x \begin{vmatrix} x+1 & -2 \\ -2 & x+2 \end{vmatrix} + 2 \begin{vmatrix} -2 & 2 \\ x+2 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & x+1 \\ 0 & -2 \end{vmatrix}$

$$= x[(x+1)(x+2) - 4] + 2[0 - 2(x+2)]$$

$$= x(x^2 + 3x + 2 - 4) - 4(x+2)$$

$$= x(x^2 + 3x - 2) - 4x - 8$$

$$= x^3 + 3x^2 - 2x - 4x - 8$$

$$= x^3 + 3x^2 - 6x - 8$$

$$= (x+4)(x+1)(x-2) \quad \{\text{using technology}\}$$

But $(x+4)(x+1)(x-2) = 0$ {given}
 $\therefore x = -4, -1 \text{ or } 2$

3 $\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} -7 & 9 \\ 9 & -3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix}$

a $2\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} -4 & 6 \\ 8 & -2 \end{bmatrix} - \begin{bmatrix} -14 & 18 \\ 18 & -6 \end{bmatrix} = \begin{bmatrix} 10 & -12 \\ -10 & 4 \end{bmatrix}$

b $\mathbf{A}$ is 2×2 and $\mathbf{C}$ is 2×3 $\therefore \mathbf{AC}$ is 2×3

$$\mathbf{AC} = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2+0 & 0+6 & -6+3 \\ -4+0 & 0-2 & 12-1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -3 \\ -4 & -2 & 11 \end{bmatrix}$$

c $\mathbf{C}$ is 2×3 and $\mathbf{B}$ is 2×2 $\therefore \mathbf{CB}$ is not possible.

$$\mathbf{d} \quad \mathbf{DA} = \mathbf{B}$$

$$\therefore \mathbf{DAA}^{-1} = \mathbf{BA}^{-1} \quad \{\text{post multiplying by } \mathbf{A}^{-1}\}$$

$$\therefore \mathbf{D} = \mathbf{BA}^{-1}$$

$$\therefore \mathbf{D} = \begin{bmatrix} -7 & 9 \\ 9 & -3 \end{bmatrix} = \frac{1}{-2(-1) - 3(4)} \begin{bmatrix} -1 & -3 \\ -4 & -2 \end{bmatrix} = -\frac{1}{10} \begin{bmatrix} 7-36 & 21-18 \\ -9+12 & -27+6 \end{bmatrix}$$

$$\therefore \mathbf{D} = \begin{bmatrix} \frac{29}{10} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{21}{10} \end{bmatrix}$$

$$4 \quad \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \mathbf{X} = \begin{bmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \\ -1 & 13 & 18 \end{bmatrix}$$

$$= \frac{1}{11} \begin{bmatrix} 11 & 33 & 22 \\ -11 & 11 & 33 \end{bmatrix}$$

$$\therefore \mathbf{X} = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 3 \end{bmatrix}$$

$$5 \quad 5\mathbf{A}^2 - 6\mathbf{B} = 3\mathbf{I}$$

$$\therefore \mathbf{A}(5\mathbf{A} - 6\mathbf{I}) = 3\mathbf{I}$$

$$\therefore \mathbf{A} \times \frac{1}{3}(5\mathbf{A} - 6\mathbf{I}) = \mathbf{I}$$

$$\therefore \mathbf{A}^{-1} = \frac{5}{3}\mathbf{A} - 2\mathbf{I}$$

$$6 \quad \mathbf{a} \quad \mathbf{i} \quad \text{If } \mathbf{AB} = \mathbf{B} \text{ then } \mathbf{ABB}^{-1} = \mathbf{BB}^{-1} \text{ provided } \mathbf{B}^{-1} \text{ exists.}$$

$$\therefore \mathbf{A} = \mathbf{I} \text{ provided } \mathbf{B}^{-1} \text{ exists, i.e., provided that } |\mathbf{B}| \neq 0.$$

$$\mathbf{ii} \quad (\mathbf{A} + \mathbf{B})^2 = (\mathbf{A} + \mathbf{B})(\mathbf{A} + \mathbf{B})$$

$$= \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$$

$$= \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2 \text{ provided that } \mathbf{AB} = \mathbf{BA}.$$

$$\mathbf{b} \quad \mathbf{M} = \begin{bmatrix} k & 2 \\ 2 & k \end{bmatrix} \begin{bmatrix} k-1 & -2 \\ -3 & -k \end{bmatrix} \text{ and } \mathbf{M}^{-1} \text{ exists provided that } |\mathbf{M}| \neq 0.$$

$$\text{Now } |\mathbf{M}| = \begin{vmatrix} k & 2 \\ 2 & k \end{vmatrix} \begin{vmatrix} k-1 & -2 \\ -3 & k \end{vmatrix} = (k^2 - 4)(k^2 - k - 6) \\ = (k+2)(k-2)(k+2)(k-3)$$

$$\therefore \mathbf{M}^{-1} \text{ exists provided that } k \neq 3 \text{ or } \pm 2$$

7 In augmented matrix form, the system is:

$$\begin{bmatrix} 1 & -1 & -2 & -3 \\ t & 1 & -1 & 3t \\ 1 & 3 & t & 13 \end{bmatrix} \quad \begin{array}{cccc} t & 1 & -1 & 3t \\ -t & t & 2t & 3t \\ \hline 0 & 1+t & -1+2t & 6t \\ & 1 & 3 & t \\ & -1 & 1 & 2 \\ & \hline 0 & 4 & t+2 & 16 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 1+t & -1+2t & 6t \\ 0 & 4 & t+2 & 16 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - tR_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 1+t & -1+2t & 6t \\ 0 & 0 & t^2-5t+6 & 16-8t \end{bmatrix} \quad R_3 \rightarrow (1+t)R_3 - 4R_2$$

$$\begin{array}{cccc} 0 & 4+4t & t^2+3t+2 & 16+16t \\ 0 & -4-4t & 4-8t & -24t \\ \hline 0 & 0 & t^2-5t+6 & 16-8t \end{array}$$

Row 3 gives $(t-3)(t-2)z = -8(t-2) \quad \therefore z = \frac{-8(t-2)}{(t-3)(t-2)}$

If $t = 3$, there are no solutions, $\therefore$ there is no unique solution.

If $t = 2$, row 3 becomes a row of zeros so there are infinitely many solutions, i.e., there is no unique solution.

The reduced matrix becomes
$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -3 \\ 0 & 3 & 3 & 12 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Row 2 gives $3y + 3z = 12$ i.e., $y + z = 4$. Let $z = s$, then $y = 4 - s$

and substituting in row 1 gives $x - (4 - s) - 2s = -3$

$$\therefore x - 4 + s - 2s = -3$$

$$\text{i.e., } x = 1 + s$$

i.e., when $t = 2$, the solutions are of the form $x = 1 + s$, $y = 4 - s$, $z = s$ where s is in $\mathcal{R}$.

If $t \neq 2$ and $t \neq 3$, then $z = \frac{-8}{t-3}$.

Substituting in row 2 gives

$$(1+t)y - \frac{8(-1+2t)}{t-3} = 6t \quad \text{and} \quad x - \frac{2(3t-4)}{t-3} + \frac{2 \times 8}{t-3} = -3$$

$$\therefore (t-3)(1+t)y + 8 - 16t = 6t(t-3) \quad \therefore (t-3)x - 6t + 8 + 16 = -3t + 9$$

$$\therefore (t-3)(1+t)y = 6t^2 - 18t + 16t - 8 \quad \therefore (t-3)x = 3t - 15$$

$$\therefore y = \frac{2(3t^2 - t - 4)}{(t-3)(t+1)} \quad \therefore x = \frac{3(t-5)}{t-3}$$

$$= \frac{2(3t-4)(t+1)}{(t-3)(t+1)}$$

$$= \frac{2(3t-4)}{t-3}$$

So if $t \neq 2$ or 3 , then the unique solution is:

$$x = \frac{3(t-5)}{t-3}, \quad y = \frac{2(3t-4)}{t-3}, \quad z = \frac{-8}{t-3} \quad \text{for } t \text{ in } \mathcal{R}$$

8 a $s = at^2 + bt + c$

At $t = 1$, $s(1) = 63 \quad \therefore a + b + c = 63$

At $t = 2$, $s(2) = 72 \quad \therefore 4a + 2b + c = 72$

At $t = 7$, $s(7) = 27 \quad \therefore 49a + 7b + c = 27$

Using technology, $a = -3$, $b = 18$, $c = 48$ and $\therefore s(t) = -3t^2 + 18t + 48$

b $s(0) = 48 \quad \therefore$ the height of the cliff is 48 m

c The rock reaches sea level when $s(t) = 0$

$$\therefore -3(t^2 - 6t - 16) = 0$$

$$\therefore -3(t-8)(t+2) = 0$$

$$\therefore t = 8 \text{ or } -2$$

but $t \geq 0$, i.e., it reaches sea level after 8 seconds.

REVIEW SET 14D

1 This system in matrix form is
$$\begin{bmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 9 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ -3 \\ 9 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \text{So, } x = 1, \quad y = -1, \quad z = 2. \quad \{\text{using technology}\}$$

$$\begin{aligned} 2 \quad \begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} &= (a+b) \begin{vmatrix} b+c & a \\ c+a & b \end{vmatrix} + c \begin{vmatrix} a & a \\ c+a & b \end{vmatrix} + c \begin{vmatrix} a & b+c \\ b & b \end{vmatrix} \\ &= (a+b)[(b+c)(c+a) - ab] + c[ab - a(c+a)] + c[ab - (b+c)b] \\ &= (a+b)(bc + ab + c^2 + ac - ab) + abc - ac^2 - a^2c + abc - b^2c - bc^2 \\ &= abc + b^2c + ac^2 + bc^2 + a^2c + abc + abc - ac^2 - a^2c + abc - b^2c - bc^2 \\ &= 4abc \end{aligned}$$

$$3 \quad \text{If } \mathbf{A}^2 = 5\mathbf{A} + 2\mathbf{I},$$

$$\begin{array}{lll} \mathbf{A}^3 = \mathbf{A}(5\mathbf{A} + 2\mathbf{I}) & \mathbf{A}^4 = \mathbf{A}(27\mathbf{A} + 10\mathbf{I}) & \mathbf{A}^5 = \mathbf{A}(145\mathbf{A} + 54\mathbf{I}) \\ = 5\mathbf{A}^2 + 2\mathbf{A} & = 27\mathbf{A}^2 + 10\mathbf{A}\mathbf{I} & = 145\mathbf{A}^2 + 54\mathbf{A}\mathbf{I} \\ = 5(5\mathbf{A} + 2\mathbf{I}) + 2\mathbf{A} & = 27(5\mathbf{A} + 2\mathbf{I}) + 10\mathbf{A} & = 145(5\mathbf{A} + 2\mathbf{I}) + 54\mathbf{A} \\ = 25\mathbf{A} + 10\mathbf{I} + 2\mathbf{A} & = 135\mathbf{A} + 54\mathbf{I} + 10\mathbf{A} & = 725\mathbf{A} + 290\mathbf{I} + 54\mathbf{A} \\ = 27\mathbf{A} + 10\mathbf{I} & = 145\mathbf{A} + 54\mathbf{I} & = 779\mathbf{A} + 290\mathbf{I} \end{array}$$

$$\begin{aligned} \mathbf{A}^6 &= \mathbf{A}(779\mathbf{A} + 290\mathbf{I}) \\ &= 779\mathbf{A}^2 + 290\mathbf{A}\mathbf{I} \\ &= 779(5\mathbf{A} + 2\mathbf{I}) + 290\mathbf{A}\mathbf{I} \\ &= 4185\mathbf{A} + 1558\mathbf{I} \end{aligned}$$

$$4 \quad \mathbf{a} \quad C(0) = 80 \quad \therefore a(0) + b(0) + c(0) + d = 80 \quad \therefore d = 80$$

$$\begin{array}{ll} \mathbf{b} \quad C(1) = 100 & \therefore a + b + c + 80 = 100 \\ C(2) = 148 & 8a + 4b + 2c + 80 = 148 \\ C(4) = 376 & 64a + 16b + 4c + 80 = 376 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 64 & 16 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 20 \\ 68 \\ 296 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 8 & 4 & 2 \\ 64 & 16 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 68 \\ 296 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

{technology}

$$\therefore a = 2, \quad b = 8, \quad c = 10$$

$$\begin{array}{ll} 5 \quad \mathbf{AXB} = \mathbf{C} & \text{So } \mathbf{X} = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -12 & -11 \\ -10 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}^{-1} \\ \therefore \mathbf{A}^{-1}\mathbf{AXB}\mathbf{B}^{-1} = \mathbf{A}^{-1}\mathbf{CB}^{-1} & \\ \therefore \mathbf{IXI} = \mathbf{A}^{-1}\mathbf{CB}^{-1} & \therefore \mathbf{X} = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \quad \{\text{using technology}\} \\ \therefore \mathbf{X} = \mathbf{A}^{-1}\mathbf{CB}^{-1} & \end{array}$$

$$\mathbf{6} \quad \mathbf{a} \quad \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 5 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 19 \\ 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 5 \\ 1 & 1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 19 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore x = 6, \quad y = -2, \quad z = 1$$

$$\mathbf{b} \quad \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 1 & -3 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{7}{6} \\ -\frac{7}{6} \end{bmatrix}$$

$$\therefore x = \frac{3}{2}, \quad y = -\frac{7}{6}, \quad z = -\frac{7}{6}$$

7 In augmented matrix form, the system is:

$$\begin{bmatrix} k & 2 & 1 \\ 2 & k & -2 \end{bmatrix} \sim \begin{bmatrix} k & 2 & 1 \\ 0 & (k+2)(k-2) & -2(k+1) \end{bmatrix} \quad R_2 \rightarrow kR_2 - 2R_1 \quad \begin{array}{ccc} 2k & k^2 & -2k \\ -2k & -4 & -2 \\ \hline 0 & k^2 - 4 & -2k - 2 \end{array}$$

$$\text{Row 2 gives } (k+2)(k-2)y = -2(k+1) \quad \text{i.e., } y = \frac{-2(k+1)}{(k+2)(k-2)}$$

If $k \neq \pm 2$, then substituting in row 1 gives

$$kx + 2 \left(\frac{-2(k+1)}{(k+2)(k-2)} \right) = 1$$

$$\therefore k(k+2)(k-2)x - 4k - 4 = k^2 - 4$$

$$\therefore x = \frac{k^2 + 4k}{k(k+2)(k-2)}$$

$$\therefore x = \frac{k+4}{(k+2)(k-2)}$$

$$\therefore \text{ if } k \neq \pm 2, \text{ then the unique solution is } x = \frac{k+4}{(k+2)(k-2)}, \quad y = \frac{-2(k+1)}{(k+2)(k-2)}$$

If $k = \pm 2$, then row 2 becomes $0z = -6$ or $0z = 2$ which has no solution, i.e., the system has no solutions.

8 a In augmented matrix form, the system is:

$$\begin{bmatrix} 2 & 1 & -1 & 9 \\ 3 & 2 & 5 & 19 \\ 1 & 1 & -3 & 1 \end{bmatrix} \quad \text{Using technology, } x = 6, \quad y = -2, \quad z = 1.$$

b In augmented matrix form, the system is:

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 3 & 2 & 1 & 1 \\ 1 & -3 & 0 & 5 \end{bmatrix} \quad \text{Using technology, } x = \frac{3}{2}, \quad y = -\frac{7}{6}, \quad z = -\frac{7}{6}.$$

9 In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & 10 \\ 4 & -6 & k & m \end{array} \right] \sim \left[\begin{array}{ccc|c} 2 & -3 & 1 & 10 \\ 0 & 0 & k-2 & m-20 \end{array} \right] \quad R_2 \rightarrow R_2 - 2R_1$$

$$\begin{array}{cccc} 4 & -6 & k & m \\ -4 & 6 & -2 & -20 \\ \hline 0 & 0 & k-2 & m-20 \end{array}$$

If $k = 2$ and $m = 20$, the last row is all zeros so there are an infinite number of solutions.

Let $y = s$ and $z = t$ in equation 1.

The solutions are of the form $x = \frac{10 + 3s - t}{2}$, $y = s$, $z = t$ for all s, t in $\mathcal{R}$.

If $k = 2$ and $m \neq 20$, the system is inconsistent.

If $k \neq 2$, then from row 2, $z = \frac{m-20}{k-2}$.

Let $y = t$ in equation 1, then $2x - 3t + \frac{m-20}{k-2} = 10$

$$\therefore 2(k-2)x - 3(k-2)t + m - 20 = 10k - 20$$

$$\therefore 2(k-2)x = 10k - m + 3(k-2)t$$

$$\therefore x = \frac{10k - m + 3(k-2)t}{2(k-2)}$$

i.e., there are an infinite number of solutions of the form

$$x = \frac{10k - m + 3(k-2)t}{2(k-2)}, \quad y = t, \quad z = \frac{m-20}{k-2}, \quad t \text{ is in } \mathcal{R}.$$

10 a In augmented matrix form, the system is:

$$\left[\begin{array}{ccc|c} 1 & 5 & -6 & 2 \\ k & 1 & -1 & 3 \\ 5 & -k & 3 & 7 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 5 & -6 & 2 \\ 0 & 1-5k & 6k-1 & 3-2k \\ 0 & -k-25 & 33 & -3 \end{array} \right] \quad \begin{array}{l} R_2 \rightarrow R_2 - kR_1 \\ R_3 \rightarrow R_3 - 5R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & -6 & 2 \\ 0 & 1-5k & 6k-1 & 3-2k \\ 0 & 0 & 2(k-2)(3k-2) & -2(k-2)(k+18) \end{array} \right] \quad R_3 \rightarrow (1-5k)R_3 + (k+25)R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & -6 & 2 \\ 0 & 1-5k & 6k-1 & 3-2k \\ 0 & 0 & (k-2)(3k-2) & -(k-2)(k+18) \end{array} \right] \quad R_3 \rightarrow R_3 \div 2 \quad \text{as required}$$

$$\begin{array}{cccc} k & 1 & -1 & 3 \\ -k & -5k & 6k & -2k \\ \hline 0 & 1-5k & 6k-1 & 3-2k \end{array} \quad \begin{array}{cccc} 5 & -k & 3 & 7 \\ -5 & -25 & 30 & -10 \\ \hline 0 & -k-25 & 33 & -3 \end{array}$$

$$\begin{array}{cccc} 0 & 5k^2 + 124k - 25 & 33 - 165k & -3 + 15k \\ 0 & -5k^2 - 124k + 25 & 6k^2 + 149k - 25 & -2k^2 - 47k + 75 \\ \hline 0 & 0 & 6k^2 - 16k + 8 & -2k^2 - 32k + 72 \\ & & = 2(3k^2 - 8k + 4) & = -2(k^2 + 16k - 36) \end{array}$$

b k has a unique solution provided $k \neq 2$ or $\frac{2}{3}$

c If $k = 2$, then row 3 becomes a row of zeros so there are infinitely many solutions.

The system becomes
$$\left[\begin{array}{ccc|c} 1 & 5 & -6 & 2 \\ 0 & -9 & 11 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Let $z = t$ in row 2, Substituting in row 1 gives $x + 5\left(\frac{1+11t}{9}\right) - 6t = 2$

then $-9y + 11t = -1$

$$\therefore 9x + 5 + 55t - 54t = 18$$

$$\text{i.e., } y = \frac{1+11t}{9}$$

$$\therefore x = \frac{13-t}{9}$$

i.e., the solutions are of the form $x = \frac{13-t}{9}$, $y = \frac{1+11t}{9}$, $z = t$, where t is in $\mathcal{R}$.

d If $k = \frac{2}{3}$, then the system is inconsistent and there are no solutions.

REVIEW SET 14E

- 1 a** Let $\$x$ be the cost of an opera ticket $3x + 2y + 5z = 267$
 $\$y$ be the cost of a play ticket $2x + 3y + z = 145$
 $\$z$ be the cost of a concert ticket $x + 5y + 4z = 230$

b So,
$$\begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 267 \\ 145 \\ 230 \end{bmatrix} \quad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 3 & 1 \\ 1 & 5 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 267 \\ 145 \\ 230 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ 18 \\ 27 \end{bmatrix} \quad \text{using technology}$$

$\therefore$ cost of each ticket is \$32 for opera, \$18 for play, \$27 for concert.

c Total cost = $4 \times \$32 + 1 \times \$18 + 2 \times \$27$
 $= \$128 + \$18 + \$54 = \200

2
$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -7 & 3 \\ 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ 1 \end{bmatrix} \quad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -7 & 3 \\ 3 & -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\therefore x = 2, y = 1, z = 3$ {using technology}

3 a $3A = 3 \begin{bmatrix} -3 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ **b** $AB = \begin{bmatrix} -3 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -10 & -6 \\ 5 & 3 \end{bmatrix}$
 $= \begin{bmatrix} -9 & 6 & 6 \\ 3 & -3 & 0 \end{bmatrix}$

c BA

$$= \begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & 2 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 4 \\ 10 & -7 & -6 \\ -1 & 0 & 2 \end{bmatrix}$$

d A is 2×3
 and C is 2×2
 $\therefore AC$ does not exist.

e $BC = \begin{bmatrix} 2 & 4 \\ -3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 1 & 3 \end{bmatrix}$
 $= \begin{bmatrix} 0 & 22 \\ 7 & -12 \\ 0 & 11 \end{bmatrix}$

4 $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 3 & 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 2 & -3 \\ 2 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix}$

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 6 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & -3 \\ 2 & -1 & 4 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -1+4+3 & 2-2+4 & -3+8+1 \\ -2+8+18 & 4-4+24 & -6+16+6 \\ -3+2+6 & 6-1+8 & -9+4+2 \end{bmatrix}$$

$$\therefore \mathbf{AB} = \begin{bmatrix} 6 & 4 & 6 \\ 24 & 24 & 16 \\ 5 & 13 & -3 \end{bmatrix}$$

$$\begin{aligned} \text{and } \det(\mathbf{AB}) &= 6 \begin{vmatrix} 24 & 16 \\ 13 & -3 \end{vmatrix} + 4 \begin{vmatrix} 16 & 24 \\ -3 & 5 \end{vmatrix} + 6 \begin{vmatrix} 24 & 24 \\ 5 & 13 \end{vmatrix} \\ &= 6(-72 - 208) + 4(80 + 72) + 6(312 - 120) \\ &= 80 \end{aligned}$$

$$\det \mathbf{A} = 1 \begin{vmatrix} 4 & 6 \\ 1 & 2 \end{vmatrix} + 2 \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}$$

$$= 1(8 - 6) + 2(18 - 4) + 1(2 - 12)$$

$$= 2 + 28 - 10$$

$$= 20$$

$$\text{and } \det \mathbf{B} = -1 \begin{vmatrix} -1 & 4 \\ 4 & 1 \end{vmatrix} + 2 \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} + (-3) \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$$

$$= -1(-1 - 16) + 2(12 - 2) - 3(8 - -3)$$

$$= 4 \quad \therefore \det \mathbf{A} \times \det \mathbf{B} = 20 \times 4 = 80 = \det(\mathbf{AB})$$

5 If $\mathbf{A}^2 = \mathbf{A} - \mathbf{I}$ then $\mathbf{A}^3 = \mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{A}^2 - \mathbf{AI} = \mathbf{A} - \mathbf{I} - \mathbf{A} = -\mathbf{I}$
 $\mathbf{A}^4 = \mathbf{AA}^3 = \mathbf{A}(-\mathbf{I}) = -\mathbf{A}$
 $\mathbf{A}^5 = \mathbf{AA}^4 = \mathbf{A}(-\mathbf{A}) = -\mathbf{A}^2 = -(\mathbf{A} - \mathbf{I}) = \mathbf{I} - \mathbf{A}$
 $\mathbf{A}^6 = \mathbf{AA}^5 = \mathbf{A}(\mathbf{I} - \mathbf{A}) = \mathbf{AI} - \mathbf{A}^2 = \mathbf{A} - (\mathbf{A} - \mathbf{I}) = \mathbf{I}$
 $\mathbf{A}^7 = \mathbf{AA}^6 = \mathbf{AI} = \mathbf{A}$
 $\mathbf{A}^8 = \mathbf{AA}^7 = \mathbf{AA} = \mathbf{A}^2 = \mathbf{A} - \mathbf{I}$

a $\mathbf{A}^{6n+3} = -\mathbf{I}$

$\mathbf{A}^{6n+5} = \mathbf{I} - \mathbf{A}$

b Now $\mathbf{A}^2 = \mathbf{A} - \mathbf{I}$

$\therefore \mathbf{A}^{-1}\mathbf{AA} = \mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}\mathbf{I} \quad \{\text{premultiplying by } \mathbf{A}^{-1}\}$

$\therefore \mathbf{IA} = \mathbf{I} - \mathbf{A}^{-1}$

$\therefore \mathbf{A}^{-1} = \mathbf{I} - \mathbf{A}$

c P_n is “ $\mathbf{A}^{6n+5} = \mathbf{I} - \mathbf{A}$ for all n in $\mathbb{Z}^+$ ”.

Proof: (By the Principle of Mathematical Induction)

(1) If $n = 1$, $\mathbf{A}^{11} = \mathbf{A}^5\mathbf{A}^6$

$= (\mathbf{I} - \mathbf{A})\mathbf{I}$

$= \mathbf{I} - \mathbf{A} \quad \therefore P_1 \text{ is true}$

(2) If P_k is true, then $\mathbf{A}^{6k+5} = \mathbf{I} - \mathbf{A}$

$\therefore \mathbf{A}^{6(k+1)+5} = \mathbf{A}^{6k+6+5}$

$= \mathbf{A}^{6k+5}\mathbf{A}^6$

$= (\mathbf{I} - \mathbf{A})\mathbf{I}$

$= \mathbf{I} - \mathbf{A}$

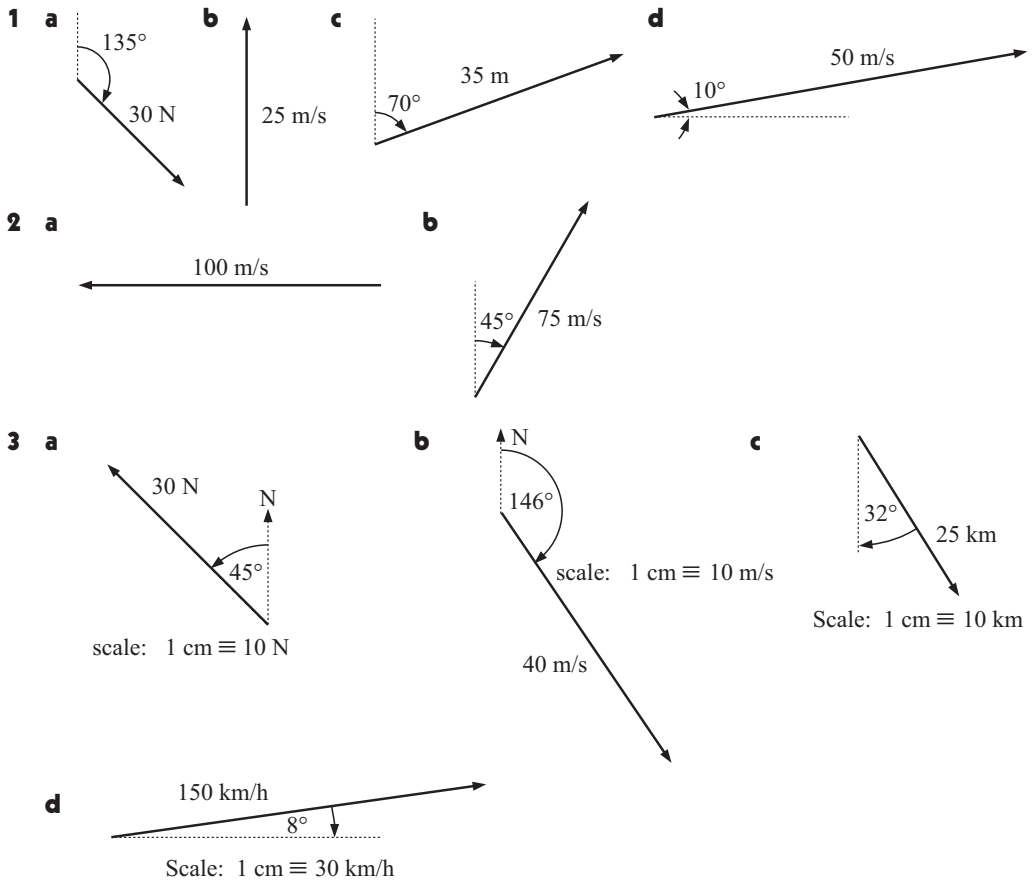
Thus P_{k+1} is true whenever P_k is true and P_1 is true.

$\Rightarrow P_n$ is true $\{\text{Principle of Mathematical Induction}\}$

Chapter 15

VECTORS IN 2 AND 3 DIMENSIONS

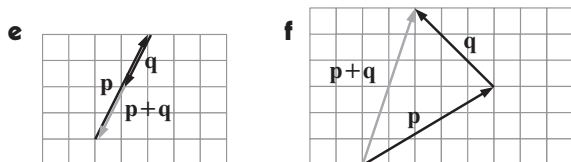
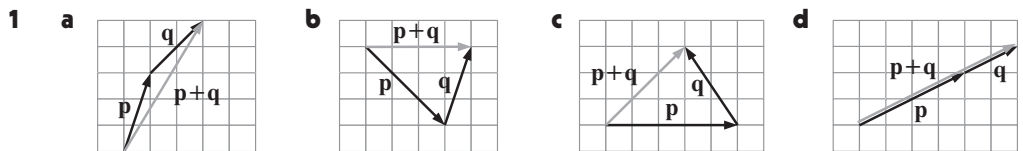
EXERCISE 15A.1



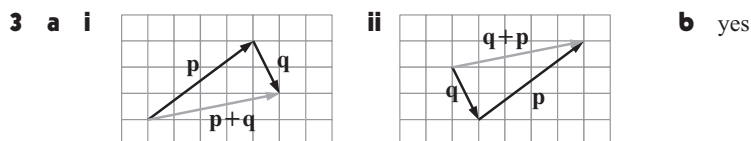
EXERCISE 15A.2

- 1
 - a If they are equal in magnitude, they have the same length. These are **p**, **q**, **s** and **t**.
 - b Those parallel are **p**, **q**, **r** and **t**.
 - c Those in the same direction are: **p** and **r**, **q** and **t**.
 - d To be equal they must have the same direction and be equal in length $\therefore$ **q** = **t**.
 - e **p** and **q** are negatives (equal length, but opposite direction). Likewise **p** and **t** are negatives. We write **p** = -**q** and **p** = -**t**.
- 2
 - a True, as they have the same length and are parallel.
 - b True, as they are sides of an equilateral triangle.
 - c False, as they do not have the same direction.
 - d False, as they have opposite directions.
 - e True, as they have the same length and direction.
 - f False, as they do not have the same direction.

EXERCISE 15B.1

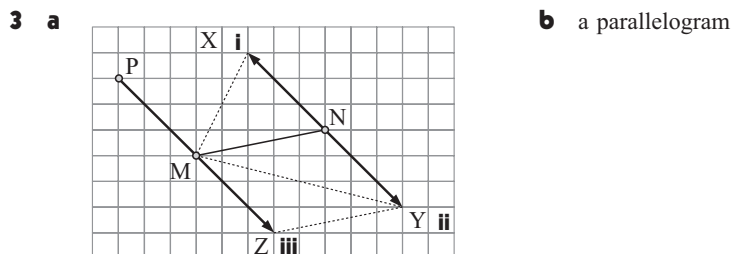
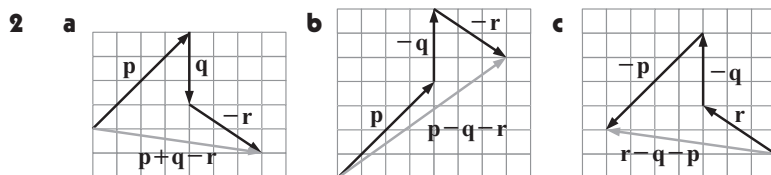
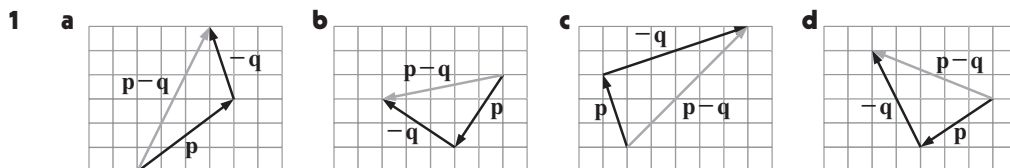


2 a $\vec{AB} + \vec{BC} = \vec{AC}$ b $\vec{BC} + \vec{CD} = \vec{BD}$ c $\vec{AB} + \vec{BC} + \vec{CD} = \vec{AC} + \vec{CD} = \vec{AD}$ d $\vec{AC} + \vec{CB} + \vec{BD} = \vec{AB} + \vec{BD} = \vec{AD}$



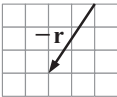
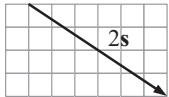
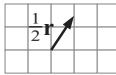
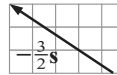
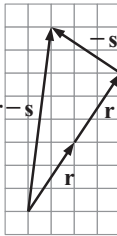
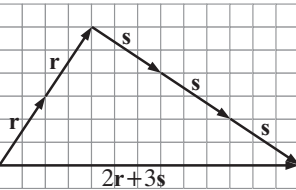
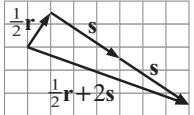
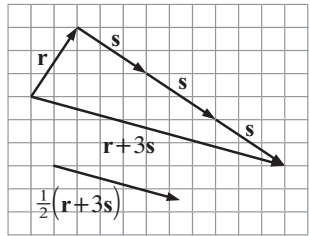
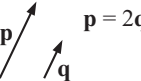
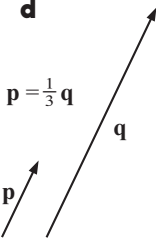
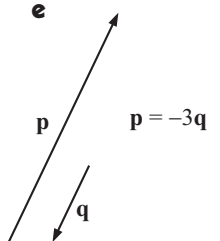
4 $\vec{PS} = \vec{PR} + \vec{RS} = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ But $\vec{PS} = \vec{PQ} + \vec{QS} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
 $\therefore (\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ {as both are equal to $\vec{PS}$ }
 Note: $\vec{PS} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ also

EXERCISE 15B.2

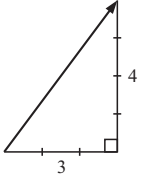
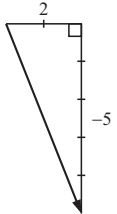
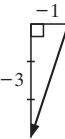


- 4 a** $\vec{AC} + \vec{CB} = \vec{AB}$
b $\vec{AD} - \vec{BD} = \vec{AD} + \vec{DB} = \vec{AB}$
c $\vec{AC} + \vec{CA} = \vec{AA} = \mathbf{0}$
d $\vec{AB} + \vec{BC} + \vec{CD} = \vec{AC} + \vec{CD} = \vec{AD}$
- e** $\vec{BA} - \vec{CA} + \vec{CB} = \vec{BA} + \vec{AC} + \vec{CB} = \vec{BC} + \vec{CB} = \vec{BB} = \mathbf{0}$
f $\vec{AB} - \vec{CB} - \vec{DC} = \vec{AB} + \vec{BC} + \vec{CD} = \vec{AC} + \vec{CD} = \vec{AD}$
- 5 a** $\mathbf{t} = \mathbf{r} + \mathbf{s}$
b $\mathbf{r} = -\mathbf{s} - \mathbf{t}$
c $\mathbf{r} = -\mathbf{p} - \mathbf{q} - \mathbf{s}$
d $\mathbf{r} = \mathbf{q} - \mathbf{p} + \mathbf{s}$
- e** $\mathbf{p} = \mathbf{t} + \mathbf{s} + \mathbf{r} - \mathbf{q}$
f $\mathbf{p} = -\mathbf{u} + \mathbf{t} + \mathbf{s} - \mathbf{r} - \mathbf{q}$
- 6 a i** $\vec{OB} = \vec{OA} + \vec{AB} = \mathbf{r} + \mathbf{s}$
ii $\vec{CA} = \vec{CB} + \vec{BA} = -\vec{BC} - \vec{AB} = -\mathbf{t} - \mathbf{s}$
iii $\vec{OC} = \vec{OA} + \vec{AB} + \vec{BC} = \mathbf{r} + \mathbf{s} + \mathbf{t}$
- b i** $\vec{AD} = \vec{AB} + \vec{BD} = \mathbf{p} + \mathbf{q}$
ii $\vec{BC} = \vec{BD} + \vec{DC} = \mathbf{q} + \mathbf{r}$
iii $\vec{AC} = \vec{AB} + \vec{BD} + \vec{DC} = \mathbf{p} + \mathbf{q} + \mathbf{r}$

EXERCISE 15B.3

- 1 a** 
b 
c 
d 
e 
f 
g 
h 
- 2 a** 
b 
c 
d 
e 

EXERCISE 15C.1

- 1 a** 
b 
c 
d 

$$\mathbf{2} \quad \mathbf{a} \quad \begin{bmatrix} 7 \\ 3 \end{bmatrix} \quad \mathbf{b} \quad \begin{bmatrix} -6 \\ 0 \end{bmatrix} \quad \mathbf{c} \quad \begin{bmatrix} 2 \\ -5 \end{bmatrix} \quad \mathbf{d} \quad \begin{bmatrix} 0 \\ 6 \end{bmatrix} \quad \mathbf{e} \quad \begin{bmatrix} -6 \\ 3 \end{bmatrix} \quad \mathbf{f} \quad \begin{bmatrix} -5 \\ -5 \end{bmatrix}$$

EXERCISE 15C.2

$$\mathbf{1} \quad \mathbf{a} \quad \mathbf{a} + \mathbf{b}$$

$$= \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\ = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$\mathbf{b} \quad \mathbf{b} + \mathbf{a}$$

$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} \\ = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$\mathbf{c} \quad \mathbf{b} + \mathbf{c}$$

$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} \\ = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\mathbf{d} \quad \mathbf{c} + \mathbf{b}$$

$$= \begin{bmatrix} -2 \\ -5 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\ = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\mathbf{e} \quad \mathbf{a} + \mathbf{c}$$

$$= \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} \\ = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$$

$$\mathbf{f} \quad \mathbf{c} + \mathbf{a}$$

$$= \begin{bmatrix} -2 \\ -5 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} \\ = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$$

$$\mathbf{g} \quad \mathbf{a} + \mathbf{a}$$

$$= \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} \\ = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$$

$$\mathbf{h} \quad \mathbf{b} + \mathbf{a} + \mathbf{c}$$

$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} \\ = \begin{bmatrix} -2 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} \\ = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

$$\mathbf{2} \quad \mathbf{a} \quad \mathbf{p} - \mathbf{q}$$

$$= \begin{bmatrix} -4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ -5 \end{bmatrix} \\ = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

$$\mathbf{b} \quad \mathbf{q} - \mathbf{r}$$

$$= \begin{bmatrix} -1 \\ -5 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

$$\mathbf{c} \quad \mathbf{p} + \mathbf{q} - \mathbf{r}$$

$$= \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ -5 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ = \begin{bmatrix} -8 \\ -1 \end{bmatrix}$$

$$\mathbf{d} \quad \mathbf{p} - \mathbf{q} - \mathbf{r}$$

$$= \begin{bmatrix} -4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ -5 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\ = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$$

$$\mathbf{e} \quad \mathbf{q} - \mathbf{r} - \mathbf{p}$$

$$= \begin{bmatrix} -1 \\ -5 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \begin{bmatrix} -4 \\ 2 \end{bmatrix} \\ = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$

$$\mathbf{f} \quad \mathbf{r} + \mathbf{q} - \mathbf{p}$$

$$= \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ -5 \end{bmatrix} - \begin{bmatrix} -4 \\ 2 \end{bmatrix} \\ = \begin{bmatrix} 6 \\ -9 \end{bmatrix}$$

$$\mathbf{3} \quad \mathbf{a} \quad \overrightarrow{AC}$$

$$= \overrightarrow{AB} + \overrightarrow{BC} \\ = -\overrightarrow{BA} + \overrightarrow{BC} \\ = -\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} \\ = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$$

$$\mathbf{b} \quad \overrightarrow{CB}$$

$$= \overrightarrow{CA} + \overrightarrow{AB} \\ = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} \\ = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{c} \quad \overrightarrow{SP}$$

$$= \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP} \\ = -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ} \\ = -\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\ = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

EXERCISE 15C.3

$$\mathbf{1} \quad \mathbf{a} \quad -3\mathbf{p}$$

$$= -3\begin{bmatrix} 1 \\ 5 \end{bmatrix} \\ = \begin{bmatrix} -3 \\ -15 \end{bmatrix}$$

$$\mathbf{b} \quad \frac{1}{2}\mathbf{q}$$

$$= \frac{1}{2}\begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\mathbf{c} \quad 2\mathbf{p} + \mathbf{q}$$

$$= 2\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ = \begin{bmatrix} 2 \\ 10 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ = \begin{bmatrix} 0 \\ 14 \end{bmatrix}$$

$$\mathbf{d} \quad \mathbf{p} - 2\mathbf{q}$$

$$= \begin{bmatrix} 1 \\ 5 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ = \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \begin{bmatrix} -4 \\ 8 \end{bmatrix} \\ = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

$$\mathbf{e} \quad \mathbf{p} - \frac{1}{2}\mathbf{r}$$

$$= \begin{bmatrix} 1 \\ 5 \end{bmatrix} - \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \end{bmatrix} \\ = \begin{bmatrix} 2\frac{1}{2} \\ 5\frac{1}{2} \end{bmatrix}$$

$$\mathbf{f} \quad 2\mathbf{p} + 3\mathbf{r}$$

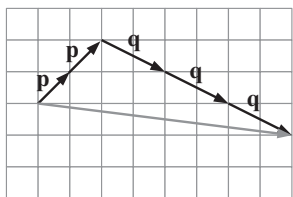
$$= 2\begin{bmatrix} 1 \\ 5 \end{bmatrix} + 3\begin{bmatrix} -3 \\ -1 \end{bmatrix} \\ = \begin{bmatrix} 2 \\ 10 \end{bmatrix} + \begin{bmatrix} -9 \\ -3 \end{bmatrix} \\ = \begin{bmatrix} -7 \\ 7 \end{bmatrix}$$

$$\mathbf{g} \quad 2\mathbf{q} - 3\mathbf{r}$$

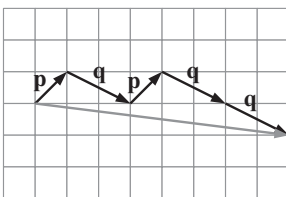
$$= 2\begin{bmatrix} -2 \\ 4 \end{bmatrix} - 3\begin{bmatrix} -3 \\ -1 \end{bmatrix} \\ = \begin{bmatrix} -4 \\ 8 \end{bmatrix} - \begin{bmatrix} -9 \\ -3 \end{bmatrix} \\ = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$\mathbf{h} \quad 2\mathbf{p} - \mathbf{q} + \frac{1}{3}\mathbf{r}$$

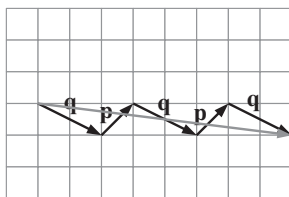
$$= \begin{bmatrix} 2 \\ 10 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ -\frac{1}{3} \end{bmatrix} \\ = \begin{bmatrix} 3 \\ 5\frac{2}{3} \end{bmatrix}$$

2 a

$$= \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

b

$$= \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

c

$$= \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

All vector expressions are equal {each consists of $2\mathbf{p}$ and $3\mathbf{q}$ }.

Each expression is equal to $2\mathbf{p} + 3\mathbf{q}$.

EXERCISE 15C.4**1****a** $|\mathbf{r}|$

$$= \sqrt{2^2 + 3^2}$$

$$= \sqrt{13} \text{ units}$$

b $|\mathbf{s}|$

$$= \sqrt{(-1)^2 + 4^2}$$

$$= \sqrt{17} \text{ units}$$

c $\mathbf{r} + \mathbf{s}$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\therefore |\mathbf{r} + \mathbf{s}|$$

$$= \sqrt{1^2 + 7^2}$$

$$= \sqrt{50} \text{ units}$$

d $\mathbf{r} - \mathbf{s}$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\therefore |\mathbf{r} - \mathbf{s}|$$

$$= \sqrt{3^2 + (-1)^2}$$

$$= \sqrt{10} \text{ units}$$

e $\mathbf{s} - 2\mathbf{r}$

$$= \begin{bmatrix} -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ -2 \end{bmatrix}$$

$$\therefore |\mathbf{s} - 2\mathbf{r}|$$

$$= \sqrt{(-5)^2 + (-2)^2}$$

$$= \sqrt{29} \text{ units}$$

2**a** $|\mathbf{p}|$

$$= \sqrt{1^2 + 3^2}$$

$$= \sqrt{10} \text{ units}$$

b $2\mathbf{p} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$$\therefore |2\mathbf{p}| = \sqrt{2^2 + 6^2}$$

$$= \sqrt{40}$$

$$= 2\sqrt{10} \text{ units}$$

c $-2\mathbf{p} = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$

$$\therefore |-2\mathbf{p}| = \sqrt{(-2)^2 + (-6)^2}$$

$$= \sqrt{4 + 36}$$

$$= \sqrt{40}$$

$$= 2\sqrt{10} \text{ units}$$

d $3\mathbf{p} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

$$\therefore |3\mathbf{p}| = \sqrt{3^2 + 9^2}$$

$$= \sqrt{9 + 81}$$

$$= \sqrt{90}$$

$$= 3\sqrt{10} \text{ units}$$

e $-3\mathbf{p} = \begin{bmatrix} -3 \\ -9 \end{bmatrix}$

$$\therefore |-3\mathbf{p}| = \sqrt{(-3)^2 + (-9)^2}$$

$$= \sqrt{9 + 81}$$

$$= \sqrt{90}$$

$$= 3\sqrt{10} \text{ units}$$

f $|\mathbf{q}| = \sqrt{(-2)^2 + 4^2}$

$$= \sqrt{4 + 16}$$

$$= \sqrt{20}$$

$$= 2\sqrt{5} \text{ units}$$

g $4\mathbf{q} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$

$$\therefore |4\mathbf{q}| = \sqrt{(-8)^2 + 16^2}$$

$$= \sqrt{64 + 256}$$

$$= \sqrt{320}$$

$$= 8\sqrt{5} \text{ units}$$

h $-4\mathbf{q} = \begin{bmatrix} 8 \\ -16 \end{bmatrix}$

$$\therefore |-4\mathbf{q}| = \sqrt{8^2 + (-16)^2}$$

$$= \sqrt{64 + 256}$$

$$= \sqrt{320}$$

$$= 8\sqrt{5} \text{ units}$$

i $\frac{1}{2}\mathbf{q} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\therefore \left| \frac{1}{2}\mathbf{q} \right| = \sqrt{(-1)^2 + 2^2}$$

$$= \sqrt{5} \text{ units}$$

j $-\frac{1}{2}\mathbf{q} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\therefore \left| -\frac{1}{2}\mathbf{q} \right| = \sqrt{1^2 + (-2)^2}$$

$$= \sqrt{5} \text{ units}$$

3 $k\mathbf{a} = \begin{bmatrix} ka_1 \\ ka_2 \end{bmatrix}$

$$\therefore |k\mathbf{a}| = \sqrt{(ka_1)^2 + (ka_2)^2}$$

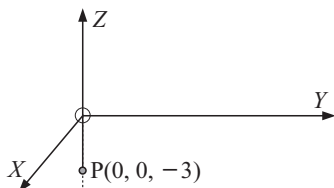
$$= \sqrt{k^2 a_1^2 + k^2 a_2^2}$$

$$= \sqrt{k^2 (a_1^2 + a_2^2)}$$

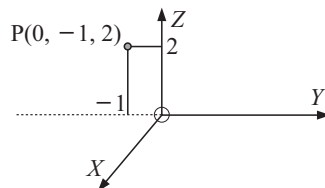
$$= \sqrt{k^2} \sqrt{a_1^2 + a_2^2}$$

$$= |k| \sqrt{a_1^2 + a_2^2}$$

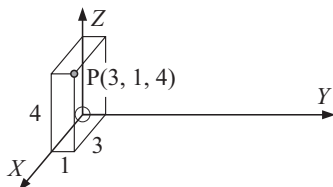
$$= |k| |\mathbf{a}|$$

EXERCISE 15D**1 a**

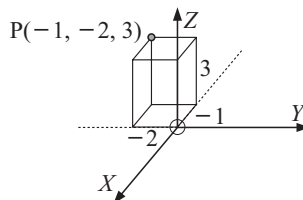
$$OP = \sqrt{0^2 + 0^2 + (-3)^2} = 3 \text{ units}$$

b

$$OP = \sqrt{0^2 + (-1)^2 + 2^2} = \sqrt{5} \text{ units}$$

c

$$\begin{aligned} OP &= \sqrt{3^2 + 1^2 + 4^2} \\ &= \sqrt{26} \text{ units} \end{aligned}$$

d

$$\begin{aligned} OP &= \sqrt{(-1)^2 + (-2)^2 + 3^2} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

2 a i

$$\begin{aligned} AB &= \sqrt{(0 - (-1))^2 + (-1 - 2)^2 + (1 - 3)^2} \\ &= \sqrt{1 + 9 + 4} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{ii Midpoint is at } &\left(\frac{-1+0}{2}, \frac{2-1}{2}, \frac{3+1}{2}\right) \\ \text{i.e., } &\left(-\frac{1}{2}, \frac{1}{2}, 2\right) \end{aligned}$$

b i

$$\begin{aligned} AB &= \sqrt{(2-0)^2 + (-1-0)^2 + (3-0)^2} \\ &= \sqrt{4 + 1 + 9} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{ii Midpoint is at } &\left(\frac{0+2}{2}, \frac{0-1}{2}, \frac{0+3}{2}\right) \\ \text{i.e., } &\left(1, -\frac{1}{2}, \frac{3}{2}\right) \end{aligned}$$

c i

$$\begin{aligned} AB &= \sqrt{(-1-3)^2 + (0-1)^2 + (1-1)^2} \\ &= \sqrt{16 + 1 + 4} \\ &= \sqrt{21} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{ii Midpoint is at } &\left(\frac{3-1}{2}, \frac{-1+0}{2}, \frac{-1+1}{2}\right) \\ \text{i.e., } &\left(1, -\frac{1}{2}, 0\right) \end{aligned}$$

d i

$$\begin{aligned} AB &= \sqrt{(0-2)^2 + (1-0)^2 + (0-3)^2} \\ &= \sqrt{4 + 1 + 9} \\ &= \sqrt{14} \text{ units} \end{aligned}$$

$$\begin{aligned} \text{ii Midpoint is at } &\left(\frac{2+0}{2}, \frac{0+1}{2}, \frac{-3+0}{2}\right) \\ \text{i.e., } &\left(1, \frac{1}{2}, -\frac{3}{2}\right) \end{aligned}$$

3 P(0, 4, 4) Q(2, 6, 5) R(1, 4, 3)

$$\begin{aligned} PQ &= \sqrt{(2-0)^2 + (6-4)^2 + (5-4)^2} \\ &= \sqrt{4 + 4 + 1} \\ &= 3 \end{aligned}$$

$$\begin{aligned} QR &= \sqrt{(1-2)^2 + (4-6)^2 + (3-5)^2} \\ &= \sqrt{1 + 4 + 4} \\ &= 3 \end{aligned}$$

$$\begin{aligned} PR &= \sqrt{(1-0)^2 + (4-4)^2 + (3-4)^2} \\ &= \sqrt{1 + 0 + 1} \\ &= \sqrt{2} \end{aligned}$$

$\therefore PQ = QR$ and so $\triangle PQR$ is isosceles.

4 a $A(2, -1, 7)$ $B(3, 1, 4)$ $C(5, 4, 5)$

$$\begin{aligned} AB &= \sqrt{(3-2)^2 + (1-(-1))^2 + (4-7)^2} & AC &= \sqrt{(5-2)^2 + (4-(-1))^2 + (5-7)^2} \\ &= \sqrt{1+4+9} & &= \sqrt{9+25+4} \\ &= \sqrt{14} & &= \sqrt{38} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(5-3)^2 + (4-1)^2 + (5-4)^2} \\ &= \sqrt{4+9+1} \\ &= \sqrt{14} \end{aligned}$$

$\therefore$ as $AB = BC$, $\triangle ABC$ is isosceles.

b $A(0, 0, 3)$ $B(2, 8, 1)$ $C(-9, 6, 18)$

$$\begin{aligned} AB &= \sqrt{(2-0)^2 + (8-0)^2 + (1-3)^2} & AC &= \sqrt{(-9-0)^2 + (6-0)^2 + (18-3)^2} \\ &= \sqrt{4+64+4} & &= \sqrt{81+36+225} \\ &= \sqrt{72} & &= \sqrt{342} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(-9-2)^2 + (6-8)^2 + (18-1)^2} \\ &= \sqrt{121+4+289} \\ &= \sqrt{414} \end{aligned}$$

Since $BC^2 = AB^2 + AC^2$, $\triangle ABC$ is right-angled.

c $A(5, 6, -2)$ $B(6, 12, 9)$ $C(2, 4, 2)$

$$\begin{aligned} AB &= \sqrt{(6-5)^2 + (12-6)^2 + (9-(-2))^2} & AC &= \sqrt{(2-5)^2 + (4-6)^2 + (2-(-2))^2} \\ &= \sqrt{1+36+121} & &= \sqrt{9+4+16} \\ &= \sqrt{158} & &= \sqrt{29} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(2-6)^2 + (4-12)^2 + (2-9)^2} \\ &= \sqrt{16+64+49} \\ &= \sqrt{129} \end{aligned}$$

Since $AB^2 = AC^2 + BC^2$, $\triangle ABC$ is right-angled.

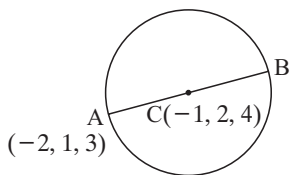
d $A(1, 0, -3)$ $B(2, 2, 0)$ $C(4, 6, 6)$

$$\begin{aligned} AB &= \sqrt{(2-1)^2 + (2-0)^2 + (0-(-3))^2} & AC &= \sqrt{(4-1)^2 + (6-0)^2 + (6-(-3))^2} \\ &= \sqrt{1^2+2^2+3^2} & &= \sqrt{3^2+6^2+9^2} \\ &= \sqrt{14} & &= \sqrt{126} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(4-2)^2 + (6-2)^2 + (6-0)^2} & &= 3\sqrt{14} \\ &= \sqrt{2^2+4^2+6^2} \\ &= \sqrt{56} \\ &= 2\sqrt{14} \end{aligned}$$

Since $AB + BC = AC$, the points A, B and C lie on a straight line, i.e., do not form a triangle.

5



If B is (a, b, c) then $\frac{a-2}{2} = -1$, $\frac{b+1}{2} = 2$, $\frac{c+3}{2} = 4$

$$\therefore a = 0, \quad b = 3, \quad c = 5$$

$$\therefore B \text{ is } (0, 3, 5)$$

$$\begin{aligned} r = AC &= \sqrt{(-1-(-2))^2 + (2-1)^2 + (4-3)^2} \\ &= \sqrt{1+1+1} \\ &= \sqrt{3} \text{ units} \end{aligned}$$

6 a $(0, y, 0)$ for any y **b** Distance between $(0, y, 0)$ and $B(-1, -1, 2)$ is $\sqrt{(-1)^2 + (-1 - y)^2 + 2^2}$

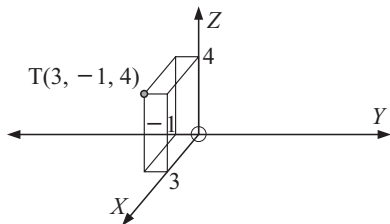
$$\therefore \sqrt{1 + (y + 1)^2 + 4} = \sqrt{14}$$

$$\therefore (y + 1)^2 = 9$$

$$\therefore y + 1 = \pm 3$$

$$\therefore y = -1 \pm 3$$

$$\therefore y = -4 \text{ or } 2 \quad \therefore \text{the two points are } (0, -4, 0) \text{ and } (0, 2, 0)$$

EXERCISE 15E.1**1 a**

$$\mathbf{b} \quad \vec{OT} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

$$\begin{aligned} \mathbf{c} \quad OT &= \sqrt{(3-0)^2 + (-1-0)^2 + (4-0)^2} \\ &= \sqrt{9 + 1 + 16} \\ &= \sqrt{26} \text{ units} \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad \vec{AB} = \begin{bmatrix} 1 - (-3) \\ 0 - 1 \\ -1 - 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$$

$$\vec{BA} = \begin{bmatrix} -3 - 1 \\ 1 - 0 \\ 2 - (-1) \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$$

$$\mathbf{b} \quad |\vec{AB}| = \sqrt{4^2 + (-1)^2 + (-3)^2} = \sqrt{26} \text{ units}$$

$$|\vec{BA}| = \sqrt{(-4)^2 + 1^2 + 3^2} = \sqrt{26} \text{ units}$$

$$\mathbf{3} \quad \vec{OA} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \quad \vec{OB} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \quad \vec{AB} = \begin{bmatrix} -1 - 3 \\ 1 - 1 \\ 2 - 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 2 \end{bmatrix}$$

4 a The position vector of M from N
 $= \vec{NM}$

$$= \begin{bmatrix} 4 - (-1) \\ -2 - 2 \\ -1 - 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ -1 \end{bmatrix}$$

b The position vector of N from M
 $= \vec{MN}$

$$= \begin{bmatrix} -1 - 4 \\ 2 - (-2) \\ 0 - (-1) \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$$

$$\mathbf{c} \quad |\vec{MN}| = \sqrt{(-5)^2 + 4^2 + 1^2} = \sqrt{25 + 16 + 1} = \sqrt{42} \text{ units}$$

5 a The position vector of A from O

$$= \vec{OA} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{aligned} |\vec{OA}| &= \sqrt{(-1)^2 + 2^2 + 5^2} \\ &= \sqrt{1 + 4 + 25} \\ &= \sqrt{30} \text{ units} \end{aligned}$$

b The position vector of C from A

$$= \vec{AC} = \begin{bmatrix} -3 - (-1) \\ 1 - 2 \\ 0 - 5 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ -5 \end{bmatrix}$$

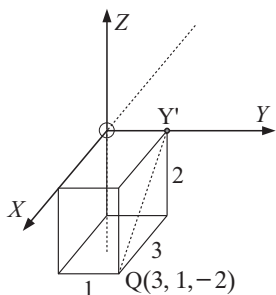
$$\begin{aligned} |\vec{AC}| &= \sqrt{(-2)^2 + (-1)^2 + (-5)^2} \\ &= \sqrt{4 + 1 + 25} \\ &= \sqrt{30} \text{ units} \end{aligned}$$

c The position vector of B from C

$$= \vec{CB} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{aligned} \text{and } |\vec{CB}| &= \sqrt{5^2 + (-1)^2 + 3^2} \\ &= \sqrt{25 + 1 + 9} \\ &= \sqrt{35} \text{ units} \end{aligned}$$

6



- a** The distance from Q to the Y-axis is the distance from Q to $Y'(0, 1, 0)$

$$\begin{aligned}\therefore QY' &= \sqrt{(3-0)^2 + (1-1)^2 + (-2-0)^2} \\ &= \sqrt{9+4} \\ &= \sqrt{13} \text{ units}\end{aligned}$$

- b** The distance from Q to the origin is

$$\begin{aligned}QO &= \sqrt{(3-0)^2 + (1-0)^2 + (-2-0)^2} \\ &= \sqrt{9+1+4} \\ &= \sqrt{14} \text{ units}\end{aligned}$$

- c** The distance from Q to the ZOY plane is the distance from Q to $(0, 1, -2)$, i.e., 3 units.

EXERCISE 15E.2

1 a $\begin{bmatrix} a-4 \\ b-3 \\ c+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$

$$\begin{aligned}\therefore a-4 &= 1, \\ b-3 &= 3, \\ c+2 &= -4\end{aligned}$$

$$\therefore a = 5, \quad b = 6, \quad c = -6$$

b $\begin{bmatrix} a-5 \\ b-2 \\ c+3 \end{bmatrix} = \begin{bmatrix} 3-a \\ 2-b \\ 5-c \end{bmatrix}$

$$\begin{aligned}\therefore a-5 &= 3-a, \\ b-2 &= 2-b, \\ c+3 &= 5-c, \\ \therefore 2a &= 8, \quad 2b = 4, \quad 2c = 2 \\ \therefore a &= 4, \quad b = 2, \quad c = 1\end{aligned}$$

2 a $2 \begin{bmatrix} 1 \\ 0 \\ 3a \end{bmatrix} = \begin{bmatrix} b \\ c-1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 0 \\ 6a \end{bmatrix} = \begin{bmatrix} b \\ c-1 \\ 2 \end{bmatrix}$$

$$\therefore 6a = 2, \quad b = 2, \quad c-1 = 0 \quad \therefore a = \frac{1}{3}, \quad b = 2, \quad c = 1$$

b $\begin{bmatrix} 2 \\ a \\ 3 \end{bmatrix} = \begin{bmatrix} b \\ a^2 \\ a+b \end{bmatrix}$

$$\begin{aligned}\therefore b &= 2, \quad a^2 = a, \quad a+b = 3 \\ \text{i.e., } a &= 1, \quad b = 2\end{aligned}$$

c $a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$

$$\therefore a + 2b = -1 \quad \dots (1), \quad a + c = 3 \quad \dots (2) \quad \text{and} \quad -b + c = 3 \quad \dots (3)$$

$$(1) - (2) \text{ gives: } 2b - c = -4 \quad \dots (4)$$

$$\text{Adding (3) and (4) gives } b = -1$$

$$\therefore \text{ using (3), } c = 2$$

$$\text{and using (2), } a = 1$$

$$\therefore a = 1, \quad b = -1, \quad c = 2$$

3 $A(-1, 3, 4) \quad B(2, 5, -1) \quad C(-1, 2, -2) \quad D(r, s, t)$

a If $\overrightarrow{AC} = \overrightarrow{BD}$ then $\begin{bmatrix} -1 - (-1) \\ 2 - 3 \\ -2 - 4 \end{bmatrix} = \begin{bmatrix} r - 2 \\ s - 5 \\ t + 1 \end{bmatrix}$

$$\therefore r - 2 = 0, \quad s - 5 = -1 \quad \text{and} \quad t + 1 = -6 \quad \therefore r = 2, \quad s = 4 \quad \text{and} \quad t = -7$$

b If $\overrightarrow{AB} = \overrightarrow{DC}$ then $\begin{bmatrix} 2 - (-1) \\ 5 - 3 \\ -1 - 4 \end{bmatrix} = \begin{bmatrix} -1 - r \\ 2 - s \\ -2 - t \end{bmatrix}$

$$\therefore -1 - r = 3, \quad 2 - s = 2 \quad \text{and} \quad -2 - t = -5 \quad \therefore r = -4, \quad s = 0 \quad \text{and} \quad t = 3$$

$$4 \quad a \quad \overrightarrow{AB} = \begin{bmatrix} 3-1 \\ -3-2 \\ 2-3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} \quad \text{and} \quad \overrightarrow{DC} = \begin{bmatrix} 7-5 \\ -4-1 \\ 5-6 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$$

b ABCD is a parallelogram since its opposite sides are parallel and equal in length.

5 a Suppose S is at (x, y, z) $\overrightarrow{PQ} = \overrightarrow{SR}$ {opposite sides are parallel and equal in length}

$$\begin{array}{lcl} P(-1, 2, 3) & \xrightarrow{\quad} & Q(1, -2, 5) \\ & \nwarrow & \nearrow \\ & S(x, y, z) & \xrightarrow{\quad} R(0, 4, -1) \end{array} \quad \therefore \begin{bmatrix} 1-(-1) \\ -2-2 \\ 5-3 \end{bmatrix} = \begin{bmatrix} 0-x \\ 4-y \\ -1-z \end{bmatrix} \quad \text{i.e.,} \quad \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -x \\ 4-y \\ -1-z \end{bmatrix}$$

$$\therefore \begin{array}{lll} -x = 2 & 4-y = -4 & -1-z = 2 \\ \therefore x = -2 & y = 8 & z = -3 \end{array}$$

b The midpoint of PR is $\left(\frac{-1+0}{2}, \frac{2+4}{2}, \frac{3+(-1)}{2} \right)$ i.e., $(-\frac{1}{2}, 3, 1)$

The midpoint of QS is $\left(\frac{1+(-2)}{2}, \frac{-2+8}{2}, \frac{5+(-3)}{2} \right)$ i.e., $(-\frac{1}{2}, 3, 1)$

i.e., PR and QS have the same midpoint.

EXERCISE 15F

$$\begin{array}{lll} 1 \quad a \quad 2x = q & b \quad \frac{1}{2}x = n & c \quad -3x = p \\ \therefore \frac{1}{2}(2x) = \frac{1}{2}q & \therefore 2(\frac{1}{2}x) = 2n & \therefore 3x = -p \\ \therefore x = \frac{1}{2}q & \therefore x = 2n & \therefore \frac{1}{3}(3x) = -\frac{1}{3}p \\ & & \therefore x = -\frac{1}{3}p \\ d \quad q + 2x = r & e \quad 4s - 5x = t & f \quad 4m - \frac{1}{3}x = n \\ \therefore 2x = r - q & \therefore -5x = t - 4s & \therefore 4m - n = \frac{1}{3}x \\ \therefore x = \frac{1}{2}(r - q) & \therefore 5x = 4s - t & \therefore x = 12m - 3n \\ & \therefore x = \frac{4}{5}s - \frac{1}{5}t & \end{array}$$

$$\begin{array}{lll} 2 \quad a \quad 2y = r & b \quad \frac{1}{2}y = s & c \quad r + 2y = s \\ \therefore y = \frac{1}{2}r & \therefore y = 2s & \therefore 2y = s - r \\ = \frac{1}{2} \begin{bmatrix} -2 \\ 3 \end{bmatrix} & = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \therefore y = \frac{1}{2}s - \frac{1}{2}r \\ = \begin{bmatrix} -1 \\ \frac{3}{2} \end{bmatrix} & = \begin{bmatrix} 2 \\ 4 \end{bmatrix} & = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ \frac{3}{2} \end{bmatrix} \\ & & = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix} \\ d \quad 3s - 4y = r & & \therefore 3s - r = 4y \\ \therefore 3s - r = 4y & & \therefore y = \frac{3}{4}s - \frac{1}{4}r \\ = \begin{bmatrix} \frac{3}{4} \\ \frac{3}{2} \end{bmatrix} - \begin{bmatrix} -\frac{2}{4} \\ \frac{3}{4} \end{bmatrix} & & = \begin{bmatrix} \frac{3}{4} \\ \frac{3}{2} \end{bmatrix} - \begin{bmatrix} -\frac{2}{4} \\ \frac{3}{4} \end{bmatrix} \\ = \begin{bmatrix} \frac{5}{4} \\ \frac{3}{4} \end{bmatrix} & & \end{array}$$

$$3 \quad kx = a \quad \therefore k \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\therefore kx_1 = a_1 \quad \text{and} \quad kx_2 = a_2$$

$$\therefore x_1 = \frac{1}{k}a_1 \quad \text{and} \quad x_2 = \frac{1}{k}a_2 \quad \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{k}a_1 \\ \frac{1}{k}a_2 \end{bmatrix} = \frac{1}{k} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$4 \quad a \quad 2a + x = b$$

$$\therefore x = b - 2a = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -5 \end{bmatrix}$$

b $3x - a = 2b$

$$\therefore 3x = a + 2b$$

$$\therefore x = \frac{1}{3}(a + 2b) = \frac{1}{3} \left(\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right) = \frac{1}{3} \left(\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} \right) = \frac{1}{3} \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{3} \\ \frac{5}{3} \end{bmatrix}$$

$$\mathbf{c} \quad 2\mathbf{b} - 2\mathbf{x} = -\mathbf{a}$$

$$\therefore \mathbf{a} + 2\mathbf{b} = 2\mathbf{x}$$

$$\therefore \mathbf{x} = \frac{1}{2}(\mathbf{a} + 2\mathbf{b}) = \frac{1}{2} \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \quad \{\text{using } \mathbf{b}\} = \begin{bmatrix} \frac{3}{2} \\ -1 \\ \frac{5}{2} \end{bmatrix}$$

$$5 \quad \overrightarrow{\text{AB}} = \overrightarrow{\text{AO}} + \overrightarrow{\text{OB}} = -\overrightarrow{\text{OA}} + \overrightarrow{\text{OB}} = - \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

$$\therefore |\overrightarrow{\text{AB}}| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{9 + 16 + 4} = \sqrt{29} \text{ units}$$

$$6 \quad \overrightarrow{\text{OA}} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad \overrightarrow{\text{OB}} = \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix}, \quad \overrightarrow{\text{OC}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \overrightarrow{\text{OD}} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$$

$$\therefore \overrightarrow{\text{BD}} = \overrightarrow{\text{BO}} + \overrightarrow{\text{OD}} = -\overrightarrow{\text{OB}} + \overrightarrow{\text{OD}} = - \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix} + \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ 6 \end{bmatrix}$$

$$\text{and } \overrightarrow{\text{AC}} = \overrightarrow{\text{AO}} + \overrightarrow{\text{OC}} = - \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} \quad \therefore \overrightarrow{\text{BD}} = \begin{bmatrix} -2 \\ -6 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} = 2\overrightarrow{\text{AC}}$$

$$7 \quad \mathbf{a} \quad \overrightarrow{\text{BD}} = \frac{1}{2}\overrightarrow{\text{OA}} \\ = \frac{1}{2}\mathbf{a}$$

$$\mathbf{b} \quad \overrightarrow{\text{AB}} = \overrightarrow{\text{AO}} + \overrightarrow{\text{OB}} \\ = -\mathbf{a} + \mathbf{b} \\ = \mathbf{b} - \mathbf{a}$$

$$\mathbf{c} \quad \overrightarrow{\text{BA}} = -\overrightarrow{\text{AB}} \\ = -(\mathbf{b} - \mathbf{a}) \\ = -\mathbf{b} + \mathbf{a} \quad \text{or} \quad \mathbf{a} - \mathbf{b}$$

$$\mathbf{d} \quad \overrightarrow{\text{OD}} = \overrightarrow{\text{OB}} + \overrightarrow{\text{BD}} \\ = \mathbf{b} + \frac{1}{2}\mathbf{a}$$

$$\mathbf{e} \quad \overrightarrow{\text{AD}} = \overrightarrow{\text{AO}} + \overrightarrow{\text{OD}} \\ = -\mathbf{a} + \mathbf{b} + \frac{1}{2}\mathbf{a} \\ = -\frac{1}{2}\mathbf{a} + \mathbf{b}$$

$$\mathbf{f} \quad \overrightarrow{\text{DA}} = -\overrightarrow{\text{AD}} \\ = \frac{1}{2}\mathbf{a} - \mathbf{b}$$

$$8 \quad \mathbf{a} \quad \overrightarrow{\text{AD}} = \overrightarrow{\text{AB}} + \overrightarrow{\text{BD}} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -1 \end{bmatrix}$$

$$\mathbf{b} \quad \overrightarrow{\text{CB}} = \overrightarrow{\text{CA}} + \overrightarrow{\text{AB}} = -\overrightarrow{\text{AC}} + \overrightarrow{\text{AB}} = - \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix}$$

$$\mathbf{c} \quad \overrightarrow{\text{CD}} = \overrightarrow{\text{CB}} + \overrightarrow{\text{BD}} = \begin{bmatrix} -3 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \quad (\text{using } \mathbf{b}) = \begin{bmatrix} -3 \\ 6 \\ -5 \end{bmatrix}$$

$$9 \quad \mathbf{a} \quad \mathbf{a} + \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \quad \mathbf{b} \quad \mathbf{a} - \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$$

$$\mathbf{c} \quad \mathbf{b} + 2\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -9 \end{bmatrix}$$

$$\mathbf{d} \quad \mathbf{a} - 3\mathbf{c} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 10 \end{bmatrix}$$

$$\mathbf{e} \quad \mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

$$\mathbf{f} \quad \mathbf{c} - \frac{1}{2}\mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{3}{2} \\ -\frac{7}{2} \end{bmatrix}$$

$$\mathbf{g} \quad \mathbf{a} - \mathbf{b} - \mathbf{c} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix}$$

$$\mathbf{h} \quad 2\mathbf{b} - \mathbf{c} + \mathbf{a} = 2 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$$

$$\mathbf{10} \quad \mathbf{a} \quad |\mathbf{a}| = \sqrt{(-1)^2 + 1^2 + 3^2} = \sqrt{11} \text{ units}$$

$$\mathbf{b} \quad |\mathbf{b}| = \sqrt{1^2 + (-3)^2 + 2^2} = \sqrt{14} \text{ units}$$

$$\begin{aligned} \mathbf{c} \quad |\mathbf{b} + \mathbf{c}| &= \left| \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} \right| = \left| \begin{bmatrix} -1 \\ -1 \\ 6 \end{bmatrix} \right| \\ &= \sqrt{(-1)^2 + (-1)^2 + 6^2} \\ &= \sqrt{1 + 1 + 36} \\ &= \sqrt{38} \text{ units} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad |\mathbf{a} - \mathbf{c}| &= \left| \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} \right| = \left| \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right| \\ &= \sqrt{1^2 + (-1)^2 + (-1)^2} \\ &= \sqrt{3} \text{ units} \end{aligned}$$

$$\mathbf{e} \quad |\mathbf{a}|\mathbf{b} = \sqrt{11} \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{11} \\ -3\sqrt{11} \\ 2\sqrt{11} \end{bmatrix}$$

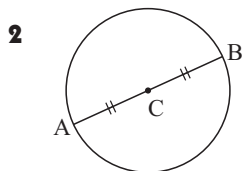
$$\mathbf{f} \quad \frac{1}{|\mathbf{a}|} \times \mathbf{a} = \frac{1}{\sqrt{11}} \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{bmatrix}$$

EXERCISE 15G

$$\mathbf{1} \quad \mathbf{a} \quad \mathbf{M} \text{ is } \left(\frac{3+(-1)}{2}, \frac{6+2}{2} \right) \\ \text{i.e., } \mathbf{M} \text{ is } (1, 4)$$

$$\begin{aligned} \mathbf{b} \quad \overrightarrow{\mathbf{CA}} &= \begin{bmatrix} 3 & -4 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \\ \overrightarrow{\mathbf{CM}} &= \begin{bmatrix} 1 & -4 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \\ \overrightarrow{\mathbf{CB}} &= \begin{bmatrix} -1 & -4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \frac{1}{2}\overrightarrow{\mathbf{CA}} + \frac{1}{2}\overrightarrow{\mathbf{CB}} &= \frac{1}{2} \begin{bmatrix} 7 \\ 5 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 3 \end{bmatrix} \text{ which is } \overrightarrow{\mathbf{CM}} \end{aligned}$$



$$\mathbf{a} \quad \overrightarrow{\mathbf{AC}} = \begin{bmatrix} 1 & -3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \quad \therefore \mathbf{B} \text{ is } (1 - 2, 4 + 6), \text{ i.e., } (-1, 10)$$

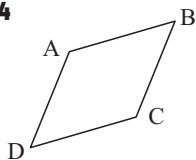
$$\mathbf{b} \quad \overrightarrow{\mathbf{AC}} = \begin{bmatrix} -1 & -0 \\ -2 & -5 \end{bmatrix} = \begin{bmatrix} -1 \\ -7 \end{bmatrix} \quad \therefore \mathbf{B} \text{ is } (-1 - 1, -2 - 7), \text{ i.e., } (-2, -9)$$

$$\mathbf{c} \quad \overrightarrow{\mathbf{AC}} = \begin{bmatrix} 3 & -1 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \quad \therefore \mathbf{B} \text{ is } (3 + 4, 0 + 4), \text{ i.e., } (7, 4)$$

$$\mathbf{3} \quad \overrightarrow{\mathbf{AB}} = \begin{bmatrix} 2 & -1 \\ 3 & -5 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix}$$

$$\begin{aligned} \therefore \mathbf{C} \text{ is } (2 + 3, 3 - 2, -3 - 5) \text{ i.e., } (5, 1, -8) \\ \mathbf{D} \text{ is } (5 + 3, 1 - 2, -8 - 5) \text{ i.e., } (8, -1, -13) \\ \mathbf{E} \text{ is } (8 + 3, -1 - 2, -13 - 5) \text{ i.e., } (11, -3, -18) \end{aligned}$$

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$$\begin{aligned} \text{a} \quad \overrightarrow{AB} &= \begin{bmatrix} 4-3 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \overrightarrow{DC} &= \begin{bmatrix} -1-(-2) \\ 4-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{aligned}$$

$$\text{Now } \overrightarrow{AB} = \overrightarrow{DC}$$

$\therefore$ sides AB and DC are equal in length and parallel.

This is sufficient to deduce that ABCD is a parallelogram.

$$\begin{aligned} \text{b} \quad \overrightarrow{AB} &= \begin{bmatrix} -1-5 \\ 2-0 \\ 4-3 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix} \\ \overrightarrow{DC} &= \begin{bmatrix} 4-10 \\ -3-5 \\ 6-5 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix} \end{aligned}$$

$$\text{So } \overrightarrow{AB} = \overrightarrow{DC}$$

$\therefore$ sides AB and DC are equal in length and parallel.

This is sufficient to deduce that ABCD is a parallelogram.

$$\begin{aligned} \text{c} \quad \overrightarrow{AB} &= \begin{bmatrix} 1-2 \\ 4-3 \\ -1-2 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -3 \end{bmatrix} \\ \overrightarrow{DC} &= \begin{bmatrix} -2-(-1) \\ 6-1 \\ -2-2 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ -4 \end{bmatrix} \end{aligned}$$

$$\text{So, } \overrightarrow{AB} \neq \overrightarrow{DC}$$

$\therefore$ ABCD cannot be a parallelogram.

 5 a Let D be (a, b)

$$\text{Now } \overrightarrow{CD} = \overrightarrow{BA}$$

$$\therefore \begin{bmatrix} a-8 \\ b-2 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 0-1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a-8 \\ b+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore a = 9, b = -1$$

So, D is $(9, -1)$

 b Let R be (a, b, c)

$$\text{Now } \overrightarrow{SR} = \overrightarrow{PQ}$$

$$\therefore \begin{bmatrix} a-4 \\ b-0 \\ c-7 \end{bmatrix} = \begin{bmatrix} -2-(-1) \\ 5-4 \\ 2-3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a-4 \\ b \\ c-7 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\therefore a = 3, b = 1, c = 6$$

So, R is $(3, 1, 6)$

 c Let X be (a, b, c)

$$\text{Now } \overrightarrow{WX} = \overrightarrow{ZY}$$

$$\therefore \begin{bmatrix} a-1 \\ b-5 \\ c-8 \end{bmatrix} = \begin{bmatrix} 3-0 \\ -2-4 \\ -2-6 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a+1 \\ b-5 \\ c-8 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -8 \end{bmatrix}$$

$$\therefore a = 2, b = -1, c = 0$$

So, X is $(2, -1, 0)$

6 a

$$r \begin{bmatrix} 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -8 \\ -27 \end{bmatrix}$$

$$\therefore \begin{bmatrix} r+2s \\ -r+5s \end{bmatrix} = \begin{bmatrix} -8 \\ -27 \end{bmatrix}$$

$$\therefore r + 2s = -8 \quad \dots\dots (1)$$

$$-r + 5s = -27$$

$$\text{adding } 7s = -35$$

$$\therefore s = -5$$

$$\text{and in (1) } r + 2(-5) = -8$$

$$\therefore r - 10 = -8$$

$$\therefore r = 2$$

So, $r = 2, s = -5$

b

$$r \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -19 \\ 2 \end{bmatrix}$$

$$\therefore 2r + s = 7 \quad \dots\dots (1)$$

$$-3r + 7s = -19 \quad \dots\dots (2)$$

$$r + 2s = 2 \quad \dots\dots (3)$$

$$\text{From (1) and (2) } -4r - 2s = -14$$

$$r + 2s = 2$$

$$\therefore -3r = -12$$

$$\therefore r = 4$$

$$\text{In (1) } 2(4) + s = 7$$

$$\therefore s = -1$$

$$\text{Checking in (3) } r + 2s = 4 + 2(-1) = 2$$

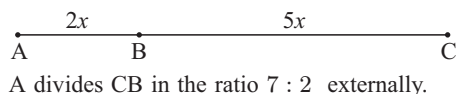
$\therefore r = 4, s = -1$ satisfies all equations

7

$$\text{a} \quad \overrightarrow{AB} = \begin{bmatrix} 4-(-2) \\ 3-1 \\ 0-4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

$$\overrightarrow{BC} = \begin{bmatrix} 19-4 \\ 8-3 \\ -10-0 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ -10 \end{bmatrix} = 5 \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

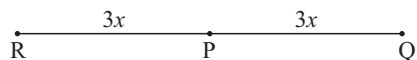
$\therefore \overrightarrow{AB}$ is parallel to $\overrightarrow{BC}$, and since B is a common point, A, B and C are collinear.



$$\mathbf{b} \quad \vec{RP} = \begin{bmatrix} 2 - (-1) \\ 1 - 7 \\ 1 - 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \text{ and}$$

$$\vec{PQ} = \begin{bmatrix} 5 - 2 \\ -5 - 1 \\ -2 - 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$\therefore \vec{RP}$ is parallel to $\vec{PQ}$, and since P is a common point, P, Q and R are collinear.



$\therefore$ Q divides PR in the ratio 1 : 2 externally.

$$\mathbf{8} \quad \mathbf{a} \quad \begin{array}{c} \bullet \\ C(-13, a, b) \end{array} \quad \begin{array}{c} \bullet \\ A(2, -3, 4) \end{array} \quad \begin{array}{c} \bullet \\ B(11, -9, 7) \end{array}$$

Since A, B and C are collinear, $\vec{CA}$ is parallel to $\vec{AB}$.

$$\therefore \begin{bmatrix} 15 \\ -3 - a \\ 4 - b \end{bmatrix} = k \begin{bmatrix} 9 \\ -6 \\ 3 \end{bmatrix}$$

$$\therefore 9k = 15,$$

$$-3 - a = -6k$$

$$\text{and } 4 - b = 3k$$

$$\therefore k = \frac{5}{3},$$

$$\text{and so } a = -3 + 6k$$

$$= -3 + 10$$

$$= 7$$

$$\text{and } b = 4 - 3k = 4 - 5 = -1$$

$$\mathbf{b} \quad \begin{array}{c} \bullet \\ L(4, -3, 7) \end{array} \quad \begin{array}{c} \bullet \\ K(1, -1, 0) \end{array} \quad \begin{array}{c} \bullet \\ M(a, 2, b) \end{array}$$

Since K, L and M are collinear, $\vec{LK}$ is parallel to $\vec{KM}$.

$$\therefore \begin{bmatrix} -3 \\ 2 \\ -7 \end{bmatrix} = k \begin{bmatrix} a - 1 \\ 3 \\ b \end{bmatrix}$$

$$\therefore 3k = 2,$$

$$k(a - 1) = -3$$

$$\text{and } kb = -7$$

$$\therefore k = \frac{2}{3},$$

$$a - 1 = -\frac{3}{k} = -\frac{9}{2}$$

$$\text{i.e., } a = -\frac{7}{2}$$

$$\text{and } b = -\frac{7}{k} = -\frac{21}{2}$$

EXERCISE 15H

$$\mathbf{1} \quad \text{Since } \mathbf{a} \text{ and } \mathbf{b} \text{ are parallel, then } \mathbf{b} = k\mathbf{a} \quad \therefore \begin{bmatrix} -6 \\ r \\ s \end{bmatrix} = k \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ 3k \end{bmatrix}$$

$$\therefore 2k = -6, \quad r = -k, \quad s = 3k \quad \therefore k = -3, \quad r = 3, \quad s = -9$$

$$\mathbf{2} \quad \text{If } \begin{bmatrix} a \\ 2 \\ b \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \text{ are parallel, then } \begin{bmatrix} a \\ 2 \\ b \end{bmatrix} = k \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

$$\text{i.e., } 2 = -k, \quad a = 3k, \quad b = 2k \quad \therefore k = -2, \quad a = -6 \quad \text{and} \quad b = -4$$

$$\mathbf{3} \quad \mathbf{a} \quad \text{Let the vector parallel to } \mathbf{a} \text{ be } k\mathbf{a}$$

$$\text{i.e., } k\mathbf{a} = k \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2k \\ -k \\ -2k \end{bmatrix}$$

$$k\mathbf{a} \text{ has length} = 1,$$

$$\text{so } \sqrt{(2k)^2 + (-k)^2 + (-2k)^2} = 1$$

$$\therefore 4k^2 + k^2 + 4k^2 = 1$$

$$\therefore 9k^2 = 1$$

$$\text{Choosing } k = \frac{1}{3}, \quad \therefore k = \pm \frac{1}{3}$$

$$\text{the vector is } \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$$

$$\mathbf{b} \quad \text{Let the vector parallel to } \mathbf{b} \text{ be } k\mathbf{b}$$

$$\text{i.e., } k\mathbf{b} = k \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2k \\ -k \\ 2k \end{bmatrix}$$

$$k\mathbf{b} \text{ has length} = 2,$$

$$\text{so } \sqrt{(-2k)^2 + (-k)^2 + (2k)^2} = 2$$

$$\therefore 4k^2 + k^2 + 4k^2 = 4$$

$$\therefore 9k^2 = 4$$

$$\text{Choosing } k = \frac{2}{3}, \quad \therefore k = \pm \frac{2}{3}$$

$$\text{the vector is } \begin{bmatrix} -\frac{4}{3} \\ -\frac{2}{3} \\ \frac{4}{3} \end{bmatrix}$$

- 4 a** $\vec{AB} = 3\vec{CD}$ means that $\vec{AB}$ is parallel to $\vec{CD}$ and 3 times its length.

c 
 $\vec{AB} = 2\vec{BC}$ means that A, B and C are collinear and the length of $\vec{AB}$ is twice the length of $\vec{BC}$ (i.e., B cuts $\vec{AC}$ internally in the ratio 2 : 1).

- b** $\vec{RS} = -\frac{1}{2}\vec{KL}$ means that $\vec{RS}$ is parallel to $\vec{KL}$, half its length and in the opposite direction.

d 
 $\vec{BC} = \frac{1}{3}\vec{AC}$ means that A, B and C are collinear and the length of $\vec{BC}$ is one third the length of $\vec{AC}$ (i.e., B cuts $\vec{AC}$ internally in the ratio 2 : 1).

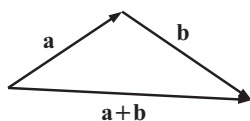
5 $\vec{OP} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $\vec{OQ} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$, $\vec{OR} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $\vec{OS} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$

a $\vec{PR} = \vec{PO} + \vec{OR} = -\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$ and

$\vec{QS} = \vec{QO} + \vec{OS} = -\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \\ 6 \end{bmatrix} = 2\begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix} = 2\vec{PR}$ and so $QS \parallel PR$

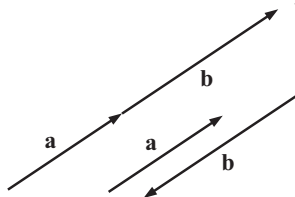
- b** $\therefore$ As $\vec{QS} = 2\vec{PR}$ then $|\vec{QS}| = 2|\vec{PR}|$, i.e., QS is twice as long as PR.

- 6 a** Consider **a** not parallel to **b**:



Clearly, $|\mathbf{a}| + |\mathbf{b}| > |\mathbf{a} + \mathbf{b}|$

- b** Consider **a** parallel to **b**:



$$|\mathbf{a}| + |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$$

or

$$|\mathbf{a}| + |\mathbf{b}| > |\mathbf{a} + \mathbf{b}|$$

- c** If $\mathbf{a} = \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$, then $\mathbf{a} + \mathbf{b} = \mathbf{b}$ $\therefore |\mathbf{a}| + |\mathbf{b}| = 0 + |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$

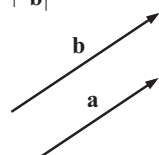
Similarly if $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} = \mathbf{0}$, then $\mathbf{a} + \mathbf{b} = \mathbf{a}$

$$\therefore |\mathbf{a}| + |\mathbf{b}| = |\mathbf{a}| + 0 = |\mathbf{a}| = |\mathbf{a} + \mathbf{b}|$$

If $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$, then $\mathbf{a} + \mathbf{b} = \mathbf{0}$

$$\therefore |\mathbf{a}| + |\mathbf{b}| = |\mathbf{a} + \mathbf{b}|$$

Combining all possibilities, $|\mathbf{a}| + |\mathbf{b}| \geq |\mathbf{a} + \mathbf{b}|$ i.e., $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$



EXERCISE 15I

1 a $\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
 $\therefore |\mathbf{i} - \mathbf{j} + \mathbf{k}| = \sqrt{3}$ units

b $3\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$
 $\therefore |3\mathbf{i} - \mathbf{j} + \mathbf{k}| = \sqrt{9 + 1 + 1} = \sqrt{11}$ units

c $\mathbf{i} - 5\mathbf{k} = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$
 $\therefore |\mathbf{i} - 5\mathbf{k}| = \sqrt{1 + 25} = \sqrt{26}$ units

d $\frac{1}{2}(\mathbf{j} + \mathbf{k}) = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ $\therefore \left| \frac{1}{2}(\mathbf{j} + \mathbf{k}) \right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ units

$$\begin{array}{lll} \mathbf{2} \quad \mathbf{a} & \text{length} = 1 & \mathbf{b} \quad \text{length} = 1 & \mathbf{c} \quad \text{length} = 1 \\ \therefore \sqrt{0^2 + k^2} = 1 & & \therefore \sqrt{k^2 + 0} = 1 & \therefore \sqrt{k^2 + 1} = 1 \\ \therefore k^2 = 1 & & \therefore k^2 = 1 & \therefore k^2 + 1 = 1 \\ \therefore k = \pm 1 & & \therefore k = \pm 1 & \therefore k^2 = 0 \\ & & & \therefore k = 0 \end{array}$$

$$\begin{array}{ll} \mathbf{d} & \text{length} = 1 \\ \therefore \sqrt{\left(\frac{1}{4}\right)^2 + k^2 + \frac{1}{16}} = 1 & \mathbf{e} \quad \text{length} = 1 \\ \therefore \sqrt{k^2 + \frac{5}{16}} = 1 & \therefore \sqrt{k^2 + \frac{4}{9} + \frac{1}{9}} = 1 \\ \therefore k^2 = \frac{11}{16} & \therefore \sqrt{k^2 + \frac{5}{9}} = 1 \\ \therefore k = \pm \frac{\sqrt{11}}{4} & \therefore k^2 = \frac{4}{9} \\ & \therefore k = \pm \frac{2}{3} \end{array}$$

$$\begin{array}{lll} \mathbf{3} \quad \mathbf{a} & \text{length} & \mathbf{b} \quad \text{length} & \mathbf{c} \quad \text{length} \\ = \sqrt{3^2 + 4^2} & & = \sqrt{2^2 + (-1)^2 + 1^2} & = \sqrt{1^2 + 2^2 + (-2)^2} \\ = 5 \text{ units} & & = \sqrt{4 + 1 + 1} & = \sqrt{1 + 4 + 4} \\ & & = \sqrt{6} \text{ units} & = \sqrt{9} \\ & & & = 3 \text{ units} \end{array}$$

$$\mathbf{d} \quad \text{length} = \sqrt{(-2.36)^2 + (5.65)^2} \div 6.12 \text{ units}$$

$$\mathbf{4} \quad \mathbf{a} \quad \mathbf{i} + 2\mathbf{j} \text{ has length } \sqrt{1^2 + 2^2} = \sqrt{5} \text{ units} \quad \therefore \text{unit vector} = \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$$

$$\mathbf{b} \quad 2\mathbf{i} - 3\mathbf{k} \text{ has length } \sqrt{2^2 + 0^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13} \text{ units} \\ \therefore \text{unit vector is } \frac{1}{\sqrt{13}}(2\mathbf{i} - 3\mathbf{k})$$

$$\mathbf{c} \quad -2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k} \text{ has length } \sqrt{(-2)^2 + (-5)^2 + (-2)^2} = \sqrt{4 + 25 + 4} = \sqrt{33} \text{ units} \\ \therefore \text{unit vector is } \frac{1}{\sqrt{33}}(-2\mathbf{i} - 5\mathbf{j} - 2\mathbf{k})$$

$$\mathbf{5} \quad \mathbf{a} \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ has length } \sqrt{2^2 + (-1)^2} = \sqrt{5} \text{ units} \\ \therefore \text{the unit vector in the same direction is } \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ \therefore \text{the vector of length 3 units in the same direction is } \frac{3}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{6}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}} \end{bmatrix}$$

$$\mathbf{b} \quad \begin{bmatrix} -1 \\ -4 \end{bmatrix} \text{ has length } \sqrt{(-1)^2 + (-4)^2} = \sqrt{17} \text{ units} \\ \therefore \text{the unit vector in the opposite direction is } -\frac{1}{\sqrt{17}} \begin{bmatrix} -1 \\ -4 \end{bmatrix} = \frac{1}{\sqrt{17}} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\ \therefore \text{the vector of length 2 units in the opposite direction is } \frac{2}{\sqrt{17}} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{17}} \\ \frac{8}{\sqrt{17}} \end{bmatrix}$$

$$\mathbf{c} \quad \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \text{ has length } \sqrt{(-1)^2 + 4^2 + 1^2} = \sqrt{18} = 3\sqrt{2} \text{ units} \\ \therefore \text{the unit vector in the same direction is } \frac{1}{3\sqrt{2}} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \\ \therefore \text{the vector of length 6 units in the same direction is } \frac{6}{3\sqrt{2}} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 4\sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$\mathbf{d} \quad \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} \text{ has length } \sqrt{(-1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3 \text{ units} \\ \therefore \text{the unit vector in the opposite direction is } -\frac{1}{3} \begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\ \therefore \text{the vector of length 5 units in the opposite direction is } \frac{5}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{10}{3} \\ \frac{10}{3} \end{bmatrix}$$

EXERCISE 15J.1

$$\begin{array}{llll}
 \mathbf{1} & \mathbf{a} & \mathbf{q} \bullet \mathbf{p} & \mathbf{b} & \mathbf{q} \bullet \mathbf{r} & \mathbf{c} & \mathbf{q} \bullet (\mathbf{p} + \mathbf{r}) & \mathbf{d} & 3\mathbf{r} \bullet \mathbf{q} \\
 & & = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 2 \end{bmatrix} & & = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 4 \end{bmatrix} & & = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 6 \end{bmatrix} & & = \begin{bmatrix} -6 \\ 12 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 5 \end{bmatrix} \\
 & & = -3 + 10 & & = 2 + 20 & & = -1 + 30 & & = 6 + 60 \\
 & & = 7 & & = 22 & & = 29 & & = 66
 \end{array}$$

$$\begin{array}{llll}
 \mathbf{e} & 2\mathbf{p} \bullet 2\mathbf{p} & \mathbf{f} & \mathbf{i} \bullet \mathbf{p} & \mathbf{g} & \mathbf{q} \bullet \mathbf{j} & \mathbf{h} & \mathbf{i} \bullet \mathbf{i} \\
 & = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 6 \\ 4 \end{bmatrix} & & = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 2 \end{bmatrix} & & = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \end{bmatrix} & & = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
 & = 36 + 16 & & = 3 + 0 & & = 0 + 5 & & = 1 + 0 \\
 & = 52 & & = 3 & & = 5 & & = 1
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{2} & \mathbf{a} \quad \mathbf{a} \bullet \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \\
 & = 2(-1) + 1(1) + 3(1) \\
 & = -2 + 1 + 3 \\
 & = 2 \\
 & \mathbf{b} \quad \mathbf{b} \bullet \mathbf{a} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\
 & = (-1)(2) + 1(1) + 1(3) \\
 & = -2 + 1 + 3 \\
 & = 2
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{c} & |\mathbf{a}|^2 = (\sqrt{2^2 + 1^2 + 3^2})^2 \\
 & = 14 \\
 \mathbf{d} & \mathbf{a} \bullet \mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \\
 & = 2(2) + 1(1) + 3(3) \\
 & = 14
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{e} & \mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) \\
 & = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right) \\
 & = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \\
 & = 2(-1) + 1(0) + 3(2) \\
 & = 4 \\
 \mathbf{f} & \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} \\
 & = 2 + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \{\text{using } \mathbf{a}\} \\
 & = 2 + 2(0) + 1(-1) + 3(1) \\
 & = 4
 \end{array}$$

$$\begin{array}{lll}
 \mathbf{3} & \mathbf{a} & (\mathbf{i} + \mathbf{j} - \mathbf{k}) \bullet (2\mathbf{j} + \mathbf{k}) \\
 & = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \\
 & = 1(0) + 1(2) - 1(1) \\
 & = 1 \\
 & \mathbf{b} & \mathbf{i} \bullet \mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
 & & = 1 \\
 & \mathbf{c} & \mathbf{i} \bullet \mathbf{j} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
 & & = 0
 \end{array}$$

$$\begin{array}{l}
 \mathbf{4} \quad \mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet \left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right) \\
 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \bullet \begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{bmatrix} \\
 = a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\
 = a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \\
 = (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\
 = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}
 \end{array}$$

$$\begin{array}{l}
 \therefore \mathbf{p} \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{p} \bullet \mathbf{c} + \mathbf{p} \bullet \mathbf{d} \quad \text{and if we let } \mathbf{p} = \mathbf{a} + \mathbf{b}, \\
 \text{then } (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{p} \bullet (\mathbf{c} + \mathbf{d}) \\
 = \mathbf{p} \bullet \mathbf{c} + \mathbf{p} \bullet \mathbf{d}
 \end{array}$$

$$\begin{aligned}
&= (\mathbf{a} + \mathbf{b}) \bullet \mathbf{c} + (\mathbf{a} + \mathbf{b}) \bullet \mathbf{d} \\
&= \mathbf{c} \bullet (\mathbf{a} + \mathbf{b}) + \mathbf{d} \bullet (\mathbf{a} + \mathbf{b}) \\
&= \mathbf{c} \bullet \mathbf{a} + \mathbf{c} \bullet \mathbf{b} + \mathbf{d} \bullet \mathbf{a} + \mathbf{d} \bullet \mathbf{b} \\
&= \mathbf{a} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{d}
\end{aligned}$$

5

a $\begin{bmatrix} 3 \\ t \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0$ $\therefore -6 + t = 0$ $\therefore t = 6$

b $\begin{bmatrix} t \\ t+2 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -4 \end{bmatrix} = 0$ $\therefore 3t - 4(t+2) = 0$ $\therefore 3t - 4t - 8 = 0$ $\therefore -t = 8$ $\therefore t = -8$

c $\begin{bmatrix} t \\ t+2 \end{bmatrix} \bullet \begin{bmatrix} 2-3t \\ t \end{bmatrix} = 0$ $\therefore 2t - 3t^2 + t^2 + 2t = 0$ $\therefore -2t^2 + 4t = 0$ $\therefore t^2 - 2t = 0$ $\therefore t(t-2) = 0$ $\therefore t = 0 \text{ or } 2$

d $\begin{bmatrix} 3 \\ -1 \\ t \end{bmatrix} \bullet \begin{bmatrix} 2t \\ -3 \\ -4 \end{bmatrix} = 0$ $\therefore 3(2t) + (-1)(-3) + t(-4) = 0$ $\therefore 6t + 3 - 4t = 0$ $\therefore 2t + 3 = 0$ $\therefore t = -\frac{3}{2}$

6

a If $\mathbf{p} \parallel \mathbf{q}$ then $\begin{bmatrix} 3 \\ t \end{bmatrix} = k \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ where $k \neq 0$ $\therefore 3 = -2k$ and $t = k$ $\therefore k = -\frac{3}{2}$ and $t = -\frac{3}{2}$

b If $\mathbf{r} \parallel \mathbf{s}$ then $\begin{bmatrix} t \\ t+2 \end{bmatrix} = k \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ where $k \neq 0$ $\therefore t = 3k$ and $t+2 = -4k$ $\therefore t+2 = -4\left(\frac{t}{3}\right)$

c If $\mathbf{a} \parallel \mathbf{b}$ then $\begin{bmatrix} t \\ t+2 \end{bmatrix} = k \begin{bmatrix} 2-3t \\ t \end{bmatrix}$ $\therefore 3t+6 = -4t$ $\therefore 7t = -6$ $\therefore t = -\frac{6}{7}$

$\therefore t = k(2-3t)$ and $t+2 = kt$ $\therefore \frac{t}{2-3t} = \frac{t+2}{t}$ {equating k 's}

$\therefore t^2 = (t+2)(2-3t)$

$\therefore t^2 = 2t - 3t^2 + 4 - 6t$

$\therefore 4t^2 + 4t - 4 = 0$

$\therefore t^2 + t - 1 = 0$ which has $\Delta = 1^2 - 4(1)(-1) = 5$

$\therefore t = \frac{-1 \pm \sqrt{5}}{2}$

d If $\mathbf{a} \parallel \mathbf{b}$ then $\begin{bmatrix} 3 \\ -1 \\ t \end{bmatrix} = k \begin{bmatrix} 2t \\ -3 \\ -4 \end{bmatrix}$ where $k \neq 0$

$\therefore 3 = 2kt, -1 = -3k$ and $t = -4k$

$\therefore k = \frac{1}{3}$ and $3 = \frac{2}{3}t, t = -\frac{4}{3}$

i.e., $t = \frac{9}{2}$ and $-\frac{4}{3}$ simultaneously which is impossible.

$\therefore$ the vectors can never be parallel.

7 **a** $\bullet$ **b**

$= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$= 3(-1) + 1(1) + 2(1)$

$= 0$

b $\bullet$ **c**

$= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix}$

$= (-1)(1) + 1(5) + 1(-4)$

$= 0$

a $\bullet$ **c**

$= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix}$

$= (3)(1) + 1(5) + 2(-4)$

$= 0$

$\therefore$ **a**, **b** and **c** are mutually perpendicular.

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad & \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \\
 &= 1(2) + 1(3) + 5(-1) \\
 &= 0 \\
 &\therefore \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \text{ are perpendicular} \\
 \mathbf{b} \quad & \begin{bmatrix} 3 \\ t \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 1-t \\ -3 \\ 4 \end{bmatrix} = 0 \\
 &\therefore 3(1-t) + t(-3) + (-2)4 = 0 \\
 &\therefore 3 - 3t - 3t - 8 = 0 \\
 &\therefore -6t = 5 \\
 &\therefore t = -\frac{5}{6}
 \end{aligned}$$

9 We have three points: A(5, 1, 2) B(6, -1, 0) C(3, 2, 0)

$$\text{Then } \overrightarrow{AB} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \quad \overrightarrow{AC} = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \overrightarrow{BC} = \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix}$$

$$\text{Now } \overrightarrow{AB} \bullet \overrightarrow{AC} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} = (-2) + (-2) + 4 = 0$$

$\therefore \overrightarrow{AB}$ is perpendicular to $\overrightarrow{AC}$ and so $\triangle ABC$ is right-angled at A.

$$\begin{aligned}
 \mathbf{10} \quad \mathbf{a} \quad & \begin{array}{ccc} \text{A}(2, 4, 2) & \text{B}(-1, 2, 3) & \overrightarrow{AB} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \\ \text{D}(0, 5, 5) & \text{C}(-3, 3, 6) & \overrightarrow{DC} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \end{array} \\
 & \overrightarrow{BC} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \quad \therefore \overrightarrow{AB} \text{ is parallel to } \overrightarrow{DC} \text{ and} \\
 & \overrightarrow{BC} \text{ is parallel to } \overrightarrow{AD}. \\
 & \therefore ABCD \text{ is a parallelogram.}
 \end{aligned}$$

$$\mathbf{b} \quad |\overrightarrow{AB}| = \sqrt{14} \text{ units and } |\overrightarrow{BC}| = \sqrt{14} \text{ units} \quad \therefore ABCD \text{ is a rhombus.}$$

$$\mathbf{c} \quad \overrightarrow{AC} \bullet \overrightarrow{BD} = \begin{bmatrix} -5 \\ -1 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = (-5) \times 1 + (-1) \times 3 + 4(2) = 0$$

$\therefore \overrightarrow{AC}$ is perpendicular to $\overrightarrow{BD}$ which illustrates that the diagonals of a rhombus are perpendicular.

11 a $x - y = 3$ has gradient $+\frac{1}{1}$ and so has direction vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$3x + 2y = 11$ has gradient $-\frac{3}{2}$ and so has direction vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

$$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \sqrt{1+1}\sqrt{4+9}\cos\theta$$

$$\therefore 2 - 3 = \sqrt{2}\sqrt{13}\cos\theta$$

$$\therefore \frac{-1}{\sqrt{26}} = \cos\theta$$

$$\therefore \theta \doteq 101.3 \quad \therefore \text{the angle is } 101.3^\circ \text{ or } 78.7^\circ$$

b $y = x + 2$ has slope $1 = \frac{1}{1}$ $\therefore$ direction vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$y = 1 - 3x$ has slope $-3 = \frac{-3}{1}$ $\therefore$ direction vector is $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

$$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \sqrt{1+1}\sqrt{1+9}\cos\theta$$

$$\therefore 1 - 3 = \sqrt{2}\sqrt{10}\cos\theta$$

$$\therefore \frac{-2}{\sqrt{20}} = \cos\theta$$

$$\therefore \theta \doteq 116.6 \quad \therefore \text{the angle is } 116.6^\circ \text{ or } 63.4^\circ$$

c $y + x = 7$ has slope $-1 = \frac{-1}{1}$ $\therefore$ direction vector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$x - 3y + 2 = 0$ has slope $\frac{1}{3}$ $\therefore$ direction vector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

$$\begin{aligned}\therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 1 \end{bmatrix} &= \sqrt{1+1}\sqrt{9+1}\cos\theta \\ \therefore 3-1 &= \sqrt{2}\sqrt{10}\cos\theta \\ \therefore \frac{2}{\sqrt{20}} &= \cos\theta \\ \therefore \theta &\doteq 63.4^\circ \quad \therefore \text{the angle is } 63.4^\circ \text{ or } 116.6^\circ\end{aligned}$$

d $y = 2 - x$ has slope $-1 = \frac{-1}{1}$ $\therefore$ has direction vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$x - 2y = 7$ has slope $\frac{1}{2}$ $\therefore$ has direction vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\begin{aligned}\therefore \begin{bmatrix} 1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \end{bmatrix} &= \sqrt{1+1}\sqrt{4+1}\cos\theta \\ \therefore 2-1 &= \sqrt{2}\sqrt{5}\cos\theta \\ \therefore \cos\theta &= \frac{1}{\sqrt{10}} \\ \therefore \theta &= \cos^{-1}\left(\frac{1}{\sqrt{10}}\right) \doteq 71.6^\circ \quad \therefore \text{the angle is } 71.6^\circ \text{ or } 108.4^\circ\end{aligned}$$

12 a $\mathbf{p} \bullet \mathbf{q} = |\mathbf{p}| |\mathbf{q}| \cos\theta$ **b** $\mathbf{p} \bullet \mathbf{q} = |\mathbf{p}| |\mathbf{q}| \cos\theta$
 $= 2 \times 5 \times \cos 60^\circ$ $= 6 \times 3 \times \cos 120^\circ$
 $= 5$ $= -9$

13 a $\begin{bmatrix} 5 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 5 \end{bmatrix} = -10 + 10 = 0$, so $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ is one such vector.
 $\therefore$ required vectors have form $k \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ where $k \neq 0$.

Note: $k \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, $k \neq 0$ is also ok.

b $\begin{bmatrix} -1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 2 - 2 = 0$, so $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is one such vector.
 $\therefore$ required vectors have form $k \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $k \neq 0$.

c $\begin{bmatrix} 3 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3 - 3 = 0$, so $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is one such vector.
 $\therefore$ required vectors have form $k \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $k \neq 0$.

d $\begin{bmatrix} -4 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 4 \end{bmatrix} = -12 + 12 = 0$, so $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is one such vector.
 $\therefore$ required vectors have form $k \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $k \neq 0$

e $\begin{bmatrix} 2 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 + 0 = 0$, so $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is one such vector.
 $\therefore$ required vectors have form $k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $k \neq 0$.

EXERCISE 15J.2

1 a $\mathbf{a} = -\mathbf{i} - \mathbf{j} + \mathbf{k} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{aligned}\mathbf{a} \bullet \mathbf{b} &= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= (-1) + (-1) + 1 \\ &= -1\end{aligned}$$

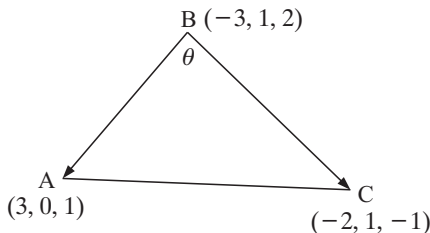
b $\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$
 $= \frac{-1}{\sqrt{3} \times \sqrt{3}}$
 $= -\frac{1}{3}$
 $\therefore \theta \doteq 109.5^\circ$
 (acute 70.5°)

c the projection vector of $\mathbf{a}$ on $\mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} \right) \frac{1}{|\mathbf{b}|} \mathbf{b} = \frac{-1}{\sqrt{3}} \times \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

d the projection vector has length $\left| -\frac{1}{3} \right| \sqrt{1^2 + 1^2 + 1^2} = \frac{1}{3} \sqrt{3} = \frac{1}{\sqrt{3}}$ units (as expected)

2 Given $A(3, 0, 1)$, $B(-3, 1, 2)$ and $C(-2, 1, -1)$,

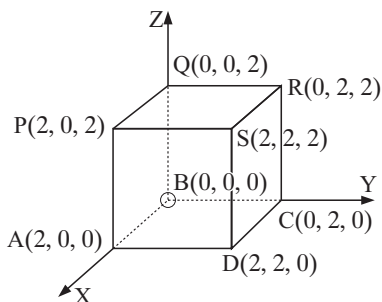
$$\overrightarrow{BC} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \quad \text{and} \quad \overrightarrow{BA} = \begin{bmatrix} 6 \\ -1 \\ -1 \end{bmatrix}$$



$$\begin{aligned} \therefore \cos \theta &= \frac{\overrightarrow{BC} \cdot \overrightarrow{BA}}{|\overrightarrow{BC}| |\overrightarrow{BA}|} \\ &= \frac{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -1 \\ -1 \end{bmatrix}}{\sqrt{1+9} \times \sqrt{36+1+1}} \\ &= \frac{6+0+3}{\sqrt{10} \times \sqrt{38}} \\ &= \frac{9}{\sqrt{380}} \quad \text{and so } \theta \doteq 62.5^\circ \end{aligned}$$

If $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are used we would find the exterior angle of the triangle at B, i.e., 117.5° .

3



a Suppose the origin is at B.

Now $\overrightarrow{BA} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $\overrightarrow{BS} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$$\therefore \overrightarrow{BA} \cdot \overrightarrow{BS} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 4 + 0 + 0 = 4$$

$$\begin{aligned} \therefore \cos \angle ABS &= \frac{4}{\sqrt{4+0+0} \times \sqrt{4+4+4}} \\ &= \frac{4}{2 \times 2\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

$$\therefore \angle ABS \doteq 54.7^\circ$$

b Consider vectors away from B.

$$\overrightarrow{BR} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \overrightarrow{BP} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$\overrightarrow{BR} \cdot \overrightarrow{BP} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 0 + 0 + 4 = 4$$

$$\begin{aligned} \therefore \cos \angle RBP &= \frac{4}{\sqrt{0+4+4} \sqrt{4+0+4}} \\ &= \frac{4}{\sqrt{8} \times \sqrt{8}} \\ &= \frac{1}{2} \quad \text{and so } \angle RBP = 60^\circ \end{aligned}$$

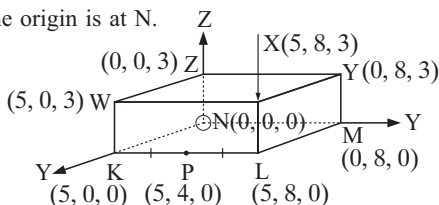
c $\overrightarrow{BP} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ and $\overrightarrow{BS} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$$\begin{aligned} \therefore \overrightarrow{BP} \cdot \overrightarrow{BS} &= \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \\ &= 4 + 0 + 4 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \therefore \cos \angle PBS &= \frac{8}{\sqrt{4+4} \sqrt{4+4+4}} \\ &= \frac{8}{\sqrt{96}} \\ \therefore \angle PBS &\doteq 35.3^\circ \end{aligned}$$

4 Suppose the origin is at N.

a



b $\vec{NY} = \begin{bmatrix} 0 \\ 8 \\ 3 \end{bmatrix}$ and $\vec{NP} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$

$$\vec{NY} \cdot \vec{NP} = \begin{bmatrix} 0 \\ 8 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = 0 + 32 + 0 = 32$$

$$\vec{NY} = \begin{bmatrix} 0 \\ 8 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{NX} = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

$$\vec{NY} \cdot \vec{NX} = \begin{bmatrix} 0 \\ 8 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix} = 0 + 64 + 9 = 73$$

$$\therefore \cos \angle YNP = \frac{32}{\sqrt{64+9}\sqrt{25+16}} = \frac{32}{\sqrt{73}\sqrt{41}} \quad \therefore \angle YNP \doteq 54.2^\circ$$

$$\therefore \cos \angle YNX = \frac{73}{\sqrt{64+9}\sqrt{25+64+9}} = \frac{73}{\sqrt{73}\sqrt{98}} = \sqrt{\frac{73}{98}} \quad \therefore \angle YNX \doteq 30.3^\circ$$

5 a M is the midpoint of BC $\therefore$ M is at $\left(\frac{2+1}{2}, \frac{2+3}{2}, \frac{2+1}{2}\right)$ i.e., $\left(\frac{3}{2}, \frac{5}{2}, \frac{3}{2}\right)$

b

Now $\vec{MD} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$ and $\vec{MA} = \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}$

$$\therefore \cos \theta = \frac{\vec{MD} \cdot \vec{MA}}{|\vec{MD}| |\vec{MA}|} = \frac{\begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}}{\sqrt{\frac{9}{4} + \frac{1}{4} + \frac{9}{4}} \sqrt{\frac{1}{4} + \frac{9}{4} + \frac{1}{4}}}$$

$$\therefore \cos \theta = \frac{\frac{3}{4} + \frac{3}{4} + \frac{3}{4}}{\sqrt{\frac{19}{4}} \sqrt{\frac{11}{4}}} = \frac{\frac{9}{4}}{\frac{\sqrt{201}}{4}} = \frac{9}{\sqrt{201}} \quad \text{and so} \quad \theta \doteq 51.5^\circ$$

6 a $\begin{bmatrix} 2 \\ t \\ t-2 \end{bmatrix} \cdot \begin{bmatrix} t \\ 3 \\ t \end{bmatrix} = 0 \quad \therefore \quad 2t + 3t + t(t-2) = 0$
 $\therefore \quad 5t + t^2 - 2t = 0$
 $\therefore \quad t^2 + 3t = 0$
 $\therefore \quad t(t+3) = 0 \quad \text{and so} \quad t = 0 \quad \text{or} \quad t = -3$

b Given that $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ r \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} s \\ t \\ 1 \end{bmatrix}$ are mutually perpendicular

$$\therefore \mathbf{a} \cdot \mathbf{b} = 0, \quad \mathbf{b} \cdot \mathbf{c} = 0 \quad \text{and} \quad \mathbf{a} \cdot \mathbf{c} = 0$$

$$\therefore \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ r \end{bmatrix} = 0 \quad \therefore \quad 2 + 4 + 3r = 0$$

$$\therefore \quad 3r = -6$$

$$\therefore \quad r = -2$$

$$\text{and} \quad \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} s \\ t \\ 1 \end{bmatrix} = 0 \quad \therefore \quad 2s + 2t - 2 = 0$$

$$\therefore \quad s + t = 1 \quad \text{.....(1)}$$

$$\text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} s \\ t \\ 1 \end{bmatrix} = 0 \quad \therefore \quad s + 2t + 3 = 0$$

$$\therefore \quad s + 2t = -3 \quad \text{.....(2)}$$

$$(2) - (1) \text{ gives } t = -4 \quad \text{and so} \quad s = 5 \quad \text{i.e., } r = -2, \quad s = 5 \quad \text{and} \quad t = -4$$

- 7 a** Choose any vector in the direction of

the X -axis, e.g., $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{aligned} \text{Then } \cos \theta &= \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\left| \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right| \left| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right|}} \\ &= \frac{1}{\sqrt{1}\sqrt{1+4+9}} \\ &= \frac{1}{\sqrt{14}} \quad \text{and so } \theta = 74.5^\circ \end{aligned}$$

- b** A line parallel to the Y -axis is $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$$\begin{aligned} \text{Then } \cos \theta &= \frac{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}}{\sqrt{1}\sqrt{1+1+9}} \\ &= \frac{1}{\sqrt{11}} \\ \therefore \theta &\doteq 72.45^\circ \end{aligned}$$

- 8** We want vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ such that $\mathbf{a} \neq \mathbf{0}$, $\mathbf{a} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{c}$ and $\mathbf{b} \neq \mathbf{c}$

For example, $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

In this case, $\mathbf{a} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{c} = 0$

- 9 a** $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b}) + (\mathbf{a} - \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b})$
 $= \mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{a} - \mathbf{a} \bullet \mathbf{b} - \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b}$
 $= 2\mathbf{a} \bullet \mathbf{a} + 2\mathbf{b} \bullet \mathbf{b}$
 $= 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$ as required

- b** $|\mathbf{a} + \mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b}) - (\mathbf{a} - \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b})$
 $= \mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b} - \mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} - \mathbf{b} \bullet \mathbf{b}$
 $= 2\mathbf{b} \bullet \mathbf{a} + 2\mathbf{a} \bullet \mathbf{b} \dots\dots(2)$
 $= 4\mathbf{a} \bullet \mathbf{b}$ as required {since $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$ }

- 10** We are given that $\mathbf{a} \neq \mathbf{b}$, $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$

a

Now if $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$

then $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2$

i.e., $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} + \mathbf{b}) = (\mathbf{a} - \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b})$

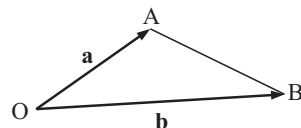
$$\therefore \mathbf{a} \bullet \mathbf{a} + \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{a} - \mathbf{a} \bullet \mathbf{b} - \mathbf{b} \bullet \mathbf{a} + \mathbf{b} \bullet \mathbf{b}$$

$$\therefore 2\mathbf{a} \bullet \mathbf{b} + 2\mathbf{b} \bullet \mathbf{a} = 0$$

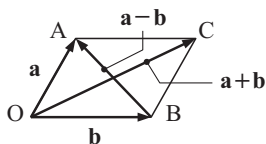
$$\therefore 4\mathbf{a} \bullet \mathbf{b} = 0 \quad \{\text{as } \mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}\}$$

$$\therefore \mathbf{a} \bullet \mathbf{b} = 0, \text{ and since neither } \mathbf{a} \text{ nor } \mathbf{b} = \mathbf{0}$$

$\mathbf{a}$ is perpendicular to $\mathbf{b}$.



- b** Consider the following diagram representing $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$:



We let $\overrightarrow{AC} = \mathbf{b}$

so $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + \mathbf{b}$

and $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -\mathbf{b} + \mathbf{a} = \mathbf{a} - \mathbf{b}$

$\therefore \mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ represent the diagonals of the parallelogram OACB

But if $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, then the diagonals must be equal in length.

This is only possible if OACB is a square, i.e., $\mathbf{a}$ is perpendicular to $\mathbf{b}$.

$$\begin{aligned}
 11 \quad (\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b}) &= \mathbf{a} \bullet \mathbf{a} - \mathbf{a} \bullet \mathbf{b} + \mathbf{b} \bullet \mathbf{a} - \mathbf{b} \bullet \mathbf{b} \\
 &= \mathbf{a} \bullet \mathbf{a} - \mathbf{b} \bullet \mathbf{b} \quad \{\text{since } \mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}\} \\
 &= |\mathbf{a}|^2 - |\mathbf{b}|^2 \\
 &= 9 - 16 \\
 &= -7
 \end{aligned}$$

12 The scalar product is only defined between two *vectors*.

Hence $(\mathbf{a} \bullet \mathbf{b}) \bullet \mathbf{c}$ is meaningless.

$\swarrow$ $\searrow$
 a scalar a vector

EXERCISE 15K.1

1 a $[2, -3, 1] \times [1, 4, -2]$

$$\begin{aligned}
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & 4 & -2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -3 & 1 \\ 4 & -2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} \\
 &= (6 - 4)\mathbf{i} + (1 - (-4))\mathbf{j} + (8 - (-3))\mathbf{k} \\
 &= 2\mathbf{i} + 5\mathbf{j} + 11\mathbf{k}
 \end{aligned}$$

b $[-1, 0, 2] \times [3, -1, -2]$

$$\begin{aligned}
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 2 \\ 3 & -1 & -2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 2 \\ -1 & -2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 2 & -1 \\ -2 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 0 \\ 3 & -1 \end{vmatrix} \\
 &= (0 - (-2))\mathbf{i} + (6 - 2)\mathbf{j} + (1 - 0)\mathbf{k} \\
 &= 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}
 \end{aligned}$$

c $(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \times (\mathbf{i} - \mathbf{k})$

$$\begin{aligned}
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \\
 &= (-1 + 0)\mathbf{i} + (-2 - (-1))\mathbf{j} + (0 - 1)\mathbf{k} \\
 &= -\mathbf{i} - \mathbf{j} - \mathbf{k}
 \end{aligned}$$

d $(2\mathbf{i} - \mathbf{k}) \times (\mathbf{j} + 3\mathbf{k})$

$$\begin{aligned}
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 1 & 3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & 2 \\ 3 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\
 &= (0 - (-1))\mathbf{i} + (0 - 6)\mathbf{j} + (2 - 0)\mathbf{k} \\
 &= \mathbf{i} - 6\mathbf{j} + 2\mathbf{k}
 \end{aligned}$$

2 If $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$,

$$\begin{aligned}
 \text{then } \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -1 & 3 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 3 & 1 \\ -1 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} \\
 &= -11\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}
 \end{aligned}$$

$$\therefore \mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -11 \\ -2 \\ 5 \end{bmatrix} = -11 - 4 + 15, \quad \mathbf{b} \bullet (\mathbf{a} \times \mathbf{b}) = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} -11 \\ -2 \\ 5 \end{bmatrix} = 11 - 6 - 5$$

$\therefore \mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{3} \quad \mathbf{a} \quad \mathbf{i} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{0} \quad \mathbf{j} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{0} \quad \mathbf{k} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{0}$$

$$\mathbf{b} \quad \mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k} \quad \mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i} \quad \mathbf{k} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}$$

$$\mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\mathbf{j} \quad \mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$$

If $\mathbf{a}$ is space vector, then $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ and $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

$$\mathbf{4} \quad \mathbf{a} \quad \mathbf{a} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ a_2 & a_3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} a_3 & a_1 \\ a_3 & a_1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} \\ = \mathbf{i} \times 0 + \mathbf{j} \times 0 + \mathbf{k} \times 0 \\ = \mathbf{0}$$

$$\mathbf{b} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ = \mathbf{i}(a_2b_3 - a_3b_2) + \mathbf{j}(a_3b_1 - a_1b_3) + \mathbf{k}(a_1b_2 - a_2b_1) \\ = -[\mathbf{i}(a_3b_2 - a_2b_3) + \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_2b_1 - a_1b_2)] \\ = - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\ = -\mathbf{b} \times \mathbf{a}$$

$$\mathbf{5} \quad \mathbf{a} \quad \mathbf{b} \times \mathbf{c} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \\ = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \\ = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} \\ = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$\mathbf{b} \quad \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = 1 + 12 + 4 = 17$$

$$\mathbf{c} \quad \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 1 \\ 0 & 1 & -2 \end{vmatrix} = 1 \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} + 3 \begin{vmatrix} 1 & 2 \\ -2 & 0 \end{vmatrix} + 2 \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \\ = 1(1) + 3(4) + 2(2) \\ = 1 + 12 + 4 \\ = 17$$

$$\begin{aligned} \mathbf{7} \quad \mathbf{a} \quad \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 0 & -1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \\ &= 2\mathbf{i} - \mathbf{j} - \mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \mathbf{a} \times \mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & 0 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} \\ &= 0\mathbf{i} + 5\mathbf{j} + 0\mathbf{k} \\ &= 5\mathbf{j} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}) &= 2\mathbf{i} - \mathbf{j} - \mathbf{k} + 5\mathbf{j} \\ &= 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} \quad \{\text{using } \mathbf{a} \text{ and } \mathbf{b}\} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times (2\mathbf{i} - \mathbf{j}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -1 & 0 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \\ &= 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} \end{aligned}$$

8 We suspect that $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

$$\begin{aligned} \mathbf{9} \quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 + c_2 & b_3 + c_3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} a_3 & a_1 \\ b_3 + c_3 & b_1 + c_1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 + c_1 & b_2 + c_2 \end{vmatrix} \\ &= \mathbf{i}(a_2(b_3 + c_3) - a_3(b_2 + c_2)) + \mathbf{j}(a_3(b_1 + c_1) - a_1(b_3 + c_3)) + \mathbf{k}(a_1(b_2 + c_2) - a_2(b_1 + c_1)) \\ &= \mathbf{i}(a_2b_3 - a_3b_2) + \mathbf{j}(a_3b_1 - a_1b_3) + \mathbf{k}(a_1b_2 - a_2b_1) \\ &\quad + \mathbf{i}(a_2c_3 - a_3c_2) + \mathbf{j}(a_3c_1 - a_1c_3) + \mathbf{k}(a_1c_2 - a_2c_1) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \quad \text{as required} \end{aligned}$$

10 Now $\mathbf{p} \times (\mathbf{c} + \mathbf{d}) = \mathbf{p} \times \mathbf{c} + \mathbf{p} \times \mathbf{d}$

$\therefore$ if we let $\mathbf{p} = (\mathbf{a} + \mathbf{b})$,

$$\begin{aligned} \text{then } (\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) &= \mathbf{p} \times (\mathbf{c} + \mathbf{d}) \\ &= \mathbf{p} \times \mathbf{c} + \mathbf{p} \times \mathbf{d} \\ &= (\mathbf{a} + \mathbf{b}) \times \mathbf{c} + (\mathbf{a} + \mathbf{b}) \times \mathbf{d} \\ &= -\mathbf{c} \times (\mathbf{a} + \mathbf{b}) - \mathbf{d} \times (\mathbf{a} + \mathbf{b}) \quad \{\text{since } \mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}\} \\ &= -\mathbf{c} \times \mathbf{a} - \mathbf{c} \times \mathbf{b} - \mathbf{d} \times \mathbf{a} - \mathbf{d} \times \mathbf{b} \quad \{\text{since } \mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}\} \\ &= \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{d} + \mathbf{b} \times \mathbf{d} \quad \{\text{since } \mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}\} \end{aligned}$$

$$\begin{array}{lll}
 \mathbf{11} \quad \mathbf{a} & \mathbf{a} \times (\mathbf{a} + \mathbf{b}) & \mathbf{b} \quad (\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) \quad \mathbf{c} \quad \mathbf{a} \text{ is perpendicular to } (\mathbf{a} \times \mathbf{b}), \\
 & = \mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} & = \mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{b} \quad 2\mathbf{a} \text{ is perpendicular to } (\mathbf{a} \times \mathbf{b}). \\
 & = \mathbf{0} + \mathbf{a} \times \mathbf{b} & = \mathbf{0} + \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} + \mathbf{0} \quad \therefore 2\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) = 0 \\
 & = \mathbf{a} \times \mathbf{b} & = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} \\
 & & = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{b} \\
 & & = \mathbf{0}
 \end{array}$$

$$\mathbf{12} \quad \mathbf{a} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = -4\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

$\therefore$ the vectors are $k(-4\mathbf{i} + \mathbf{j} + 3\mathbf{k})$, $k \neq 0$, $k \in \mathbf{R}$

$$\mathbf{b} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 4 \\ 5 & 0 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 4 & -1 \\ 2 & 5 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 3 \\ 5 & 0 \end{vmatrix} = 6\mathbf{i} + 22\mathbf{j} - 15\mathbf{k}$$

$\therefore$ the vectors are $k(6\mathbf{i} + 22\mathbf{j} - 15\mathbf{k})$, $k \neq 0$, $k \in \mathbf{R}$

$$\mathbf{c} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

$\therefore$ the vectors are $n(-\mathbf{i} + \mathbf{j} - 2\mathbf{k})$, $n \neq 0$, $n \in \mathbf{R}$

$$\mathbf{d} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 2 & -3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & -1 \\ 2 & -3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & 1 \\ -3 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = 5\mathbf{i} + \mathbf{j} + 4\mathbf{k}$$

$\therefore$ the vectors are $n(5\mathbf{i} + \mathbf{j} + 4\mathbf{k})$, $n \neq 0$, $n \in \mathbf{R}$

$$\mathbf{13} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = 4\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$$

$\therefore$ The vectors $k(4\mathbf{i} - 5\mathbf{j} - 7\mathbf{k})$, $k \neq 0$ are all perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

However, we require the vector to have length 5.

$$\therefore \sqrt{(4k)^2 + (-5k)^2 + (-7k)^2} = 5$$

$$\therefore 16k^2 + 25k^2 + 49k^2 = 25$$

$$\therefore 90k^2 = 25$$

$$\therefore k^2 = \frac{25}{90} = \frac{5}{18}$$

$$\therefore k = \pm \frac{\sqrt{5}}{3\sqrt{2}} = \pm \frac{\sqrt{10}}{6}$$

$\therefore$ the possible vectors are $\pm \frac{\sqrt{10}}{6}[4, -5, -7]$

$$\mathbf{14} \quad \mathbf{a} \quad \text{Given } A(1, 3, 2) \quad B(0, 2, -5) \quad \text{and} \quad C(3, 1, -4),$$

$$\overrightarrow{AB} = \begin{bmatrix} -1 \\ -1 \\ -7 \end{bmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix}$$

$$\therefore \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -7 \\ 2 & -2 & -6 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & -7 \\ -2 & -6 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -7 & -1 \\ -6 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & -1 \\ 2 & -2 \end{vmatrix}$$

$$= (6 - 14)\mathbf{i} + (-14 - 6)\mathbf{j} + (2 + 2)\mathbf{k}$$

$$= -8\mathbf{i} - 20\mathbf{j} + 4\mathbf{k}$$

$$= -4(2\mathbf{i} + 5\mathbf{j} - \mathbf{k})$$

$\therefore 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$ will do.

b Given $P(2, 0, -1)$, $Q(0, 1, 3)$ and $R(1, -1, 1)$, $\vec{PQ} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ and $\vec{PR} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

$$\begin{aligned}\therefore \mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 4 \\ -1 & -1 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 4 & -2 \\ 2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 1 \\ -1 & -1 \end{vmatrix} \\ &= 6\mathbf{i} + 3\mathbf{k} \\ &= 3(2\mathbf{i} + \mathbf{k}) \text{ and so } 2\mathbf{i} + \mathbf{k} \text{ will do.}\end{aligned}$$

EXERCISE 15K.2

1 a $\mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0\mathbf{i} - \mathbf{j} + 0\mathbf{k} = -\mathbf{j}$

$$\mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0\mathbf{i} + \mathbf{j} + 0\mathbf{k} = \mathbf{j}$$

b Using the RH rule these two results check.

c $\mathbf{i} \times \mathbf{k} = |\mathbf{i}| |\mathbf{k}| \sin 90^\circ \times (-\mathbf{j}) \quad \{\text{using the RH rule}\}$
 $= 1 \times 1 \times 1 \times (-\mathbf{j})$
 $= -\mathbf{j}$

$\mathbf{k} \times \mathbf{i} = |\mathbf{k}| |\mathbf{i}| \sin 90^\circ \times \mathbf{j} \quad \{\text{using the RH rule}\}$
 $= 1 \times 1 \times 1 \times \mathbf{j}$
 $= \mathbf{j}$

2 a $\mathbf{a} \cdot \mathbf{b} = 2 \times 1 + (-1) \times 0 + 3 \times (-1)$
 $= -1$

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 0 & -1 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} \\ &= \mathbf{i} + 5\mathbf{j} + \mathbf{k}\end{aligned}$$

c $\sin^2 \theta + \cos^2 \theta = 1$

$$\therefore \sin^2 \theta + \frac{1}{28} = 1$$

$$\therefore \sin^2 \theta = \frac{27}{28}$$

$$\therefore \sin \theta = \pm \sqrt{\frac{27}{28}}$$

But $0 \leq \theta \leq \pi$

$$\therefore \sin \theta = \sqrt{\frac{27}{28}}$$

3 $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Leftrightarrow |\mathbf{a}| |\mathbf{b}| \sin \theta \times \mathbf{u} = \mathbf{0}$

$$\Leftrightarrow |\mathbf{a}| |\mathbf{b}| \sin \theta = 0 \quad \{\text{since } |\mathbf{a}| \neq 0 \text{ and } |\mathbf{b}| \neq 0, \mathbf{u} \text{ exists and } \neq \mathbf{0}\}$$

$$\Leftrightarrow \sin \theta = 0 \quad \{\text{since } |\mathbf{a}| \neq 0, \text{ and } |\mathbf{b}| \neq 0\}$$

$$\Leftrightarrow \theta = 0 \text{ or } \pi$$

$$\Leftrightarrow \mathbf{a} \text{ is parallel to } \mathbf{b}$$

b $\cos \theta$

$$\begin{aligned}&= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \\ &= \frac{-1}{\sqrt{4+1+9}\sqrt{1+1}} \\ &= \frac{-1}{\sqrt{14}\sqrt{2}} \\ &= \frac{-1}{\sqrt{28}}\end{aligned}$$

d $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$

$$\begin{aligned}\therefore \sin \theta &= \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}| |\mathbf{b}|} \\ &= \frac{\sqrt{1^2 + 5^2 + 1^2}}{\sqrt{14}\sqrt{2}} \\ &= \sqrt{\frac{27}{28}}\end{aligned}$$

4 a $\vec{OA} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ and $\vec{OB} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

b
$$\vec{OA} \times \vec{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix}$$

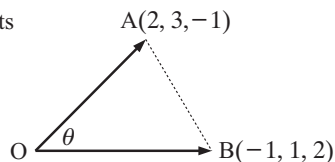
$$= 7\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

$\therefore |\vec{OA} \times \vec{OB}| = \sqrt{7^2 + (-3)^2 + 5^2} = \sqrt{83}$ units

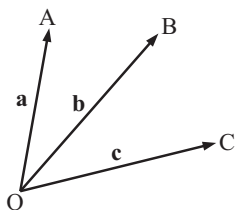
c Area $\triangle AOB = \frac{1}{2} |\vec{OA}| |\vec{OB}| \sin \theta$

$$= \frac{1}{2} |\vec{OA} \times \vec{OB}|$$

$$= \frac{1}{2} \sqrt{83} \text{ units}^2$$



5



b Since $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$,

$$\mathbf{b} \times (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{b} \times \mathbf{0}$$

$$\therefore \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$$

$$\therefore \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$$

$$\therefore \mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$$

 {since $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ }

a
$$\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$$

$$\therefore \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{c} = \mathbf{0}$$

$$\therefore (\mathbf{a} - \mathbf{b}) \times \mathbf{c} = \mathbf{0}$$

$$\vec{BA} \times \mathbf{c} = \mathbf{0}$$

$$\therefore \vec{OC} \text{ must be parallel to } \vec{AB}.$$

c
$$\mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} \text{ where } \mathbf{c} \neq \mathbf{0}$$

$$\therefore \mathbf{b} \times \mathbf{c} = -\mathbf{a} \times \mathbf{c}$$

$$\therefore \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} = \mathbf{0}$$

$$\therefore (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{0}$$

$$\therefore \mathbf{a} + \mathbf{b} \text{ is parallel to } \mathbf{c}$$

 i.e., $\mathbf{a} + \mathbf{b} = k\mathbf{c}$ for some scalar k .

EXERCISE 15K.3

1 a Given $A(2, 1, 1)$, $B(4, 3, 0)$ and $C(1, 3, -2)$, $\vec{AB} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{AC} = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$

$$\therefore \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ -1 & 2 & -3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & 2 \\ -3 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix}$$

$$= -4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$$

$$\therefore \text{Area} = \frac{1}{2} |[-4\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}]| \quad \{\text{as area} = \frac{1}{2} |\vec{AB} \times \vec{AC}|\}$$

$$= \frac{1}{2} \sqrt{(-4)^2 + 7^2 + 6^2}$$

$$= \frac{1}{2} \sqrt{101} \text{ units}^2$$

b Given $A(0, 0, 0)$, $B(-1, 2, 3)$ and $C(1, 2, 6)$, $\vec{AB} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{AC} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$

$$\therefore \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ 1 & 2 & 6 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & 3 \\ 2 & 6 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 3 & -1 \\ 6 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= 6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}$$

$$\therefore \text{Area} = \frac{1}{2} |[6\mathbf{i} + 9\mathbf{j} - 4\mathbf{k}]| \quad \{\text{as area} = \frac{1}{2} |\vec{AB} \times \vec{AC}|\}$$

$$= \frac{1}{2} \sqrt{6^2 + 9^2 + (-4)^2}$$

$$= \frac{1}{2} \sqrt{133} \text{ units}^2$$

c Given $A(1, 3, 2)$ $B(2, -1, 0)$ and $C(1, 10, 6)$, $\vec{AB} = \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix}$ and $\vec{AC} = \begin{bmatrix} 0 \\ 7 \\ 4 \end{bmatrix}$

$$\therefore \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & -2 \\ 0 & 7 & 4 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -4 & -2 \\ 7 & 4 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -2 & 1 \\ 4 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -4 \\ 0 & 7 \end{vmatrix}$$

$$= -2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$$

$$\therefore \text{area} = \frac{1}{2} |[-2\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}]| = \frac{1}{2} \sqrt{(-2)^2 + (-4)^2 + 7^2} = \frac{1}{2} \sqrt{69} \text{ units}^2$$

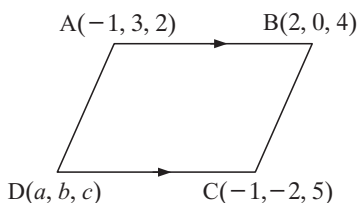
2 Given $A(-1, 2, 2)$ $B(2, -1, 4)$ and $C(0, 1, 0)$, $\vec{AB} = \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}$ and $\vec{AC} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

$$\therefore \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & 2 \\ 1 & -1 & -2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -3 & 2 \\ -1 & -2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 3 & -3 \\ 1 & -1 \end{vmatrix}$$

$$= 8\mathbf{i} + 8\mathbf{j}$$

$$\therefore \text{area of parallelogram} = |[8\mathbf{i} + 8\mathbf{j}]| = \sqrt{8^2 + 8^2} = 8\sqrt{2} \text{ units}^2$$

3 a



Suppose D is at (a, b, c)

Since $\vec{AB} = \vec{DC}$,

$$\begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 - a \\ -2 - b \\ 5 - c \end{bmatrix}$$

$$\therefore -1 - a = 3, \quad -2 - b = -3 \quad \text{and} \quad 5 - c = 2$$

$$\therefore a = -4, \quad b = 1 \quad \text{and} \quad c = 3$$

i.e., D is at $(-4, 1, 3)$.

b $\vec{BC} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$ and $\vec{BA} = \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix}$

$$\therefore \vec{BC} \times \vec{BA} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & 1 \\ -3 & 3 & -2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 1 & -3 \\ -2 & -3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -3 & -2 \\ -3 & 3 \end{vmatrix}$$

$$= [\mathbf{i} - 9\mathbf{j} - 15\mathbf{k}]$$

$$\therefore \text{area} = |[\mathbf{i} - 9\mathbf{j} - 15\mathbf{k}]| = \sqrt{1^2 + (-9)^2 + (-15)^2} = \sqrt{307} \text{ units}^2$$

4 a Now $\vec{AB} = [1, 2, -1]$, $\vec{AC} = [-1, 2, -3]$ and $\vec{AD} = [-2, 2, 2]$

$\therefore$ the volume of the tetrahedron

$$= \frac{1}{6} \left| \vec{AB} \bullet (\vec{AC} \times \vec{AD}) \right|$$

$$= \frac{1}{6} \left| \begin{vmatrix} 1 & 2 & -1 \\ -1 & 2 & -3 \\ -2 & 2 & 2 \end{vmatrix} \right| = \frac{1}{6} \left| 1 \begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix} + 2 \begin{vmatrix} -3 & -1 \\ 2 & -2 \end{vmatrix} - 1 \begin{vmatrix} -1 & 2 \\ -2 & 2 \end{vmatrix} \right|$$

$$= \frac{1}{6} |10 + 2(8) - 1(2)|$$

$$= \frac{1}{6} |24|$$

$$= 4 \text{ units}^3$$

- b** The total surface area of the tetrahedron is the sum of the four triangular faces forming it.

$$\begin{aligned}\text{Face 1} \quad \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -1 & 2 & -3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & 1 \\ -3 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \\ &= -4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}\end{aligned}$$

$$\therefore A_1 = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \sqrt{4^2 + 4^2 + 4^2} = \frac{1}{2} (4\sqrt{3}) = 2\sqrt{3} \text{ units}^2$$

$$\begin{aligned}\text{Face 2} \quad \overrightarrow{AB} \times \overrightarrow{AD} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \\ &= 6\mathbf{i} + 6\mathbf{k}\end{aligned}$$

$$\therefore A_2 = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AD}| = \frac{1}{2} \sqrt{6^2 + 6^2} = \frac{1}{2} 6\sqrt{2} = 3\sqrt{2} \text{ units}^2$$

$$\begin{aligned}\text{Face 3} \quad \overrightarrow{AC} \times \overrightarrow{AD} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -3 \\ -2 & 2 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -3 & -1 \\ 2 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 2 \\ -2 & 2 \end{vmatrix} \\ &= 10\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}\end{aligned}$$

$$\therefore A_3 = \frac{1}{2} |\overrightarrow{AC} \times \overrightarrow{AD}| = \frac{1}{2} \sqrt{10^2 + 64 + 4} = \frac{1}{2} \sqrt{168} = \sqrt{42} \text{ units}^2$$

$$\begin{aligned}\text{Face 4} \quad \overrightarrow{BC} \times \overrightarrow{BD} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -2 \\ -3 & 0 & 3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & -2 \\ 0 & 3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -2 & -2 \\ 3 & -3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 0 \\ -3 & 0 \end{vmatrix} \\ &= 12\mathbf{j}\end{aligned}$$

$$\therefore A_4 = \frac{1}{2} |\overrightarrow{BC} \times \overrightarrow{BD}| = \frac{1}{2} \times 12 = 6 \text{ units}^2$$

$$\therefore \text{total surface area} = (\sqrt{42} + 2\sqrt{3} + 3\sqrt{2} + 6) \text{ units}^2$$

- 5 a** Given $A(3, 0, 0)$ $B(0, 1, 0)$ $C(1, 2, 3)$ $O(0, 0, 0)$,

We label the other vertices as shown below.

$$\overrightarrow{OX} = \overrightarrow{OB} + \overrightarrow{OC} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

so X is at $(1, 3, 3)$

$$\overrightarrow{OY} = \overrightarrow{OA} + \overrightarrow{OC} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

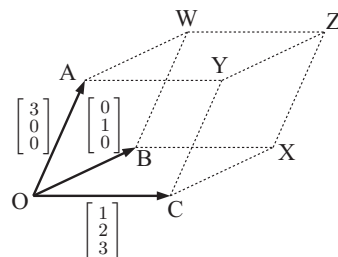
so Y is at $(4, 2, 3)$

$$\overrightarrow{OW} = \overrightarrow{OA} + \overrightarrow{OB} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

so W is at $(3, 1, 0)$

$$\overrightarrow{OZ} = \overrightarrow{OW} + \overrightarrow{OC} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}$$

so Z is at $(4, 3, 3)$.



b $\overrightarrow{BA} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$ and $\overrightarrow{BC} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

$$\therefore \cos \angle ABC = \frac{\overrightarrow{BA} \bullet \overrightarrow{BC}}{|\overrightarrow{BA}| |\overrightarrow{BC}|} = \frac{2}{\sqrt{110}} \quad \therefore \theta \doteq 79.01^\circ$$

$$\text{c Volume} = \left| \begin{vmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} \right| = \left| 3 \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \right| + 0 \left| \begin{vmatrix} 0 & 0 \\ 3 & 1 \end{vmatrix} \right| + 0 \left| \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \right| = 3 \times 3 = 9 \text{ units}^3$$

$$6 \quad \text{Now } \vec{AB} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{AC} = \begin{bmatrix} k+1 \\ 1 \\ -3 \end{bmatrix}$$

$$\text{Area of } \triangle ABC = \frac{1}{2} |\vec{AC} \times \vec{AB}|$$

$$\therefore \sqrt{88} = \frac{1}{2} \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ k+1 & 1 & -3 \\ 3 & -1 & -1 \end{vmatrix} \right|$$

$$\begin{aligned} 2\sqrt{88} &= \left| \mathbf{i} \begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -3 & k+1 \\ -1 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} k+1 & 1 \\ 3 & -1 \end{vmatrix} \right| \\ &= |(-1-3)\mathbf{i} + (-9+(k+1))\mathbf{j} + (-(k+1)-3)\mathbf{k}| \\ &= |-4\mathbf{i} + (k-8)\mathbf{j} + (-k-4)\mathbf{k}| \end{aligned}$$

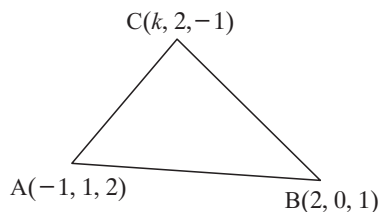
$$\therefore \sqrt{352} = \sqrt{16 + (k-8)^2 + (-k-4)^2}$$

$$\therefore 352 = 16 + k^2 - 16k + 64 + k^2 + 8k + 16$$

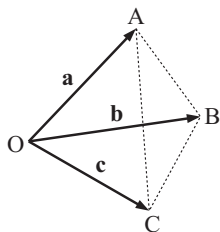
$$\therefore 2k^2 - 8k - 256 = 0$$

$$\therefore k^2 - 4k - 128 = 0$$

$$\therefore k = \frac{4 \pm \sqrt{16 + 4(1)(128)}}{2} = 2 \pm \sqrt{132} = 2 \pm 2\sqrt{33}$$



7



Total surface area S of the tetrahedron is the sum of the areas of the 4 triangular faces.

$$\text{Now } \vec{AB} = \vec{AO} + \vec{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$$

$$\text{and } \vec{AC} = \vec{AO} + \vec{OC} = -\mathbf{a} + \mathbf{c} = \mathbf{c} - \mathbf{a}$$

$$\therefore S = \frac{1}{2} |\mathbf{a} \times \mathbf{b}| + \frac{1}{2} |\mathbf{a} \times \mathbf{c}| + \frac{1}{2} |\mathbf{b} \times \mathbf{c}| + \frac{1}{2} |(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|$$

8

$$\begin{aligned} \vec{BA} &= \vec{BO} + \vec{OA} \\ &= -\mathbf{b} + \mathbf{a} = \mathbf{a} - \mathbf{b} \end{aligned}$$

$$\begin{aligned} \text{and } \vec{BC} &= \vec{BO} + \vec{OC} \\ &= -\mathbf{b} + \mathbf{c} = \mathbf{c} - \mathbf{b} \end{aligned}$$

$$\begin{aligned} \therefore \text{Area } \triangle ABC &= \frac{1}{2} |\vec{BA} \times \vec{BC}| \\ &= \frac{1}{2} |(\mathbf{a} - \mathbf{b}) \times (\mathbf{c} - \mathbf{b})| \\ &= \frac{1}{2} |(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b})| \end{aligned}$$

Now if A, B and C are collinear, then area $\triangle ABC = 0$

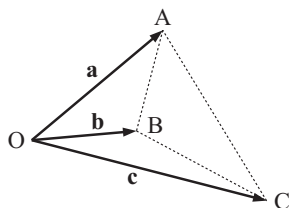
$$\therefore (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b}) = \mathbf{0}$$

$$\text{or } \vec{AB} = \vec{AO} + \vec{OB} = \mathbf{b} - \mathbf{a}$$

$$\vec{BC} = \vec{BO} + \vec{OC} = \mathbf{c} - \mathbf{b}$$

If A, B and C are collinear, then $\vec{AB}$ is parallel to $\vec{BC}$

$$\therefore \vec{AB} \times \vec{BC} = \mathbf{0}, \text{ i.e., } (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{b}) = \mathbf{0}$$



9

$$\mathbf{a} \quad \mathbf{b} - \mathbf{a} = \overrightarrow{AB} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \quad \mathbf{c} - \mathbf{a} = \overrightarrow{AC} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{d} - \mathbf{a} = \overrightarrow{AD} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

$$\therefore (\mathbf{b} - \mathbf{a}) \bullet (\mathbf{c} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a})$$

$$= \begin{vmatrix} 1 & 3 & -2 \\ 2 & 0 & -1 \\ 3 & -1 & -1 \end{vmatrix} = 1 \begin{vmatrix} 0 & -1 \\ -1 & -1 \end{vmatrix} + 3 \begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix}$$

$$= (-1) + 3(-1) - 2(-2)$$

$$= 0$$

and so A, B, C and D are coplanar

$$\mathbf{b} \quad \mathbf{q} - \mathbf{p} = \overrightarrow{PQ} = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix} \quad \mathbf{r} - \mathbf{p} = \overrightarrow{PR} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} \quad \mathbf{s} - \mathbf{p} = \overrightarrow{PS} = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}$$

$$\therefore (\mathbf{q} - \mathbf{p}) \bullet (\mathbf{r} - \mathbf{p}) \times (\mathbf{s} - \mathbf{p})$$

$$= \begin{vmatrix} -2 & -1 & -1 \\ 0 & 1 & -5 \\ -1 & 1 & -4 \end{vmatrix} = -2 \begin{vmatrix} 1 & -5 \\ 1 & -4 \end{vmatrix} - 1 \begin{vmatrix} -5 & 0 \\ -4 & -1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}$$

$$= -2(1) - 1(5) - 1(1)$$

$$= -8$$

$$\neq 0$$

and so P, Q, R and S are not coplanar

$$10 \quad \mathbf{b} - \mathbf{a} = \overrightarrow{AB} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \quad \mathbf{c} - \mathbf{a} = \overrightarrow{AC} = \begin{bmatrix} -2 \\ k-1 \\ -1 \end{bmatrix} \quad \mathbf{d} - \mathbf{a} = \overrightarrow{AD} = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}$$

$$\therefore (\mathbf{b} - \mathbf{a}) \bullet (\mathbf{c} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a})$$

$$= \begin{vmatrix} 2 & -1 & -2 \\ -2 & k-1 & -1 \\ -1 & 1 & -4 \end{vmatrix} = 2 \begin{vmatrix} k-1 & -1 \\ 1 & -4 \end{vmatrix} - 1 \begin{vmatrix} -1 & -2 \\ -4 & -1 \end{vmatrix} - 2 \begin{vmatrix} -2 & k-1 \\ -1 & 1 \end{vmatrix}$$

$$= 2(-4k + 4 + 1) - 1(1 - 8) - 2(-2 + k - 1)$$

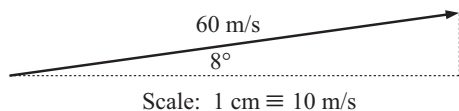
$$= -8k + 10 + 7 - 2k + 6$$

$$= -10k + 23$$

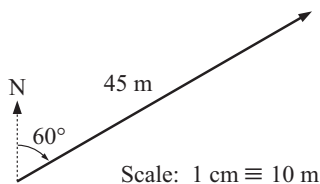
$\therefore$ A, B, C and D are coplanar when $-10k + 23 = 0$, i.e., when $k = \frac{23}{10}$.

REVIEW SET 15A

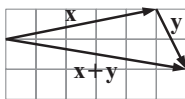
1 a



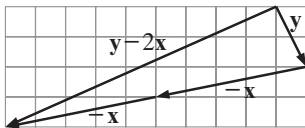
b



2 a



b



3 a

$$\overrightarrow{PR} + \overrightarrow{RQ} = \overrightarrow{PQ}$$

$$\begin{aligned} \mathbf{b} \quad \overrightarrow{PS} + \overrightarrow{SQ} + \overrightarrow{QR} \\ = \overrightarrow{PQ} + \overrightarrow{QR} \\ = \overrightarrow{PR} \end{aligned}$$

4 Dino's first displacement vector is $9 \begin{bmatrix} \cos 246^\circ \\ \sin 246^\circ \end{bmatrix}$, Dino's second displacement vector is $6 \begin{bmatrix} \cos 96^\circ \\ \sin 96^\circ \end{bmatrix}$

$\therefore$ Dino's resultant displacement vector is $\begin{bmatrix} 9 \cos 246^\circ \\ 9 \sin 246^\circ \end{bmatrix} + \begin{bmatrix} 6 \cos 96^\circ \\ 6 \sin 96^\circ \end{bmatrix} \div \begin{bmatrix} -4.288 \\ -2.255 \end{bmatrix}$

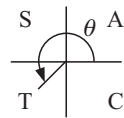
which has length $\sqrt{(-4.288)^2 + (-2.255)^2} \div 4.845$

$$\therefore \text{ the resultant displacement vector} = 4.845 \begin{bmatrix} -0.8850 \\ -0.4654 \end{bmatrix} \begin{matrix} \leftarrow \cos \theta \\ \leftarrow \sin \theta \end{matrix}$$

If $\cos \theta = -0.8850$ and $\sin \theta = -0.4654$

θ is in Quadrant 3 $\therefore \theta = 180^\circ + \cos^{-1}(0.8850) \div 207.7^\circ$

$\therefore$ Dino is 4.845 km from the start at bearing 208° .

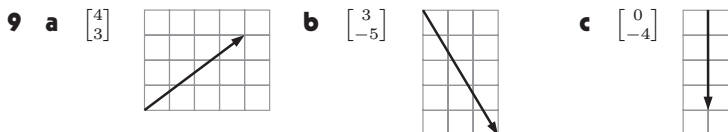


$$\begin{array}{ll} \mathbf{5} \quad \mathbf{a} & \overrightarrow{AB} - \overrightarrow{CB} \\ & = \overrightarrow{AB} + \overrightarrow{BC} \\ & = \overrightarrow{AC} \end{array} \quad \begin{array}{ll} \mathbf{b} & \overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC} \\ & = \overrightarrow{AC} + \overrightarrow{CD} \\ & = \overrightarrow{AD} \end{array}$$

$$\begin{array}{ll} \mathbf{6} \quad \mathbf{a} & \text{If } \overrightarrow{AB} = \frac{1}{2}\overrightarrow{CD} \text{ then} \\ & AB \parallel CD \text{ and } AB = \frac{1}{2}(CD) \\ \mathbf{b} & \text{If } \overrightarrow{AB} = 2\overrightarrow{AC} \text{ then} \\ & AB \parallel AC \text{ and } AB = 2(AC) \\ & \text{i.e., } A, B \text{ and } C \text{ are collinear and} \\ & AB = 2(AC). \\ & \text{So, } C \text{ is the midpoint of } AB. \end{array}$$

$$\begin{array}{ll} \mathbf{7} \quad \mathbf{a} & \mathbf{p} + \mathbf{r} - \mathbf{q} = \mathbf{0} \\ & \therefore \mathbf{p} + \mathbf{r} = \mathbf{q} \\ \mathbf{b} & \mathbf{l} + \mathbf{m} - \mathbf{n} + \mathbf{j} - \mathbf{k} = \mathbf{0} \\ & \therefore \mathbf{l} + \mathbf{m} + \mathbf{j} = \mathbf{n} + \mathbf{k} \end{array}$$

$$\begin{array}{ll} \mathbf{8} \quad \mathbf{a} & \overrightarrow{OQ} = \overrightarrow{OR} + \overrightarrow{RQ} = \mathbf{r} + \mathbf{q} \\ \mathbf{b} & \overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OR} + \overrightarrow{RQ} = -\mathbf{p} + \mathbf{r} + \mathbf{q} \\ \mathbf{c} & \overrightarrow{ON} = \overrightarrow{OR} + \overrightarrow{RN} = \mathbf{r} + \frac{1}{2}\mathbf{q} \\ \mathbf{d} & \overrightarrow{MN} = \overrightarrow{MQ} + \overrightarrow{QN} \\ & = \frac{1}{2}\overrightarrow{PQ} + \frac{1}{2}\overrightarrow{QR} \\ & = \frac{1}{2}(-\mathbf{p} + \mathbf{r} + \mathbf{q}) + \frac{1}{2}(-\mathbf{q}) \\ & = -\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{r} + \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{q} \\ & = \frac{1}{2}\mathbf{r} - \frac{1}{2}\mathbf{p} \end{array}$$

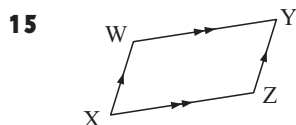


$$\begin{array}{llll} \mathbf{10} \quad \mathbf{a} & 2\mathbf{p} + \mathbf{q} & \mathbf{b} & \mathbf{q} - 3\mathbf{r} \\ & = 2\begin{bmatrix} -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix} & & = \begin{bmatrix} 2 \\ -4 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ & = \begin{bmatrix} -6 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix} & & = \begin{bmatrix} 2 \\ -4 \end{bmatrix} - \begin{bmatrix} 3 \\ 9 \end{bmatrix} \\ & = \begin{bmatrix} -4 \\ -2 \end{bmatrix} & & = \begin{bmatrix} -1 \\ -13 \end{bmatrix} \\ \mathbf{c} & \mathbf{p} - \mathbf{q} + \mathbf{r} & \mathbf{11} & \overrightarrow{SP} \\ & = \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} & & = \overrightarrow{SR} + \overrightarrow{RQ} + \overrightarrow{QP} \\ & = \begin{bmatrix} -5 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} & & = -\overrightarrow{RS} + \overrightarrow{RQ} - \overrightarrow{PQ} \\ & = \begin{bmatrix} -4 \\ 8 \end{bmatrix} & & = -\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ 1 \end{bmatrix} \\ & & & = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \end{array}$$

$$\begin{array}{llll} \mathbf{12} \quad \mathbf{a} & |\mathbf{r}| & \mathbf{b} & |\mathbf{s}| \\ & = \sqrt{4^2 + 1^2} & & = \sqrt{(-3)^2 + 2^2} \\ & = \sqrt{17} \text{ units} & & = \sqrt{13} \text{ units} \\ \mathbf{c} & \mathbf{r} + \mathbf{s} & \mathbf{d} & 2\mathbf{s} - \mathbf{r} \\ & = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} & & = 2\begin{bmatrix} -3 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ & = \begin{bmatrix} 1 \\ 3 \end{bmatrix} & & = \begin{bmatrix} -10 \\ 3 \end{bmatrix} \\ & \therefore |\mathbf{r} + \mathbf{s}| & & \therefore |2\mathbf{s} - \mathbf{r}| \\ & = \sqrt{1^2 + 3^2} & & = \sqrt{100 + 9} \\ & = \sqrt{10} \text{ units} & & = \sqrt{109} \text{ units} \end{array}$$

$$\begin{array}{ll} \mathbf{13} \quad \mathbf{a} & \overrightarrow{BC} = 2\overrightarrow{OA} = 2\mathbf{p} \\ \text{Now} & \overrightarrow{AC} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{BC} \\ & = -\mathbf{p} + \mathbf{q} + 2\mathbf{p} \\ & = \mathbf{p} + \mathbf{q} \\ \mathbf{b} & \overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} \\ & = \mathbf{p} + \frac{1}{2}\overrightarrow{AC} \\ & = \mathbf{p} + \frac{1}{2}(\mathbf{p} + \mathbf{q}) \\ & = \frac{3}{2}\mathbf{p} + \frac{1}{2}\mathbf{q} \end{array}$$

$$\begin{array}{ll}
 \mathbf{14} \quad \mathbf{a} & \mathbf{p} - 3\mathbf{x} = \mathbf{0} \\
 & \therefore \mathbf{p} = 3\mathbf{x} \\
 & \therefore \frac{1}{3}\mathbf{p} = \mathbf{x} \\
 & \therefore \mathbf{x} = \begin{bmatrix} -1 \\ \frac{1}{3} \end{bmatrix} \\
 \mathbf{b} & 2\mathbf{q} - \mathbf{x} = \mathbf{r} \\
 & \therefore 2\mathbf{q} - \mathbf{r} = \mathbf{x} \\
 & \therefore \mathbf{x} = 2\begin{bmatrix} 2 \\ -4 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\
 & \therefore \mathbf{x} = \begin{bmatrix} 1 \\ -10 \end{bmatrix}
 \end{array}$$



$$\begin{aligned}
 \overrightarrow{WY} &= \begin{bmatrix} 3-3 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix} \\
 \overrightarrow{XZ} &= \begin{bmatrix} 4-2 \\ 10-5 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}
 \end{aligned}$$

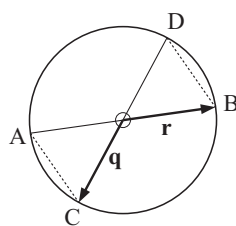
$$\text{So, } \overrightarrow{WY} = \overrightarrow{XZ}$$

$\therefore$ WY is parallel to XZ and they are equal in length. This is sufficient to deduce that WYZX is a parallelogram.

$$\begin{array}{l}
 \mathbf{16} \quad r \begin{bmatrix} -2 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 13 \\ -24 \end{bmatrix} \\
 \therefore \begin{bmatrix} -2r+3s \\ r-4s \end{bmatrix} = \begin{bmatrix} 13 \\ -24 \end{bmatrix} \\
 \therefore \begin{array}{l} -2r+3s = 13 \\ r-4s = -24 \end{array} \quad \left| \begin{array}{l} \times 2 \dots (1) \\ \end{array} \right. \\
 \therefore \begin{array}{l} -2r+3s = 13 \\ 2r-8s = -48 \end{array} \\
 \hline
 -5s = -35 \\
 \therefore s = 7
 \end{array}$$

$$\begin{aligned}
 \text{and in (1)} \quad r - 4(7) &= -24 \\
 \therefore r &= -24 + 28 \\
 \therefore r &= 4 \\
 \text{i.e., } r &= 4 \quad \text{and} \quad s = 7
 \end{aligned}$$

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$$\begin{array}{ll}
 \mathbf{a} & \overrightarrow{DB} = \overrightarrow{DO} + \overrightarrow{OB} \\
 & = \overrightarrow{OC} + \overrightarrow{OB} \\
 & = \mathbf{q} + \mathbf{r} \\
 \mathbf{b} & \overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} \\
 & = \overrightarrow{OB} + \overrightarrow{OC} \\
 & = \mathbf{r} + \mathbf{q}
 \end{array}$$

$$\text{We see that } \overrightarrow{DB} = \overrightarrow{AC}$$

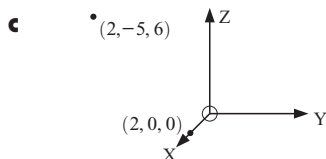
$\therefore$ DB is parallel to AC and equal in length.

This is sufficient to deduce that ACBD is a parallelogram.

REVIEW SET 15B

$$\mathbf{1} \quad \mathbf{a} \quad \overrightarrow{PQ} = \begin{bmatrix} -1-2 \\ 7-5 \\ 9-6 \end{bmatrix} = \begin{bmatrix} -3 \\ 12 \\ 3 \end{bmatrix}$$

$$\begin{aligned}
 \mathbf{b} \quad PQ &= \sqrt{(-3)^2 + 12^2 + 3^2} \\
 &= \sqrt{162} \text{ units}
 \end{aligned}$$



$$\begin{aligned}
 \therefore \text{distance} &= \sqrt{(2-2)^2 + (0-(-5))^2 + (0-6)^2} \\
 &= \sqrt{0+25+36} \\
 &= \sqrt{61} \text{ units}
 \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad \mathbf{m} - \mathbf{n} + \mathbf{p} = \begin{bmatrix} 6 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 11 \end{bmatrix}$$

$$\mathbf{b} \quad 2\mathbf{n} - 3\mathbf{p} = 2 \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -8 \end{bmatrix} - \begin{bmatrix} -3 \\ 9 \\ 18 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -26 \end{bmatrix}$$

$$\begin{aligned}
 \mathbf{c} \quad \mathbf{m} + \mathbf{p} &= \begin{bmatrix} 6 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix} \\
 \therefore |\mathbf{m} + \mathbf{p}| &= \sqrt{25+0+49} \\
 &= \sqrt{74} \text{ units}
 \end{aligned}$$

$$\mathbf{3} \quad \overrightarrow{CB} = \overrightarrow{CA} + \overrightarrow{AB} = -\overrightarrow{AC} + \overrightarrow{AB} = \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ -8 \\ 7 \end{bmatrix}$$

$$\begin{aligned}
 \mathbf{4} \quad \text{The vectors are parallel } \therefore \begin{bmatrix} -12 \\ -20 \\ 2 \end{bmatrix} &= k \begin{bmatrix} 3 \\ m \\ n \end{bmatrix} \\
 \therefore 3k &= -12, \quad km = -20, \quad kn = 2 \\
 \therefore k &= -4, \quad m = 5, \quad n = -\frac{1}{2}
 \end{aligned}$$

$$5 \quad \vec{PQ} = \begin{bmatrix} 4-6 \\ 6-8 \\ 8-2 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} \quad \therefore \text{Both } \vec{PQ} \text{ and } \vec{QR} \text{ are parallel to } \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$$

$$\vec{QR} = \begin{bmatrix} 19-4 \\ 3-6 \\ 17-8 \end{bmatrix} = \begin{bmatrix} 15 \\ -3 \\ 9 \end{bmatrix} = 3 \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} \quad \therefore PQ \parallel QR \text{ with } Q \text{ common to both.}$$

$\therefore P, Q, R$ are collinear.

$$\vec{PQ} : \vec{QR} = 2 \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} : 3 \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} = 2 : 3$$

$\therefore Q$ divides PR internally in the ratio $2 : 3$.

6 As the vectors are perpendiculars,

$$\begin{bmatrix} -4 \\ t+2 \\ t \end{bmatrix} \bullet \begin{bmatrix} t \\ 1+t \\ -3 \end{bmatrix} = 0$$

$$\therefore 4t + (t+2)(1+t) - 3t = 0$$

$$\therefore -4t + t + t^2 + 2 + 2t - 3t = 0$$

$$\therefore t^2 - 4t + 2 = 0$$

$$\therefore t = \frac{4 \pm \sqrt{16 - 4(1)(2)}}{2}$$

$$\therefore t = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

$$7 \quad \text{If } \theta \text{ is the angle then} \quad \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \sqrt{4+16+9}\sqrt{1+1+9}\cos\theta$$

$$\therefore -2 - 4 + 9 = \sqrt{29}\sqrt{11}\cos\theta$$

$$\therefore \frac{3}{\sqrt{29 \times 11}} = \cos\theta \quad \text{and so } \theta \doteq 80.3^\circ$$

8 If D is the origin, DA the x -axis, DC the y -axis and DE the z -axis, then A is $(4, 0, 0)$, C is $(0, 8, 0)$ and G is $(4, 8, 5)$

$$\vec{AG} = \begin{bmatrix} 4-4 \\ 8-0 \\ 5-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 5 \end{bmatrix} \quad \vec{AC} = \begin{bmatrix} 0-4 \\ 8-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ 0 \end{bmatrix}$$

$$\text{If the required angle is } \theta \text{ then} \quad \begin{bmatrix} 0 \\ 8 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} -4 \\ 8 \\ 0 \end{bmatrix} = \sqrt{0+64+25}\sqrt{16+64+0}\cos\theta$$

$$\therefore 0 + 64 + 0 = \sqrt{89}\sqrt{80}\cos\theta$$

$$\therefore \cos\theta = \frac{64}{\sqrt{89 \times 80}} \quad \text{and so } \theta \doteq 40.7^\circ$$

$$9 \quad \text{a} \quad \vec{PQ} = \begin{bmatrix} -4-2 \\ 4-3 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix}$$

$$\text{b} \quad PQ = |\vec{PQ}| = \sqrt{36+1+9} = \sqrt{46} \text{ units}$$

$$\text{c} \quad M \text{ is } \left(\frac{2-4}{2}, \frac{3+4}{2}, \frac{-1+2}{2} \right) \quad \text{i.e., } \left(-1, \frac{7}{2}, \frac{1}{2} \right)$$

$$10 \quad \text{a} \quad \mathbf{p} \bullet \mathbf{q} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = -3 - 2 + 4 = -1$$

$$\text{b} \quad \mathbf{p} + 2\mathbf{q} - \mathbf{r} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}$$

$$\text{c} \quad \mathbf{p} \bullet \mathbf{r} = |\mathbf{p}| |\mathbf{r}| \cos\theta \quad \therefore \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \sqrt{1+4+1}\sqrt{1+1+4}\cos\theta$$

$$\therefore -1 + 2 + 2 = \sqrt{6}\sqrt{6} \cos \theta$$

$$\therefore 3 = 6 \cos \theta$$

$$\therefore \cos \theta = \frac{1}{2}$$

$$\therefore \theta = 60^\circ$$

11

$$\overrightarrow{KL} = \begin{bmatrix} -2-3 \\ 1-1 \\ 3-4 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}$$

$$\overrightarrow{LK} = \begin{bmatrix} 3-(-2) \\ 1-1 \\ 4-3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$$\overrightarrow{KM} = \begin{bmatrix} 4-3 \\ 1-1 \\ 3-4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\overrightarrow{LM} = \begin{bmatrix} 4-(-2) \\ 1-1 \\ 3-3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

$$\overrightarrow{KL} \bullet \overrightarrow{KM} = |\overrightarrow{KL}| |\overrightarrow{KM}| \cos K$$

$$\therefore \overrightarrow{LK} \bullet \overrightarrow{LM} = |\overrightarrow{LK}| |\overrightarrow{LM}| \cos L$$

$$\therefore -5 + 0 + 1 = \sqrt{25+0+1} \sqrt{1+0+1} \cos K \quad \therefore 30 + 0 + 0 = \sqrt{25+0+1} \sqrt{36+0+0} \cos L$$

$$\therefore -4 = \sqrt{26} \sqrt{2} \cos K$$

$$\therefore 30 = \sqrt{26} \times 6 \cos L$$

$$\therefore \cos K = -\frac{4}{\sqrt{52}}$$

$$\therefore \frac{5}{\sqrt{26}} = \cos L$$

$$\therefore K \doteq 123.7^\circ$$

$$\therefore L \doteq 11.3^\circ$$

$$\text{and } M \doteq 180^\circ - 123.7^\circ - 11.3^\circ \doteq 45.0^\circ$$

$$\text{and } M \doteq 180^\circ - 123.7^\circ - 11.3^\circ \doteq 45.0^\circ$$

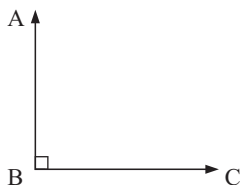
12

If the angle is θ then, $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \sqrt{9+1+4} \sqrt{4+25+1} \cos \theta$

$$\therefore 6 + 5 - 2 = \sqrt{14} \sqrt{30} \cos \theta$$

$$\therefore \frac{9}{\sqrt{14 \times 30}} = \cos \theta$$

$$\therefore \theta \doteq 63.95^\circ$$

13


$$\overrightarrow{BA} = \begin{bmatrix} 4-(-1) \\ 2-5 \\ -1-2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -3 \end{bmatrix}$$

$$\overrightarrow{BC} = \begin{bmatrix} 3-(-1) \\ -3-5 \\ c-2 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ c-2 \end{bmatrix}$$

$$\text{But } \overrightarrow{BA} \bullet \overrightarrow{BC} = 0$$

$$\therefore 20 + 24 - 3(c-2) = 0$$

$$\therefore 44 = 3(c-2)$$

$$\therefore 3c - 6 = 44$$

$$\therefore 3c = 50$$

$$\therefore c = \frac{50}{3}$$

14

a $\mathbf{a} \bullet \mathbf{b}$ is a scalar, so in $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ we would be trying to find the scalar product of a scalar and a vector which is impossible.

b $\mathbf{a} \bullet \mathbf{b} \times \mathbf{c}$ does not need a bracket about the $\mathbf{b} \times \mathbf{c}$ as this must be performed first before it can be 'dotted' with $\mathbf{a}$.

15

a Is a unit vector if

$$\sqrt{\left(\frac{4}{7}\right)^2 + \left(\frac{1}{k}\right)^2} = 1$$

$$\therefore \frac{16}{49} + \frac{1}{k^2} = 1$$

$$\therefore \frac{1}{k^2} = \frac{33}{49}$$

$$\therefore k = \pm \frac{7}{\sqrt{33}}$$

b Is a unit vector if

$$\sqrt{k^2 + k^2} = 1$$

$$\therefore 2k^2 = 1$$

$$\therefore k^2 = \frac{1}{2}$$

$$\therefore k = \pm \frac{1}{\sqrt{2}}$$

REVIEW SET 15C

$$\begin{array}{lll}
 \mathbf{1} \quad \mathbf{a} \quad \mathbf{p} \bullet \mathbf{q} & \mathbf{b} \quad \mathbf{p} - \mathbf{r} & \therefore \mathbf{q} \bullet (\mathbf{p} - \mathbf{r}) \\
 = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 5 \end{bmatrix} & = \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix} & = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 6 \\ -6 \end{bmatrix} \\
 = -3 + (-10) & = \begin{bmatrix} 6 \\ -6 \end{bmatrix} & = -6 - 30 \\
 = -13 & & = -36
 \end{array}$$

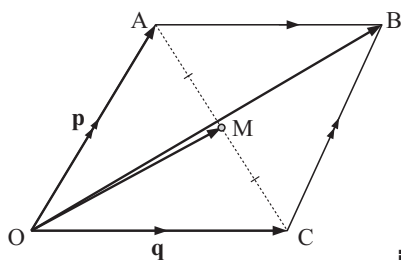
$$\begin{array}{ll}
 \mathbf{2} \quad \text{LHS} = \mathbf{p} \bullet (\mathbf{q} - \mathbf{r}) & \text{RHS} = \mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r} \\
 = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \left(\begin{bmatrix} -2 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right) & = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -3 \end{bmatrix} \\
 = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ 8 \end{bmatrix} & = (-6 - 10) - (3 + 6) \\
 = -9 - 16 & = -16 - 9 \\
 = -25 & = -25 \quad \therefore \text{LHS} = \text{RHS} \quad \checkmark
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{3} \quad \text{Since they are perpendicular} & \mathbf{4} \quad \overrightarrow{\text{AB}} = \begin{bmatrix} -1-2 \\ 4-3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \\
 \begin{bmatrix} 3 \\ 3-2t \end{bmatrix} \bullet \begin{bmatrix} t^2+t \\ -2 \end{bmatrix} = 0 & \overrightarrow{\text{AC}} = \begin{bmatrix} 3-2 \\ k-3 \end{bmatrix} = \begin{bmatrix} 1 \\ k-3 \end{bmatrix} \\
 \therefore 3(t^2+t) - 2(3-2t) = 0 & \text{Now } \overrightarrow{\text{AB}} \bullet \overrightarrow{\text{AC}} = 0 \quad \{\text{as } \angle \text{BAC} = 90^\circ\} \\
 \therefore 3t^2 + 3t - 6 + 4t = 0 & \therefore \begin{bmatrix} -3 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ k-3 \end{bmatrix} = 0 \\
 \therefore 3t^2 + 7t - 6 = 0 & \therefore -3 + k - 3 = 0 \\
 \therefore (3t-2)(t+3) = 0 & \therefore k = 6 \\
 \therefore t = \frac{2}{3} \text{ or } -3 &
 \end{array}$$

$$\begin{array}{l}
 \mathbf{5} \quad \text{One vector perpendicular to } \begin{bmatrix} -4 \\ 5 \end{bmatrix} \text{ is } \begin{bmatrix} 5 \\ 4 \end{bmatrix} \text{ as dot product} = -20 + 20 = 0 \\
 \therefore \text{all vectors have form } k \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad k \neq 0
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{6} \quad \overrightarrow{\text{KL}} = \begin{bmatrix} 3-2 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \overrightarrow{\text{LK}} = -\overrightarrow{\text{KL}} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\
 \overrightarrow{\text{KM}} = \begin{bmatrix} 1-2 \\ -3-1 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix} & \overrightarrow{\text{LM}} = \begin{bmatrix} 1-3 \\ -3-2 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \end{bmatrix} \\
 \text{Now } \overrightarrow{\text{KL}} \bullet \overrightarrow{\text{KM}} = |\overrightarrow{\text{KL}}| |\overrightarrow{\text{KM}}| \cos K & \text{Now } \overrightarrow{\text{LK}} \bullet \overrightarrow{\text{LM}} = |\overrightarrow{\text{LK}}| |\overrightarrow{\text{LM}}| \cos L \\
 \therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ -4 \end{bmatrix} = \sqrt{25+1} \sqrt{9+16} \cos K & \therefore \begin{bmatrix} -1 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \sqrt{25+1} \sqrt{4+25} \cos L \\
 \therefore 15 - 4 = \sqrt{26} \sqrt{25} \cos K & \therefore 10 + 5 = \sqrt{26} \sqrt{29} \cos L \\
 \therefore \frac{11}{5\sqrt{26}} = \cos K & \therefore \cos L = \frac{15}{\sqrt{26 \times 29}} \\
 \therefore K = \cos^{-1} \left(\frac{11}{5\sqrt{26}} \right) & \therefore L \doteq 56.9^\circ \\
 \therefore K = 64.4^\circ & \text{and } \angle M \doteq 180^\circ - 56.89^\circ - 64.44^\circ \doteq 58.7^\circ
 \end{array}$$

$$\begin{array}{l}
 \mathbf{7} \quad 4x - 5y = 11 \quad \text{has gradient } \frac{4}{5} \quad \therefore \text{direction vector } \begin{bmatrix} 5 \\ 4 \end{bmatrix} \\
 2x + 3y = 7 \quad \text{has gradient } -\frac{2}{3} \quad \therefore \text{direction vector is } \begin{bmatrix} 3 \\ -2 \end{bmatrix} \\
 \text{If the angle is } \theta, \quad \begin{bmatrix} 5 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \sqrt{5^2+4^2} \sqrt{3^2+(-2)^2} \cos \theta \\
 \therefore 15 - 8 = \sqrt{41} \sqrt{13} \cos \theta \\
 \therefore \frac{7}{\sqrt{41 \times 13}} = \cos \theta \\
 \therefore \theta \doteq 72.3 \quad \therefore \text{the angle is } 72.3^\circ \text{ (or } 107.7^\circ)
 \end{array}$$

8 a

i (1) $\vec{OB}$

$$= \vec{OA} + \vec{AB}$$

$$= \vec{OA} + \vec{OC}$$

$$= \mathbf{p} + \mathbf{q}$$

 (2) $\vec{OM}$

$$= \vec{OA} + \vec{AM}$$

$$= \vec{OA} + \frac{1}{2}\vec{AC}$$

$$= \mathbf{p} + \frac{1}{2}(\vec{AO} + \vec{OC})$$

$$= \mathbf{p} + \frac{1}{2}(-\mathbf{p} + \mathbf{q})$$

$$= \mathbf{p} - \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$$

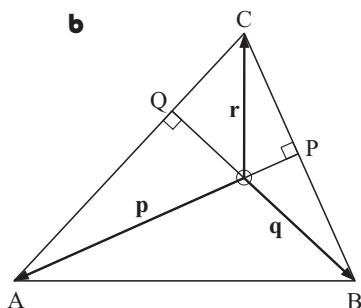
$$= \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}$$

ii We notice that $\vec{OM} = \frac{1}{2}\vec{OB}$

$$\therefore \vec{OM} \parallel \vec{OB} \text{ and } OM = \frac{1}{2}(OB)$$

i.e., O, M and B are collinear (as O is common)

 and $\therefore$ M is the midpoint of $\vec{OB}$.

b

i $\vec{AC} = \vec{AO} + \vec{OC}$

$$= -\mathbf{p} + \mathbf{r}$$

$$= \mathbf{r} - \mathbf{p}$$

 $\vec{BC} = \vec{BO} + \vec{OC}$

$$= -\mathbf{q} + \mathbf{r}$$

$$= \mathbf{r} - \mathbf{q}$$

ii
 $AP \perp BC$

$$\therefore \mathbf{p} \perp \mathbf{r} - \mathbf{q}$$

$$\therefore \mathbf{p} \bullet (\mathbf{r} - \mathbf{q}) = 0$$

$$\therefore \mathbf{p} \bullet \mathbf{r} - \mathbf{p} \bullet \mathbf{q} = 0$$

$$\therefore \mathbf{p} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q}$$

 and $BQ \perp AC$

$$\therefore \mathbf{q} \perp (\mathbf{r} - \mathbf{p})$$

$$\therefore \mathbf{q} \bullet (\mathbf{r} - \mathbf{p}) = 0$$

$$\therefore \mathbf{q} \bullet \mathbf{r} - \mathbf{q} \bullet \mathbf{p} = 0$$

$$\therefore \mathbf{q} \bullet \mathbf{r} = \mathbf{p} \bullet \mathbf{q}$$

iii $\mathbf{r} \bullet \vec{AB} = \mathbf{r} \bullet (-\mathbf{p} + \mathbf{q})$

$$= -\mathbf{r} \bullet \mathbf{p} + \mathbf{r} \bullet \mathbf{q}$$

$$= -\mathbf{p} \bullet \mathbf{q} + \mathbf{p} \bullet \mathbf{q} \quad \{\text{from ii}\}$$

$$= 0$$

 and so $\mathbf{r} \perp \vec{AB}$ i.e., $OC \perp AB$
9 a

$$2\mathbf{a} - 3\mathbf{b} = 2 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 2 \end{bmatrix} - \begin{bmatrix} -3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 7 \\ -12 \\ -7 \end{bmatrix}$$

b

$$\mathbf{a} - 3\mathbf{x} = \mathbf{b} \quad \therefore \mathbf{a} - \mathbf{b} = 3\mathbf{x} \quad \therefore \mathbf{x} = \frac{1}{3}(\mathbf{a} - \mathbf{b}) = \frac{1}{3} \begin{bmatrix} 3 \\ -5 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{5}{3} \\ -\frac{2}{3} \end{bmatrix}$$

c

 The projection of vector $\mathbf{a}$ on $\mathbf{b}$

$$= \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|} \frac{1}{|\mathbf{b}|} \mathbf{b} \quad \text{where } \mathbf{a} \bullet \mathbf{b} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = -2 - 6 + 3 = -5$$

$$= \frac{-5}{\sqrt{14}} \frac{1}{\sqrt{14}} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

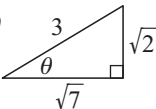
$$= \frac{-5}{14} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

10

$$|\mathbf{a}| = 3, \quad |\mathbf{b}| = \sqrt{7} \quad \text{and} \quad \mathbf{a} \times \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} \mathbf{a} \quad |\mathbf{a} \times \mathbf{b}| &= |\mathbf{a}| |\mathbf{b}| \sin \theta \\ \sqrt{1+4+9} &= 3 \times \sqrt{7} \sin \theta \\ \sin \theta &= \frac{\sqrt{14}}{3\sqrt{7}} = \frac{\sqrt{2}}{3} \end{aligned}$$

But $\cos^2 \theta = 1 - \sin^2 \theta$



$$\therefore \cos \theta = \pm \frac{\sqrt{7}}{3}$$

$$\begin{aligned} \text{Hence } \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}| |\mathbf{b}| \cos \theta \\ &= 3 \times \sqrt{7} \times \left(\pm \frac{\sqrt{7}}{3}\right) \\ &= \pm 7 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \text{Area } \triangle OAB &= \frac{1}{2} |\mathbf{a} \times \mathbf{b}| \\ &= \frac{1}{2} \sqrt{14} \text{ units}^2 \end{aligned}$$

$$\mathbf{c} \quad V = \frac{1}{6} |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|$$

$$\text{Now } \mathbf{c} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \text{and } \mathbf{a} \times \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$\begin{aligned} \therefore V &= \frac{1}{6} \left| \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right| \\ &= \frac{1}{6} |1 - 2 - 6| \\ &= \frac{7}{6} \text{ units}^3 \end{aligned}$$

REVIEW SET 15D

$$1 \quad \mathbf{a} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} \quad \mathbf{b} = \mathbf{i} - 2\mathbf{k}$$

$$\begin{aligned} \mathbf{a} \quad 3\mathbf{a} - 2\mathbf{b} &= 3[3\mathbf{i} - \mathbf{j} + 2\mathbf{k}] - 2[\mathbf{i} - 2\mathbf{k}] \\ &= 9\mathbf{i} - 3\mathbf{j} + 6\mathbf{k} - 2\mathbf{i} + 4\mathbf{k} \\ &= 7\mathbf{i} - 3\mathbf{j} + 10\mathbf{k} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \mathbf{a} - \mathbf{b} &= 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} - \mathbf{i} + 2\mathbf{k} \\ &= 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} \\ |\mathbf{a} - \mathbf{b}| &= \sqrt{2^2 + (-1)^2 + 4^2} \\ &= \sqrt{21} \text{ units} \end{aligned}$$

$$\mathbf{c} \quad \text{The vector projection of } \mathbf{b} \text{ on } \mathbf{a} \text{ has length} = \frac{|\mathbf{b} \cdot \mathbf{a}|}{|\mathbf{a}|} = \frac{|3 + 0 + (-4)|}{\sqrt{14}} = \frac{1}{\sqrt{14}} \text{ units}$$

$$2 \quad \text{Let } \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \therefore k\mathbf{a} = \begin{bmatrix} ka_1 \\ ka_2 \\ ka_3 \end{bmatrix}$$

$$\begin{aligned} \therefore |k\mathbf{a}| &= \sqrt{(ka_1)^2 + (ka_2)^2 + (ka_3)^2} \\ &= \sqrt{k^2(a_1^2 + a_2^2 + a_3^2)} \\ &= \sqrt{k^2} \sqrt{a_1^2 + a_2^2 + a_3^2} \\ &= |k| |\mathbf{a}| \end{aligned}$$

$$3 \quad \text{Given } P(-1, 2, 3) \text{ and } Q(4, 0, -1)$$

$$\mathbf{a} \quad \overrightarrow{PQ} = \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix}$$

$$\mathbf{b} \quad \overrightarrow{PQ} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 5$$

$$\therefore |\overrightarrow{PQ}| \sqrt{1^2 + 0^2 + 0^2} \cos \alpha = 5$$

$$\therefore \sqrt{25 + 4 + 16} \cos \alpha = 5$$

$$\therefore \cos \alpha = \frac{5}{\sqrt{45}}$$

$$\therefore \cos \alpha \doteq 41.81^\circ$$

$$\mathbf{c} \quad \begin{array}{c} \xrightarrow{\quad 2x \quad \quad \quad x \quad} \\ Q(4, 0, -1) \quad \quad R(a, b, c) \quad P(-1, 2, 3) \end{array}$$

$$\overrightarrow{QR} = 2\overrightarrow{RP}$$

$$[a - 4, b, c + 1] = 2[-1 - a, 2 - b, 3 - c]$$

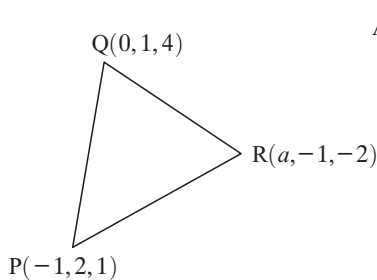
$$a - 4 = -2 - 2a, \quad b = 4 - 2b, \quad c + 1 = 6 - 2c$$

$$\therefore 3a = 2 \quad \therefore 3b = 4 \quad \therefore 3c = 5$$

$$\therefore a = \frac{2}{3} \quad \therefore b = \frac{4}{3} \quad \therefore c = \frac{5}{3}$$

$$\therefore R \text{ is at } \left(\frac{2}{3}, \frac{4}{3}, \frac{5}{3}\right)$$

4



$$\begin{aligned}
 \text{Area of } \Delta &= \frac{1}{2} | \vec{PQ} \times \vec{PR} | \\
 &= \frac{1}{2} \left| \begin{bmatrix} 0 & -1 \\ 1 & -2 \\ 4 & -1 \end{bmatrix} \times \begin{bmatrix} a & -1 \\ -1 & -2 \\ -2 & -1 \end{bmatrix} \right| \\
 &= \frac{1}{2} \left| \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} a+1 \\ -3 \\ -3 \end{bmatrix} \right|
 \end{aligned}$$

$$\therefore \frac{1}{2} \left| \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ a+1 & -3 & -3 \end{bmatrix} \right| = \sqrt{118}$$

$$\therefore \left| \begin{bmatrix} -1 & 3 \\ -3 & -3 \end{bmatrix} \mathbf{i} + \begin{bmatrix} 3 & 1 \\ -3 & a+1 \end{bmatrix} \mathbf{j} + \begin{bmatrix} 1 & -1 \\ a+1 & -3 \end{bmatrix} \mathbf{k} \right| = 2\sqrt{118}$$

$$|12\mathbf{i} + (3a+3+3)\mathbf{j} + (-3+a+1)\mathbf{k}| = 2\sqrt{118}$$

$$\therefore \sqrt{144 + (3a+6)^2 + (a-2)^2} = 2\sqrt{118}$$

$$\therefore 144 + 9a^2 + 36a + 36 + a^2 - 4a + 4 = 472$$

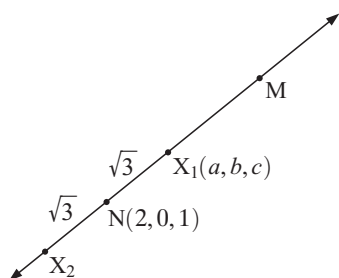
$$\therefore 10a^2 + 32a - 288 = 0$$

$$\therefore 5a^2 + 16a - 144 = 0$$

$$\therefore (5a+36)(a-4) = 0$$

$$\therefore a = -\frac{36}{5} \text{ or } 4$$

5 a



$$\vec{MN} = \begin{bmatrix} 2 & -1 \\ 0 & -3 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\vec{NX} = \begin{bmatrix} a-2 \\ b-0 \\ c-1 \end{bmatrix} \text{ and } |\vec{NX}| = \sqrt{3}$$

$$\therefore \begin{bmatrix} a-2 \\ b-0 \\ c-1 \end{bmatrix} = \pm \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \quad \{\text{as this vector has length } \sqrt{3}\}$$

$$\therefore a-2 = \pm 1, \quad b = \mp 1, \quad c-1 = \mp 1$$

$$\therefore a = 3 \text{ or } 1, \quad b = -1 \text{ or } 1, \quad c = 0 \text{ or } 2$$

$$\therefore X \text{ is at } (3, -1, 0) \text{ or } (1, 1, 2)$$

b The unit vector in the direction of $\vec{MN}$ is $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$$\therefore \text{the required vector is } \frac{2}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \text{ i.e., } \begin{bmatrix} \frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} \\ -\frac{2}{\sqrt{3}} \end{bmatrix}$$

6 $\vec{BC} = \begin{bmatrix} -1 & -2 \\ -9 & -0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \\ 3 \end{bmatrix}$ and $\vec{CA} = \begin{bmatrix} 1 & -1 \\ -3 & -9 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}$

$$\therefore -\frac{1}{3}\vec{BC} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \text{ and } \frac{1}{2}\vec{CA} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

$$\text{So, } -\frac{1}{3}\vec{BC} = \frac{1}{2}\vec{CA}$$

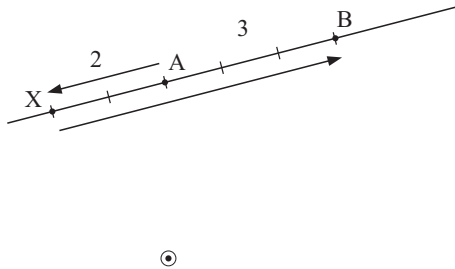
$$\text{or } \vec{BC} = -\frac{3}{2}\vec{CA}$$

$$\therefore \vec{BC} \parallel \vec{CA} \text{ i.e., } B, C \text{ and } A \text{ are collinear}$$

For C divides BA we need $\vec{BC} : \vec{CA} = -3 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} : 2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = -3 : 2$

$\therefore$ C divides BA externally in the ratio 3 : 2.

7



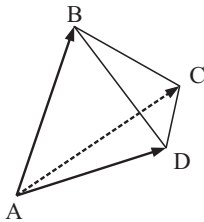
Let X divide AB externally in the ratio 2 : 5.

$$\therefore \vec{AX} : \vec{XB} = -2 : 5$$

$$\begin{aligned} \text{Now } \vec{OX} &= \vec{OA} + \vec{AX} \\ &= \vec{OA} + \frac{2}{3}\vec{BA} \\ &= \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} -2-3 \\ 3-1 \\ 5-1 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} -5 \\ 4 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -2 - \frac{10}{3} \\ 3 + \frac{8}{3} \\ 5 + \frac{8}{3} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -\frac{16}{3} \\ \frac{17}{3} \\ \frac{23}{3} \end{bmatrix} \end{aligned}$$

$$\therefore X \text{ is } \left(-\frac{16}{3}, \frac{17}{3}, \frac{23}{3}\right)$$

8

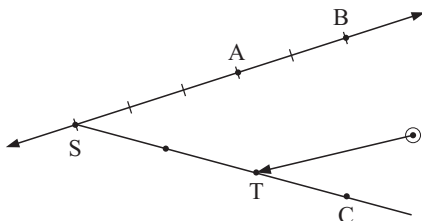


$$\begin{aligned} \vec{AB} &= \begin{bmatrix} -1-3 \\ 2-1 \\ 1-2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix} \\ \vec{AC} &= \begin{bmatrix} -2-3 \\ 0-1 \\ 3-2 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix} \\ \vec{AD} &= \begin{bmatrix} 4-3 \\ 3-1 \\ -1-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \end{aligned}$$

Volume

$$\begin{aligned} &= \frac{1}{6} |\vec{AB} \bullet (\vec{AC} \times \vec{AD})| \\ &= \frac{1}{6} \left| \begin{vmatrix} -4 & 1 & -1 \\ -5 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix} \right| \text{units}^3 \\ &= \frac{1}{6} \left| -4 \begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -5 \\ -3 & 1 \end{vmatrix} - 1 \begin{vmatrix} -5 & -1 \\ 1 & 2 \end{vmatrix} \right| \\ &= \frac{1}{6} |-4(1) + 1(-14) - 1(-10 + 1)| \\ &= \frac{1}{6} |-9| \\ &= 1\frac{1}{2} \text{units}^3 \end{aligned}$$

9 $\vec{AS} : \vec{SB} = -3 : 5$ and $\vec{CT} : \vec{TS} = 1 : 2$



$$\begin{aligned} \vec{OT} &= \vec{OC} + \vec{CT} \\ &= \mathbf{c} + \frac{1}{3}\vec{CS} \\ &= \mathbf{c} + \frac{1}{3}[\vec{CB} + \vec{BS}] \\ &= \mathbf{c} + \frac{1}{3}[\vec{CO} + \vec{OB}] + \frac{1}{3} \times \frac{5}{2}\vec{BA} \\ &= \mathbf{c} + \frac{1}{3}[-\mathbf{c} + \mathbf{b}] + \frac{5}{6}[\vec{BO} + \vec{OA}] \\ &= \mathbf{c} - \frac{1}{3}\mathbf{c} + \frac{1}{3}\mathbf{b} + \frac{5}{6}[-\mathbf{b} + \mathbf{a}] \\ &= \mathbf{c} - \frac{1}{3}\mathbf{c} + \frac{1}{3}\mathbf{b} - \frac{5}{6}\mathbf{b} + \frac{5}{6}\mathbf{a} \\ &= \frac{5}{6}\mathbf{a} - \frac{1}{2}\mathbf{b} + \frac{2}{3}\mathbf{c} \end{aligned}$$

$$\begin{aligned}
 \mathbf{10} \quad \begin{bmatrix} 1 \\ r \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} &= 0 & \therefore 2 + 2r - 2 &= 0 \\
 & & \therefore 2r &= 0 \\
 & & \therefore r &= 0
 \end{aligned}$$

So, we want a unit vector parallel to $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ (which has length $\sqrt{1+0+4} = \sqrt{5}$)

$$\therefore \text{vectors are } \pm \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ i.e., } \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k} \text{ or } \frac{-1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{k}$$

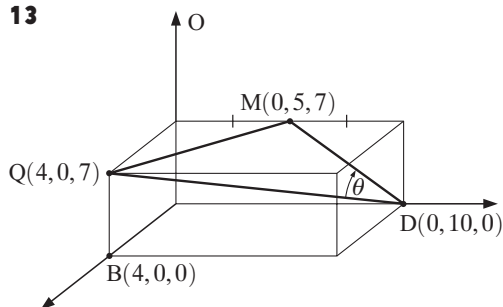
$$\begin{aligned}
 \mathbf{11} \quad |\mathbf{u} \times \mathbf{v}| &= |\mathbf{u}| |\mathbf{v}| \sin \theta & \text{But } \cos^2 \theta + \sin^2 \theta &= 1 \\
 \therefore \sqrt{1+9+16} &= 3 \times 5 \times \sin \theta & \therefore \cos^2 \theta + \frac{26}{225} &= 1 \\
 \therefore \frac{\sqrt{26}}{15} &= \sin \theta & \therefore \cos^2 \theta &= \frac{199}{225} \\
 & & \therefore \cos \theta &= \pm \frac{\sqrt{199}}{15}
 \end{aligned}$$

So, if θ is acute, $\cos \theta = \frac{\sqrt{149}}{15}$ and $\mathbf{u} \bullet \mathbf{v} = |\mathbf{a}| |\mathbf{b}| \cos \theta = 3 \times 5 \times \frac{\sqrt{199}}{15} = \sqrt{199}$
 and, if θ is obtuse, $\cos \theta = -\frac{\sqrt{199}}{15}$ and $\mathbf{u} \bullet \mathbf{v} = -\sqrt{199}$

12 3 vectors are coplanar if the volume of the tetrahedron defined by them is 0.

$$\begin{aligned}
 \therefore \begin{vmatrix} 1 & 2 & -3 \\ 2 & 2 & 3 \\ 1 & 2-t & t+1 \end{vmatrix} &= 0 \quad \text{i.e., } 1 \begin{vmatrix} 2 & 3 \\ 2-t & t+1 \end{vmatrix} + 2 \begin{vmatrix} 3 & 2 \\ t+1 & 1 \end{vmatrix} - 3 \begin{vmatrix} 2 & 2 \\ 1 & 2-t \end{vmatrix} = 0 \\
 \therefore 1(2t+2-6+3t) + 2(3-2t-2) - 3(4-2t-2) &= 0 \\
 \therefore 5t-4+2-4t-6+6t &= 0 \\
 \therefore 7t &= 8 \\
 \therefore t &= \frac{8}{7}
 \end{aligned}$$

13



Placing a set of axes with origin at A, as showing gives $Q(4, 0, 7)$, $M(0, 5, 7)$, $D(0, 10, 0)$.

$$\overrightarrow{DQ} = \begin{bmatrix} 4-0 \\ 0-10 \\ 7-0 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 7 \end{bmatrix}$$

$$\overrightarrow{DM} = \begin{bmatrix} 0-0 \\ 5-10 \\ 7-0 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$$

$$\overrightarrow{DQ} \bullet \overrightarrow{DM} = |\overrightarrow{DQ}| |\overrightarrow{DM}| \cos \theta$$

$$\therefore 0 + 50 + 49 = \sqrt{16+100+49} \sqrt{0+25+49} \cos \theta$$

$$\therefore 99 = \sqrt{165} \sqrt{74} \cos \theta$$

$$\therefore \cos \theta = \frac{99}{\sqrt{165 \times 74}}$$

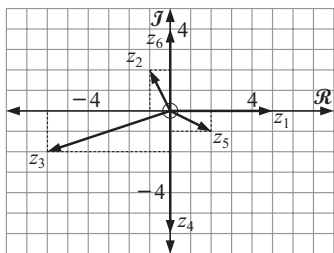
$$\therefore \theta \doteq 26.4^\circ$$

Chapter 16

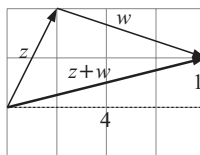
COMPLEX NUMBERS

EXERCISE 16A.1

1

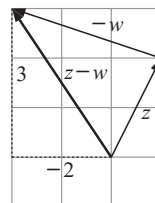


2 a



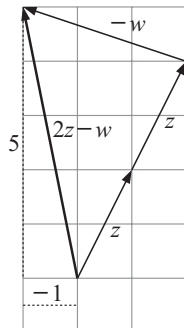
$$\begin{aligned} z + w &= (1 + 2i) + (3 - i) \\ &= 4 + i \end{aligned}$$

b



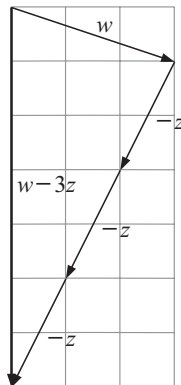
$$\begin{aligned} z - w &= (1 + 2i) - (3 - i) \\ &= -2 + 3i \end{aligned}$$

c



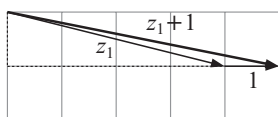
$$\begin{aligned} 2z - w &= 2(1 + 2i) - (3 - i) \\ &= 2 + 4i - 3 + i \\ &= -1 + 5i \end{aligned}$$

d



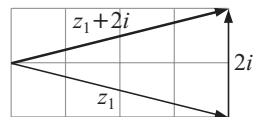
$$\begin{aligned} w - 3z &= (3 - i) - 3(1 + 2i) \\ &= 3 - i - 3 - 6i \\ &= -7i \end{aligned}$$

3 a



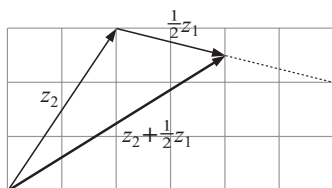
$$\begin{aligned} z_1 + 1 &= 4 - i + 1 \\ &= 5 - i \end{aligned}$$

b



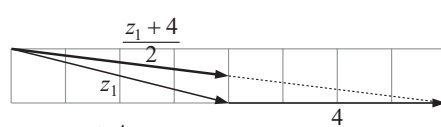
$$\begin{aligned} z_1 + 2i &= 4 - i + 2i \\ &= 4 + i \end{aligned}$$

c

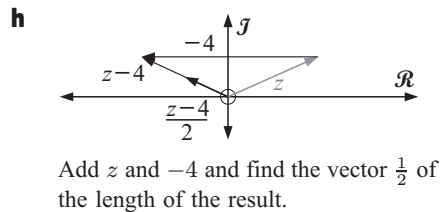
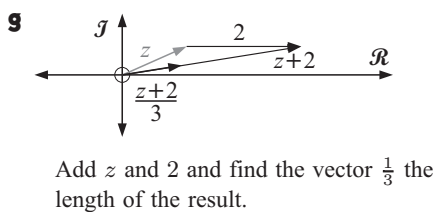
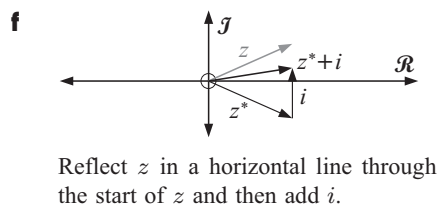
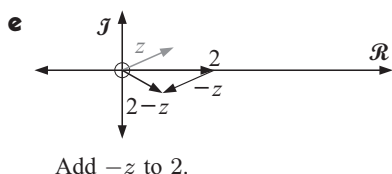
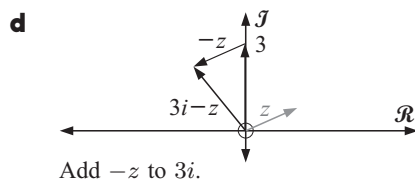
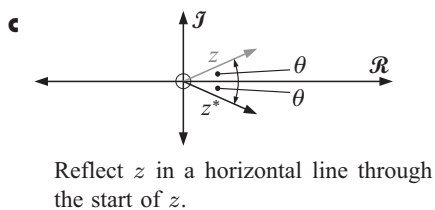
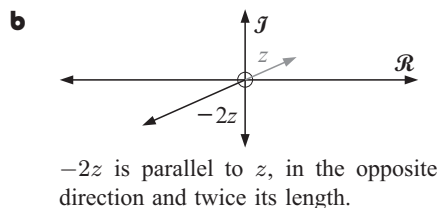
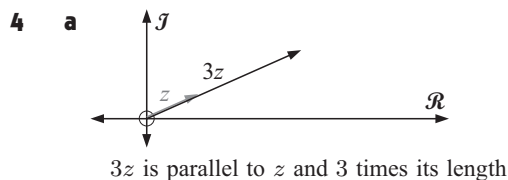


$$\begin{aligned} z_2 + \frac{1}{2}z_1 &= (2 + 3i) + \frac{1}{2}(4 - i) \\ &= 2 + 3i + 2 - \frac{1}{2}i \\ &= 4 + \frac{5}{2}i \end{aligned}$$

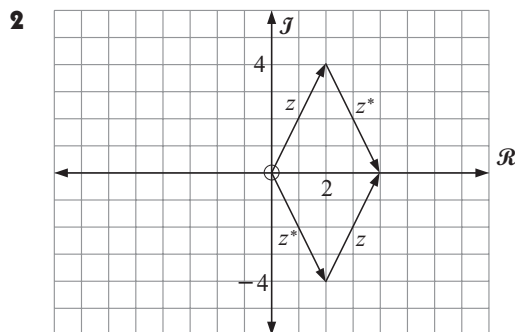
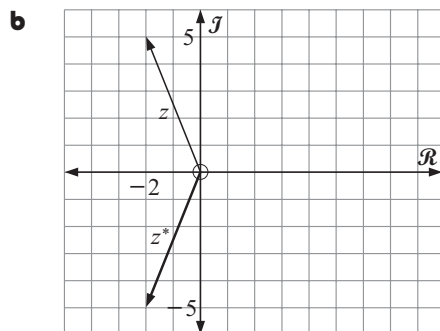
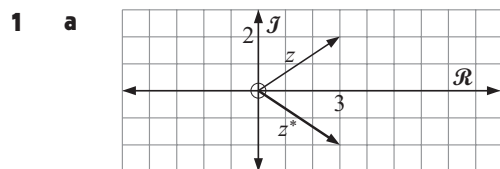
d



$$\begin{aligned} \frac{z_1 + 4}{2} &= \frac{4 - i + 4}{2} \\ &= 4 - \frac{1}{2}i \end{aligned}$$

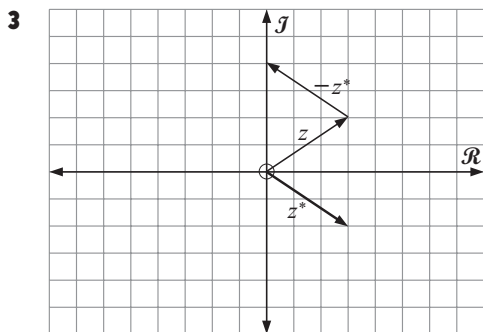


EXERCISE 16A.2



The figure formed by $z + z^*$ and $z^* + z$ is a rhombus.

No matter where z is placed, $z + z^*$ lies on the Real axis. i.e., is always real.



The resulting vector will always lie along the Imaginary axis.

i.e., will be purely imaginary *or* zero

i.e., when $z = 0$, $z - z^* = 0$.

4 If z is real then z^* will also be real. In fact if $z = a$, $z^* = a$ i.e., $z^* = z$ if z is real.

EXERCISE 16B.1

1 a $|3 - 4i|$
 $= \sqrt{3^2 + (-4)^2}$
 $= \sqrt{9 + 16}$
 $= \sqrt{25}$
 $= 5$

b $|5 + 12i|$
 $= \sqrt{5^2 + 12^2}$
 $= \sqrt{25 + 144}$
 $= \sqrt{169}$
 $= 13$

c $|-8 + 2i|$
 $= \sqrt{(-8)^2 + 2^2}$
 $= \sqrt{64 + 4}$
 $= \sqrt{68}$
 $= 2\sqrt{17}$

d $|3i|$
 $= \sqrt{0^2 + 3^2}$
 $= \sqrt{9}$
 $= 3$

e $|-4|$
 $= \sqrt{(-4)^2 + 0^2}$
 $= \sqrt{16}$
 $= 4$

2 a $|z|$
 $= |2 + i|$
 $= \sqrt{2^2 + 1^2}$
 $= \sqrt{5}$

b $|z^*|$
 $= |2 - i|$
 $= \sqrt{2^2 + (-1)^2}$
 $= \sqrt{5}$

c $|z^*|^2$
 $= (\sqrt{5})^2$
 $= 5$

d $z \times z^*$
 $= (2 + i)(2 - i)$
 $= 4 - i^2$
 $= 5$

e $|zw|$
 $= |(2 + i)(-1 + 3i)|$
 $= |-2 + 6i - i + 3i^2|$
 $= |-5 + 5i|$
 $= \sqrt{(-5)^2 + 5^2}$
 $= \sqrt{50} \text{ or } 5\sqrt{2}$

f $|z||w|$
 $= \sqrt{2^2 + 1^2} \sqrt{(-1)^2 + 3^2}$
 $= \sqrt{5} \times \sqrt{10}$
 $= \sqrt{50}$
 $= 5\sqrt{2}$

g $\left| \frac{z}{w} \right|$
 $= \left| \frac{2 + i}{-1 + 3i} \right|$
 $= \left| \frac{(2 + i)}{(-1 + 3i)} \times \frac{(-1 - 3i)}{(-1 - 3i)} \right|$
 $= \left| \frac{-2 - 6i - i - 3i^2}{(-1)^2 - (3i)^2} \right|$
 $= \left| \frac{-2 + 3 - 7i}{10} \right|$
 $= \left| \frac{1}{10} - \frac{7}{10}i \right|$
 $= \sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{-7}{10}\right)^2}$
 $= \sqrt{\frac{1+49}{100}}$
 $= \sqrt{\frac{50}{100}}$
 $= \frac{1}{\sqrt{2}}$

h $\frac{|z|}{|w|}$
 $= \frac{\sqrt{5}}{\sqrt{10}}$
 $= \sqrt{\frac{5}{10}}$
 $= \frac{1}{\sqrt{2}}$

i $z^2 = (2 + i)^2$
 $= 4 + 4i + i^2$
 $= 3 + 4i$
 $\therefore |z^2| = \sqrt{3^2 + 4^2}$
 $= \sqrt{25}$
 $= 5$

$$\begin{aligned} \mathbf{j} \quad |z|^2 &= (\sqrt{5})^2 \quad \{\text{from a}\} \\ &= 5 \end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad z^3 &= z^2 \times z \\ &= (3 + 4i)(2 + i) \\ &= 6 + 3i + 8i + 4i^2 \\ &= 2 + 11i \\ \therefore |z^3| &= \sqrt{2^2 + 11^2} \\ &= \sqrt{4 + 121} \\ &= \sqrt{125} \\ &= 5\sqrt{5} \end{aligned}$$

$$\begin{aligned} \mathbf{l} \quad |z|^3 &= (\sqrt{5})^3 \quad \{\text{from a}\} \\ &= \sqrt{125} \\ &= 5\sqrt{5} \end{aligned}$$

3 Rules for Modulus

$$(1) \quad z \times z^* = |z|^2$$

$$(2) \quad |z^*| = |z|$$

$$(3) \quad |zw| = |z||w|$$

$$(4) \quad \left| \frac{z}{w} \right| = \frac{|z|}{|w|}$$

$$(5) \quad |z^n| = |z|^n$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \text{Let } z &= a + bi \\ \therefore |z^*| &= |a - bi| \\ &= \sqrt{a^2 + (-b)^2} \\ &= \sqrt{a^2 + b^2} \\ &= |a + bi| \\ &= |z| \quad \text{as required} \end{aligned}$$

$$\begin{aligned} \text{Let } z &= a + bi \\ \therefore z \times z^* &= (a + bi)(a - bi) \\ &= a^2 - b^2i^2 \\ &= a^2 + b^2 \\ &= (\sqrt{a^2 + b^2})^2 \\ &= |z|^2 \quad \text{as required} \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad |z_1 z_2 z_3| &= |(z_1 z_2) z_3| \\ &= |z_1 z_2| |z_3| \quad \{\text{as } |zw| = |z||w|\} \\ &= |z_1| |z_2| |z_3| \quad \{|zw| = |z||w| \text{ again}\} \end{aligned}$$

$$\begin{aligned} \text{Letting } z_1 = z_2 = z_3 = z \quad \text{we get the result that } |z^3| &= |z \times z \times z| \\ &= |z| \times |z| \times |z| \\ &= |z|^3 \end{aligned}$$

b Now extending this result by the same argument

$$\begin{aligned} |z_1 z_2 z_3 z_4| &= |(z_1 z_2 z_3) z_4| \\ &= |z_1 z_2 z_3| |z_4| \\ &= |z_1| |z_2| |z_3| |z_4| \end{aligned}$$

and putting $z_1 = z_2 = z_3 = z_4 = z$ we get

$$|z^4| = |z \times z \times z \times z| = |z| \times |z| \times |z| \times |z| = |z|^4$$

6 The generalisation of **5** is: $|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$ and that $|z^n| = |z|^n$

$$\begin{aligned} \mathbf{7} \quad \left| \frac{z}{w} \right| \times |w| &= \left| \frac{z}{w} \times w \right| \quad \{\text{using } |z_1| |z_2| = |z_1 z_2|\} \\ &= |z| \end{aligned}$$

$$\text{i.e., } \left| \frac{z}{w} \right| \times |w| = |z|$$

$$\therefore \left| \frac{z}{w} \right| = \frac{|z|}{|w|} \quad \text{provided } w \neq 0 \quad \{\text{dividing both sides by } |w|\}$$

$$\begin{aligned} \mathbf{8} \quad \mathbf{a} \quad |2z| &= |2| |z| \\ &= 2 \times 3 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad |-3z| &= |-3| |z| \\ &= 3 \times 3 \\ &= 9 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad |(1 + 2i)z| &= |1 + 2i| \times |z| \\ &= \sqrt{1 + 4} \times 3 \\ &= 3\sqrt{5} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad |iz| &= |i| |z| \\ &= 1 \times 3 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad \left| \frac{1}{z} \right| &= \frac{|1|}{|z|} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad \left| \frac{2i}{z^2} \right| &= \frac{|2i|}{|z|^2} \\ &= \frac{2}{3^2} \\ &= \frac{2}{9} \end{aligned}$$

$$\begin{aligned} \mathbf{9} \quad z = \cos \theta + i \sin \theta, \quad \therefore |z| &= \sqrt{\cos^2 \theta + \sin^2 \theta} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{10} \quad \text{If } z = 1 - i\sqrt{3} \quad \text{then } |z| &= \sqrt{1^2 + (-\sqrt{3})^2} \quad \therefore |z^{20}| = |z|^{20} \\ &= \sqrt{4} \\ &= 2 \quad \quad \quad = 2^{20} \\ & \quad \quad \quad = 1\,048\,576 \end{aligned}$$

$$\begin{aligned} \mathbf{11} \quad \mathbf{a} \quad w = \frac{z+1}{z-1}. \quad \text{Let } z = a+ib, \quad \therefore w &= \frac{a+bi+1}{a+bi-1} \\ &= \frac{(a+1)+bi}{(a-1)+bi} \\ &= \frac{(a+1)+bi}{(a-1)+bi} \times \frac{(a-1)-bi}{(a-1)-bi} \\ &= \frac{(a+1)(a-1) - b(a+1)i + b(a-1)i - b^2i^2}{(a-1)^2 - (bi)^2} \\ &= \frac{a^2 - 1 - abi - bi + abi - bi + b^2}{(a-1)^2 + b^2} \\ &= \left(\frac{a^2 + b^2 - 1}{(a-1)^2 + b^2} \right) + \left(\frac{-2b}{(a-1)^2 + b^2} \right) i \end{aligned}$$

$$\mathbf{b} \quad \therefore \operatorname{Re}(w) = \frac{a^2 + b^2 - 1}{(a-1)^2 + b^2} = \frac{a^2 + b^2 - 1}{a^2 - 2a + 1 + b^2} = \frac{a^2 + b^2 - 1}{a^2 + b^2 - 2a + 1}$$

$$\text{Since } |z| = 1 \quad \sqrt{a^2 + b^2} = 1 \quad \therefore a^2 + b^2 = 1$$

$$\therefore \operatorname{Re}(w) = \frac{1-1}{1-2a+1} = 0 \quad \text{provided } a \neq 1$$

if $a = 1$, then $\operatorname{Re}(w)$ is undefined

EXERCISE 16B.2

$$\mathbf{1} \quad \mathbf{a} \quad A(3, 6) \quad B(-1, 2) \quad z = 3 + 6i \quad w = -1 + 2i$$

$$\begin{aligned} \mathbf{i} \quad z - w &= (3 + 6i) - (-1 + 2i) \\ &= 4 + 4i \\ |z - w| &= \sqrt{4^2 + 4^2} \\ &= \sqrt{32} \\ &= 4\sqrt{2} \\ \therefore AB &= 4\sqrt{2} \text{ units} \end{aligned} \quad \begin{aligned} \mathbf{ii} \quad \frac{z+w}{2} &= \frac{(3+6i)+(-1+2i)}{2} \\ &= \frac{2+8i}{2} \\ &= 1+4i, \quad \text{and so M is at } (1, 4) \end{aligned}$$

b $A(-4, 7) \quad B(1, -3) \quad z = -4 + 7i \quad w = 1 - 3i$

$$\begin{aligned} \text{i} \quad z - w &= (-4 + 7i) - (1 - 3i) \\ &= -5 + 10i \\ |z - w| &= \sqrt{5^2 + 10^2} \\ &= \sqrt{125} \\ &= 5\sqrt{5} \\ \therefore AB &= 5\sqrt{5} \text{ units} \end{aligned} \quad \begin{aligned} \text{ii} \quad \frac{z + w}{2} &= \frac{(-4 + 7i) + (1 - 3i)}{2} \\ &= \frac{-3 + 4i}{2} \\ &= -\frac{3}{2} + 2i, \text{ and so } M \text{ is at } \left(-\frac{3}{2}, 2\right) \end{aligned}$$

2 a i $\vec{OQ} = z + w$ **ii** $\vec{PR} = w - z$

b In $\triangle OPQ$ $|z + w|$ represents the length of OQ , $|z|$ = length of OP and $|w|$ the length of PQ .
Now if w, z are not parallel we will form the $\triangle OPQ$ and this means $OQ < OP + PQ$

i.e., $|z + w| < |z| + |w|$

if w and z are parallel then we form a straight line and $OQ = OP + PQ$

i.e., $|z + w| = |z| + |w|$

Consequently $|z + w| \leq |z| + |w|$

c In $\triangle OPR$, the length of RP is represented by $|z - w|$ provided w and z not parallel we form a triangle and $OR + OP > OR$

$|z - w| + |z| > |w|$

$\therefore |z - w| > |w| - |z|$

Equality will occur when w is parallel to z , in which case a straight line OPR is formed and

$|z - w| = |w| - |z|$

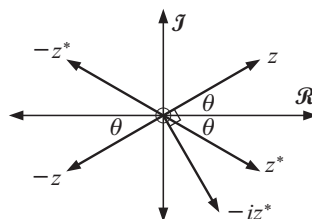
Consequently $|z - w| \geq |w| - |z|$

3 a $z \rightarrow z^*$. Reflection in the Real Axis.

b $z \rightarrow -z^*$. Reflection in the Imaginary Axis.

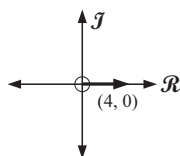
c $z \rightarrow -z$. Rotation about O , anticlockwise π .

d $z \rightarrow -iz$. Rotation about O , clockwise $\frac{\pi}{2}$.

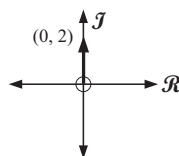


EXERCISE 16B.3

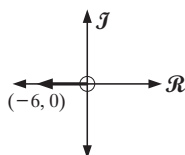
1 a $z = 4$ $\arg z = 0$
 $|z| = 4$
 $\therefore z = 4 \text{ cis } 0$



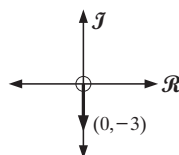
b $z = 2i$ $\arg z = \frac{\pi}{2}$
 $|z| = 2$
 $\therefore z = 2 \text{ cis } \left(\frac{\pi}{2}\right)$



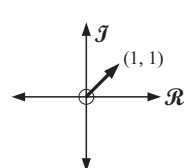
c $z = -6$ $\arg z = \pi$
 $|z| = 6$
 $\therefore z = 6 \text{ cis } \pi$



d $z = -3i$ $\arg z = -\frac{\pi}{2}$
 $|z| = 3$
 $\therefore z = 3 \text{ cis } \left(-\frac{\pi}{2}\right)$



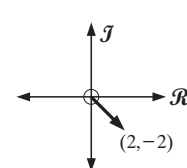
e $z = 1 + i$ $\arg z = \frac{\pi}{4}$



$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\therefore z = \sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} \right)$$

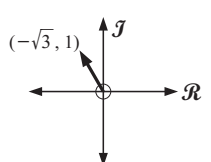
f $z = 2 - 2i$ $\arg z = -\frac{\pi}{4}$



$$|z| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

$$\therefore z = 2\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right)$$

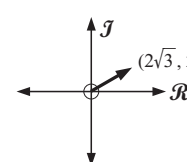
g $z = -\sqrt{3} + i$, $\arg z = \frac{5\pi}{6}$



$$|z| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

$$\therefore z = 2 \operatorname{cis} \left(\frac{5\pi}{6} \right)$$

h $z = 2\sqrt{3} + 2i$ $\arg z = \frac{\pi}{6}$



$$|z| = \sqrt{12 + 4} = \sqrt{16} = 4$$

$$\therefore z = 4 \operatorname{cis} \left(\frac{\pi}{6} \right)$$

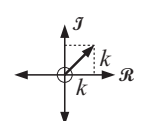
2 $z = 0$, i.e., $0 + 0i$ cannot be written in polar form. The vector representing $\overrightarrow{OP}$ has length zero, i.e., is the origin and an argument is not defined (i.e., no angle can be formed with the positive x -axis).

3 If $k = 0$ it is not possible

If $k > 0$ $|z| = \sqrt{k^2 + k^2} = k\sqrt{2}$

$\arg z = \frac{\pi}{4}$

$\therefore z = k\sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} \right)$

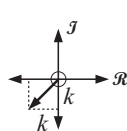


If $k < 0$ $|z| = \sqrt{k^2 + k^2} = |k|\sqrt{2}$

Since $k < 0$ $|z| = -k\sqrt{2}$

$\arg z = -\frac{3\pi}{4}$

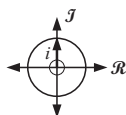
$\therefore z = -k\sqrt{2} \operatorname{cis} \left(-\frac{3\pi}{4} \right)$



4 a $2 \operatorname{cis} \left(\frac{\pi}{2} \right)$

$= 2 \left(\cos \left(\frac{\pi}{2} \right) + i \sin \left(\frac{\pi}{2} \right) \right)$

$= 2i$

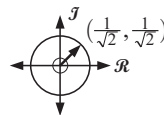


b $8 \operatorname{cis} \left(\frac{\pi}{4} \right)$

$= 8 \left(\cos \left(\frac{\pi}{4} \right) + i \sin \left(\frac{\pi}{4} \right) \right)$

$= 8 \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right)$

$= 4\sqrt{2} + 4\sqrt{2}i$

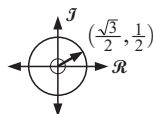


c $4 \operatorname{cis} \left(\frac{\pi}{6} \right)$

$= 4 \left(\cos \left(\frac{\pi}{6} \right) + i \sin \left(\frac{\pi}{6} \right) \right)$

$= 4 \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i \right)$

$= 2\sqrt{3} + 2i$

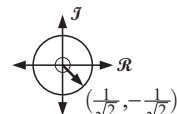


d $\sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right)$

$= \sqrt{2} \left(\cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right) \right)$

$= \sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right)$

$= 1 - i$

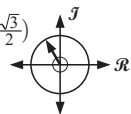


e $\sqrt{3} \operatorname{cis} \left(\frac{2\pi}{3} \right)$

$= \sqrt{3} \left(\cos \left(\frac{2\pi}{3} \right) + i \sin \left(\frac{2\pi}{3} \right) \right)$

$= \sqrt{3} \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right)$

$= -\frac{\sqrt{3}}{2} + \frac{3}{2}i$

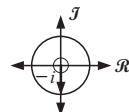


f $5 \operatorname{cis} \pi$

$= 5(\cos \pi + i \sin \pi)$

$= 5(-1)$

$= -5$



$$\begin{aligned} \mathbf{5} \quad \mathbf{a} \quad & \text{cis } 0 \\ &= \cos 0 + i \sin 0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & |\text{cis } \theta| \\ &= |\cos \theta + i \sin \theta| \\ &= \sqrt{\cos^2 \theta + \sin^2 \theta} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \text{cis } \alpha \times \text{cis } \beta &= (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \\ &= \cos \alpha \cos \beta + i \cos \alpha \sin \beta + i \sin \alpha \cos \beta + i^2 \sin \alpha \sin \beta \\ &= [\cos \alpha \cos \beta - \sin \alpha \sin \beta] + i [\sin \alpha \cos \beta + \sin \beta \cos \alpha] \\ &= \cos(\alpha + \beta) + i \sin(\alpha + \beta) \\ &= \text{cis }(\alpha + \beta) \end{aligned}$$

EXERCISE 16B.4

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad & \text{cis } \theta \text{ cis } 2\theta \\ &= \text{cis }(\theta + 2\theta) \\ &= \text{cis } 3\theta \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \frac{\text{cis } 3\theta}{\text{cis } \theta} \\ &= \text{cis } (3\theta - \theta) \\ &= \text{cis } 2\theta \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & [\text{cis } \theta]^3 \\ &= (\text{cis } \theta)(\text{cis } \theta)(\text{cis } \theta) \\ &= (\text{cis } 2\theta)(\text{cis } \theta) \\ &= \text{cis } 3\theta \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \text{cis } \left(\frac{\pi}{18}\right) \times \text{cis } \left(\frac{\pi}{9}\right) \\ &= \text{cis } \left(\frac{\pi}{18} + \frac{\pi}{9}\right) \\ &= \text{cis } \left(\frac{3\pi}{18}\right) \\ &= \text{cis } \left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2} + \frac{1}{2}i \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & 2 \text{cis } \left(\frac{\pi}{12}\right) \text{cis } \left(\frac{\pi}{6}\right) \\ &= 2 \text{cis } \left(\frac{\pi}{12} + \frac{\pi}{6}\right) \\ &= 2 \text{cis } \left(\frac{3\pi}{12}\right) \\ &= 2 \text{cis } \left(\frac{\pi}{4}\right) \\ &= 2 \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\ &= \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}i \\ &= \sqrt{2} + i\sqrt{2} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & 2 \text{cis } \left(\frac{2\pi}{5}\right) \times 4 \text{cis } \left(\frac{8\pi}{5}\right) \\ &= 8 \text{cis } \left(\frac{2\pi}{5} + \frac{8\pi}{5}\right) \\ &= 8 \text{cis } 2\pi \\ &= 8 \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \frac{4 \text{cis } \left(\frac{\pi}{12}\right)}{2 \text{cis } \left(\frac{7\pi}{12}\right)} \\ &= 2 \text{cis } \left(\frac{\pi}{12} - \frac{7\pi}{12}\right) \\ &= 2 \text{cis } \left(-\frac{6\pi}{12}\right) \\ &= 2 \text{cis } \left(-\frac{\pi}{2}\right) \\ &= -2i \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \frac{\sqrt{32} \text{cis } \left(\frac{\pi}{8}\right)}{\sqrt{2} \text{cis } \left(\frac{-7\pi}{8}\right)} \\ &= \frac{\sqrt{32}}{\sqrt{2}} \text{cis } \left(\frac{\pi}{8} - \frac{-7\pi}{8}\right) \\ &= \sqrt{16} \text{cis } \left(\frac{8\pi}{8}\right) \\ &= 4 \text{cis } (\pi) \\ &= -4 \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \left[\sqrt{2} \text{cis } \left(\frac{\pi}{8}\right)\right]^4 \\ &= \sqrt{2} \text{cis } \left(\frac{\pi}{8}\right) \times \sqrt{2} \text{cis } \left(\frac{\pi}{8}\right) \times \sqrt{2} \text{cis } \left(\frac{\pi}{8}\right) \times \sqrt{2} \text{cis } \left(\frac{\pi}{8}\right) \\ &= (\sqrt{2})^4 \text{cis } \left(\frac{\pi}{8} + \frac{\pi}{8} + \frac{\pi}{8} + \frac{\pi}{8}\right) \\ &= 4 \text{cis } \left(\frac{\pi}{2}\right) \\ &= 4(0 + 1i) \text{ i.e., } 4i \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad & \text{cis } 17\pi \\ &= \text{cis } (\pi + 8(2\pi)) \\ &= \text{cis } \pi \\ &= -1 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \text{cis } -37\pi \\ &= \text{cis } (\pi - 19(2\pi)) \\ &= \text{cis } (\pi) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \text{cis } \left(\frac{91\pi}{3}\right) \\ &= \text{cis } \left(\frac{\pi}{3} + 15(2\pi)\right) \\ &= \text{cis } \left(\frac{\pi}{3}\right) \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}i \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & z = 2 \text{cis } \theta \\ & |z| = 2 \\ & \arg z = \theta \end{aligned} \quad \begin{aligned} \mathbf{b} \quad & z^* = 2 \text{cis } (-\theta) \end{aligned} \quad \begin{aligned} \mathbf{c} \quad & -z = 2 \text{cis } (\theta + \pi) \end{aligned} \quad \begin{aligned} \mathbf{d} \quad & -z^* = 2 \text{cis } (\pi - \theta) \end{aligned}$$

$$4 \quad \mathbf{a} \quad i = 1 \operatorname{cis} \left(\frac{\pi}{2} \right) = \operatorname{cis} \left(\frac{\pi}{2} \right)$$

$$\mathbf{b} \quad \therefore iz = \operatorname{cis} \left(\frac{\pi}{2} \right) \times r \operatorname{cis} \theta \\ = r \operatorname{cis} \left(\theta + \frac{\pi}{2} \right)$$

$\mathbf{c}$ iz has the same length as z and its argument is $\frac{\pi}{2}$ more than the argument of z , i.e., an anticlockwise rotation about O of $\frac{\pi}{2}$.

$$\mathbf{d} \quad z \rightarrow -iz \quad -iz = \operatorname{cis} \left(-\frac{\pi}{2} \right) \times r \operatorname{cis} \theta \\ = r \operatorname{cis} \left(\theta - \frac{\pi}{2} \right) \quad \text{i.e., a clockwise rotation of } \frac{\pi}{2} \text{ about O.}$$

$$5 \quad \mathbf{a} \quad \cos \theta - i \sin \theta \\ = \cos(-\theta) + i \sin(-\theta) \\ = \operatorname{cis}(-\theta)$$

$$\mathbf{b} \quad \sin \theta - i \cos \theta \quad \text{If } z = r \operatorname{cis} \theta \\ = -i \cos \theta - i^2 \sin \theta \quad \text{then } z^* = r \operatorname{cis}(-\theta) \\ = -i(\cos \theta + i \sin \theta) \\ = \operatorname{cis} \left(-\frac{\pi}{2} \right) \operatorname{cis} \theta \\ = \operatorname{cis} \left(\theta - \frac{\pi}{2} \right)$$

$$6 \quad \mathbf{a} \quad \text{Consider} \quad \cos \left(\frac{\pi}{12} \right) + i \sin \left(\frac{\pi}{12} \right) = \operatorname{cis} \left(\frac{\pi}{12} \right) \\ = \operatorname{cis} \left(\frac{4\pi}{12} - \frac{3\pi}{12} \right) \\ = \operatorname{cis} \left(\frac{\pi}{3} - \frac{\pi}{4} \right) \\ = \frac{\operatorname{cis} \left(\frac{\pi}{3} \right)}{\operatorname{cis} \left(\frac{\pi}{4} \right)} \\ = \left(\frac{\frac{1}{2} + i\frac{\sqrt{3}}{2}}{\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}} \right) \times \frac{2}{2} \\ = \left(\frac{1 + i\sqrt{3}}{\sqrt{2} + i\sqrt{2}} \right) \left(\frac{\sqrt{2} - i\sqrt{2}}{\sqrt{2} - i\sqrt{2}} \right) \\ = \frac{\sqrt{2} - i\sqrt{2} + i\sqrt{6} - i^2\sqrt{6}}{2 + 2} \\ = \left(\frac{\sqrt{6} + \sqrt{2}}{4} \right) + \left(\frac{\sqrt{6} - \sqrt{2}}{4} \right) i \\ \therefore \cos \left(\frac{\pi}{12} \right) = \frac{\sqrt{6} + \sqrt{2}}{4} \quad \sin \left(\frac{\pi}{12} \right) = \frac{\sqrt{6} - \sqrt{2}}{4} \quad \{\text{equating real and imaginary parts}\}$$

$$\mathbf{b} \quad \text{Consider} \quad \cos \left(\frac{11\pi}{12} \right) + i \sin \left(\frac{11\pi}{12} \right) = \operatorname{cis} \left(\frac{11\pi}{12} \right) \\ = \operatorname{cis} \left(\frac{3\pi}{12} + \frac{8\pi}{12} \right) \\ = \operatorname{cis} \left(\frac{\pi}{4} + \frac{2\pi}{3} \right) \\ = \operatorname{cis} \left(\frac{\pi}{4} \right) \times \operatorname{cis} \left(\frac{2\pi}{3} \right) \\ = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \right) \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) \quad \text{Note: } \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} \\ = -\frac{\sqrt{2}}{4} + i\frac{\sqrt{6}}{4} - i\frac{\sqrt{2}}{4} + i^2\frac{\sqrt{6}}{4} \\ = \left(\frac{-\sqrt{2} - \sqrt{6}}{4} \right) + \left(\frac{\sqrt{6} - \sqrt{2}}{4} \right) i \\ \therefore \cos \left(\frac{11\pi}{12} \right) = \frac{-\sqrt{2} - \sqrt{6}}{4} \quad \sin \left(\frac{11\pi}{12} \right) = \frac{\sqrt{6} - \sqrt{2}}{4} \quad \{\text{equating real and imaginary parts}\}$$

EXERCISE 16B.5

- 1** Let $z = R \operatorname{cis} \theta$ and $w = r \operatorname{cis} \phi$ $w \neq 0$

$$\frac{z}{w} = \frac{R \operatorname{cis} \theta}{r \operatorname{cis} \phi} = \frac{R}{r} \operatorname{cis} (\theta - \phi) \quad \dots (*)$$

$$\therefore \left| \frac{z}{w} \right| = \frac{R}{r} = \frac{|z|}{|w|}$$

also from * $\arg \left(\frac{z}{w} \right) = \theta - \phi = \arg z - \arg w$ if $w \neq 0$

$$\therefore \arg \left(\frac{z}{w} \right) = \arg z - \arg w$$

- 2 a** $z = 3 \operatorname{cis} \theta$

$$-z = -1 \times 3 \operatorname{cis} \theta$$

$$= \operatorname{cis} (\pi) \times 3 \operatorname{cis} \theta$$

$$= 3 \operatorname{cis} (\theta + \pi) \quad \therefore | -z | = 3 \text{ and } \arg (-z) = \theta + \pi \quad -\pi \leq \arg z \leq \pi$$

b $z^* = 3 \operatorname{cis} (-\theta) \quad \therefore |z^*| = 3 \text{ and } \arg z^* = -\theta$

c $iz = \operatorname{cis} \left(\frac{\pi}{2} \right) \times 3 \operatorname{cis} \theta$

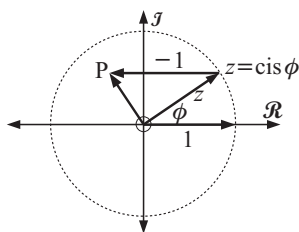
$$= 3 \operatorname{cis} \left(\frac{\pi}{2} + \theta \right) \quad \therefore |iz| = 3 \text{ and } \arg iz = \theta + \frac{\pi}{2}$$

d $(1+i)z = \sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} \right) \times 3 \operatorname{cis} \theta$

$$= 3\sqrt{2} \operatorname{cis} \left(\theta + \frac{\pi}{4} \right)$$

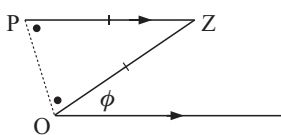
$$\therefore |(1+i)z| = 3\sqrt{2} \text{ and } \arg [(1+i)z] = \theta + \frac{\pi}{4}$$

- 3 a** $z - 1$
 $= z + (-1)$



$\therefore z - 1$ is represented by the vector $\vec{OQ}$

Considering the $\triangle OPQ$



$$\angle PZO = \phi \quad \{\text{alternate } \angle \text{'s}\}$$

$$OZ = ZP = 1$$

$$\therefore \triangle OPZ \text{ is isosceles}$$

$$\therefore \angle POZ = \angle OPZ = \alpha$$

$$\therefore 2\alpha + \phi = \pi \text{ and so } \alpha = \frac{\pi - \phi}{2}$$

$$\therefore \arg(z - 1) = \frac{\pi - \phi}{2} + \phi$$

$$= \frac{\pi}{2} - \frac{\phi}{2} + \phi$$

$$\arg(z - 1) = \frac{\pi}{2} + \frac{\phi}{2} \quad \dots (*)$$

Using the Cosine Rule in $\triangle OPZ$

$$OP^2 = 1^2 + 1^2 = 2(1)(1) \cos \phi$$

$$\therefore OP^2 = 2 - 2 \cos \phi$$

$$\therefore OP^2 = 2 - 2(1 - 2 \sin^2 \left(\frac{\phi}{2} \right))$$

$$\therefore OP^2 = 2 - 2 + 4 \sin^2 \left(\frac{\phi}{2} \right)$$

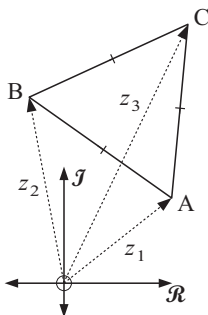
$$\therefore OP^2 = 4 \sin^2 \left(\frac{\phi}{2} \right)$$

$$\therefore |z - 1| = 2 \sin \left(\frac{\phi}{2} \right) \quad \dots (**)$$

b $\therefore z - 1 = 2 \sin \left(\frac{\phi}{2} \right) \operatorname{cis} \left(\frac{\pi}{2} + \frac{\phi}{2} \right) \quad \{\text{using } * \text{ and } **\}$

c $\therefore (z - 1)^* = 2 \sin \left(\frac{\phi}{2} \right) \operatorname{cis} \left(-\frac{\pi}{2} - \frac{\phi}{2} \right)$

4 a Now $z_2 - z_1 = \overrightarrow{AB}$
 $z_3 - z_2 = \overrightarrow{BC}$



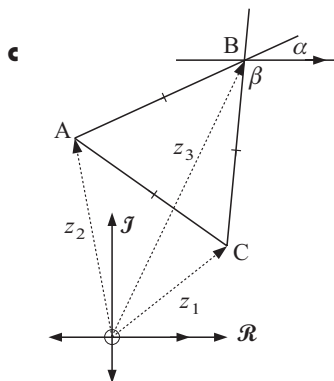
b
$$\left| \frac{z_2 - z_1}{z_3 - z_2} \right| = \frac{|z_2 - z_1|}{|z_3 - z_2|}$$

$$= \frac{|\overrightarrow{AB}|}{|\overrightarrow{BC}|}$$

But $\triangle ABC$ is equilateral

$$\therefore |\overrightarrow{AB}| = |\overrightarrow{BC}|$$

$$\therefore \left| \frac{z_2 - z_1}{z_3 - z_2} \right| = 1$$



Now $\arg \left(\frac{z_2 - z_1}{z_3 - z_2} \right) = \arg(z_2 - z_1) - \arg(z_3 - z_2)$

Since $z_2 - z_1 = \overrightarrow{AB}$ and $z_3 - z_2 = \overrightarrow{BC}$,

$$\arg(z_2 - z_1) = \alpha$$

and $\arg(z_3 - z_2) = -\beta$

$$\therefore \arg \left(\frac{z_2 - z_1}{z_3 - z_2} \right) = \alpha - (-\beta)$$

$$= \alpha + \beta$$

But $\angle ABC = \frac{\pi}{3}$ since the triangle is equilateral

$$\therefore \alpha + \beta = \frac{2\pi}{3} \quad \{\text{angles in a line}\}$$

$$\therefore \arg \left(\frac{z_2 - z_1}{z_3 - z_2} \right) = \frac{2\pi}{3}$$

d
$$\left(\frac{z_2 - z_1}{z_3 - z_2} \right)^3 = \left(1 \operatorname{cis} \frac{2\pi}{3} \right)^3$$

$$= 1(\operatorname{cis} 2\pi)$$

$$= 1$$

EXERCISE 16B.6

1 Using technology

a $\sqrt{3} \operatorname{cis} (2.5187)$
 $= -1.4068 + 1.0105i$

b $\sqrt{11} \operatorname{cis} \left(-\frac{3\pi}{8} \right)$
 $= 1.2692 - 3.0642i$

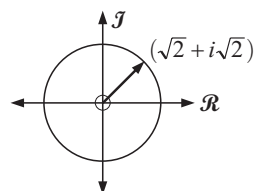
c $2.83649 \operatorname{cis} (-2.68432)$
 $= -2.5451 - 1.2523i$

2 a $3 - 4i$
 $= 5 \operatorname{cis} (-0.9273)$

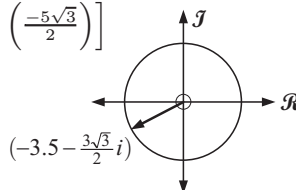
b $-5 - 12i$
 $= 13 \operatorname{cis} (-1.9656)$

c $-11.6814 + 13.2697i$
 $= 17.6788 \operatorname{cis} (2.2926)$

3 a $3 \operatorname{cis} \left(\frac{\pi}{4} \right) + \operatorname{cis} \left(\frac{-3\pi}{4} \right)$
 $= 3 \cos \left(\frac{\pi}{4} \right) + 3i \sin \left(\frac{\pi}{4} \right) + \cos \left(\frac{-3\pi}{4} \right) + i \sin \left(\frac{-3\pi}{4} \right)$
 $= 3 \cos \left(\frac{\pi}{4} \right) + \cos \left(\frac{-3\pi}{4} \right) + i \left[3 \sin \left(\frac{\pi}{4} \right) + \sin \left(\frac{-3\pi}{4} \right) \right]$
 $= \left[\frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right] + i \left[\frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right]$
 $= \sqrt{2} + i\sqrt{2}$
 $= 2 \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$
 $= 2 \operatorname{cis} \frac{\pi}{4}$



b $2 \operatorname{cis} \left(\frac{2\pi}{3} \right) + 5 \operatorname{cis} \left(\frac{-2\pi}{3} \right)$
 $= 2 \cos \left(\frac{2\pi}{3} \right) + 5 \cos \left(\frac{-2\pi}{3} \right) + i \left[2 \sin \left(\frac{2\pi}{3} \right) + 5 \sin \left(\frac{-2\pi}{3} \right) \right]$
 $= \left[-1 + \left(-\frac{5}{2} \right) \right] + i \left[\sqrt{3} + \left(\frac{-5\sqrt{3}}{2} \right) \right]$
 $= -\frac{7}{2} - \frac{3\sqrt{3}}{2}i$
 $= \sqrt{19} \operatorname{cis} (-2.5030)$



4 a Sum of roots

$$\begin{aligned}
 &= 2 \operatorname{cis} \left(\frac{2\pi}{3} \right) + 2 \operatorname{cis} \left(\frac{4\pi}{3} \right) \\
 &= 2 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) + 2 \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \\
 &= -1 + \sqrt{3}i - 1 - \sqrt{3}i \\
 &= -2
 \end{aligned}$$

Product of roots

$$\begin{aligned}
 &= 2 \operatorname{cis} \left(\frac{2\pi}{3} \right) \times 2 \operatorname{cis} \left(\frac{4\pi}{3} \right) \\
 &= 4 \operatorname{cis} \left(\frac{6\pi}{3} \right) \\
 &= 4 \operatorname{cis} 0 \\
 &= 4
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{ equations are } k(x^2 - (-2)x + 4) &= 0 \\
 \text{i.e., } k(x^2 + 2x + 4) &= 0, \quad k \neq 0
 \end{aligned}$$

$$\begin{aligned}
 \text{b Sum of roots} &= \sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} \right) + \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right) \\
 &= \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) + \sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) \\
 &= (1 + i) + (1 - i) \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \text{Product of roots} &= \sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} \right) \times \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right) \\
 &= 2 \operatorname{cis} 0 \\
 &= 2
 \end{aligned}$$

$$\therefore \text{ equations are } k(x^2 - 2x + 2) = 0, \quad k \neq 0$$

EXERCISE 16C

$$\begin{aligned}
 \text{1 a } (\sqrt{2} \operatorname{cis} \left(\frac{\pi}{5} \right))^{10} \\
 &= (\sqrt{2})^{10} \operatorname{cis} \left(\frac{10\pi}{5} \right) \\
 &= 2^5 \operatorname{cis} 2\pi \\
 &= 2^5 \\
 &= 32
 \end{aligned}$$

$$\begin{aligned}
 \text{b } (\operatorname{cis} \left(\frac{\pi}{12} \right))^{36} \\
 &= \operatorname{cis} \left(\frac{36\pi}{12} \right) \\
 &= \operatorname{cis} 3\pi \\
 &= \operatorname{cis} \pi \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{c } (\sqrt{2} \operatorname{cis} \left(\frac{\pi}{8} \right))^{12} \\
 &= 2^6 \operatorname{cis} \left(\frac{12\pi}{8} \right) \\
 &= 64 \operatorname{cis} \left(\frac{3\pi}{2} \right) \\
 &= -64i
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \sqrt{5 \operatorname{cis} \left(\frac{\pi}{7} \right)} \\
 &= (5 \operatorname{cis} \left(\frac{\pi}{7} \right))^{\frac{1}{2}} \\
 &= \sqrt{5} \operatorname{cis} \left(\frac{1}{2} \times \frac{\pi}{7} \right) \\
 &= \sqrt{5} \operatorname{cis} \left(\frac{\pi}{14} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{e } \sqrt[3]{8 \operatorname{cis} \left(\frac{\pi}{2} \right)} \\
 &= (8 \operatorname{cis} \left(\frac{\pi}{2} \right))^{\frac{1}{3}} \\
 &= 2 \operatorname{cis} \left(\frac{\pi}{6} \right) \\
 &= 2 \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i \right) \\
 &= \sqrt{3} + i
 \end{aligned}$$

$$\begin{aligned}
 \text{f } (8 \operatorname{cis} \left(\frac{\pi}{5} \right))^{\frac{5}{3}} \\
 &= 8^{\frac{5}{3}} \operatorname{cis} \left(\frac{5}{3} \times \frac{\pi}{5} \right) \\
 &= 2^5 \operatorname{cis} \left(\frac{\pi}{3} \right) \\
 &= 32 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \\
 &= 16 + 16\sqrt{3}i
 \end{aligned}$$

$$\begin{aligned}
 \text{2 a } (1 + i)^{15} \\
 &= \left(\sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) \right)^{15} \\
 &= (\sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} \right))^{15} \\
 &= \sqrt{2}^{14} \sqrt{2} \operatorname{cis} \left(\frac{15\pi}{4} \right) \\
 &= 2^7 \sqrt{2} \operatorname{cis} \left(\frac{7\pi}{4} \right) \\
 &= 128\sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) \\
 &= 128 - 128i
 \end{aligned}$$

$$\begin{aligned}
 \text{b } (1 - i\sqrt{3})^{11} \\
 &= \left(2 \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \right)^{11} \\
 &= (2 \operatorname{cis} \left(-\frac{\pi}{3} \right))^{11} \\
 &= 2^{11} \operatorname{cis} \left(-\frac{11\pi}{3} \right) \\
 &= 2048 \operatorname{cis} \left(\frac{\pi}{3} \right) \\
 &= 2048 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) \\
 &= 1024 + 1024\sqrt{3}i
 \end{aligned}$$

$$\begin{aligned}
 \text{c } (\sqrt{2} - i\sqrt{2})^{-19} \\
 &= \left(2 \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) \right)^{-19} \\
 &= (2 \operatorname{cis} \left(-\frac{\pi}{4} \right))^{-19} \\
 &= 2^{-19} \operatorname{cis} \left(\frac{19\pi}{4} \right) \\
 &= 2^{-19} \operatorname{cis} \left(\frac{3\pi}{4} \right) \\
 &= 2^{-19} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) \\
 &= 2^{-19} 2^{-\frac{1}{2}} (-1 + i) \\
 &= 2^{-\frac{39}{2}} (-1 + i)
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & (-1 + i)^{-11} \\
 &= \sqrt{2} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right)^{-11} \\
 &= (\sqrt{2} \operatorname{cis} \left(\frac{3\pi}{4} \right))^{-11} \\
 &= (\sqrt{2})^{-11} \operatorname{cis} \left(-\frac{33\pi}{4} \right) \\
 &= (\sqrt{2})^{-11} \operatorname{cis} \left(-\frac{\pi}{4} \right) \\
 &= (\sqrt{2})^{-11} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) \\
 &= (\sqrt{2})^{-12} (1 - i) \\
 &= \frac{1}{64} (1 - i)
 \end{aligned}$$

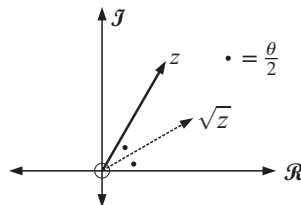
$$\begin{aligned}
 \text{e} \quad & (\sqrt{3} - i)^{\frac{1}{2}} \\
 &= 2 \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i \right)^{\frac{1}{2}} \\
 &= (2 \operatorname{cis} \left(-\frac{\pi}{6} \right))^{\frac{1}{2}} \\
 &= 2^{\frac{1}{2}} \operatorname{cis} \left(-\frac{\pi}{12} \right) \\
 &= \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{12} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & (2 + 2i\sqrt{3})^{-\frac{5}{2}} \\
 &= 4 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right)^{-\frac{5}{2}} \\
 &= (4 \operatorname{cis} \left(\frac{\pi}{3} \right))^{-\frac{5}{2}} \\
 &= 2^{-\frac{5}{2} \times 2} \operatorname{cis} \left(-\frac{5}{2} \times \frac{\pi}{3} \right) \\
 &= 2^{-5} \operatorname{cis} \left(-\frac{5\pi}{6} \right) \\
 &= \frac{1}{32} \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right) \\
 &= -\frac{1}{64} (\sqrt{3} + i)
 \end{aligned}$$

$$\begin{aligned}
 4 \quad \text{a} \quad & z = |z| \operatorname{cis} \theta \\
 & \sqrt{z} = (|z| \operatorname{cis} \theta)^{\frac{1}{2}} \\
 & \sqrt{z} = |z|^{\frac{1}{2}} \operatorname{cis} \left(\frac{\theta}{2} \right) \quad \{\text{De Moivre}\}
 \end{aligned}$$

$$\text{b} \quad -\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$$

c Yes: This can be seen in the given illustration.



5 If $z = r \operatorname{cis} \theta$, then

De Moivre's Theorem states that $z^n = r^n \operatorname{cis} n\theta$ for all rational values of n .

and $\therefore |z^n| = r^n = |z|^n$,

$$\arg(z^n) = n\theta = n \arg z$$

$$\begin{aligned}
 6 \quad & \operatorname{cis}(-\theta) = \cos(-\theta) + i \sin(-\theta) \\
 &= \cos \theta - i \sin \theta
 \end{aligned}$$

$$\begin{aligned}
 \therefore (\cos \theta - i \sin \theta)^{-3} &= [\operatorname{cis}(-\theta)]^{-3} \\
 &= \operatorname{cis} 3\theta \quad \{\text{De Moivre}\}
 \end{aligned}$$

$$\begin{aligned}
 7 \quad & z = 1 + i = \sqrt{2} \operatorname{cis} \left(\frac{\pi}{4} \right) \\
 \therefore z^n &= (\sqrt{2})^n \operatorname{cis} \left(\frac{n\pi}{4} \right) \quad \{\text{De Moivre}\} \\
 &= (\sqrt{2})^n \left[\cos \left(\frac{n\pi}{4} \right) + i \sin \left(\frac{n\pi}{4} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 \text{a} \quad & \text{If } z^n \text{ is real then } \sin \left(\frac{n\pi}{4} \right) = 0 \quad \therefore \frac{n\pi}{4} = 0 + k\pi \\
 & \therefore n = 4k \quad \text{where } k \text{ is any integer}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \text{If } z^n \text{ is purely imaginary then } \cos \left(\frac{n\pi}{4} \right) = 0 \\
 & \text{i.e., } \frac{n\pi}{4} = \frac{\pi}{2} + k\pi \\
 & n = 2 + 4k \quad \text{where } k \text{ is any integer}
 \end{aligned}$$

$$\begin{aligned}
 8 \quad \text{a} \quad & z = 2 \operatorname{cis} \theta \\
 \therefore z^3 &= 2^3 \operatorname{cis} 3\theta \\
 \therefore |z^3| &= 8 \\
 \text{and } \arg z^3 &= 3\theta
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & z = 2 \operatorname{cis} \theta \\
 \therefore z^2 &= 4 \operatorname{cis} 2\theta \\
 \therefore iz^2 &= i(4 \operatorname{cis} 2\theta) \\
 \therefore iz^2 &= \operatorname{cis} \left(\frac{\pi}{2} \right) (4 \operatorname{cis} 2\theta) \\
 \therefore iz^2 &= 4 \operatorname{cis} \left(\frac{\pi}{2} + 2\theta \right) \\
 \therefore |iz^2| &= 4 \\
 \text{and } \arg(iz^2) &= \frac{\pi}{2} + 2\theta
 \end{aligned}$$

c $z = 2 \operatorname{cis} \theta$

Now $\frac{1}{z} = z^{-1},$

so $\frac{1}{z} = (2 \operatorname{cis} \theta)^{-1}$
 $= \frac{1}{2} \operatorname{cis} (-\theta)$

$\therefore \left| \frac{1}{z} \right| = \frac{1}{2}$

and $\arg \left(\frac{1}{z} \right) = -\theta$

d $z = 2 \operatorname{cis} \theta$

Now $-\frac{i}{z^2} = -i \times z^{-2},$

so $-\frac{i}{z^2} = \operatorname{cis} \left(-\frac{\pi}{2} \right) \times (2 \operatorname{cis} \theta)^{-2}$
 $= 2^{-2} \operatorname{cis} \left(-\frac{\pi}{2} \right) \operatorname{cis} (-2\theta)$

$= \frac{1}{4} \operatorname{cis} \left(-\frac{\pi}{2} - 2\theta \right)$

$\therefore \left| \frac{-i}{z^2} \right| = \frac{1}{4}$ and $\arg \left(\frac{-i}{z^2} \right) = -\frac{\pi}{2} - 2\theta$

9 If $z = \operatorname{cis} \theta,$ then $\frac{z^2 - 1}{z^2 + 1} = \frac{(\operatorname{cis} \theta)^2 - 1}{(\operatorname{cis} \theta)^2 + 1}$
 $= \frac{\operatorname{cis} 2\theta - 1}{\operatorname{cis} 2\theta + 1} \quad \{\text{De Moivre}\}$
 $= \frac{\cos 2\theta + i \sin 2\theta - 1}{\cos 2\theta + i \sin 2\theta + 1}$
 $= \frac{-2 \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta \right) + i \sin 2\theta}{2 \left(\frac{1}{2} + \frac{1}{2} \cos 2\theta \right) + i \sin 2\theta}$
 $= \frac{-2 \sin^2 \theta + 2i \cos \theta \sin \theta}{2 \cos^2 \theta + 2i \cos \theta \sin \theta}$
 $= \frac{2 \sin \theta (i \cos \theta + i^2 \sin \theta)}{2 \cos \theta (\cos \theta + i \sin \theta)}$
 $= \frac{i \sin \theta \operatorname{cis} \theta}{\cos \theta \operatorname{cis} \theta}$
 $= i \tan \theta$

10 $\operatorname{cis} 3\theta = \cos 3\theta + i \sin 3\theta$

But $\operatorname{cis} 3\theta = (\operatorname{cis} \theta)^3 \quad \{\text{De Moivre's Theorem}\}$

$= (\cos \theta + i \sin \theta)^3$
 $= \cos^3 \theta + 3 \cos^2 \theta (i \sin \theta) + 3 \cos \theta (i \sin \theta)^2 + (i \sin \theta)^3$
 $= [\cos^3 \theta - 3 \cos \theta \sin^2 \theta] + i [3 \cos^2 \theta \sin \theta - \sin^3 \theta]$

a $\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta \quad \{\text{equating imaginary parts}\}$
 $= 3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta$
 $= 3 \sin \theta - 3 \sin^3 \theta - \sin^3 \theta$
 $= 3 \sin \theta - 4 \sin^3 \theta$

b $\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta \quad \{\text{equating real parts}\}$
 $= \cos^3 \theta - 3 \cos \theta (1 - \cos^2 \theta)$
 $= \cos^3 \theta - 3 \cos \theta + 3 \cos^3 \theta$
 $= 4 \cos^3 \theta - 3 \cos \theta$

11 a If $z = \operatorname{cis} \theta,$ then

$z^n + \frac{1}{z^n} = z^n + z^{-n}$
 $= (\operatorname{cis} \theta)^n + (\operatorname{cis} \theta)^{-n}$
 $= \operatorname{cis} (n\theta) + \operatorname{cis} (-n\theta)$
 $= (\cos n\theta + i \sin n\theta) + (\cos(-n\theta) + i \sin(-n\theta))$
 $= (\cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta) \quad \{\cos(-\alpha) = \cos \alpha, \quad \sin(-\alpha) = -\sin \alpha\}$
 $= 2 \cos n\theta$

b In **a** if we let $n = 1$ we get $z + \frac{1}{z} = 2 \cos \theta$

$$\begin{aligned} \mathbf{c} \quad \left(z + \frac{1}{z}\right)^3 &= z^3 + 3z^2 \left(\frac{1}{z}\right) + 3z \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 \\ &= z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} \end{aligned}$$

$$\mathbf{d} \quad \text{From } \mathbf{c} \quad \left(z + \frac{1}{z}\right)^3 = \left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)$$

$$\text{Using } \mathbf{a} \text{ and } \mathbf{b} \quad (2 \cos \theta)^3 = 2 \cos 3\theta + 3(2 \cos \theta)$$

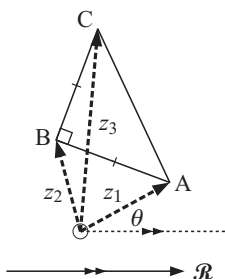
$$\therefore 8 \cos^3 \theta = 2 \cos 3\theta + 6 \cos \theta$$

$$\therefore \cos^3 \theta = \frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta$$

e If we let $\theta = \frac{13\pi}{12}$ in **d** we get

$$\begin{aligned} \cos^3 \left(\frac{13\pi}{12}\right) &= \frac{1}{4} \cos \left(\frac{39\pi}{12}\right) + \frac{3}{4} \cos \left(\frac{13\pi}{12}\right) \\ &= \frac{1}{4} \cos \left(\frac{13\pi}{4}\right) + \frac{3}{4} \cos \left(\frac{3\pi}{4} + \frac{\pi}{3}\right) \\ &= \frac{1}{4} \cos \left(\frac{5\pi}{4}\right) + \frac{3}{4} \left[\cos \left(\frac{3\pi}{4}\right) \cos \left(\frac{\pi}{3}\right) - \sin \left(\frac{3\pi}{4}\right) \sin \left(\frac{\pi}{3}\right) \right] \\ &= \frac{1}{4} \left(-\frac{1}{\sqrt{2}}\right) + \frac{3}{4} \left[\left(-\frac{1}{\sqrt{2}}\right) \left(\frac{1}{2}\right) - \left(\frac{1}{\sqrt{2}}\right) \left(\frac{\sqrt{3}}{2}\right) \right] \\ &= -\frac{1}{4\sqrt{2}} - \frac{3}{8\sqrt{2}} - \frac{3\sqrt{3}}{8\sqrt{2}} \\ &= -\frac{1}{4\sqrt{2}} \left(\frac{2\sqrt{2}}{2\sqrt{2}}\right) - \frac{3}{8\sqrt{2}} \left(\frac{\sqrt{2}}{\sqrt{2}}\right) - \frac{3\sqrt{3}}{8\sqrt{2}} \left(\frac{\sqrt{2}}{\sqrt{2}}\right) \\ &= \frac{-2\sqrt{2}-3\sqrt{2}-3\sqrt{6}}{16} \\ &= \frac{-5\sqrt{2}-3\sqrt{6}}{16} \end{aligned}$$

12 a



$$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -z_2 + z_3 = z_3 - z_2$$

$$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = -z_2 + z_1 = z_1 - z_2$$

$$\text{Now } \arg(\overrightarrow{BC}) - \arg(\overrightarrow{BA}) = \frac{\pi}{2}$$

$$\therefore \arg(z_3 - z_2) - \arg(z_1 - z_2) = \frac{\pi}{2}$$

$$\therefore \arg\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \frac{\pi}{2}$$

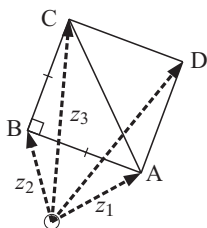
$$\text{Also } \left| \frac{z_3 - z_2}{z_1 - z_2} \right| = \frac{|z_3 - z_2|}{|z_1 - z_2|} = \frac{BC}{AB} = 1$$

So, $\frac{z_3 - z_2}{z_1 - z_2}$ has modulus 1 and argument $\frac{\pi}{2}$

$$\therefore \frac{z_3 - z_2}{z_1 - z_2} = 1 \operatorname{cis} \frac{\pi}{2} = i$$

$$\therefore \left(\frac{z_3 - z_2}{z_1 - z_2}\right)^2 = i^2 = -1 \quad \text{and so} \quad \therefore (z_1 - z_2)^2 = -(z_3 - z_2)^2$$

b



$$\overrightarrow{BA} = \overrightarrow{CD} \quad \text{and} \quad \overrightarrow{BA} = z_1 - z_2$$

$$\therefore \overrightarrow{CD} = z_1 - z_2$$

$$\text{Now } \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$$

$$\therefore \overrightarrow{OD} = z_3 + z_1 - z_2$$

$$\therefore z_3 + z_1 - z_2 \text{ represents } D.$$

13 a $\cos 4\theta + i \sin 4\theta$

$$= \text{cis } 4\theta$$

$$= (\text{cis } \theta)^4 \quad \{\text{De Moivre's theorem in reverse}\}$$

$$= (\cos \theta + i \sin \theta)^4$$

$$= \cos^4 \theta + 4 \cos^3 \theta (i \sin \theta) + 6 \cos^2 \theta (i \sin \theta)^2 + 4 \cos \theta (i \sin \theta)^3 + (i \sin \theta)^4$$

$$= \cos^4 \theta + [4 \cos^3 \theta \sin \theta]i - 6 \cos^2 \theta \sin^2 \theta - [4 \cos \theta \sin^3 \theta]i + \sin^4 \theta$$

$$= [\cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta] + [4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta]i$$

i Equating real parts gives $\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta (1 - \cos^2 \theta) + (1 - \cos^2 \theta)^2$

$$\therefore \cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta + 6 \cos^4 \theta + 1 - 2 \cos^2 \theta + \cos^4 \theta$$

$$\text{i.e., } \cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$$

ii Equating imaginary parts gives $\sin 4\theta = 4 \cos^3 \theta \sin \theta - 4 \cos \theta \sin^3 \theta$

b If $z = \text{cis } \theta$, $z^n - \frac{1}{z^n} = z^n - z^{-n}$

$$= [\text{cis } \theta]^n - [\text{cis } \theta]^{-n}$$

$$= \text{cis } n\theta - \text{cis } (-n\theta) \quad \{\text{De Moivre}\}$$

$$= \cos n\theta + i \sin n\theta - [\cos(-n\theta) + i \sin(-n\theta)]$$

$$= \cos n\theta + i \sin n\theta - \cos(-n\theta) - i \sin(-n\theta)$$

$$= \cos n\theta + i \sin n\theta - \cos n\theta + i \sin n\theta$$

$$\{\text{as } \cos(-\phi) = \cos \phi, \sin(-\phi) = -\sin \phi\}$$

$$= 2i \sin n\theta \quad \dots (*)$$

If we let $n = 1$, $z - \frac{1}{z} = 2i \sin \theta$

$$\therefore [2i \sin \theta]^3 = \left(z - \frac{1}{z}\right)^3$$

$$\therefore 8i^3 \sin^3 \theta = z^3 + 3z^2 \left(-\frac{1}{z}\right) + 3z \left(-\frac{1}{z}\right)^2 + \left(-\frac{1}{z}\right)^3$$

$$= z^3 - \frac{1}{z^3} - 3 \left(z - \frac{1}{z}\right)$$

$$= 2i \sin 3\theta - 3 \times 2i \sin \theta \quad \{\text{using } (*)\}$$

$$\therefore -8i \sin^3 \theta = 2i \sin 3\theta - 6i \sin \theta$$

$$\therefore \sin^3 \theta = -\frac{1}{4} \sin 3\theta + \frac{3}{4} \sin \theta$$

$$\text{i.e., } \sin^3 \theta = \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta$$

EXERCISE 16D

1 a The cube roots of 1 are solutions to $z^3 = 1$ i.e., $z^3 - 1 = 0$

Now $z = 1$ is a solution, so $z - 1$ is a factor.

$$\therefore (z - 1)(z^2 + z + 1) = 0$$

$$\therefore z = 1 \quad \text{or} \quad \frac{-1 \pm \sqrt{1 - 4}}{2}$$

$$\therefore z = 1 \quad \text{or} \quad -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$1 \begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 \end{array}$$

b $z^3 = 1$

$$\therefore z^3 = 1 \text{ cis } (0 + k2\pi)$$

$$\therefore z = (1 \text{ cis } (k2\pi))^{\frac{1}{3}}$$

$$\therefore z = 1 \text{ cis } \left(\frac{k2\pi}{3}\right) \quad \{\text{De Moivre}\}$$

$$\therefore z = \text{cis } 0, \text{ cis } \left(\frac{2\pi}{3}\right) \text{ or } \text{cis } \left(\frac{4\pi}{3}\right) \quad \{\text{when } k = 0, 1, 2\}$$

$$\text{i.e., } z = 1, -\frac{1}{2} + i \frac{\sqrt{3}}{2}, -\frac{1}{2} - i \frac{\sqrt{3}}{2}$$

2 a $z^3 = -8i$

$$z^3 = 8 \operatorname{cis} \left(-\frac{\pi}{2} + k2\pi \right)$$

$$\therefore z = \left(8 \operatorname{cis} \left(-\frac{\pi}{2} + k2\pi \right) \right)^{\frac{1}{3}}$$

$$\therefore z = 8^{\frac{1}{3}} \operatorname{cis} \left(-\frac{\pi}{6} + \frac{k2\pi}{3} \right) \quad \{\text{De Moivre}\}$$

$$\therefore z = 2 \operatorname{cis} \left(-\frac{\pi}{6} \right), 2 \operatorname{cis} \left(\frac{3\pi}{6} \right) \text{ or } 2 \operatorname{cis} \left(\frac{7\pi}{6} \right) \quad \{\text{when } k = 0, 1, 2\}$$

$$\therefore z = 2 \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i \right), 2 \operatorname{cis} \frac{\pi}{2} \text{ or } 2 \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right)$$

$$\therefore z = \sqrt{3} - i, 2i \text{ or } -\sqrt{3} - i$$

b $z^3 = -27i$

$$\therefore z^3 = 27 \operatorname{cis} \left(-\frac{\pi}{2} + k2\pi \right)$$

$$\therefore z = 3 \left(\operatorname{cis} \left(-\frac{\pi}{2} + k2\pi \right) \right)^{\frac{1}{3}}$$

$$\therefore z = 3 \operatorname{cis} \left(-\frac{\pi}{6} + \frac{k4\pi}{6} \right) \quad \{\text{De Moivre}\}$$

$$\therefore z = 3 \operatorname{cis} \left(-\frac{\pi}{6} \right), 3 \operatorname{cis} \left(\frac{3\pi}{6} \right) \text{ or } 3 \operatorname{cis} \left(\frac{7\pi}{6} \right) \quad \{\text{when } k = 0, 1, 2\}$$

$$\therefore z = 3 \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i \right), 3 \operatorname{cis} \left(\frac{\pi}{2} \right) \text{ or } 3 \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right)$$

$$\therefore z = \frac{3\sqrt{3}}{2} - \frac{3}{2}i, 3i \text{ or } -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

3 $z^3 = -1$

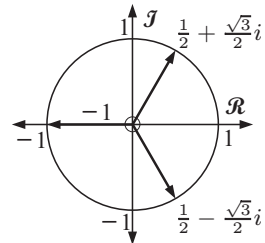
$$\therefore z^3 = 1 \operatorname{cis} (\pi + k2\pi)$$

$$\therefore z = \left(\operatorname{cis} (\pi + k2\pi) \right)^{\frac{1}{3}}$$

$$\therefore z = \operatorname{cis} \left(\frac{\pi}{3} + \frac{k2\pi}{3} \right) \quad \{\text{De Moivre}\}$$

$$\therefore z = \operatorname{cis} \left(\frac{\pi}{3} \right), \operatorname{cis} \pi, \operatorname{cis} \left(\frac{5\pi}{3} \right) \quad \{\text{when } k = 0, 1, 2\}$$

$$\therefore z = \frac{1}{2} + i\frac{\sqrt{3}}{2}, -1, \frac{1}{2} - i\frac{\sqrt{3}}{2}$$



4 a $z^4 = 16$

$$\therefore z^4 = 16 \operatorname{cis} (0 + k2\pi)$$

$$\therefore z = \left(16 \operatorname{cis} (k2\pi) \right)^{\frac{1}{4}}$$

$$\therefore z = 16^{\frac{1}{4}} \operatorname{cis} \left(\frac{k\pi}{2} \right) \quad \{\text{De Moivre}\}$$

$$\therefore z = 2 \operatorname{cis} \left(\frac{k\pi}{2} \right)$$

$$\therefore z = 2 \operatorname{cis} 0, 2 \operatorname{cis} \left(\frac{\pi}{2} \right), 2 \operatorname{cis} \pi, 2 \operatorname{cis} \left(\frac{3\pi}{2} \right) \quad \{\text{when } k = 0, 1, 2 \text{ or } 3\}$$

$$\therefore z = \pm 2 \text{ and } \pm 2i$$

b $z^4 = 16$

$$\therefore z^4 = 16 \operatorname{cis} (\pi + k2\pi)$$

$$\therefore z = \left(16 \operatorname{cis} (\pi + k2\pi) \right)^{\frac{1}{4}}$$

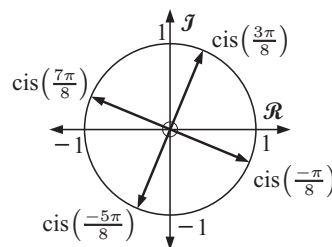
$$\therefore z = 16^{\frac{1}{4}} \operatorname{cis} \left(\frac{\pi}{4} + \frac{k2\pi}{4} \right) \quad \{\text{De Moivre}\}$$

$$\therefore z = 2 \operatorname{cis} \left(\frac{\pi}{4} + \frac{k2\pi}{4} \right)$$

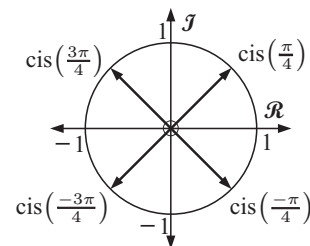
$$\therefore z = 2 \operatorname{cis} \left(\frac{\pi}{4} \right), 2 \operatorname{cis} \left(\frac{3\pi}{4} \right), 2 \operatorname{cis} \left(\frac{5\pi}{4} \right), 2 \operatorname{cis} \left(\frac{7\pi}{4} \right) \quad \{\text{when } k = 0, 1, 2 \text{ or } 3\}$$

$$\therefore z = \sqrt{2} \pm i\sqrt{2}, -\sqrt{2} \pm i\sqrt{2}$$

$$\begin{aligned}
 5 \quad & z^4 = -i \\
 \therefore & z^4 = \text{cis} \left(-\frac{\pi}{2} + k2\pi \right) \\
 \therefore & z = \left(\text{cis} \left(-\frac{\pi}{2} + k2\pi \right) \right)^{\frac{1}{4}} \\
 \therefore & z = \text{cis} \left(-\frac{\pi}{8} + \frac{k\pi}{2} \right) \quad \{\text{De Moivre}\} \\
 \therefore & z = \text{cis} \left(-\frac{\pi}{8} + \frac{k4\pi}{8} \right) \\
 \therefore & z = \text{cis} \left(-\frac{\pi}{8} \right), \text{cis} \left(\frac{3\pi}{8} \right), \text{cis} \left(\frac{7\pi}{8} \right), \text{cis} \left(\frac{11\pi}{8} \right) \quad \{\text{when } k = 0, 1, 2 \text{ or } 3\}
 \end{aligned}$$



$$\begin{aligned}
 6 \quad & z^4 + 1 = 0 \\
 \therefore & z^4 = -1 \\
 \therefore & z^4 = \text{cis} (\pi + k2\pi) \\
 \therefore & z = [\text{cis} (\pi + k2\pi)]^{\frac{1}{4}} \\
 \therefore & z = \text{cis} \left(\frac{\pi}{4} + \frac{k\pi}{2} \right) \quad \{\text{De Moivre}\} \\
 \therefore & z = \text{cis} \left(\frac{\pi}{4} + \frac{k2\pi}{4} \right) \\
 \therefore & z = \text{cis} \left(\frac{\pi}{4} \right), \text{cis} \left(\frac{3\pi}{4} \right), \text{cis} \left(\frac{5\pi}{4} \right), \text{cis} \left(\frac{7\pi}{4} \right) \quad \{\text{when } k = 0, 1, 2 \text{ or } 3\} \\
 \therefore & z = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i
 \end{aligned}$$



For the pair of roots $\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$,

$$\begin{aligned}
 \text{sum} &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \\
 &= \frac{2}{\sqrt{2}} \\
 &= \sqrt{2} \\
 \text{product} &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) \\
 &= \frac{1}{2} + \frac{1}{2} \\
 &= 1
 \end{aligned}$$

and so we have a quadratic factor of $z^2 - \sqrt{2}z + 1$

For the pair of roots $-\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$,

$$\text{sum} = -\sqrt{2} \quad \text{and} \quad \text{product} = \frac{1}{2} + \frac{1}{2} = 1$$

and so we have a quadratic factor of $z^2 + \sqrt{2}z + 1$

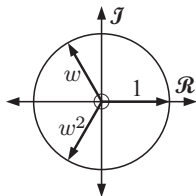
EXERCISE 16E

$$\begin{aligned}
 1 \quad \text{a} \quad \text{i} \quad & (z+3)^3 = 1 \\
 \therefore & z+3 = 1, w \text{ or } w^2 \text{ where } w = \text{cis} \left(\frac{2\pi}{3} \right) \\
 \therefore & z+3 = w^n \text{ where } n = 0, 1, 2 \\
 \therefore & z = w^n - 3 \text{ where } n = 0, 1, 2 \text{ and } w = \text{cis} \left(\frac{2\pi}{3} \right)
 \end{aligned}$$

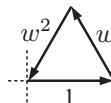
$$\begin{aligned}
 \text{ii} \quad & (z-1)^3 = 8 \\
 \therefore & \left[\frac{z-1}{2} \right]^3 = 1 \\
 \therefore & \frac{z-1}{2} = 1, w \text{ or } w^2 \text{ where } w = \text{cis} \left(\frac{2\pi}{3} \right) \\
 \therefore & \frac{z-1}{2} = w^n \text{ where } n = 0, 1, 2 \\
 \therefore & z = 2w^n + 1 \text{ where } n = 0, 1, 2 \text{ and } w = \text{cis} \left(\frac{2\pi}{3} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{iii} \quad & (2z - 1)^3 = -1 \\
 \therefore & (2z - 1)^3 = (-1)^3 \\
 \therefore & \left(\frac{2z - 1}{-1} \right)^3 = 1 \\
 \therefore & (1 - 2z)^3 = 1 \\
 \therefore & 1 - 2z = 1, w, w^2 \text{ where } w = \text{cis} \left(\frac{2\pi}{3} \right) \\
 \therefore & 1 - 2z = w^n \text{ where } n = 0, 1, 2 \\
 \therefore & 2z = 1 - w^n \\
 \therefore & z = \frac{1 - w^n}{2} \text{ where } n = 0, 1, 2 \text{ and } w = \text{cis} \left(\frac{2\pi}{3} \right)
 \end{aligned}$$

- 2** The following represents the cube roots of unity:



Adding these vectorially



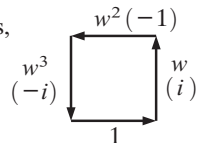
the resultant vector is **0**

$$\therefore 1 + w + w^2 = 0$$

- 3 a** Yes, as if $z^4 = 1$
 $z^4 - 1 = 0$
 $\therefore (z^2 + 1)(z^2 - 1) = 0$
 $\therefore (z + i)(z - i)(z + 1)(z - 1) = 0$
 $\therefore z = \pm i, \pm 1$

b $1 + w + w^2 + w^3 = 1 + (i) + (-1) + (-i)$
 $= 1 + i - 1 - i$
 $= 0$

- c** Adding the four vectors, vectorially



- 4 a** The 5th roots of unity are the solutions to $z^5 = 1$.

$$\therefore z^5 = \text{cis} (0 + k2\pi)$$

$$\therefore z^5 = \text{cis} (k2\pi)$$

$$\therefore z = [\text{cis} (k2\pi)]^{\frac{1}{5}}$$

$$\therefore z = [\text{cis} \left(\frac{k2\pi}{5} \right)] \quad \{\text{De Moivre}\}$$

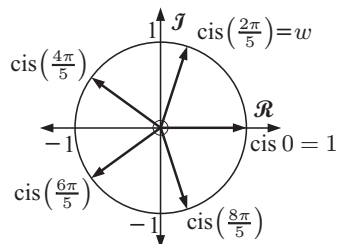
$$\therefore z = \text{cis} 0 = 1 \text{ or}$$

$$\text{cis} \left(\frac{2\pi}{5} \right) = w \text{ or}$$

$$\text{cis} \left(\frac{4\pi}{5} \right) = (\text{cis} \frac{2\pi}{5})^2 = w^2 \text{ or}$$

$$\text{cis} \left(\frac{6\pi}{5} \right) = (\text{cis} \frac{2\pi}{5})^3 = w^3 \text{ or}$$

$$\text{cis} \left(\frac{8\pi}{5} \right) = (\text{cis} \frac{2\pi}{5})^4 = w^4 \quad \{\text{when } k = 0, 1, 2, 3, 4\}$$



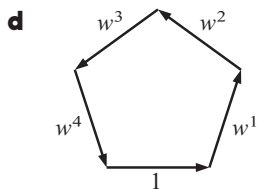
- b** Hence the five roots can be expressed as $1, w, w^2, w^3, w^4$ where $w = \text{cis} \left(\frac{2\pi}{5} \right)$

$$\begin{aligned}
 \text{c} \quad & (1 + w + w^2 + w^3 + w^4)(1 - w) \\
 &= 1 + w + w^2 + w^3 + w^4 - w - w^2 - w^3 - w^4 - w^5 \\
 &= 1 - w^5
 \end{aligned}$$

Since w is a solution of $z^5 = 1$, $1 - w^5 = 0$

$$\therefore (1 + w + w^2 + w^3 + w^4)(1 - w) = 0$$

But $w \neq 1$, so $1 + w + w^2 + w^3 + w^4 = 0$



- 5 a** The n th roots of unity are the solutions to $z^n = 1$

$$z^n = 1$$

$$\therefore z^n = \text{cis } (0 + k2\pi)$$

$$\therefore z^n = \text{cis } (k2\pi)$$

$$\therefore z = [\text{cis } (k2\pi)]^{\frac{1}{n}}$$

$$\therefore z = \text{cis } \left(\frac{k2\pi}{n} \right) \quad \{\text{De Moivre}\}$$

$$\therefore z = \text{cis } 0 = 1 \quad \text{or}$$

$$\text{cis } \left(\frac{2\pi}{n} \right) = w \quad \text{or} \quad \{\text{letting } k = 0, 1, 2, 3, \dots, 0, n-1\}$$

$$\text{cis } \left(\frac{4\pi}{n} \right) = \left(\text{cis } \left(\frac{2\pi}{n} \right) \right)^2 = w^2 \quad \text{or}$$

$$\vdots$$

$$\text{cis } \left(\frac{2\pi}{n}(n-1) \right) = \left[\text{cis } \left(\frac{2\pi}{n} \right) \right]^{n-1} = w^{n-1}$$

i.e., the n roots of $z^n = 1$ are $1, w, w^2, w^3, \dots, w^{n-1}$ where $w = \text{cis } \frac{2\pi}{n}$

- b** Now $(1 + w + w^2 + \dots + w^{n-1})(w - 1) = w^n - 1$

But w is a solution to $z^n - 1 = 0$ so $w^n - 1 = 0$

$$\therefore (1 + w + w^2 + \dots + w^{n-1})(w - 1) = 0$$

$$\therefore \text{since } w \neq 1, \quad 1 + w + w^2 + \dots + w^{n-1} = 0$$

REVIEW SET 16A

- 1** $(i - \sqrt{3})^2 = i^2 - 2\sqrt{3}i + 3 = 2 - 2\sqrt{3}i$ $\therefore (i - \sqrt{3})^5 = (-8 - 8\sqrt{3}i)(i - \sqrt{3})$
 $= -8i + 8\sqrt{3} + 8\sqrt{3} + 24i = 16\sqrt{3} + 16i$
 $\therefore (i - \sqrt{3})^4 = (2 - 2\sqrt{3}i)^2 = 4 - 8\sqrt{3}i + 12i^2 = -8 - 8\sqrt{3}i$ $\therefore$ Real part is $16\sqrt{3}$.
 Imaginary part is 16.

- 2 a** $|z - i| = |z + 1 + i|$ Since $z = x + iy$,

$$|x + iy - i| = |x + iy + 1 + i|$$

$$\therefore |x + i(y - 1)| = |(x + 1) + i(y + 1)|$$

$$\therefore \sqrt{x^2 + (y - 1)^2} = \sqrt{(x + 1)^2 + (y + 1)^2}$$

$$\therefore x^2 + (y - 1)^2 = (x + 1)^2 + (y + 1)^2$$

$$\therefore x^2 + y^2 - 2y + 1 = x^2 + 2x + 1 + y^2 + 2y + 1$$

$$\therefore 1 - 2y = 2x + 2y + 2$$

$$\therefore 2x + 4y = -1$$

- b** $z^* + iz = 0$

Since $z = x + iy$,

$$x - iy + i(x + iy) = 0$$

$$\therefore x - iy + ix - y = 0$$

$$\therefore (x - y) + i(x - y) = 0$$

$$\therefore x - y = 0$$

$$\therefore y = x$$

- 3** $|z + 16| = 4|z + 1|$

$$\therefore |z + 16|^2 = 16|z + 1|^2$$

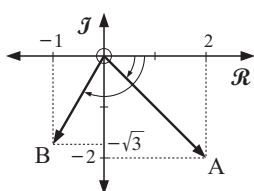
$$\therefore (z + 16)(z + 16)^* = 16(z + 1)(z + 1)^*$$

$$\therefore (z + 16)(z^* + 16^*) = 16(z + 1)(z^* + 1^*)$$

$$\therefore zz^* + 16z + 16z^* + 256 = 16(zz^* + z^* + z + 1) \quad \{16^* = 16, 1^* = 1\}$$

$$\begin{aligned}\therefore |z|^2 + 16z + 16z^* + 256 &= 16|z|^2 + 16z^* + 16z + 16 \\ 240 &= 15|z|^2 \\ \therefore |z|^2 &= 16 \\ \therefore |z| &= 4 \quad \text{as } |z| \geq 0\end{aligned}$$

4



$$\begin{aligned}\mathbf{a} \quad \arg \overrightarrow{OA} &= \frac{-\pi}{4} \\ \arg \overrightarrow{OB} &= \frac{-2\pi}{3} \\ \therefore \angle AOB &= \frac{2\pi}{3} - \frac{\pi}{4} \\ &= \frac{5\pi}{12}\end{aligned}$$

b

$$\begin{aligned}zw &= 2\sqrt{2} \operatorname{cis} \left(\frac{-\pi}{4} \right) \times 2 \operatorname{cis} \left(\frac{-2\pi}{3} \right) \\ &= 4\sqrt{2} \operatorname{cis} \left(\frac{-\pi}{4} + \frac{-2\pi}{3} \right) \\ &= 4\sqrt{2} \operatorname{cis} \left(\frac{-11\pi}{12} \right) \\ \therefore \arg(zw) &= \frac{-11\pi}{12}\end{aligned}$$

5

$$\mathbf{a} \quad -5i = 5 \operatorname{cis} \left(-\frac{\pi}{2} \right)$$

$$\begin{aligned}\mathbf{b} \quad 2 - 2i\sqrt{3} &= 4 \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \\ &= 4 \operatorname{cis} \left(-\frac{\pi}{3} \right)\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad k - ki &= -k\sqrt{2} \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) \\ &= -k\sqrt{2} \operatorname{cis} \left(\frac{3\pi}{4} \right) \quad \text{which is in polar form since } k < 0\end{aligned}$$

6

$$z = (1 + bi)^2 = 1 + 2bi - b^2$$

$$\therefore z = [1 - b^2] + 2bi$$

$$\tan \left(\frac{\pi}{3} \right) = \frac{2b}{1 - b^2}$$

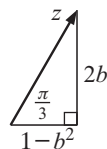
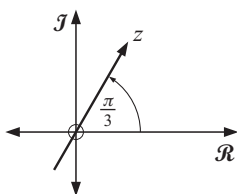
$$\therefore \sqrt{3} = \frac{2b}{1 - b^2}$$

$$\therefore \sqrt{3} - \sqrt{3}b^2 = 2b$$

$$\therefore \sqrt{3}b^2 + 2b - \sqrt{3} = 0$$

$$\therefore b = \frac{-2 \pm \sqrt{4 - 4(\sqrt{3})(-\sqrt{3})}}{2\sqrt{3}}$$

$$= \frac{-2 \pm 4}{2\sqrt{3}} \quad \text{but } b > 0 \quad \therefore b = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$



7

$$\mathbf{a} \quad \operatorname{cis} \theta \times \operatorname{cis} \phi$$

$$= (\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi)$$

$$= (\cos \theta \cos \phi - \sin \theta \sin \phi) + i(\sin \theta \cos \phi + \cos \theta \sin \phi)$$

$$= \cos(\theta + \phi) + i \sin(\theta + \phi)$$

$$= \operatorname{cis}(\theta + \phi) \quad \text{as required}$$

b

$$1 - i = \sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right) = \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right)$$

$$\therefore (1 - i)z = \sqrt{2} \operatorname{cis} \left(-\frac{\pi}{4} \right) \times 2\sqrt{2} \operatorname{cis} \alpha$$

$$= 4 \operatorname{cis} \left(\alpha - \frac{\pi}{4} \right) \quad \{\text{using } \mathbf{a}\}$$

$$\therefore \arg[(1 - i)z] = \alpha - \frac{\pi}{4}$$

$$\begin{aligned} \mathbf{8} \quad \mathbf{a} \quad & \left| \frac{z_1^2}{z_2^2} \right| = \frac{|z_1|^2}{|z_2|^2} \quad \text{But } |z_1| = |z_2| \text{ since the triangle is isosceles} \\ & \therefore \left| \frac{z_1^2}{z_2^2} \right| = 1 \end{aligned}$$

$$\begin{aligned} \text{Also, } \arg \left(\frac{z_1^2}{z_2^2} \right) &= \arg(z_1^2) - \arg(z_2^2) \\ &= 2 \arg z_1 - 2 \arg z_2 \\ &= 2(\arg z_1 - \arg z_2) \\ &= 2 \times \frac{\pi}{2} \text{ since } z_1 \text{ and } z_2 \text{ are perpendicular} \\ &= \pi \end{aligned}$$

$$\mathbf{b} \quad \frac{z_1^2}{z_2^2} = \text{cis } \pi = -1 \quad \therefore z_1^2 = -z_2^2 \quad \therefore z_1^2 + z_2^2 = 0$$

REVIEW SET 16B

$$\mathbf{1} \quad z_1 = \text{cis} \left(\frac{\pi}{6} \right) \text{ and } z_2 = \text{cis} \left(\frac{\pi}{4} \right)$$

$$\therefore \left(\frac{z_1}{z_2} \right)^3 = \left[\frac{\text{cis} \left(\frac{\pi}{6} \right)}{\text{cis} \left(\frac{\pi}{4} \right)} \right]^3 = [\text{cis} \left(\frac{\pi}{6} - \frac{\pi}{4} \right)]^3$$

$$\therefore \left(\frac{z_1}{z_2} \right)^3 = [\text{cis} \left(-\frac{\pi}{12} \right)]^3 = \text{cis} \left(-\frac{3\pi}{12} \right) \quad \{\text{De Moivre}\}$$

$$\therefore \left(\frac{z_1}{z_2} \right)^3 = \text{cis} \left(-\frac{\pi}{4} \right) = \cos \left(-\frac{\pi}{4} \right) + i \sin \left(-\frac{\pi}{4} \right)$$

$$\therefore \left(\frac{z_1}{z_2} \right)^3 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

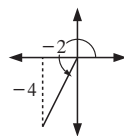
$$\mathbf{2} \quad z = 4 + i \quad w = 2 - 3i$$

$$\begin{aligned} \mathbf{a} \quad 2w^* - iz &= 2(2 + 3i) - i(4 + i) \\ &= 4 + 6i - 4i - i^2 \\ &= 5 + 2i \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad |w - z^*| &= |(2 - 3i) - (4 - i)| \\ &= |2 - 3i - 4 + i| \\ &= |-2 - 2i| \\ &= \sqrt{(-2)^2 + (-2)^2} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad |z^{10}| &= |z|^{10} \\ &= |4 + i|^{10} \\ &= (\sqrt{16 + 1})^{10} \\ &= \sqrt{17}^{10} \\ &= 17^5 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \arg(w - z) &= \arg[(2 - 3i) - (4 + i)] \\ &= \arg[-2 - 4i] \\ &\div -2.034 \end{aligned}$$



$$\mathbf{3} \quad \text{If } \frac{2 - 3i}{2a + bi} = 3 + 2i, \text{ then } \frac{2 - 3i}{3 + 2i} = 2a + bi$$

$$\therefore 2a + bi = \left(\frac{2 - 3i}{3 + 2i} \right) \left(\frac{3 - 2i}{3 - 2i} \right) = \frac{6 - 4i - 9i - 6}{9 + 4}$$

$$\therefore 2a + bi = \frac{0 - 13i}{13} = 0 - i$$

$$\therefore 2a = 0 \text{ and } b = -1 \quad \text{i.e., } a = 0, \quad b = -1$$

- 4 a** If $\arg z = \frac{\pi}{2}$, then we have a ray vertically upwards beginning at the origin.
 If $\arg(z - i) = \frac{\pi}{2}$, the graph is translated $[0, 1]$, and we have a ray vertically upwards beginning at i .

Since $\frac{y-1}{x} = \tan \frac{\pi}{2}$, we have $x = 0$,

and geometrically we require $y > 1$.

$$\mathbf{b} \quad \left| \frac{z+2}{z-2} \right| = 2, \quad \therefore \frac{|z+2|}{|z-2|} = 2$$

$$\therefore |z+2| = 2|z-2|$$

If $z = x + iy$, then $\sqrt{(x+2)^2 + y^2} = 2\sqrt{(x-2)^2 + y^2}$

$$\therefore (x+2)^2 + y^2 = 4(x-2)^2 + 4y^2$$

$$\therefore x^2 + 4x + 4 + y^2 = 4x^2 - 16x + 16 + 4y^2$$

i.e., $3x^2 + 3y^2 - 20x + 12 = 0$, which is a circle

5 Now $2 - 2\sqrt{3}i = 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$$= 4 \operatorname{cis}\left(-\frac{\pi}{3}\right)$$

$\therefore (2 - 2\sqrt{3}i)^n = 4^n \operatorname{cis}\left(-\frac{n\pi}{3}\right)$ {De Moivre}

This is real if $\sin\left(-\frac{n\pi}{3}\right) = 0$

i.e., $-\frac{n\pi}{3} = k\pi$, k an integer

$\therefore n = 3k$ where k is an integer

- 6** The cube roots of -27 are the solutions to $z^3 = -27$.

$\therefore z^3 = 27 \operatorname{cis}(\pi + k2\pi)$

$\therefore z = [27 \operatorname{cis}(\pi + k2\pi)]^{\frac{1}{3}}$ where $k = 0, 1, 2$

$\therefore z = 27^{\frac{1}{3}} \operatorname{cis}\left(\frac{\pi + k2\pi}{3}\right)$

$\therefore z = 3 \operatorname{cis}\left(\frac{\pi + k2\pi}{3}\right)$

$\therefore z = 3 \operatorname{cis}\left(\frac{\pi}{3}\right), 3 \operatorname{cis} \pi, 3 \operatorname{cis}\left(\frac{5\pi}{3}\right)$

$\therefore z = 3\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right), 3(-1 + 0i), 3\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$\therefore z = -3$ or $\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$

7 a $z = 4 \operatorname{cis} \theta$

$z^3 = (4 \operatorname{cis} \theta)^3$

$= 4^3 \operatorname{cis} 3\theta$

$\therefore |z^3| = 64$

and $\arg(z^3) = 3\theta$

b $\frac{1}{z} = z^{-1}$

$= (4 \operatorname{cis} \theta)^{-1}$

$= 4^{-1} \operatorname{cis}(-\theta)$

$\therefore \left|\frac{1}{z}\right| = \frac{1}{4}$ and $\arg\left(\frac{1}{z}\right) = -\theta$

c If $z = 4 \operatorname{cis} \theta$

$z^* = 4 \operatorname{cis}(-\theta)$

$iz^* = \left(\operatorname{cis} \frac{\pi}{2}\right)(4 \operatorname{cis}(-\theta))$

$= 4 \operatorname{cis}\left(\frac{\pi}{2} - \theta\right)$

$\therefore |iz^*| = 4$ and $\arg(iz^*) = \frac{\pi}{2} - \theta$

$$\begin{aligned}
 \text{8 a} \quad & \text{If } z = \operatorname{cis} \phi \\
 & = \cos \phi + i \sin \phi \\
 \therefore |z| &= \sqrt{\cos^2 \phi + \sin^2 \phi} \\
 &= \sqrt{1} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \text{If } z = \operatorname{cis} \phi \\
 & \text{then } z^* = \operatorname{cis} (-\phi) \\
 \therefore z z^* &= \operatorname{cis} \phi (\operatorname{cis} (-\phi)) \\
 &= \operatorname{cis} (\theta + (-\phi)) \\
 &= \operatorname{cis} 0 \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & z = \operatorname{cis} \phi \\
 \therefore z^n &= \operatorname{cis} n\phi \\
 \therefore z^{-n} &= \operatorname{cis} (-n\phi) \\
 \therefore z^n - \frac{1}{z^n} &= \operatorname{cis} n\phi - \operatorname{cis} (-n\phi) \\
 &= (\cos n\phi + i \sin n\phi) - (\cos(-n\phi) + i \sin(-n\phi)) \\
 &= \cos n\phi + i \sin n\phi - \cos n\phi + i \sin n\phi \\
 &= 2i \sin n\phi
 \end{aligned}$$

$$\therefore z^* = \frac{1}{z}$$

$$\begin{aligned}
 \text{9 a} \quad & \text{To prove: } \arg z^n = n \arg z \\
 & \text{Let } z = r \operatorname{cis} \theta \\
 \therefore z^n &= r^n \operatorname{cis} n\theta \quad \{\text{De Moivre}\} \\
 \text{and so } \arg z^n &= n\theta \\
 \text{i.e., } \arg z^n &= n \arg z \quad \text{as required}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad \left(\frac{z}{w}\right)^* &= \left(\frac{a+bi}{c+di}\right)^* \\
 &= \left(\frac{(a+bi)(c-di)}{(c+di)(c-di)}\right)^* \\
 &= \left(\frac{(ac+bd) + i(bc-ad)}{c^2+d^2}\right)^* \\
 &= \frac{(ac+bd) - i(bc-ad)}{c^2+d^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{and also } \frac{z^*}{w^*} &= \frac{a-bi}{c-di} \\
 &= \frac{(a-bi)(c+di)}{(c-di)(c+di)} \\
 &= \frac{(ac+bd) - i(bc-ad)}{c^2+d^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{10 a} \quad & \cos 3\theta + i \sin 3\theta \\
 &= \operatorname{cis} 3\theta \\
 &= (\operatorname{cis} \theta)^3 \quad \therefore n = 3
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & \frac{1}{\cos 2\theta + i \sin 2\theta} \times \frac{\cos 2\theta - i \sin 2\theta}{\cos 2\theta - i \sin 2\theta} \quad \text{or} \quad \frac{1}{\cos 2\theta + i \sin 2\theta} = \frac{1}{\operatorname{cis} 2\theta} \\
 &= \frac{\cos 2\theta - i \sin 2\theta}{\cos^2 2\theta + \sin^2 2\theta} \\
 &= \cos 2\theta - i \sin 2\theta \\
 &= \cos(-2\theta) + i \sin(-2\theta) \\
 &= \operatorname{cis} (-2\theta) \\
 &= (\operatorname{cis} \theta)^{-2} \quad \therefore n = -2
 \end{aligned}$$

$$\begin{aligned}
 &= (\operatorname{cis} 2\theta)^{-1} \\
 &= [(\operatorname{cis} \theta)^2]^{-1} \\
 &= (\operatorname{cis} \theta)^{-2} \\
 \therefore n &= -2
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & \cos \theta - i \sin \theta \\
 &= \cos(-\theta) + i \sin(-\theta) \\
 &= \operatorname{cis} (-\theta) \\
 &= (\operatorname{cis} \theta)^{-1} \quad \therefore n = -1
 \end{aligned}$$

11 The fifth roots of i are the solutions to $z^5 = i$

$$\therefore z^5 = 1 \operatorname{cis} \left(\frac{\pi}{2} + k2\pi \right)$$

$$\therefore z = 1^{\frac{1}{5}} \left[\operatorname{cis} \left(\frac{\pi}{2} + k2\pi \right) \right]^{\frac{1}{5}}$$

$$\therefore z = \operatorname{cis} \left(\frac{\pi}{10} + \frac{k2\pi}{5} \right)$$

$$\therefore z = \operatorname{cis} \left(\frac{\pi}{10} + \frac{k4\pi}{10} \right)$$

$$\therefore z_1 = \operatorname{cis} \left(\frac{\pi}{10} \right), \operatorname{cis} \left(\frac{5\pi}{10} \right) = i, \operatorname{cis} \left(\frac{9\pi}{10} \right), \operatorname{cis} \left(\frac{13\pi}{10} \right), \operatorname{cis} \left(\frac{17\pi}{10} \right) \quad (\text{letting } k = 0, 1, 2, 3, 4)$$

12 Let $z = x + iy$ $\therefore z + \frac{1}{z} = (x + iy) + \frac{1}{x + iy} \times \frac{x - iy}{x - iy}$

$$= (x + iy) + \frac{(x - iy)}{x^2 + y^2}$$

$$= \frac{(x^2 + y^2)(x + iy) + (x - iy)}{x^2 + y^2}$$

$$= \frac{x^3 + ix^2y + xy^2 + y^3i + x - iy}{x^2 + y^2}$$

$$= \frac{x(x^2 + y^2 + 1) + i(x^2 + y^2 - 1)y}{x^2 + y^2}$$

which is real if $\frac{(x^2 + y^2 - 1)y}{x^2 + y^2} = 0$ i.e., $x^2 + y^2 - 1 = 0$ or $y = 0$

i.e., $x^2 + y^2 = 1$ or $y = 0$

i.e., $|z|^2 = 1$ or $y = 0$

i.e., $|z| = 1$ or z is real

or Let $z = r \operatorname{cis} \theta$ $\therefore z + \frac{1}{z} = r \operatorname{cis} \theta + \frac{1}{r} \operatorname{cis}(-\theta)$

$$= r \cos \theta + ir \sin \theta + \frac{1}{r} \cos(-\theta) + i \times \frac{1}{r} \sin(-\theta)$$

This is real if $r \sin \theta + \frac{1}{r} \sin(-\theta) = 0$

$$\therefore r \sin \theta - \frac{1}{r} \sin \theta = 0$$

$$\therefore \sin \theta \left(r - \frac{1}{r} \right) = 0$$

$$\therefore \sin \theta = 0 \quad \text{or} \quad r - \frac{1}{r} = 0,$$

i.e., $\theta = 0$ or $r = 1$

i.e., $z = r$ (which is real) or $|z| = 1$

13 a If w is the root of $z^5 = 1$ with smallest positive argument, then $w = \operatorname{cis} \left(\frac{2\pi}{5} \right)$ and $w^4 = \operatorname{cis} \left(\frac{8\pi}{5} \right)$.

These have sum, $\alpha + \beta = \cos \left(\frac{2\pi}{5} \right) + i \sin \left(\frac{2\pi}{5} \right) + \cos \left(\frac{8\pi}{5} \right) + i \sin \left(\frac{8\pi}{5} \right)$

$$= \cos \left(\frac{2\pi}{5} \right) + i \sin \left(\frac{2\pi}{5} \right) + \cos \left(\frac{2\pi}{5} \right) - i \sin \left(\frac{2\pi}{5} \right)$$

$$= 2 \cos \left(\frac{2\pi}{5} \right)$$

and product, $\alpha\beta = \operatorname{cis} \left(\frac{2\pi}{5} \right) \times \operatorname{cis} \left(\frac{8\pi}{5} \right) = \operatorname{cis} \left(\frac{10\pi}{5} \right) = \operatorname{cis} 2\pi = 1$

$\therefore$ a real quadratic with roots w, w^4 is $a(z^2 - 2 \cos \left(\frac{2\pi}{5} \right) z + 1) = 0, \quad a \neq 0$

b Let $\alpha = w + w^4$ and $\beta = w^2 + w^3$

Now we know that $1 + w + w^2 + w^3 + w^4 = 0 \dots (*)$

$$1 + (w + w^4) + (w^2 + w^3) = 0$$

$$1 + \alpha + \beta = 0$$

$$\alpha + \beta = -1$$

$$\begin{aligned} \text{and } \alpha\beta &= (w + w^4)(w^2 + w^3) \\ &= w^3 + w^4 + w^6 + w^7 \\ &= w^3 + w^4 + w + w^2 \quad \{\text{as } w^5 = 1\} \\ &= w + w^2 + w^3 + w^4 \\ &= -1 \quad \{\text{from } *\} \end{aligned}$$

$\therefore$ the quadratic factor is $a(z^2 + z - 1) = 0, a \neq 0$

- 14** Consider the diagram which shows vectors $w, z, z + w$ and $z - w$.

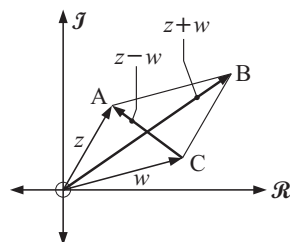
Clearly OABC is a parallelogram with $\overrightarrow{OB} = z + w$ and

$$\overrightarrow{CA} = z - w$$

If $|z + w| = |z - w|$, the diagonals are equal in length.

Hence, OABC is actually a rectangle and so angle COA

is a right angle i.e., $\arg z$ and $\arg w$ differ by $\frac{\pi}{2}$.



REVIEW SET 16C

1 a $z = r \operatorname{cis} \theta$

$$\bar{z} = r \operatorname{cis} (-\theta)$$

$\therefore T$ is a reflection in the R -axis

b $z = r \operatorname{cis} \theta$

$$-z = -r \operatorname{cis} \theta$$

$$= \operatorname{cis} \pi \times r \operatorname{cis} \theta$$

$$= r \operatorname{cis} (\theta + \pi)$$

$\therefore T$ is a rotation of π about O.

c $z = r \operatorname{cis} \theta$

$$iz = ir \operatorname{cis} \theta$$

$$= \operatorname{cis} \left(\frac{\pi}{2} \right) r \operatorname{cis} \theta$$

$$= r \operatorname{cis} \left(\theta + \frac{\pi}{2} \right)$$

$\therefore T$ is an anticlockwise rotation of $\frac{\pi}{2}$ about O.

2 Let $z = a + bi$ and $w = c + di$ $\therefore z + w = (a + c) + i(b + d)$, so $b + d = 0 \dots (1)$

$$\text{and } zw = (a + bi)(c + di)$$

$$= [ac - bd] + i[ad + bc]$$

$$\text{so, } ad + bc = 0 \dots (2)$$

From (1) $d = -b$ and in (2) $a(-b) + bc = 0$

$$\therefore b(c - a) = 0$$

$$\therefore a = c \text{ as } b \neq 0$$

$$\text{So } z = a + bi \text{ and } w = a - bi$$

$$\therefore z^* = a - bi = w$$

3

$$\text{If } (x + iy)^n = X + Yi$$

$$|(x + iy)^n| = |X + Yi|$$

$$|x + iy|^n = |X + Yi|$$

$$\left(\sqrt{x^2 + y^2} \right)^n = \sqrt{X^2 + Y^2}$$

Squaring both sides $X^2 + Y^2 = (x^2 + y^2)^n$

$$4 \quad |z - w|^2 + |z + w|^2 = 2(|z|^2 + |w|^2)$$

$$\text{Let } z = a + bi \quad w = c + di$$

$$\begin{aligned} \therefore \text{LHS} &= |(a - c) + i(b - d)|^2 + |(a + c) + i(b + d)|^2 \\ &= \left(\sqrt{(a - c)^2 + (b - d)^2} \right)^2 + \left(\sqrt{(a + c)^2 + (b + d)^2} \right)^2 \\ &= (a - c)^2 + (b - d)^2 + (a + c)^2 + (b + d)^2 \\ &= a^2 - 2ac + c^2 + b^2 - 2bd + d^2 + a^2 + 2ac + c^2 + b^2 + 2bd + d^2 \\ &= 2(a^2 + b^2) + 2(c^2 + d^2) \\ &= 2\left(\sqrt{a^2 + b^2}\right)^2 + 2\left(\sqrt{c^2 + d^2}\right)^2 \\ &= 2|z|^2 + 2|w|^2 \\ &= 2(|z|^2 + |w|^2) \end{aligned}$$

5 a Since 1 is a root of $z^5 - 1 = 0$, we find that

$$z^5 - 1 = (z - 1)(1 + z + z^2 + z^3 + z^4)$$

$\therefore$ since α is a root,

$$(\alpha - 1)(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4) = 0$$

But $\alpha \neq 1$,

$$\text{so } 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$$

$$1 \left| \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array} \right|$$

or Note that $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4$ is the sum of a geometric series.

$$\text{Then } S_n = \frac{a(1 - r^n)}{1 - r} \quad a = 1, \quad r = \alpha, \quad n = 5$$

$$\therefore S_5 = \frac{1(1 - \alpha^5)}{1 - \alpha} \quad \text{and } \alpha^5 = \text{cis}(2\pi) = 1$$

$$\therefore 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = \frac{1(1 - 1)}{1 - \alpha} = 0$$

b Now if $\left(\frac{z+2}{z-1}\right)^5 = 1$ then $\frac{z+2}{z-1} = 1, \alpha, \alpha^2, \alpha^3$ or α^4 where $\alpha = \text{cis}\left(\frac{2\pi}{5}\right)$

$$\text{i.e., } \frac{z+2}{z-1} = \alpha^n \quad \text{where } n = 0, 1, 2, 3, 4$$

$$z + 2 = \alpha^n(z - 1)$$

$$z + 2 = \alpha^n z - \alpha^n$$

$$z(\alpha^n - 1) = \alpha^n + 2 \quad \text{and so } z = \frac{\alpha^n + 2}{\alpha^n - 1}$$

$$\therefore \text{roots of } \left(\frac{z+2}{z-1}\right)^5 = 1 \text{ are } \frac{\alpha + 2}{\alpha - 1}, \frac{\alpha^2 + 2}{\alpha^2 - 1}, \frac{\alpha^3 + 2}{\alpha^3 - 1}, \frac{\alpha^4 + 2}{\alpha^4 - 1}, \text{ where } \alpha = \text{cis}\left(\frac{2\pi}{5}\right)$$

Note: The case where $n = 0$ requires $\frac{z+2}{z-1} = 1$, and so $z + 2 = z - 1$, which has no solution.

6

$$\text{Since } \left| \frac{z+1}{z-1} \right| = 1, \text{ then } |z+1| = |z-1|$$

$$\text{Letting } z = x + iy,$$

$$\therefore |(x+1) + iy| = |(x-1) + iy|$$

$$\therefore \sqrt{(x+1)^2 + y^2} = \sqrt{(x-1)^2 + y^2}$$

$$\text{Squaring both sides, we get } (x+1)^2 + y^2 = (x-1)^2 + y^2$$

Therefore since $z \neq 0$, z is purely imaginary.

$$\therefore x^2 + 2x + 1 = x^2 - 2x + 1 \quad \therefore 4x = 0 \quad \therefore x = 0$$

$$\begin{aligned}
 7 \quad w &= \frac{1+z}{1+z^*} = \frac{1 + \operatorname{cis} \phi}{1 + \operatorname{cis} (-\phi)} \times \frac{\operatorname{cis} \phi}{\operatorname{cis} \phi} \\
 &= \frac{(1 + \operatorname{cis} \phi) \operatorname{cis} \phi}{\operatorname{cis} \phi + \operatorname{cis} 0} \\
 &= \frac{(1 + \operatorname{cis} \phi) \operatorname{cis} \phi}{1 + \operatorname{cis} \phi} \\
 &= \operatorname{cis} \phi
 \end{aligned}$$

8 The cube roots of $-8i$ are solutions to $z^3 = -8i$
 $z^3 = 8 \operatorname{cis} \left(-\frac{\pi}{2} + k2\pi\right)$ where k is an integer

$$\therefore z = 8^{\frac{1}{3}} \operatorname{cis} \left(\frac{-\frac{\pi}{2} + k2\pi}{3}\right)$$

$$\therefore z = 2 \operatorname{cis} \left(-\frac{\pi}{6} + \frac{k4\pi}{6}\right)$$

$$\therefore z = 2 \operatorname{cis} \left(-\frac{\pi}{6}\right), 2 \operatorname{cis} \left(\frac{\pi}{2}\right), 2 \operatorname{cis} \left(\frac{7\pi}{6}\right) \quad \{\text{letting } k = 0, 1, 2\}$$

$$\therefore z = \sqrt{3} - i, 2i, -\sqrt{3} - i$$

9 $-1 + i\sqrt{3} = 2 \operatorname{cis} \left(\frac{2\pi}{3}\right)$
 $\therefore (-1 + i\sqrt{3})^m = 2^m \operatorname{cis} \left(\frac{m2\pi}{3}\right) \quad \{\text{De Moivre}\}$
 $= 2^m \left[\cos \left(\frac{m2\pi}{3}\right) + i \sin \left(\frac{m2\pi}{3}\right)\right]$

This is real provided $\sin \left(\frac{m2\pi}{3}\right) = 0$

$$\therefore \frac{m2\pi}{3} = 0 + k\pi$$

$$\therefore m = \frac{3k}{2} \quad \text{where } k \text{ is any integer}$$

10 We first note that $\cos(A+B) + \cos(A-B) = 2 \cos A \cos B$
and $\sin(A+B) + \sin(A-B) = 2 \sin A \cos B \dots (*)$

$$\begin{aligned}
 \text{Now } \operatorname{cis} \theta + \operatorname{cis} \phi &= (\cos \theta + i \sin \theta) + (\cos \phi + i \sin \phi) \\
 &= (\cos \theta + \cos \phi) + i(\sin \theta + \sin \phi) \\
 &= \left[\cos \left(\frac{\theta + \phi}{2} + \frac{\theta - \phi}{2}\right) + \cos \left(\frac{\theta + \phi}{2} - \frac{\theta - \phi}{2}\right)\right] \\
 &\quad + i \left[\sin \left(\frac{\theta + \phi}{2} + \frac{\theta - \phi}{2}\right) + \sin \left(\frac{\theta + \phi}{2} - \frac{\theta - \phi}{2}\right)\right] \\
 &= 2 \cos \left(\frac{\theta + \phi}{2}\right) \cos \left(\frac{\theta - \phi}{2}\right) + 2i \sin \left(\frac{\theta + \phi}{2}\right) \cos \left(\frac{\theta - \phi}{2}\right) \quad \{\text{using } *\} \\
 &= 2 \cos \left(\frac{\theta - \phi}{2}\right) \operatorname{cis} \left(\frac{\theta + \phi}{2}\right) \quad \text{as required}
 \end{aligned}$$

Now $Z^5 = 1$ has solutions $Z = 1, \alpha, \alpha^2, \alpha^3, \alpha^4$ where $\alpha = \operatorname{cis} \left(\frac{2\pi}{5}\right)$

i.e., $Z = \operatorname{cis} \left(\frac{2n\pi}{5}\right)$ where $n = 0, 1, 2, 3, 4$

$$\therefore \text{if } \left(\frac{z+1}{z-1}\right)^5 = 1, \text{ then } \frac{z+1}{z-1} = \operatorname{cis} \left(\frac{2n\pi}{5}\right)$$

$$\therefore z+1 = \operatorname{cis} \left(\frac{2n\pi}{5}\right)(z-1)$$

$$\therefore z(\operatorname{cis} \left(\frac{2n\pi}{5}\right) - 1) = \operatorname{cis} \left(\frac{2n\pi}{5}\right) + 1$$

$$\therefore z = \frac{\operatorname{cis} \left(\frac{2n\pi}{5}\right) + 1}{\operatorname{cis} \left(\frac{2n\pi}{5}\right) - 1}$$

$$\begin{aligned}
 \therefore z &= \frac{\operatorname{cis} \left(\frac{2n\pi}{5} \right) + \operatorname{cis} 0}{\operatorname{cis} \left(\frac{2n\pi}{5} \right) + \operatorname{cis} \pi} \\
 \therefore z &= \frac{2 \cos \left(\frac{\frac{2n\pi}{5} - 0}{2} \right) \operatorname{cis} \left[\frac{\frac{2n\pi}{5} + 0}{2} \right]}{2 \cos \left(\frac{\frac{2n\pi}{5} - \pi}{2} \right) \operatorname{cis} \left[\frac{\frac{2n\pi}{5} + \pi}{2} \right]} \quad \{\text{using the above identity}\} \\
 \therefore z &= \frac{\cos \frac{n\pi}{5}}{\cos \left(\frac{n\pi}{5} - \frac{\pi}{2} \right)} \operatorname{cis} \left[\frac{n\pi}{5} - \left(\frac{n\pi}{5} + \frac{\pi}{2} \right) \right] \\
 \therefore z &= \frac{\cos \frac{n\pi}{5}}{\cos \left[\left(\frac{\pi}{2} - \frac{n\pi}{5} \right) \right]} \operatorname{cis} \left(-\frac{\pi}{2} \right) \\
 \therefore z &= \frac{\cos \frac{n\pi}{5}}{\sin \frac{n\pi}{5}} (-i)
 \end{aligned}$$

$$\therefore z = -i \cot \frac{n\pi}{5}, \quad n = 1, 2, 3, 4 \quad \{\text{neglecting } n = 0 \text{ since } \cot 0 \text{ is undefined}\}$$

$$\begin{aligned}
 \text{Now } \cot \left(\frac{n\pi}{5} \right) &= -\cot \left(\pi - \frac{n\pi}{5} \right) \\
 &= -\cot \left(\frac{(5-n)\pi}{5} \right), \quad n = 1, 2, 3, 4 \\
 &= -\cot \left(\frac{n\pi}{5} \right), \quad n = 4, 3, 2, 1 \\
 \therefore z &= i \cot \left(\frac{n\pi}{5} \right), \quad n = 1, 2, 3, 4
 \end{aligned}$$

11 The 3 cube roots of $-64i$ are the solutions to $z^3 = -64i$

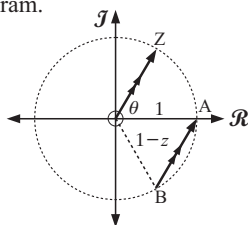
$$\begin{aligned}
 \therefore z^3 &= 64 \operatorname{cis} \left(-\frac{\pi}{2} + k2\pi \right) \text{ for integer } k \\
 \therefore z &= \left[64 \operatorname{cis} \left(-\frac{\pi}{2} + k2\pi \right) \right]^{\frac{1}{3}} \\
 \therefore z &= 64^{\frac{1}{3}} \operatorname{cis} \left(\frac{-\frac{\pi}{2} + k2\pi}{3} \right) \\
 \therefore z &= 4 \operatorname{cis} \left(-\frac{\pi}{6} + \frac{k4\pi}{6} \right) \\
 \therefore z &= 4 \operatorname{cis} \left(-\frac{\pi}{6} \right), 4 \operatorname{cis} \left(\frac{\pi}{2} \right), 4 \operatorname{cis} \left(\frac{7\pi}{6} \right) \quad \{\text{letting } k = 0, 1, 2\} \\
 \therefore z &= 4 \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i \right), 4i, 4 \operatorname{cis} \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right) \\
 \therefore z &= 2\sqrt{3} - 2i, 4i, -2\sqrt{3} - 2i
 \end{aligned}$$

12 a $(2z)^{-1} = (2 \operatorname{cis} \theta)^{-1}$
 $= 2^{-1} \operatorname{cis} (-\theta)$

$$\begin{aligned}
 \therefore |(2z)^{-1}| &= \frac{1}{2} \quad \text{and} \\
 \arg [(2z)^{-1}] &= -\theta
 \end{aligned}$$

b $1 - z = 1 - \operatorname{cis} \theta$
 $= (1 - \cos \theta) - i \sin \theta$
 $\therefore |1 - z| = \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta}$
 $= \sqrt{1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta}$
 $= \sqrt{2 - 2 \cos \theta}$
 $= \sqrt{4 \left(\frac{1}{2} - \frac{1}{2} \cos \theta \right)}$
 $= 2 \sqrt{\sin^2 \left(\frac{\theta}{2} \right)}$
 $= 2 \sin \left(\frac{\theta}{2} \right)$

w can be represented on a diagram.



$\triangle OAB$ is isosceles since $|z| = 1$,

so we let $\angle AOB = \angle ABO = \phi$

Since $OZ \parallel AB$, $\angle OAB = \theta$ {alternate $\angle$'s}

$$\therefore \phi + \phi + \theta = \pi$$

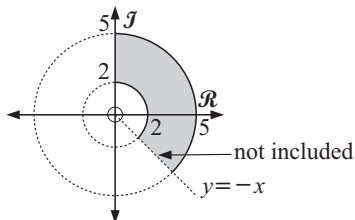
$$\therefore 2\phi = \pi - \theta$$

$$\phi = \frac{\pi}{2} - \frac{\theta}{2}$$

$$\text{But } \arg(1 - z) = -\phi,$$

$$\begin{aligned} \text{so } \arg(1 - z) &= -\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \\ &= \frac{\theta}{2} - \frac{\pi}{2} \end{aligned}$$

- 13** $\{z: 2 \leq |z| \leq 5 \text{ and } -\frac{\pi}{4} < \arg z \leq \frac{\pi}{2}\}$ **14** If $z = r \operatorname{cis} \theta$, then $|z| = r$ and $\arg z = \theta$



$$\text{Now } \frac{1}{z} = (r \operatorname{cis} \theta)^{-1} = r^{-1} \operatorname{cis} (-\theta)$$

$$= \frac{1}{r} \operatorname{cis} (-\theta)$$

$$= \frac{1}{|z|} \operatorname{cis} (-\theta)$$

$$\left| \frac{1}{z} \right| = \frac{1}{|z|} \text{ if } z \neq 0 \text{ and } \arg\left(\frac{1}{z}\right) = -\theta = -\arg z$$

14

$$z = \operatorname{cis} \alpha$$

$$\therefore 1 + z = 1 + \operatorname{cis} \alpha$$

$$= 1 + \cos \theta + i \sin \theta$$

$$= \left[1 + 2 \cos^2\left(\frac{\alpha}{2}\right) - 1\right] + i \left[2 \sin\left(\frac{\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right)\right]$$

$$= 2 \cos^2\left(\frac{\alpha}{2}\right) + i \left[2 \sin\left(\frac{\alpha}{2}\right) \cos\left(\frac{\alpha}{2}\right)\right]$$

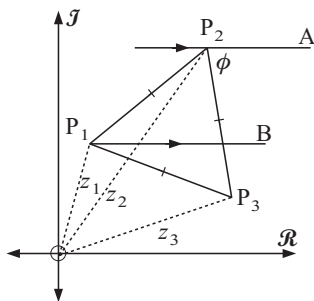
$$= 2 \cos\left(\frac{\alpha}{2}\right) \left[\cos\left(\frac{\alpha}{2}\right) + i \sin\left(\frac{\alpha}{2}\right)\right]$$

$$= 2 \cos\left(\frac{\alpha}{2}\right) \operatorname{cis}\left(\frac{\alpha}{2}\right)$$

$$\therefore |1 + z| = 2 \cos\left(\frac{\alpha}{2}\right)$$

$$\text{and } \arg(1 + z) = \frac{\alpha}{2}$$

15 a



$$\text{Now } z_2 - z_1 = \overrightarrow{P_1P_2}$$

so $\arg(z_2 - z_1) = \alpha$ as shown on the diagram alongside

$$z_3 - z_2 = \overrightarrow{P_2P_3}$$

so $\arg(z_3 - z_2) = -\phi$

Now $\angle AP_2P_1 = \alpha$ since $AP_2 \parallel P_1B$

and $\angle P_1P_2P_3 = \frac{\pi}{3}$ since the Δ is equilateral

$$\text{Hence } \alpha + \frac{\pi}{3} + \phi = \pi$$

$$\therefore \phi = -\alpha + \frac{2\pi}{3}$$

$$\therefore \arg(z_3 - z_2) = \alpha - \frac{2\pi}{3} \text{ as required}$$

b

$$\left| \frac{z_2 - z_1}{z_3 - z_2} \right| = \left| \frac{\overrightarrow{P_1P_2}}{\overrightarrow{P_2P_3}} \right|$$

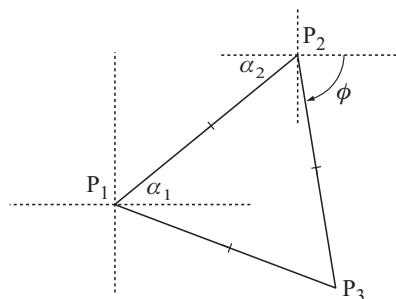
$$= 1 \text{ since the } \Delta \text{ is equilateral}$$

$$\arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) = \arg(z_2 - z_1) - \arg(z_3 - z_2)$$

$$= \alpha - \left(\alpha - \frac{2\pi}{3}\right)$$

$$= \alpha - \alpha + \frac{2\pi}{3}$$

$$= \frac{2\pi}{3}$$



Chapter 17

LINES AND PLANES IN SPACE

EXERCISE 17A.1

- 1 a i $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ ii $x = 3 + t, y = -4 + 4t, t \in \mathcal{R}$
 b i $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + t \begin{bmatrix} -8 \\ 2 \end{bmatrix}$ ii $x = 5 - 8t, y = 2 + 2t, t \in \mathcal{R}$
 c i $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ ii $x = -6 + 3t, y = 7t, t \in \mathcal{R}$
 d i $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ ii $x = -1 - 2t, y = 11 + t, t \in \mathcal{R}$

2 $x = -1 + 2\lambda, y = 4 - \lambda, \lambda \in \mathcal{R}$

When $\lambda = 0$, $x = -1 + 2(0) = -1$ and $y = 4 - 0 = 4 \therefore$ point is $(-1, 4)$

When $\lambda = 1$, $x = -1 + 2(1) = 1$ and $y = 4 - 1 = 3 \therefore$ point is $(1, 3)$

When $\lambda = 3$, $x = -1 + 2(3) = 5$ and $y = 4 - 3 = 1 \therefore$ point is $(5, 1)$

When $\lambda = -1$, $x = -1 + 2(-1) = -3$ and $y = 4 - (-1) = 5 \therefore$ point is $(-3, 5)$

When $\lambda = -4$, $x = -1 + 2(-4) = -9$ and $y = 4 - (-4) = 8 \therefore$ point is $(-9, 8)$

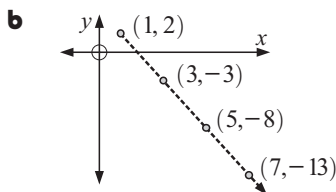
- 3 a If $t + 2 = 3$ and $1 - 3t = -2$, we get $t = 1$ and $-3t = -3$ i.e., $t = 1$
 Since $t = 1$ in each case, $(3, -2)$ lies on the line.

If $t + 2 = 0$ and $1 - 3t = 6$, we get $t = -2$ and $-3t = 5$ i.e., $t = -\frac{5}{3}$
 $\therefore (0, 6)$ does not lie on the line.

- b If $(k, 4)$ lies on $x = 1 - 2t, y = 1 + t$, then
 $k = 1 - 2t$ and $4 = 1 + t$
 $\therefore t = 3$ and $k = 1 - 6 = -5$
 i.e., $k = -5$

- 4 a $x(0) = 1$ and $y(0) = 2$,
 $\therefore$ the initial position is $(1, 2)$

- c In 1 second, the
 x -step is 2 and y -step is -5 , which is
 a distance of $\sqrt{2^2 + (-5)^2} = \sqrt{29}$
 $\therefore$ speed is $\sqrt{29}$ cm/sec.



EXERCISE 17A.2

1 a The vector equation is $\begin{bmatrix} x-1 \\ y-3 \\ z+7 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, t \in \mathcal{R}$

b The vector equation is $\begin{bmatrix} x-0 \\ y-1 \\ z-2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, t \in \mathcal{R}$

c Since the line is parallel to the X -axis, it has direction vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ the vector equation is $\begin{bmatrix} x+2 \\ y-2 \\ z-1 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, t \in \mathcal{R}$

- 2 a** The parametric equations are:

$$x = 5 + (-1)t, \quad y = 2 + 2t, \quad z = -1 + 6t$$

$$\text{i.e., } x = 5 - t, \quad y = 2 + 2t, \quad z = -1 + 6t, \quad t \in \mathcal{R}$$

- b** The parametric equations are:

$$x = 0 + 2t, \quad y = 2 + (-1)t, \quad z = -1 + 3t$$

$$\text{i.e., } x = 2t, \quad y = 2 - t, \quad z = -1 + 3t, \quad t \in \mathcal{R}$$

- c** Since the line is perpendicular to the XOY plane, it has direction vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
 $\therefore$ the parametric equations are:

$$x = 3 + 0t, \quad y = 2 + 0t, \quad z = -1 + 1t$$

$$\text{i.e., } x = 3, \quad y = 2, \quad z = -1 + t, \quad t \in \mathcal{R}$$

3 a $\overrightarrow{AB} = \begin{bmatrix} -1-1 \\ 3-2 \\ 2-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \therefore x = 1 - 2t, \quad y = 2 + t, \quad z = 1 + t, \quad t \in \mathcal{R}$

b $\overrightarrow{CD} = \begin{bmatrix} 3-0 \\ 1-1 \\ -1-3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} \quad \therefore x = 3t, \quad y = 1, \quad z = 3 - 4t, \quad t \in \mathcal{R}$

c $\overrightarrow{EF} = \begin{bmatrix} 1-1 \\ -1-2 \\ 5-5 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} \quad \therefore x = 1, \quad y = 2 - 3t, \quad z = 5, \quad t \in \mathcal{R}$

d $\overrightarrow{GH} = \begin{bmatrix} 5-0 \\ -1-1 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} \quad \therefore x = 5t, \quad y = 1 - 2t, \quad z = -1 + 4t, \quad t \in \mathcal{R}$

- 4** Given $x = 1 - \lambda$, $y = 3 + \lambda$, $z = 3 - 2\lambda$:

- a** The line meets the XOY plane when $z = 0$ i.e., $3 - 2\lambda = 0 \quad \therefore \lambda = \frac{3}{2}$

$$\text{Then } x = 1 - \frac{3}{2} = -\frac{1}{2} \quad \text{and} \quad y = 3 + \frac{3}{2} = \frac{9}{2}, \quad \text{i.e., the point is } \left(-\frac{1}{2}, \frac{9}{2}, 0\right)$$

- b** The line meets the YOZ plane when $x = 0$ i.e., $1 - \lambda = 0$

$$\therefore \lambda = 1$$

$$\text{Then } y = 3 + 1 = 4 \quad \text{and} \quad z = 3 - 2 = 1, \quad \text{i.e., the point is } (0, 4, 1)$$

- c** The line meets the XOZ plane when $y = 0$ i.e., $3 + \lambda = 0 \quad \therefore \lambda = -3$

$$\text{Then } x = 1 - (-3) = 4 \quad \text{and} \quad z = 3 - 2(-3) = 9, \quad \text{i.e., the point is } (4, 0, 9)$$

- 5** Given a line with equations $x = 2 - \lambda$, $y = 3 + 2\lambda$ and $z = 1 + \lambda$

$$\text{the distance to the point } (1, 0, -2) \text{ is } \sqrt{(2 - \lambda - 1)^2 + (3 + 2\lambda - 0)^2 + (1 + \lambda - 2)^2}.$$

$$\text{But this distance} = 5\sqrt{3} \text{ units}$$

$$\therefore \sqrt{(1 - \lambda)^2 + (3 + 2\lambda)^2 + (\lambda + 3)^2} = 5\sqrt{3}$$

$$\therefore (1 - \lambda)^2 + (3 + 2\lambda)^2 + (\lambda + 3)^2 = 75$$

$$\therefore 1 - 2\lambda + \lambda^2 + 9 + 12\lambda + 4\lambda^2 + \lambda^2 + 6\lambda + 9 = 75$$

$$\therefore 6\lambda^2 + 16\lambda - 56 = 0$$

$$\therefore 3\lambda^2 + 8\lambda - 28 = 0$$

$$\therefore (3\lambda + 14)(\lambda - 2) = 0$$

$$\therefore \lambda = -\frac{14}{3} \quad \text{or} \quad \lambda = 2$$

$$\text{When } \lambda = 2, \quad \text{the point is } (0, 7, 3)$$

$$\text{and when } \lambda = -\frac{14}{3}, \quad \text{the point is } \left(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3}\right).$$

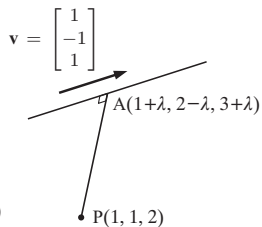
- 6 a** Let $A(1 + \lambda, 2 - \lambda, 3 + \lambda)$ be a point on the given line.

Then $\vec{PA} = \begin{bmatrix} 1 + \lambda - 1 \\ 2 - \lambda - 1 \\ 3 + \lambda - 2 \end{bmatrix} = \begin{bmatrix} \lambda \\ 1 - \lambda \\ 1 + \lambda \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is the direction vector of the line.

If $\vec{PA}$ and $\mathbf{v}$ are perpendicular, then $\vec{PA} \bullet \mathbf{v} = 0$

$$\begin{aligned} \therefore \begin{bmatrix} \lambda \\ 1 - \lambda \\ 1 + \lambda \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} &= 0 & \therefore \lambda - 1(1 - \lambda) + 1(1 + \lambda) &= 0 \\ & & \therefore \lambda - 1 + \lambda + 1 + \lambda &= 0 \\ & & \therefore 3\lambda &= 0 \\ & & \therefore \lambda &= 0 \end{aligned}$$

$\therefore$ A is at $(1, 2, 3)$ i.e., the foot of the perpendicular is $(1, 2, 3)$



- b** Let A be a point on the line such that $\vec{PA}$ is perpendicular to the line.

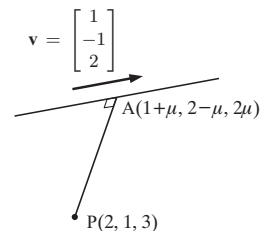
Then A is at $(1 + \mu, 2 - \mu, 2\mu)$ for some μ .

$$\text{Now } \vec{PA} = \begin{bmatrix} 1 + \mu - 2 \\ 2 - \mu - 1 \\ 2\mu - 3 \end{bmatrix} = \begin{bmatrix} \mu - 1 \\ 1 - \mu \\ 2\mu - 3 \end{bmatrix}$$

and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ is the direction vector of the line.

$$\begin{aligned} \text{Since } \vec{PA} \bullet \mathbf{v} &= 0, & \begin{bmatrix} \mu - 1 \\ 1 - \mu \\ 2\mu - 3 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} &= 0 & \therefore 1(\mu - 1) - 1(1 - \mu) + 2(2\mu - 3) &= 0 \\ & & \therefore \mu - 1 - 1 + \mu + 4\mu - 6 &= 0 \\ & & \therefore 6\mu &= 8 \\ & & \therefore \mu &= \frac{4}{3} \end{aligned}$$

$\therefore$ A is at $(\frac{7}{3}, \frac{2}{3}, \frac{8}{3})$ i.e., the foot of the perpendicular is $(\frac{7}{3}, \frac{2}{3}, \frac{8}{3})$



- 7 a** Let $A(2 + 3t, -1 + 2t, 4 + t)$ be a point on the line such that $\vec{PA}$ is perpendicular to the line.

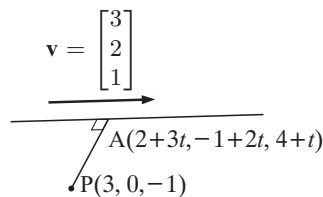
$$\text{Now } \vec{PA} = \begin{bmatrix} 2 + 3t - 3 \\ -1 + 2t - 0 \\ 4 + t - (-1) \end{bmatrix} = \begin{bmatrix} 3t - 1 \\ 2t - 1 \\ t + 5 \end{bmatrix}$$

and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is the direction vector of the line.

$$\begin{aligned} \therefore \text{ since } \vec{PA} \bullet \mathbf{v} &= 0, & \begin{bmatrix} 3t - 1 \\ 2t - 1 \\ t + 5 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} &= 0 \\ & \therefore 3(3t - 1) + 2(2t - 1) + 1(t + 5) &= 0 \\ & \therefore 9t - 3 + 4t - 2 + t + 5 &= 0 \\ & \therefore 14t &= 0 \\ & \therefore t &= 0 \end{aligned}$$

$\therefore$ A is at $(2, -1, 4)$

$$\begin{aligned} \therefore \text{ the distance } d &= \sqrt{(2 - 3)^2 + (-1 - 0)^2 + (4 - (-1))^2} = \sqrt{1 + 1 + 25} \\ &= \sqrt{27} \\ &= 3\sqrt{3} \text{ units} \end{aligned}$$



- b** Let A be a point on the line such that $\overrightarrow{PA}$ is perpendicular to the line.

Then A is at $(1 + 2t, -1 + 3t, 2 + t)$ for some t .

$$\text{Now } \overrightarrow{PA} = \begin{bmatrix} 1 + 2t - 1 \\ -1 + 3t - 1 \\ 2 + t - 3 \end{bmatrix} = \begin{bmatrix} 2t \\ 3t - 2 \\ t - 1 \end{bmatrix}$$

and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ is the direction vector of the line.

$$\therefore \text{ since } \overrightarrow{PA} \bullet \mathbf{v} = 0, \quad \therefore 4t + 3(3t - 2) + 1(t - 1) = 0$$

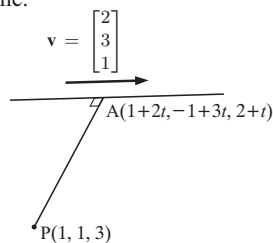
$$\begin{bmatrix} 2t \\ 3t - 2 \\ t - 1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 0 \quad \therefore 4t + 9t - 6 + t - 1 = 0$$

$$\therefore 14t = 7$$

$$\therefore t = \frac{1}{2}$$

$$\therefore A \text{ is at } (2, \frac{1}{2}, \frac{5}{2})$$

$$\therefore \text{ the distance } d = \sqrt{(2-1)^2 + (\frac{1}{2}-1)^2 + (\frac{5}{2}-3)^2} = \sqrt{1 + \frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{3}{2}} \text{ units}$$



EXERCISE 17A.3

- 1** l_1 has direction vector $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ and l_2 has direction vector $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ and if θ is the angle between them,

$$\cos \theta = \frac{\left| \begin{bmatrix} 4 \\ -3 \end{bmatrix} \bullet \begin{bmatrix} 5 \\ 4 \end{bmatrix} \right|}{\sqrt{16+9}\sqrt{25+16}} = \frac{|20 + (-12)|}{\sqrt{25 \times 41}} = \frac{8}{\sqrt{25 \times 41}}$$

$$\therefore \theta = \arccos\left(\frac{8}{\sqrt{25 \times 41}}\right) \doteq 75.5^\circ$$

$\therefore$ the required angle measures 75.5°

- 2** l_1 has direction vector $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$ and l_2 has direction vector $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and if θ is the angle between them,

$$\cos \theta = \frac{\left| \begin{bmatrix} 12 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ -4 \end{bmatrix} \right|}{\sqrt{144+25}\sqrt{9+16}} = \frac{|36 + (-20)|}{13 \times 5} = \frac{16}{65}$$

$$\therefore \theta = \arccos\left(\frac{16}{65}\right) \doteq 75.7^\circ$$

- 3** Line 1 has direction vector $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and line 2 has direction vector $\begin{bmatrix} 4 \\ 10 \end{bmatrix}$

$$\text{and } \begin{bmatrix} 5 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ 10 \end{bmatrix} = 20 + (-20) = 0$$

$\therefore l_1$ and l_2 are perpendicular

- 4** If $\frac{x-8}{3} = \frac{9-y}{16} = \frac{z-10}{7} = \lambda$ say then $x = 8 + 3\lambda$, $y = 9 - 16\lambda$, $z = 10 + 7\lambda$

$\therefore$ line 1 has vector $\begin{bmatrix} 3 \\ -16 \\ 7 \end{bmatrix}$, line 2 has vector $\begin{bmatrix} 3 \\ 8 \\ -5 \end{bmatrix}$ and if θ is the angle between them,

$$\cos \theta = \frac{\left| \begin{bmatrix} 3 \\ -16 \\ 7 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 8 \\ -5 \end{bmatrix} \right|}{\sqrt{9+256+49}\sqrt{9+64+25}} = \frac{|9 - 128 - 35|}{\sqrt{314}\sqrt{98}} = \frac{154}{\sqrt{314 \times 98}}$$

$$\therefore \theta \doteq 28.6^\circ$$

EXERCISE 17B.1

- 1 a i** When $t = 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ **ii** The velocity vector is $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$ **iii** The speed is $\sqrt{12^2 + 5^2} = 13$ m/s
 $\therefore$ the object is at $(-4, 3)$
- b i** When $t = 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}$ **ii** The velocity vector is $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ **iii** The speed is $\sqrt{3^2 + (-4)^2} = 5$ m/s
 $\therefore$ the object is at $(0, -6)$
- c i** When $t = 0$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$ **ii** The velocity vector is $\begin{bmatrix} -6 \\ -4 \end{bmatrix}$ **iii** The speed is $\sqrt{(-6)^2 + (-4)^2} = \sqrt{36 + 16} = \sqrt{52}$ m/s
 $\therefore$ the object is at $(-2, -7)$
- 2 a** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} + t \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ **i** velocity vector = $\begin{bmatrix} 8 \\ 4 \end{bmatrix}$ **ii** speed = $\sqrt{8^2 + 4^2} = \sqrt{80}$ km/h
b $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ **i** velocity vector = $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ **ii** speed = $\sqrt{6^2 + 2^2} = \sqrt{40}$ km/h
c $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -12 \\ 15 \end{bmatrix} + t \begin{bmatrix} 7 \\ 24 \end{bmatrix}$ **i** velocity vector = $\begin{bmatrix} 7 \\ 24 \end{bmatrix}$ **ii** speed = $\sqrt{7^2 + 24^2} = 25$ km/h
- 3 a** $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$ has length $\sqrt{4^2 + (-3)^2} = 5$ **b** $\begin{bmatrix} 24 \\ 7 \end{bmatrix}$ has length $\sqrt{24^2 + 7^2} = 25$
 $\therefore 30 \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ has length 150
 $\therefore$ velocity vector is $\begin{bmatrix} 120 \\ -90 \end{bmatrix}$
c $2\mathbf{i} + \mathbf{j} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ has length $\sqrt{2^2 + 1^2} = \sqrt{5}$
 $\therefore 10\sqrt{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ has length 50
 $\therefore$ velocity vector is $\begin{bmatrix} 20\sqrt{5} \\ 10\sqrt{5} \end{bmatrix}$
d $-3\mathbf{i} + 4\mathbf{j} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ has length $\sqrt{(-3)^2 + 4^2} = 5$
 $\therefore 20 \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ has length 100
 $\therefore$ velocity vector is $\begin{bmatrix} -60 \\ 80 \end{bmatrix}$.

EXERCISE 17B.2

- 1 a** $x = -3 + 2t$, $y = -2 + 4t$, $t \geq 0$ **b** when $t = 2.5$,
 $\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3+2t \\ -2+4t \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3+5 \\ -2+10 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ $\therefore$ position is $(2, 8)$.
- c i** When the car is due north,
 $x = 0$ and so $-3 + 2t = 0$
 $\therefore t = \frac{3}{2}$ sec.
ii When the car is due west,
 $y = 0$ and so $-2 + 4t = 0$
 $\therefore t = \frac{1}{2}$ sec
(O is the observation point)
- d**
- 2 a** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix} + t \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, $t \in \mathcal{R}$
b The direction vector is $\begin{bmatrix} 18-2 \\ 21-6 \end{bmatrix} = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$ $\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} + \frac{t}{10} \begin{bmatrix} 20 \\ 15 \end{bmatrix}$, $t \in \mathcal{R}$
c Since $2\mathbf{i} + \mathbf{j} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ has length $\sqrt{2^2 + 1^2} = \sqrt{5}$ then $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $t \in \mathcal{R}$
d Since $3\mathbf{i} + 4\mathbf{j} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ has length $\sqrt{3^2 + 4^2} = 5$ then $3 \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ has length 15
 $\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} + (t-1) \begin{bmatrix} 9 \\ 12 \end{bmatrix}$
or $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} - \begin{bmatrix} 9 \\ 12 \end{bmatrix} + t \begin{bmatrix} 9 \\ 12 \end{bmatrix}$ i.e., $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ -16 \end{bmatrix} + t \begin{bmatrix} 9 \\ 12 \end{bmatrix}$, $t \in \mathcal{R}$

3 $\mathbf{r} = \begin{bmatrix} -20 \\ 32 \end{bmatrix} + t \begin{bmatrix} 12 \\ -5 \end{bmatrix}$, where t is the time in hours since 6 am.

a At 6 am, $t = 0 \therefore \mathbf{r} = \begin{bmatrix} -20 \\ 32 \end{bmatrix}$ and $|\mathbf{r}| = \sqrt{(-20)^2 + (32)^2}$
 $\div 37.7 \therefore$ the ship is 37.7 km away

b Speed $= \left| \begin{bmatrix} 12 \\ -5 \end{bmatrix} \right| = \sqrt{12^2 + (-5)^2} = 13$ km/h

c When due north, $x = 0. \therefore -20 + 12t = 0$
 $\therefore t = \frac{5}{3}$ hours
 $\therefore t = 1$ hour 40 min, so the time is 7:40 am

4 Yacht A: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ Yacht B: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}, t \geq 0$

a when $t = 0, \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \therefore$ A is at (4, 5) and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$ and B is at (1, -8)

b For A, the velocity vector is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, and for B it is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

c Speed of A $= \sqrt{1^2 + (-2)^2} = \sqrt{5}$ km/h Speed of B $= \sqrt{2^2 + 1^2} = \sqrt{5}$ km/h

d The distance between them is $D = \sqrt{[(1+2t) - (4+t)]^2 + [(-8+t) - (5-2t)]^2}$
 $= \sqrt{(-3+t)^2 + (-13+3t)^2}$
 $= \sqrt{9 - 6t + t^2 + 169 - 78t + 9t^2}$
 $= \sqrt{10t^2 - 84t + 178}$

This is a minimum when $10t^2 - 84t + 178$ is a minimum. This occurs when

$t = \frac{-b}{2a} = \frac{84}{20} = 4.2$ hours. $\therefore$ the time is 4 h 12 min after 6 am i.e., 10:12 am.

e A has direction vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and B has direction vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Since $\begin{bmatrix} 1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 - 2 = 0$, the paths of the yachts are at right angles to each other.

5 **a** P has position $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and at $t = 0$, the time is 1.34 pm

$\therefore x_1(t) = -5 + 3t, y_1(t) = 4 - t.$

b Speed $= \sqrt{3^2 + (-1)^2} = \sqrt{10}$ km/min

c Q fires its torpedo after a minutes.

$\therefore$ at time t , its torpedo has travelled for $(t - a)$ minutes.

$\therefore \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 7 \end{bmatrix} + (t - a) \begin{bmatrix} -4 \\ -3 \end{bmatrix}, t > a$

i.e., $x_2(t) = 15 - 4(t - a)$ and $y_2(t) = 7 - 3(t - a)$

d They meet when $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$

$\therefore -5 + 3t = 15 - 4(t - a)$ and $4 - t = 7 - 3(t - a)$
 i.e., $7t - 4a = 20$ and $2t - 3a = 3$

Solving simultaneously, $7t - 4a = 20 \quad \times 3$
 $2t - 3a = 3 \quad \times (-4)$

$\therefore \begin{array}{r} 21t - 12a = 60 \\ -8t + 12a = -12 \\ \hline 13t = 48 \end{array}$

$\therefore t = \frac{48}{13}$ and $7(\frac{48}{13}) - 4a = 20$

i.e., $t \div 3.6923 \therefore 5.8462 = 4a$

i.e., $t \div 3$ min 41.53 $\therefore a \div 1.4615 \div 1$ min 27.7 sec

So, as $a \div 1.4615$, Q fired at 1:35:28 pm, and the explosion occurred at 1:37:42 pm.

EXERCISE 17B.3

- 1 a $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $t \geq 0$ $\therefore$ its position is $(-3 + 2t, -2 + 4t)$.
- b i It is due east when $y = 0$ $\therefore -2 + 4t = 0$ $\therefore t = \frac{1}{2}$ sec
- ii It is due north when $x = 0$ $\therefore -3 + 2t = 0$ $\therefore t = 1\frac{1}{2}$ sec

c When $y = 0$, $t = \frac{1}{2}$ and $x = -3 + 2\left(\frac{1}{2}\right) = -2$

When $x = 0$, $t = 1\frac{1}{2}$ and $y = -2 + 4\left(1\frac{1}{2}\right) = 4$

$\therefore$ axis intercepts are $(-2, 0)$ and $(0, 4)$

- 2 a $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ -3 \end{bmatrix}$, $t \geq 0$ $\therefore$ its position is $(-2 - t, 1 - 3t)$
- b Crosses the x -axis when $y = 0$ $\therefore 1 - 3t = 0$ $\therefore t = \frac{1}{3}$ sec
- c When $t = \frac{1}{3}$, the point is at $(-2\frac{1}{3}, 0)$.

- 3 a $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$ has length $\sqrt{(-3)^2 + 4^2} = 5$ $\therefore$ unit direction vector is $\frac{1}{5} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$
- $\therefore$ the velocity vector is $\frac{1}{5} \begin{bmatrix} -3 \\ 4 \end{bmatrix} \times 10 = \begin{bmatrix} -6 \\ 8 \end{bmatrix} = -6\mathbf{i} + 8\mathbf{j}$

b $\therefore$ the liner's position is given by $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix} + t \begin{bmatrix} -6 \\ 8 \end{bmatrix}$, $t \geq 0$

c The liner is due east when $y = 0$, $\therefore 0 = -6 + 8t$ $\therefore t = \frac{3}{4}$ hour

d The liner is nearest to the fishing boat when $\overrightarrow{OP} \perp \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$$\therefore \begin{bmatrix} 6-6t-0 \\ -6+8t-0 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ 4 \end{bmatrix} = 0 \quad P(6-6t, -6+8t)$$

$$\therefore -3(6-6t) + 4(-6+8t) = 0$$

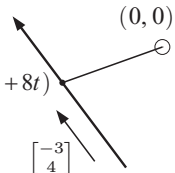
$$\therefore -18 + 18t - 24 + 32t = 0$$

$$\therefore 50t = 42$$

$$\therefore t = 0.84 \text{ hours}$$

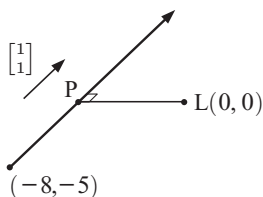
i.e., after 50 min 24 sec.

At this time, the liner is at $(6 - 6(0.84), -6 + 8(0.84))$ i.e., at $(0.96, 0.72)$.



- 4 a i Initial position vector is $\begin{bmatrix} -8-0 \\ -5-0 \end{bmatrix} = -8\mathbf{i} - 5\mathbf{j}$
- ii $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has length $\sqrt{1^2 + 1^2} = \sqrt{2}$
- $\therefore 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has length $3\sqrt{2}$
- $\therefore$ the direction vector is $\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\mathbf{i} + 3\mathbf{j}$
- iii position vector at time t is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ -5 \end{bmatrix} + t \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, or $(-8 + 3t)\mathbf{i} + (-5 + 3t)\mathbf{j}$

b



P is $(-8 + 3t, -5 + 3t)$

and $\overrightarrow{LP}$ is $\begin{bmatrix} -8+3t \\ -5+3t \end{bmatrix}$

Now when closest, $\overrightarrow{LP} \bullet \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

$$\therefore \begin{bmatrix} -8+3t \\ -5+3t \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$\therefore -8 + 3t - 5 + 3t = 0$$

$$\therefore 6t = 13$$

$$\therefore t = \frac{13}{6} = 2\frac{1}{6}$$

i.e., at 2 hours 10 min

- c** When $t = \frac{13}{6}$, P is $(-8 + 3(\frac{13}{6}), -5 + 3(\frac{13}{6}))$ i.e., $(-\frac{3}{2}, \frac{3}{2})$

$$\begin{aligned}\therefore \text{ the closest distance} &= \sqrt{\left(-\frac{3}{2} - 0\right)^2 + \left(\frac{3}{2} - 0\right)^2} \\ &= \sqrt{\frac{9}{4} + \frac{9}{4}} \\ &= \sqrt{\frac{9}{2}} \\ &\doteq 2.12 \text{ km} \quad \therefore \text{ the trawler would be breaking the law.}\end{aligned}$$

5 a $|\mathbf{b}| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$

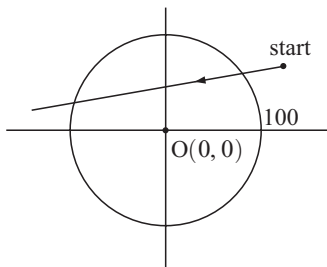
As the speed is $40\sqrt{10}$ km/h, the velocity vector is $40\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -120 \\ -40 \end{bmatrix}$.

b $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 200 \\ 100 \end{bmatrix} + t\begin{bmatrix} -120 \\ -40 \end{bmatrix}, \quad t \geq 0 \quad \{t = 0 \text{ at } 12.00 \text{ noon}\}$

c At 1:00 pm, $t = 1$ and $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 200-120 \\ 100-40 \end{bmatrix} = \begin{bmatrix} 80 \\ 60 \end{bmatrix}$

- d** The distance from $O(0, 0)$ to $P_1(80, 60)$ is $|\begin{bmatrix} 80 \\ 60 \end{bmatrix}| = \sqrt{80^2 + 60^2} = 100$ km,
which is when it becomes visible to radar. {within 100 km of $O(0, 0)$ }

e



A general point on the path is $P(200 - 120t, 100 - 40t)$.

Now $\vec{OP} = \begin{bmatrix} 200-120t \\ 100-40t \end{bmatrix}$,

and for the closest point $\vec{OP} \bullet \begin{bmatrix} -3 \\ -1 \end{bmatrix} = 0$

$$\therefore -3(200 - 120t) - 1(100 - 40t) = 0$$

$$\therefore -700 + 400t = 0$$

$$\therefore t = \frac{7}{4} = 1\frac{3}{4} \text{ hours}$$

The time when the aircraft is closest is 1:45 pm, and

at this time $\vec{OP} = \begin{bmatrix} 200-120(\frac{7}{4}) \\ 100-40(\frac{7}{4}) \end{bmatrix} = \begin{bmatrix} -10 \\ 30 \end{bmatrix}$

$$\therefore d_{\min} = \sqrt{(-10)^2 + 30^2} \doteq 31.6 \text{ km}$$

- f** It disappears from radar when $|\vec{OP}| = 100$ and $t > 1\frac{3}{4}$

$$\therefore \sqrt{(200 - 120t)^2 + (100 - 40t)^2} = 100$$

$$\therefore 40\,000 - 48\,000t + 14\,400t^2 + 10\,000 - 8000t + 1400t^2 = 10\,000$$

$$\therefore 16\,000t^2 - 56\,000t + 40\,000 = 0$$

$$\therefore 16t^2 - 56t + 40 = 0 \quad \{\div 1000\}$$

$$\therefore 2t^2 - 7t + 5 = 0 \quad \{\div 8\}$$

$$\therefore (2t - 5)(t - 1) = 0$$

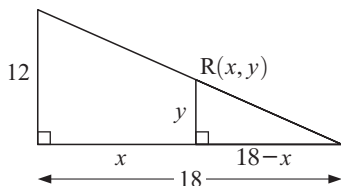
$$\therefore t = \frac{5}{2} \quad \{\text{as } t > 1\frac{3}{4}\}$$

i.e., at $2\frac{1}{2}$ hours, i.e., at 2:30 pm

- 6 a** When $x = 0$, $3y = 36$, $\therefore y = 12$ $\therefore B$ is $(0, 12)$

When $y = 0$, $2x = 36$, $\therefore x = 18$ $\therefore A$ is $(18, 0)$

b



Using the similar triangles,

$$\frac{y}{18-x} = \frac{12}{18} = \frac{2}{3}$$

$$\therefore y = \frac{2}{3}(18-x)$$

$$\therefore R \text{ is } (x, 12 - \frac{2}{3}x)$$

$$\mathbf{c} \quad \overrightarrow{PR} = \begin{bmatrix} x-4 \\ 12-\frac{2}{3}x-0 \end{bmatrix} = \begin{bmatrix} x-4 \\ 12-\frac{2}{3}x \end{bmatrix} \quad \text{and} \quad \overrightarrow{AB} = \begin{bmatrix} 0-18 \\ 12-0 \end{bmatrix} = \begin{bmatrix} -18 \\ 12 \end{bmatrix}$$

$$\mathbf{d} \quad R \text{ is located so the shortest distance } PR \text{ is when } \overrightarrow{RP} \bullet \begin{bmatrix} -18 \\ 12 \end{bmatrix} = 0$$

$$\therefore -18(x-4) + 12(12-\frac{2}{3}x) = 0$$

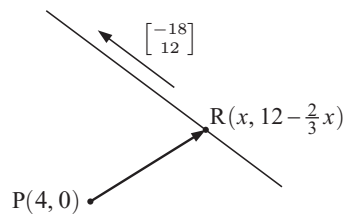
$$\therefore -18x + 72 + 144 - 8x = 0$$

$$\therefore 26x = 216$$

$$\therefore x = \frac{108}{13}$$

$$\therefore y = 12 - \frac{72}{13} = \frac{84}{13}$$

$$\text{and } y = \frac{2}{3} \left(18 - \frac{108}{13} \right)$$



$$\text{i.e., } R \text{ is at } \left(\frac{108}{13}, \frac{84}{13} \right) \quad \text{and} \quad \overrightarrow{PR} = \begin{bmatrix} \frac{108}{13}-4 \\ 12-\frac{2}{3}(\frac{108}{13}) \end{bmatrix} \div \begin{bmatrix} 4.308 \\ 6.4615 \end{bmatrix}$$

$$\therefore |\overrightarrow{PR}| \div \sqrt{(4.308)^2 + (6.4615)^2} \div 7.77 \text{ km,}$$

so the shortest distance is 7.77 km

$$\mathbf{7} \quad \mathbf{a} \quad x(0) = 10, \quad y(0) = 12 \quad \therefore P \text{ is at } (10, 12)$$

$$\mathbf{b} \quad \text{The velocity vector is } \begin{bmatrix} a \\ -3 \end{bmatrix} \quad \text{and speed} = \sqrt{a^2 + 9} = 13$$

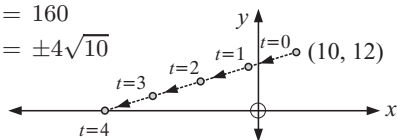
$$\therefore a^2 + 9 = 169$$

$$\therefore a^2 = 160$$

$$\therefore a = \pm 4\sqrt{10}$$

$$\mathbf{c} \quad \text{For } a < 0, \text{ i.e., } a = -4\sqrt{10},$$

$$x(t) = 10 - 4\sqrt{10}t, \quad y(t) = 12 - 3t.$$



$$\mathbf{8} \quad \text{For A, } x(t) = 3 - t, \quad y(t) = 2t - 4$$

$$\text{For B, } x(t) = 4 - 3t, \quad y(t) = 3 - 2t$$

$$\mathbf{a} \quad \text{When } t = 0, \quad x(0) = 3, \quad y(0) = -4$$

$$\therefore x(0) = 4, \quad y(0) = 3$$

$$\therefore A \text{ is at } (3, -4)$$

$$\therefore B \text{ is at } (4, 3)$$

$$\mathbf{b} \quad \text{The velocity vector of A is } \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \text{and the velocity vector of B is } \begin{bmatrix} -3 \\ -2 \end{bmatrix}.$$

$$\mathbf{c} \quad \text{If the angle is } \theta, \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \sqrt{1+4}\sqrt{9+4}\cos\theta$$

$$\therefore 3 - 4 = \sqrt{5}\sqrt{13}\cos\theta$$

$$\therefore \frac{-1}{\sqrt{65}} = \cos\theta \quad \text{and so } \theta = 97.1^\circ$$

$$\mathbf{d} \quad \text{If D is the distance between them, then}$$

$$D = \sqrt{[(4-3t) - (3-t)]^2 + [(2t-4) - (3-2t)]^2}$$

$$= \sqrt{[1-2t]^2 + [-7+4t]^2}$$

$$= \sqrt{1-4t+4t^2+49-56t+16t^2}$$

$$= \sqrt{20t^2 - 60t + 50}$$

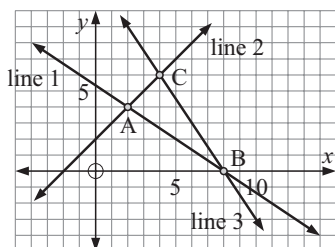
and D is a minimum when

$$t = -\frac{b}{2a} = \frac{60}{40} = 1\frac{1}{2}$$

i.e., $t = 1.5$ hours

EXERCISE 17B.4

1 a



$$\mathbf{b} \quad A \text{ is } (2, 4), \quad B \text{ is } (8, 0), \quad C \text{ is } (4, 6)$$

$$\mathbf{c} \quad BC = \sqrt{(8-4)^2 + (0-6)^2} = \sqrt{16+36} = \sqrt{52} \text{ units}$$

$$AC = \sqrt{(8-2)^2 + (0-4)^2} = \sqrt{36+16} = \sqrt{52} \text{ units}$$

$\therefore BC = AC$ and so $\triangle ABC$ is isosceles.

d Line 1 and Line 2 meet at A.

$$\therefore \begin{bmatrix} -1 \\ 6 \end{bmatrix} + \begin{bmatrix} 3r \\ -2r \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 3r-s \\ -2r-s \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$\text{i.e., } 3r - s = 1$$

$$\text{and } 2r + s = 4$$

$$\text{Adding, } \begin{array}{r} 5r = 5 \\ \hline \end{array} \therefore r = 1$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Line 1 and Line 3 meet at B.

$$\therefore \begin{bmatrix} -1 \\ 6 \end{bmatrix} + r \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 3r+2t \\ -2r-3t \end{bmatrix} = \begin{bmatrix} 11 \\ -9 \end{bmatrix}$$

$$\therefore \begin{array}{r} 3r + 2t = 11 \quad \times 3 \\ -2r - 3t = -9 \quad \times 2 \end{array}$$

$$\begin{array}{r} 3r + 2t = 11 \quad \times 3 \\ -2r - 3t = -9 \quad \times 2 \\ \hline 9r + 6t = 33 \\ -4r - 6t = -18 \\ \hline 5r = 15 \end{array}$$

$$\therefore 9r + 6t = 33$$

$$\begin{array}{r} 9r + 6t = 33 \\ -4r - 6t = -18 \\ \hline 5r = 15 \end{array}$$

$$\text{Adding, } \begin{array}{r} 5r = 15 \\ \hline \end{array}$$

$$\therefore r = 3$$

$$\text{So, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix} \quad \checkmark$$

Line 2 and Line 3 meet at C.

$$\therefore \begin{bmatrix} 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} s+2t \\ s-3t \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

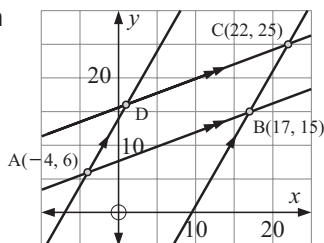
$$\text{i.e., } s + 2t = 10$$

$$-s + 3t = 5$$

$$\text{Adding, } \begin{array}{r} 5t = 15 \\ \hline \end{array} \therefore t = 3$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

2 a



b B(17, 15)

C(22, 25)

D(1, 16)

c Lines 1 and 4 meet at B

$$\therefore \begin{bmatrix} -4 \\ 6 \end{bmatrix} + r \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 25 \end{bmatrix} + u \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 7r+u \\ 3r+2u \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$$

$$\therefore \begin{array}{r} 7r + u = 26 \\ 3r + 2u = 19 \end{array}$$

$$\begin{array}{r} 7r + u = 26 \\ 3r + 2u = 19 \\ \hline -14r - 2u = -52 \\ 3r + 2u = 19 \\ \hline -11r = -33 \end{array}$$

$$\therefore -14r - 2u = -52$$

$$\begin{array}{r} 7r + u = 26 \\ 3r + 2u = 19 \\ \hline -14r - 2u = -52 \\ 3r + 2u = 19 \\ \hline -11r = -33 \end{array}$$

$$\text{Adding, } \begin{array}{r} -11r = -33 \\ \hline \end{array}$$

$$\therefore r = 3$$

$$\text{and } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix} + 3 \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ 15 \end{bmatrix} \quad \checkmark$$

Lines 2 and 3 meet at D

$$\text{and } \begin{bmatrix} -4 \\ 6 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 22 \\ 25 \end{bmatrix} + t \begin{bmatrix} -7 \\ -3 \end{bmatrix}$$

$$\therefore \begin{bmatrix} s+7t \\ 2s+3t \end{bmatrix} = \begin{bmatrix} 26 \\ 19 \end{bmatrix}$$

$$\therefore \begin{array}{r} s + 7t = 26 \\ 2s + 3t = 19 \end{array}$$

$$\begin{array}{r} s + 7t = 26 \\ 2s + 3t = 19 \\ \hline -2s - 14t = -52 \\ 2s + 3t = 19 \\ \hline -11t = -33 \end{array}$$

$$\text{Adding, } \begin{array}{r} -11t = -33 \\ \hline \end{array}$$

$$\therefore t = 3$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 22 \\ 25 \end{bmatrix} + 3 \begin{bmatrix} -7 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 16 \end{bmatrix} \quad \checkmark$$

3 a Lines 1 and 3 meet at A.

$$\therefore \begin{bmatrix} 0 \\ 2 \end{bmatrix} + r \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2r-t \\ r+t \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\therefore 2r - t = 0$$

$$r + t = 3$$

$$\text{Adding, } \begin{array}{r} 3r \\ \hline \end{array} = 3$$

$$\therefore r = 1$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\therefore$ A is (2, 3)

Lines 2 and 3 meet at C.

$$\therefore \begin{bmatrix} 8 \\ 6 \end{bmatrix} + s \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -s-t \\ -2s+t \end{bmatrix} = \begin{bmatrix} -8 \\ -1 \end{bmatrix}$$

$$\therefore -s - t = -8$$

$$-2s + t = -1$$

$$\text{Adding, } \begin{array}{r} -3s \\ \hline \end{array} = -9$$

$$\therefore s = 3$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$\therefore$ C is (5, 0)

Lines 1 and 2 meet at B.

$$\therefore \begin{bmatrix} 0 \\ 2 \end{bmatrix} + r \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} + s \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2r+s \\ r+2s \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\therefore -4r - 2s = -16$$

$$r + 2s = 4$$

$$\text{Adding, } \begin{array}{r} -3r \\ \hline \end{array} = -12$$

$$\therefore r = 4$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$\therefore$ B is (8, 6)

b A (2, 3), B(8, 6), C(5, 0)

$$AB = \sqrt{(8-2)^2 + (6-3)^2}$$

$$= \sqrt{36+9}$$

$$= \sqrt{45}$$

$$BC = \sqrt{(5-8)^2 + (0-6)^2}$$

$$= \sqrt{9+36}$$

$$= \sqrt{45}$$

The two equal sides are AB and BC and they have length $\sqrt{45}$ units.

4 a Lines QP and PR meet at P

$$\therefore \begin{bmatrix} 3 \\ -1 \end{bmatrix} + r \begin{bmatrix} 14 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \end{bmatrix} + t \begin{bmatrix} 5 \\ -7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 14r-5t \\ 10r+7t \end{bmatrix} = \begin{bmatrix} -3 \\ 19 \end{bmatrix}$$

$$\therefore 14r - 5t = -3 \quad \times 7$$

$$10r + 7t = 19 \quad \times 5$$

$$\therefore 98r - 35t = -21$$

$$50r + 35t = 95$$

$$\text{Adding, } \begin{array}{r} 148r \\ \hline \end{array} = 74$$

$$\therefore r = \frac{1}{2}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 14 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$$

$\therefore$ P is (10, 4)

Lines QP and PR meet at Q

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} + r \begin{bmatrix} 14 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 17 \\ -9 \end{bmatrix}$$

$$\therefore r \begin{bmatrix} 14 \\ 10 \end{bmatrix} = s \begin{bmatrix} 17 \\ -9 \end{bmatrix}$$

$$\therefore r = s = 0$$

$$\text{So, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$\therefore$ Q is (3, -1)

Lines QR and PR meet at R

$$\therefore \begin{bmatrix} 3 \\ -1 \end{bmatrix} + s \begin{bmatrix} 17 \\ -9 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \end{bmatrix} + t \begin{bmatrix} 5 \\ -7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 17s-5t \\ -9s+7t \end{bmatrix} = \begin{bmatrix} -3 \\ 19 \end{bmatrix}$$

$$\therefore 17s - 5t = -3 \quad \times 7$$

$$-9s + 7t = 19 \quad \times 5$$

$$\therefore 119s - 35t = -21$$

$$-45s + 35t = 95$$

$$\text{Adding, } \begin{array}{r} 74s \\ \hline \end{array} = 74$$

$$\therefore s = 1$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 17 \\ -9 \end{bmatrix} = \begin{bmatrix} 20 \\ -10 \end{bmatrix}$$

$\therefore$ R is (20, -10)

$$\mathbf{b} \quad \overrightarrow{PQ} = \begin{bmatrix} 3-10 \\ -1-4 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

$$\overrightarrow{PR} = \begin{bmatrix} 20-10 \\ -10-4 \end{bmatrix} = \begin{bmatrix} 10 \\ -14 \end{bmatrix}$$

$$\text{and } \overrightarrow{PQ} \bullet \overrightarrow{PR} = -70 + 70 = 0$$

$$\mathbf{c} \quad PQ \perp PR \quad \text{i.e., } \angle QPR = 90^\circ$$

$$\begin{aligned} \mathbf{d} \quad \text{Area} &= \frac{1}{2} | \overrightarrow{PQ} \times \overrightarrow{PR} | \\ &= \frac{1}{2} \sqrt{49 + 25} \sqrt{100 + 196} \\ &= 74 \text{ units}^2 \end{aligned}$$

5 a Lines 1 and 2 meet at B

$$\begin{aligned}\therefore \begin{bmatrix} 2 \\ 5 \end{bmatrix} + r \begin{bmatrix} 4 \\ 1 \end{bmatrix} &= \begin{bmatrix} 18 \\ 9 \end{bmatrix} + s \begin{bmatrix} -8 \\ 32 \end{bmatrix} \\ \therefore \begin{bmatrix} 4r+8s \\ r-32s \end{bmatrix} &= \begin{bmatrix} 16 \\ 4 \end{bmatrix} \\ \therefore \begin{aligned} 4r+8s &= 16 \\ r-32s &= 4 \end{aligned} \\ \therefore \begin{aligned} r+2s &= 4 \\ -r+32s &= -4 \end{aligned} \\ \text{Adding, } \frac{34s}{\therefore s} &= \frac{-4}{0} \\ \therefore s &= 0 \\ \therefore \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 18 \\ 9 \end{bmatrix} \quad \therefore \text{ B is } (18, 9)\end{aligned}$$

Lines 2 and 3 meet at C

$$\begin{aligned}\therefore \begin{bmatrix} 18 \\ 9 \end{bmatrix} + s \begin{bmatrix} -8 \\ 32 \end{bmatrix} &= \begin{bmatrix} 14 \\ 25 \end{bmatrix} + t \begin{bmatrix} -8 \\ -2 \end{bmatrix} \\ \therefore \begin{bmatrix} -8s+8t \\ 32s+2t \end{bmatrix} &= \begin{bmatrix} -4 \\ 16 \end{bmatrix} \\ \begin{aligned} 2s-2t &= 1 \\ 32s+2t &= 16 \end{aligned} \\ \text{Adding, } \frac{34s}{\therefore s} &= \frac{17}{\frac{1}{2}} \\ \therefore s &= \frac{1}{2} \\ \therefore \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 18 \\ 9 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -8 \\ 32 \end{bmatrix} = \begin{bmatrix} 14 \\ 25 \end{bmatrix} \\ \therefore \text{ C is } (14, 25)\end{aligned}$$

Lines 3 and 4 meet at D

$$\begin{aligned}\therefore \begin{bmatrix} 14 \\ 25 \end{bmatrix} + t \begin{bmatrix} -8 \\ -2 \end{bmatrix} &= \begin{bmatrix} 3 \\ 1 \end{bmatrix} + u \begin{bmatrix} -3 \\ 12 \end{bmatrix} \\ \therefore \begin{bmatrix} -8t+3u \\ -2t-12u \end{bmatrix} &= \begin{bmatrix} -11 \\ -24 \end{bmatrix} \\ \therefore \begin{aligned} -8t+3u &= -11 \\ t+6u &= 12 \end{aligned} \\ \therefore \begin{aligned} 16t-6u &= 22 \\ t+6u &= 12 \end{aligned} \\ \text{Adding, } \frac{17t}{\therefore t} &= \frac{34}{2} \\ \therefore t &= 2 \\ \therefore \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 14 \\ 25 \end{bmatrix} + 2 \begin{bmatrix} -8 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 21 \end{bmatrix} \\ \therefore \text{ D is } (-2, 21)\end{aligned}$$

b $\overrightarrow{AC} = \begin{bmatrix} 14-2 \\ 25-5 \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \end{bmatrix}$

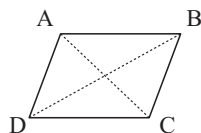
$\overrightarrow{DB} = \begin{bmatrix} 18-21 \\ 9-21 \end{bmatrix} = \begin{bmatrix} -3 \\ -12 \end{bmatrix}$

i $|\overrightarrow{AC}| = \sqrt{12^2 + 20^2} = \sqrt{544}$ units

ii $|\overrightarrow{DB}| = \sqrt{20^2 + (-12)^2} = \sqrt{544}$ units

iii $\overrightarrow{AC} \bullet \overrightarrow{DB} = 240 - 240 = 0$

- c** The diagonals are perpendicular and equal in length, so, ABCD is a square.


EXERCISE 17C

- 1 a** Line 1 has direction vector $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

Now $\begin{aligned} 1+2t &= -2+3s & 2-t &= 3-s & 3+t &= 1+2s \\ \therefore 2t-3s &= -3 \dots\dots (1) & \therefore -t+s &= 1 \dots\dots (2) & \therefore t-2s &= -2 \dots\dots (3) \end{aligned}$

Solving (2) and (3) simultaneously: $\begin{aligned} -t+s &= 1 \\ t-2s &= -2 \\ \hline -s &= -1 \end{aligned} \therefore s = 1 \text{ and } t = 0$

and in (1), $\text{LHS} = 2t - 3s = 2(0) - 3(1) = -3 \checkmark$

$\therefore s = 1, t = 0$ satisfies all three equations

$\therefore$ the two lines meet at $(1, 2, 3)$ {using $t = 0$ or $s = 1$ }

The acute angle between the lines has $\cos \theta = \frac{|6+1+2|}{\sqrt{4+1+1}\sqrt{9+1+4}} = \frac{9}{\sqrt{84}}$
and so $\theta \doteq 10.9^\circ$

- b** Line 1 has direction vector $\begin{bmatrix} 2 \\ -12 \\ 12 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{array}{lll}
 \text{Now } -1 + 2t = 4s - 3 & 2 - 12t = 3s + 2 & 4 + 12t = -s - 1 \\
 \therefore 2t - 4s = -2 & -12t - 3s = 0 & 12t + s = -5 \dots\dots (3) \\
 \therefore t - 2s = -1 \dots\dots (1) & s = -4t \dots\dots (2) &
 \end{array}$$

$$\begin{array}{l}
 \text{Solving (1) and (2) simultaneously: } t - 2(-4t) = -1 \\
 \therefore 9t = -1 \\
 \therefore t = -\frac{1}{9} \text{ and so } s = \frac{4}{9}
 \end{array}$$

$$\text{In (3), } 12t + s = 12\left(-\frac{1}{9}\right) + \frac{4}{9} = -\frac{12}{9} + \frac{4}{9} = -\frac{8}{9}, \text{ which is not } -5.$$

Since the system is inconsistent, the lines don't intersect. $\therefore$ they are skew.

$$\text{The acute angle between the lines has } \cos \theta = \frac{|8 - 36 - 12|}{\sqrt{292} \sqrt{26}} = \frac{40}{\sqrt{7592}} \text{ and so } \theta \doteq 62.7^\circ$$

c Line 1 has direction vector $\begin{bmatrix} 6 \\ 8 \\ 2 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

$$\text{As } \begin{bmatrix} 6 \\ 8 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \text{ the two lines are parallel. Hence, } \theta = 0^\circ.$$

d Line 1 has direction vector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$.

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{array}{lll}
 \text{Now } t = 1 + 3s \dots\dots (1) & 2 - t = -2 - 2s & -2 + t = 2s + \frac{1}{2} \\
 \therefore -t + 2s = -4 \dots\dots (2) & t - 2s = 2\frac{1}{2} \dots\dots (3) &
 \end{array}$$

$$\begin{array}{l}
 \text{Solving (1) and (2) simultaneously, } -(1 + 3s) + 2s = -4 \\
 \therefore -1 - 3s + 2s = -4 \\
 \therefore -s = -3 \\
 \therefore s = 3 \text{ and so } t = 1 + 3(3) = 10
 \end{array}$$

$$\text{Substituting in (3), } t - 2s = 10 - 2(3) = 4 \neq 2\frac{1}{2}$$

Since the system is inconsistent, the lines do not meet. $\therefore$ they are skew.

$$\begin{array}{l}
 \text{The acute angle between the lines has } \cos \theta = \frac{|3 + 2 + 2|}{\sqrt{1+1+1}\sqrt{9+4+4}} = \frac{7}{\sqrt{3}\sqrt{17}} \\
 \therefore \theta \doteq 11.4^\circ
 \end{array}$$

e Line 1 has direction vector $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

As one vector is not a scalar multiple of the other, the lines are not parallel.

$$\begin{array}{lll}
 1 + t = 2 + 3s & 2 - t = 3 - 2s & 3 + 2t = s - 5 \\
 t - 3s = 1 \dots\dots (1) & -t + 2s = 1 \dots\dots (2) & 3t - s = -8 \dots\dots (3)
 \end{array}$$

$$\begin{array}{l}
 \text{Solving (1) and (2) simultaneously, } t - 3s = 1 \\
 -t + 2s = 1 \\
 \text{Adding, } -s = 2 \\
 \therefore s = -2 \text{ and } t - 3(-2) = 1 \therefore t = -5
 \end{array}$$

$$\text{Checking in (3), } 2t - s = 2(-5) - (-2) = -10 + 2 = -8 \quad \checkmark$$

Since $s = -2$, $t = -5$ satisfies all three equations, the lines meet.

$$\text{They meet at } x = 1 + (-5), \quad y = 2 - (-5), \quad z = 3 + 2(-5) \text{ i.e., at } (-4, 7, -7)$$

$$\begin{aligned} \text{The acute angle between the lines has } \cos \theta &= \frac{|3+2+2|}{\sqrt{1+1+4}\sqrt{9+4+1}} \\ &= \frac{7}{\sqrt{84}} \quad \text{and so } \theta \doteq 40.2^\circ \end{aligned}$$

f Line 1 has direction vector $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and Line 2 has direction vector $\begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}$

Now $\begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, so the lines are parallel and hence $\theta = 0^\circ$.

2 $3x - y = 8 \quad \times 2 \quad \text{i.e., } 6x - 2y = 16 \quad \text{which are parallel as the slope in}$
 $6x - 2y = k \quad - \quad \quad \quad 6x - 2y = k \quad \text{each case is 3.}$

(1) If $k \neq 16$, the lines are parallel, $\therefore$ no solutions exist.

(2) If $k = 16$, the lines are coincident and so have infinitely many solutions of the form $x = t$, $y = 3t - 8$, $t \in \mathcal{R}$.

3 $4x + 8y = 1 \quad - \quad \text{i.e., } 4x + 8y = 1$
 $2x - ay = 11 \quad \times 2 \quad \quad \quad 4x - 2ay = 22$

(1) If $-2a = 8$, i.e., $a = -4$, the equations are $\begin{cases} 4x + 8y = 1 \\ 4x + 8y = 22 \end{cases}$

The lines are parallel, $\therefore$ no solutions exist.

(2) If $a \neq -4$, the lines are not parallel or coincident.

So, a unique solution exists.

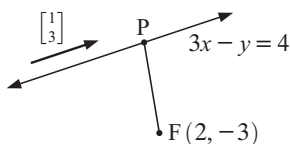
$$\begin{aligned} \left[\begin{array}{cc|c} 4 & 8 & 1 \\ 2 & -a & 11 \end{array} \right] &\sim \left[\begin{array}{cc|c} 2 & 4 & \frac{1}{2} \\ -2 & a & -11 \end{array} \right] & \begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow -R_1 \end{array} \\ &\sim \left[\begin{array}{cc|c} 2 & 4 & \frac{1}{2} \\ 0 & a+4 & -10\frac{1}{2} \end{array} \right] \end{aligned}$$

The second equation is $(a+4)y = -\frac{21}{2} \quad \therefore y = \frac{-21}{2(a+4)}$

$$\begin{aligned} \text{and } 4x + 8\left(\frac{-21}{2(a+4)}\right) &= 1 \quad \therefore 4x - \frac{84}{a+4} = 1 \\ \therefore 4x &= 1 + \frac{84}{a+4} = \frac{a+88}{a+4} \\ \therefore x &= \frac{a+88}{4(a+4)} \end{aligned}$$

$\therefore$ the unique solution is $x = \frac{a+88}{4(a+4)}, y = \frac{-21}{2(a+4)}, a \neq -4$

4



$3x - y = 4 \quad \text{i.e., } y = 3x - 4 \quad \text{has slope } 3 = \frac{3}{1}$
 $\therefore$ direction vector is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

If $x = t$, $y = 3t - 4$

So $P(t, 3t - 4)$ is a general point on the line.

Now $\overrightarrow{FP} = \begin{bmatrix} t-2 \\ 3t-4-(-3) \end{bmatrix} = \begin{bmatrix} t-2 \\ 3t-1 \end{bmatrix}$

But for shortest distance $\vec{FP} \perp \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\therefore \begin{bmatrix} t-2 \\ 3t-1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0$$

$$\therefore t-2+9t-3=0$$

$$\therefore 10t=5$$

$$\therefore t = \frac{1}{2}$$

$$\begin{aligned} \therefore \vec{FP} &= \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix} \quad \text{and} \quad |\vec{FP}| = \sqrt{\frac{9}{4} + \frac{1}{4}} \\ &= \sqrt{\frac{10}{4}} \\ &= \frac{1}{2}\sqrt{10} \text{ units} \end{aligned}$$

Thus, the shortest distance is $\frac{1}{2}\sqrt{10}$ units.

- 5** Let A $(2+3\lambda, -1+2\lambda, 4+\lambda)$ be a point on the line such that $\vec{PA}$ is perpendicular to the line.

$$\text{Now } \vec{PA} = \begin{bmatrix} 2+3\lambda-3 \\ -1+2\lambda-0 \\ 4+\lambda-(-1) \end{bmatrix} = \begin{bmatrix} 3\lambda-1 \\ 2\lambda-1 \\ \lambda+5 \end{bmatrix}$$

and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is the direction vector of the line.

$$\therefore \text{since } \vec{PA} \bullet \mathbf{v} = 0,$$

$$\begin{bmatrix} 3\lambda-1 \\ 2\lambda-1 \\ \lambda+5 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 0$$

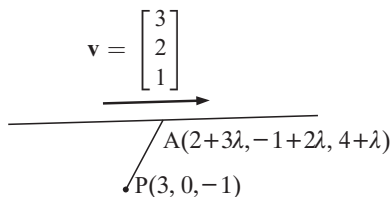
$$\therefore 3(3\lambda-1) + 2(2\lambda-1) + 1(\lambda+5) = 0$$

$$\therefore 9\lambda-3+4\lambda-2+\lambda+5=0$$

$$\therefore 14\lambda=0$$

$$\therefore \lambda=0$$

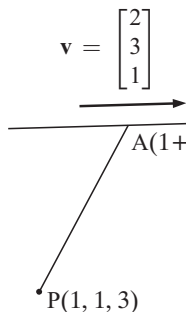
$$\begin{aligned} \therefore \text{A is at } (2, -1, 4) \quad \text{and the distance } d &= \sqrt{(2-3)^2 + (-1-0)^2 + (4-(-1))^2} \\ &= \sqrt{1+1+25} \\ &= \sqrt{27} \text{ i.e., } 3\sqrt{3} \text{ units} \end{aligned}$$



- 6** Let A be a point on the line such that $\vec{PA}$ is perpendicular to the line.

Then A is at $(1+2\lambda, -1+3\lambda, 2+\lambda)$ for some λ .

$$\begin{aligned} \text{Now } \vec{PA} &= \begin{bmatrix} 1+2\lambda-1 \\ -1+3\lambda-1 \\ 2+\lambda-3 \end{bmatrix} = \begin{bmatrix} 2\lambda \\ 3\lambda-2 \\ \lambda-1 \end{bmatrix} \\ \text{and } \mathbf{v} &= \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \text{ is the direction vector of the line.} \end{aligned}$$



$$\therefore \text{since } \vec{PA} \bullet \mathbf{v} = 0,$$

$$\begin{bmatrix} 2\lambda \\ 3\lambda-2 \\ \lambda-1 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 0$$

$$\therefore 4\lambda + 3(3\lambda-2) + 1(\lambda-1) = 0$$

$$\therefore 4\lambda + 9\lambda - 6 + \lambda - 1 = 0$$

$$\therefore 14\lambda = 7$$

$$\therefore \lambda = \frac{1}{2}$$

$$\therefore \text{A is at } (2, \frac{1}{2}, \frac{5}{2})$$

$$\begin{aligned} \therefore \text{the distance } d &= \sqrt{(2-1)^2 + (\frac{1}{2}-1)^2 + (\frac{5}{2}-3)^2} \\ &= \sqrt{1 + \frac{1}{4} + \frac{1}{4}} \\ &= \sqrt{\frac{3}{2}} \text{ units} \end{aligned}$$

- 7 a** Line 1 has direction vector $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, and letting $t = 0$, it contains the point $A(1, 0, 2)$.

Line 2 has direction vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and contains the point $B(0, 0, 0)$, $\overrightarrow{AB} = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$

$$\text{and } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = \begin{bmatrix} -4 \\ 1 \\ 3 \end{bmatrix}$$

$$\therefore \text{ the distance } d = \frac{|\overrightarrow{AB} \bullet (\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|} = \frac{|4 + 0 - 6|}{\sqrt{(-4)^2 + 1^2 + 3^2}} = \frac{2}{\sqrt{26}} \text{ units}$$

- b** Line 1 contains $A(1, 1, 3)$ and has direction vector $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$.

Line 2 contains $B(2, 1, 0)$ and has direction vector $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\overrightarrow{AB} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

$$\text{and } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -1 \\ 1 & -2 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} \\ = -\mathbf{i} + \mathbf{k}$$

$$\therefore d = \frac{|\overrightarrow{AB} \bullet (\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|} = \frac{|1 \times -1 + 0 \times 0 + -3 \times 1|}{\sqrt{(-1)^2 + 0^2 + 1^2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \text{ units}$$

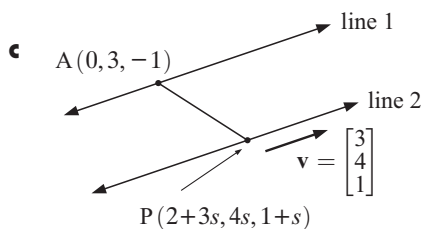
- 8 a** The lines intersect $\therefore$ shortest distance = 0 units.

- b** Line 1 contains $A(-1, 2, 4)$ and has direction vector $\mathbf{u} = \begin{bmatrix} 2 \\ -12 \\ 12 \end{bmatrix}$.

Line 2 contains $B(-3, 2, -1)$ and has $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$, $\overrightarrow{AB} = \begin{bmatrix} -3 - -1 \\ 2 - 2 \\ -1 - 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -5 \end{bmatrix}$

$$\text{and } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -12 & 12 \\ 4 & 3 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -12 & 12 \\ 3 & -1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 12 & 2 \\ -1 & 4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -12 \\ 4 & 3 \end{vmatrix} \\ = -24\mathbf{i} + 50\mathbf{j} + 54\mathbf{k}$$

$$\therefore \text{ shortest distance is } \frac{|\overrightarrow{AB} \bullet (\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|} = \frac{|(-2)(-24) + (0)(50) + (-5)(54)|}{\sqrt{(-24)^2 + 50^2 + 54^2}} = \frac{222}{\sqrt{5992}} \\ \div 2.87 \text{ units}$$



$$\overrightarrow{AP} = \begin{bmatrix} 2 + 3s - 0 \\ 4s - 3 \\ 1 + s - -1 \end{bmatrix} = \begin{bmatrix} 2 + 3s \\ 4s - 3 \\ s + 2 \end{bmatrix}$$

and for the shortest distance $\overrightarrow{AP} \bullet \mathbf{v} = 0$

$$\therefore 3(2 + 3s) + 4(4s - 3) + 1(s + 2) = 0$$

$$\therefore 6 + 9s + 16s - 12 + s + 2 = 0$$

$$\therefore 26s = 4$$

$$\therefore s = \frac{2}{13}$$

$$\therefore \overrightarrow{AP} = \begin{bmatrix} \frac{32}{13} \\ -\frac{31}{13} \\ \frac{28}{13} \end{bmatrix} \text{ and } |\overrightarrow{AP}| = \sqrt{\left(\frac{32}{13}\right)^2 + \left(-\frac{31}{13}\right)^2 + \left(\frac{28}{13}\right)^2} \div 4.05 \text{ units}$$

- d** Line 1 contains $A(0, 2, -2)$ and has direction vector $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Line 2 contains $B(1, -2, \frac{1}{2})$ and has direction vector $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$, $\overrightarrow{AB} = \begin{bmatrix} 1-0 \\ -2-2 \\ \frac{1}{2}-(-2) \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ \frac{5}{2} \end{bmatrix}$

$$\text{and } \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 3 & -2 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix}$$

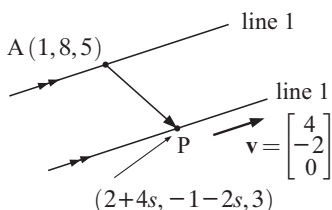
$$= 0\mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\therefore \text{shortest distance is } \frac{|\overrightarrow{AB} \bullet (\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|} = \frac{|(1)(0) + (-4)(1) + (\frac{5}{2})(1)|}{\sqrt{0^2 + 1^2 + 1^2}} = \frac{1.5}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{3\sqrt{2}}{4} \text{ units}$$

- e** The lines intersect $\therefore$ shortest distance = 0 units.

- f** The lines are parallel.



$$\overrightarrow{AP} = \begin{bmatrix} 2+4s-1 \\ -1-2s-8 \\ 3-5 \end{bmatrix} = \begin{bmatrix} 1+4s \\ -2s-9 \\ -2 \end{bmatrix}$$

and for the shortest distance

$$\overrightarrow{AP} \bullet \mathbf{v} = 0$$

$$\therefore (1+4s)(4) + (-2s-9)(-2) + 0 = 0$$

$$\therefore 4 + 16s + 4s + 18 = 0$$

$$\therefore 20s = -22 \text{ and so } s = -1.1$$

$$\text{Thus } \overrightarrow{AP} = \begin{bmatrix} -3.4 \\ -6.8 \\ 2 \end{bmatrix} \text{ and } |\overrightarrow{AP}| = \sqrt{(-3.4)^2 + (-6.8)^2 + 2^2} \div 7.86 \text{ units}$$

EXERCISE 17D

- 1 a** Since $\mathbf{n} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $(-1, 2, 4)$ lies on the plane, the equation is

$$2x - y + 3z = 2(-1) - 2 + 3(4) \quad \text{i.e., } 2x - y + 3z = 8$$

- b** A vector normal to the plane is $\overrightarrow{AB} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$, and $(2, 3, 1)$ lies on the plane.

$$\therefore 3x + 4y + z = 3(2) + 4(3) + 1$$

$$\text{i.e., } 3x + 4y + z = 19$$

- c**
 $A(1, 4, 2) \quad P(a, b, c) \quad B(4, 1, -4)$

$$\text{Now } 2\overrightarrow{AP} = \overrightarrow{PB}$$

$$\therefore \text{ if } P \text{ has coordinates } (a, b, c), \text{ then } 2 \begin{bmatrix} a-1 \\ b-4 \\ c-2 \end{bmatrix} = \begin{bmatrix} 4-a \\ 1-b \\ -4-c \end{bmatrix}$$

$$\therefore 2a-2 = 4-a \quad \text{and} \quad 2b-8 = 1-b \quad \text{and} \quad 2c-4 = -4-c$$

$$\therefore 3a = 6$$

$$3b = 9$$

$$3c = 0$$

$$\therefore a = 2$$

$$b = 3$$

$$c = 0$$

$$\therefore P \text{ is at } (2, 3, 0)$$

$$\overrightarrow{AB} = \begin{bmatrix} 3 \\ -3 \\ -6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \text{ so } \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \text{ is a normal vector to the plane.}$$

$$\therefore \text{ the plane is } x - y - 2z = 2 - 3 - 2(0) \quad \text{i.e., } x - y - 2z = -1$$

- d** The line $x = 1 + t$, $y = 2 - t$, $z = 3 + 2t$ has direction vector $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

Also, letting $t = 0$, the point $(1, 2, 3)$ lies on the plane and we call this point B.

$$\therefore \overrightarrow{AB} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \text{ and so a vector normal to the plane is } \overrightarrow{AB} \times \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \text{i.e., } \mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 2 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} \\ &= 2\mathbf{i} + 6\mathbf{j} + 2\mathbf{k} \text{ or } 2(\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \end{aligned}$$

$$\therefore \text{ since } A(3, 2, 1) \text{ lies on the plane, it has equation } x + 3y + z = 3 + 3(2) + 1$$

$$\text{i.e., } x + 3y + z = 10$$

2 a $2x + 3y - z = 8$ has $\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ **b** $3x - y + 0z = 11$ has $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$

c $0x + 0y + z = 2$ has $\mathbf{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ **d** $1x + 0y + 0z = 0$ has $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

3 a The y -axis is perpendicular to the XOZ plane $\therefore$ a normal vector is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore$ since the origin lies on the plane, it has equation $y = 0$.

b Since the plane is perpendicular to the Z -axis, it has normal vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ since $(2, -1, 4)$ lies on the plane, it has equation $z = 4$.

- 4 a** If $\mathbf{n}$ is the normal vector, then

$$\begin{aligned} \mathbf{n} &= \overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -4 \\ -1 & 0 & -2 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} 1 & -4 \\ 0 & -2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -4 & 1 \\ -2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = -2\mathbf{i} + 6\mathbf{j} + \mathbf{k} \end{aligned}$$

$\therefore$ since $A(0, 2, 6)$ lies on the plane, it has equation

$$-2x + 6y + z = -2(0) + 6(2) + 6$$

$$\text{i.e., } -2x + 6y + z = 18$$

- b** If $\mathbf{n}$ is the normal vector, then

$$\begin{aligned} \mathbf{n} &= \overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix} \times \begin{bmatrix} -3 \\ -1 \\ -1 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3 & -2 \\ -3 & -1 & -1 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} 3 & -2 \\ -1 & -1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -2 & -3 \\ -1 & -3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -3 & 3 \\ -3 & -1 \end{vmatrix} = -5\mathbf{i} + 3\mathbf{j} + 12\mathbf{k} \end{aligned}$$

$\therefore$ since $C(0, 0, 1)$ lies on the plane, it has equation $-5x + 3y + 12z = 12$.

- c** If $\mathbf{n}$ is the normal vector, then

$$\begin{aligned} \mathbf{n} &= \overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & -1 \\ 2 & -3 & -3 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} -1 & -1 \\ -3 & -3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & -2 \\ -3 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & -1 \\ 2 & -3 \end{vmatrix} = -8\mathbf{j} + 8\mathbf{k} \text{ or } -8(\mathbf{j} - \mathbf{k}) \end{aligned}$$

$\therefore$ since $A(2, 0, 3)$ lies on the plane, it has equation $y - z = -3$

- 5 a** The normal to $x - 3y + 4z = 8$ is $\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, and this is the direction vector of the line.

$\therefore$ since the line passes through $(1, -2, 0)$, it has equation

$$x = 1 + t, \quad y = -2 - 3t, \quad z = 4t, \quad t \text{ in } \mathcal{R}.$$

- b** The normal to $x - y - 2z = 11$ is $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$.

$\therefore$ since the line passes through $(3, 4, -1)$, it has equation

$$x = 3 + t, \quad y = 4 - t, \quad z = -1 - 2t, \quad t \text{ in } \mathcal{R}.$$

- 6** The line has direction vector $\overrightarrow{AB} = \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix}$

$\therefore$ since the line passes through A , it has parametric equations

$$x = 2 - t, \quad y = -1 + 3t, \quad z = 3 - 3t, \quad t \text{ in } \mathcal{R}.$$

This line meets $x + 2y - z = 5$ when $(2 - t) + 2(-1 + 3t) - (3 - 3t) = 5$

$$\therefore 2 - t - 2 + 6t - 3 + 3t = 5$$

$\therefore 8t = 8 \quad \therefore t = 1,$ and so they meet at $(1, 2, 0)$

- 7** The direction vector of the line is $\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$.

$\therefore$ since it passes through $P(1, -2, 4)$, it has parametric equations

$$x = 1 + t, \quad y = -2 + 2t, \quad z = 4 - 5t, \quad t \text{ in } \mathcal{R}.$$

- a** The line meets the YOZ plane when $x = 0$ i.e., when $t = -1$.

This corresponds to the point $(0, -4, 9)$

- b** The line meets $y + z = 2$ when $-2 + 2t + 4 - 5t = 2 \quad \therefore -3t = 0 \quad \therefore t = 0$

This corresponds to the point $(1, -2, 4)$

- c** The line meets $\frac{x-3}{2} = \frac{y+2}{3} = \frac{z-30}{-1}$ when $\frac{1+t-3}{2} = \frac{-2+2t+2}{3} = \frac{4-5t-30}{-1}$

$$\therefore \frac{t-2}{2} = \frac{2t}{3} = 5t + 26$$

$$\therefore 3t - 6 = 4t = 30t + 156$$

$$\therefore 3t - 6 = 4t \quad \text{and} \quad 4t = 30t + 156$$

$$\therefore t = -6 \quad \text{and} \quad -26t = 156$$

$$\therefore t = -6 \quad \text{is a common solution.}$$

$\therefore$ the lines meet at the point corresponding to $t = -6$, which is $(-5, -14, 34)$.

- 8 a** The plane $2x + y - 2z = -11$ has $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$\therefore$ the parametric equations of AN are $x = 1 + 2t, \quad y = 0 + t, \quad z = 2 - 2t, \quad t \text{ in } \mathcal{R}.$

This line meets the plane when $2(1 + 2t) + t - 2(2 - 2t) = -11$

$$\therefore 2 + 4t + t - 4 + 4t = -11$$

$$\therefore 9t = -9$$

$$\therefore t = -1$$

$$\text{Thus } N \text{ is } (-1, -1, 4) \text{ and } \therefore \overrightarrow{AN} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \text{ and } AN = \sqrt{(-2)^2 + (-1)^2 + 2^2} \\ = \sqrt{9} \\ = 3 \text{ units}$$

b The plane $x - y + 3z = -10$ has $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

$\therefore$ the parametric equations of AN are
 $x = 2 + t, y = -1 - t, z = 3 + 3t, t \text{ in } \mathcal{R}$

$$\text{This line meets the plane when } (2 + t) - (-1 - t) + 3(3 + 3t) = -10 \\ \therefore 2 + t + 1 + t + 9 + 9t = -10 \\ \therefore 11t = -22 \\ \therefore t = -2$$

$\therefore N$ is $(0, 1, -3)$

$$\therefore \overrightarrow{AN} = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix} \text{ and } AN = \sqrt{(-2)^2 + 2^2 + (-6)^2} = \sqrt{44} = 2\sqrt{11} \text{ units}$$

c The plane $4x - y - 2z = 8$ has $\mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$

$\therefore$ the parametric equations of AN are
 $x = 1 + 4t, y = -4 - t, z = -3 - 2t, t \text{ in } \mathcal{R}$

$$\text{This line meets the plane when } 4(1 + 4t) - (-4 - t) - 2(-3 - 2t) = 8 \\ \therefore 4 + 16t + 4 + t + 6 + 4t = 8 \\ \therefore 21t = -6 \\ \therefore t = -\frac{2}{7}$$

$$\therefore N \text{ is } \left(-\frac{1}{7}, -\frac{26}{7}, -\frac{17}{7}\right), \therefore \overrightarrow{AN} = \begin{bmatrix} -\frac{8}{7} \\ \frac{2}{7} \\ \frac{4}{7} \end{bmatrix} \text{ and so } AN = \sqrt{\left(-\frac{8}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{4}{7}\right)^2} \\ = \sqrt{\frac{84}{49}} \\ = 2\sqrt{\frac{3}{7}} \text{ units}$$

9 The mirror image lies on the normal line to the plane through the object point.

$$\text{Now } x + 2y + z = 1 \text{ has } \mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ the normal at A has parametric equations $x = 3 + t, y = 1 + 2t, z = 2 + t$

$$\text{The line meets the plane when } (3 + t) + 2(1 + 2t) + 2 + t = 1 \\ \therefore 3 + t + 2 + 4t + 2 + t = 1 \\ \therefore 6t = -6 \\ \therefore t = -1$$

$\therefore N$ is $(2, -1, 1)$

If A' is the mirror image of A, then $\overrightarrow{AN} = \overrightarrow{NA'}$

$$\therefore \text{letting } A' \text{ have coordinates } (a, b, c), \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} a - 2 \\ b + 1 \\ c - 1 \end{bmatrix}$$

$\therefore a - 2 = -1, b + 1 = -2, c - 1 = -1$ and so A' is at $(1, -3, 0)$.

- 10** The plane $x + 4y - z = -2$ has normal $\mathbf{n} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$ which passes through $(3, 4, -1)$.

$\therefore$ the normal has parametric equations: $x = 3 + t$, $y = 4 + 4t$, $z = -1 - t$
and will meet any of the coordinate axes if any two of the values of x , y , and z are zero at the same time

$\therefore$ since $x = 0$ when $t = -3$ and $y = z = 0$ when $t = -1$, the normal meets the X -axis when $t = -1$, i.e., at the point $(2, 0, 0)$.

- 11 a** Suppose the plane has normal $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Since the plane is parallel to the X -axis, $\mathbf{n} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \quad \therefore \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$

$\therefore a = 0$ and the plane has normal $\mathbf{n} = \begin{bmatrix} 0 \\ b \\ c \end{bmatrix}$

$\therefore$ the plane has equation $by + cz = d$

But A and B lie on the plane $\therefore 2b + 3c = d$ and $-b + 2c = d$

$$\therefore 2b + 3c = -b + 2c$$

$$\therefore c = -3b \quad \text{and so} \quad \mathbf{n} = \begin{bmatrix} 0 \\ b \\ -3b \end{bmatrix} = b \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

$\therefore$ a normal is $\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$, and the plane has equation $y - 3z = d$.

Using $A(1, 2, 3)$, $d = -7$ and the equation is $y - 3z = -7$

- b** Suppose $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Since the plane is parallel to the Y -axis, $\mathbf{n} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$

$\therefore b = 0$ and so $\mathbf{n} = \begin{bmatrix} a \\ 0 \\ c \end{bmatrix}$ and the plane has equation $ax + cz = d$

But A and B lie on the plane.

$$\therefore a + 3c = d \quad \text{and} \quad 2c = d$$

$$\therefore a + 3c = 2c$$

$$\therefore c = -a \quad \text{and} \quad \mathbf{n} = \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix}$$

$\therefore$ a normal is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, so the plane has equation $x - z = d$

Using $A(1, 2, 3)$, $d = -2$ and so the equation is $x - z = -2$

- c** Suppose $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Since the plane is parallel to the Z -axis, $\mathbf{n} \bullet \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$

$\therefore c = 0$ and so $\mathbf{n} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ and the plane has equation $ax + by = d$.

But A and B lie on the plane.

$$\therefore a + 2b = d \quad \text{and} \quad -b = d$$

$$\therefore a + 2b = -b$$

$$\therefore a = -3b \quad \text{and} \quad \mathbf{n} = \begin{bmatrix} -3b \\ b \\ 0 \end{bmatrix}$$

$\therefore$ a normal is $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$, so the plane has equation $-3x + y = d$.

Using $A(1, 2, 3)$, $d = -1$, $\therefore$ the equation is $-3x + y = -1$
i.e., $3x - y = 1$

12 Now $x - 1 = \frac{y-2}{2} = z + 3$ has direction vector $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

and $x + 1 = y - 3 = 2z + 5$ has direction vector $\begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{bmatrix}$ {since $2z + 5 = \frac{z + \frac{5}{2}}{\frac{1}{2}}$ }

$\therefore$ a vector perpendicular to both lines is:

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & 1 & \frac{1}{2} \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{vmatrix} + \mathbf{j} \begin{vmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$= 0\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$$

$\therefore \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ is perpendicular to both lines.

A plane with normal $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ has equation $y - 2z = c$ for some c .

Now for line 1, $\frac{y-2}{2} = z + 3$

$$\therefore y - 2 = 2z + 6$$

$$\therefore y - 2z = 8$$

And for line 2, $y - 3 = 2z + 5$

$$\therefore y - 2z = 8 \text{ also.}$$

$\therefore y - 2z = 8$ is a plane containing both lines, i.e., the lines are coplanar.

13 a Since $A(1, 2, k)$ lies on $x + 2y - 2z = 8$, $1 + 2(2) - 2k = 8$

$$\therefore 1 + 4 - 2k = 8$$

$$\therefore -2k = 3$$

$$\therefore k = -\frac{3}{2}$$

b Since $x + 2y - 2z = 8$, the plane has normal vector $\mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

$\therefore$ the normal from A has parametric equations

$$x = 1 + t, \quad y = 2 + 2t, \quad z = -\frac{3}{2} - 2t, \quad t \in \mathcal{R}.$$

$\therefore$ points of the normal that are 6 units from A have

$$\sqrt{(1+t-1)^2 + (2+2t-2)^2 + (-\frac{3}{2}-2t+\frac{3}{2})^2} = 6$$

$$\therefore \sqrt{t^2 + 4t^2 + 4t^2} = 6$$

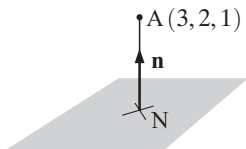
$$\therefore 9t^2 = 36$$

$$\therefore t^2 = 4$$

$$\therefore t = \pm 2$$

$$\therefore B \text{ is } (3, 6, -\frac{11}{2}) \text{ or } (-1, -2, \frac{5}{2})$$

- 14 a** The normal from A(3, 2, 1) to the plane has direction vector

$$\mathbf{n} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 4 & 2 & -2 \end{vmatrix}$$


$$= \mathbf{i} \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}$$

$$= -4\mathbf{i} + 8\mathbf{j}$$

$$\therefore \text{AN has equation } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} -4 \\ 8 \\ 0 \end{bmatrix}$$

So, N has coordinates of the form $(3 - 4t, 2 + 8t, 1)$

But N lies on the plane $\therefore \begin{bmatrix} 3 - 4t \\ 2 + 8t \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$

$$\begin{aligned} \therefore 3 - 4t &= 3 + 2\lambda + 4\mu & \therefore 2\lambda + 4\mu + 4t &= 0 \\ 2 + 8t &= 1 + \lambda + 2\mu & \lambda + 2\mu - 8t &= 1 \\ 1 &= 2 + \lambda - 2\mu & \lambda - 2\mu &= -1 \end{aligned}$$

Solving simultaneously using technology gives $\lambda = -0.4$, $\mu = 0.3$, $t = -0.1$

$\therefore$ N is $(3 - 4(-0.1), 2 + 8(-0.1), 1)$ i.e., $(3.4, 1.2, 1)$

$$\therefore \overrightarrow{\text{AN}} = \begin{bmatrix} 3.4 - 3 \\ 1.2 - 2 \\ 1 - 1 \end{bmatrix} = \begin{bmatrix} 0.4 \\ -0.8 \\ 0 \end{bmatrix} \text{ and } |\overrightarrow{\text{AN}}| = \sqrt{(0.4)^2 + (0.8)^2 + 0^2} \div \frac{2}{\sqrt{5}} \text{ units}$$

- b** The normal from A(1, 0, -2) to the plane has direction vector

$$\mathbf{n} = (3\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (-\mathbf{i} + \mathbf{j} - \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ -1 & 1 & -1 \end{vmatrix}$$

$$= \mathbf{i} \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$$\therefore \text{AN has equation } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

So, N has coordinates of the form $(1 - t, t, -2 + 2t)$

But N lies on the plane $\therefore \begin{bmatrix} 1 - t \\ t \\ -2 + 2t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

$$\begin{aligned} \therefore 1 + 3\lambda - \mu &= 1 - t & \text{and so } 3\lambda - \mu + t &= 0 \\ -1 - \lambda + \mu &= t & -\lambda + \mu - t &= 1 \\ 1 + 2\lambda - \mu &= -2 + 2t & 2\lambda - \mu - 2t &= -3 \end{aligned}$$

Solving simultaneously using technology gives $\lambda = \frac{1}{2}$, $\mu = 2\frac{1}{3}$, $t = \frac{5}{6}$

$\therefore$ N is $(1 - \frac{5}{6}, \frac{5}{6}, -2 + \frac{5}{3})$ i.e., $(\frac{1}{6}, \frac{5}{6}, -\frac{1}{3})$

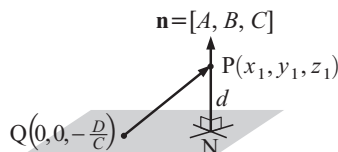
$$\therefore \overrightarrow{\text{AN}} = \begin{bmatrix} \frac{1}{6} - 1 \\ \frac{5}{6} - 0 \\ -\frac{1}{3} - (-2) \end{bmatrix} = \begin{bmatrix} -\frac{5}{6} \\ \frac{5}{6} \\ \frac{5}{3} \end{bmatrix} \text{ and } |\overrightarrow{\text{AN}}| = \sqrt{\frac{25}{36} + \frac{25}{36} + \frac{25}{9}} = \frac{5\sqrt{6}}{6} \text{ units}$$

$$\div 2.04 \text{ units}$$

- 15 a** If N is the point on the plane such that NP is a normal to it, then $\triangle NPQ$ is right-angled at N .
 Since $\vec{NP}$ is in the same direction as $\mathbf{n}$, $\vec{PN}$ is the projection vector of $\vec{PQ}$ on $-\mathbf{n}$.

$$\therefore d = PN = \frac{|\vec{PQ} \cdot (-\mathbf{n})|}{|-\mathbf{n}|} = \frac{|\vec{QP} \cdot \mathbf{n}|}{|\mathbf{n}|}$$

{property of projection vectors}



- b** Since Q is any point on the plane, it has coordinates (x, y, z) such that $Ax + By + Cz + D = 0$.
 The normal vector to the plane is $\mathbf{n} = [A, B, C]$.

$$\begin{aligned} \therefore \text{ using a, } d &= \frac{|\vec{QP} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|[x_1 - x, y_1 - y, z_1 - z] \cdot [A, B, C]|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|Ax_1 - Ax + By_1 - By + Cz_1 - Cz|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|Ax_1 + By_1 + Cz_1 - (Ax + By + Cz)|}{\sqrt{A^2 + B^2 + C^2}} \\ &= \frac{|Ax_1 + By_1 + Cz_1 - D|}{\sqrt{A^2 + B^2 + C^2}} \end{aligned}$$

- c 8 a check:** Given $A(1, 0, 2)$ and the plane $2x + y - 2z + 11 = 0$,

$$d = \frac{|2x_1 + y_1 - 2z_1 + 11|}{\sqrt{2^2 + 1^2 + (-2)^2}} = \frac{|2(1) + 0(1) - 2(2) + 11|}{\sqrt{9}} = \frac{9}{3} = 3 \text{ units}$$

- 8 b check:** Given $A(2, -1, 3)$ and the plane $x - y + 3z = -10$,

$$\begin{aligned} d &= \frac{|x_1 - y_1 + 3z_1 + 10|}{\sqrt{1^2 + (-1)^2 + 3^2}} \\ &= \frac{|2 - (-1) + 3(3) + 10|}{\sqrt{11}} \\ &= \frac{22}{\sqrt{11}} \\ &= 2\sqrt{11} \text{ units} \end{aligned}$$

- 8 c check:** Given $A(1, -4, -3)$ and the plane $4x - y - 2z = 8$,

$$\begin{aligned} d &= \frac{|4x_1 - y_1 - 2z_1 - 8|}{\sqrt{4^2 + (-1)^2 + (-2)^2}} \\ &= \frac{|4 - (-4) - 2(-3) - 8|}{\sqrt{21}} \\ &= \frac{6}{\sqrt{21}} \text{ units} \end{aligned}$$

- 16** Using the formula derived in **15 b**,

a $d = \frac{|x_1 + 2y_1 - z_1 - 10|}{\sqrt{1^2 + 2^2 + (-1)^2}} = \frac{|0 + 2(0) - 0 - 10|}{\sqrt{6}} = \frac{10}{\sqrt{6}} \text{ units}$

b $d = \frac{|x_1 + y_1 - z_1 - 2|}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{|1 + (-3) - 2 - 2|}{\sqrt{3}} = \frac{|-6|}{\sqrt{3}} = \frac{6}{\sqrt{3}} \text{ units (or } 2\sqrt{3} \text{ units)}$

- 17 a** First choose a point on the first plane, $x + y + 2z = 4$, for example, $(0, 0, 2)$.
 Then find the distance from this point to the other plane, using the formula obtained in **15 b**,

$$\therefore d = \frac{|2x_1 + 2y_1 + 4z_1 + 11|}{\sqrt{2^2 + 2^2 + 4^2}} = \frac{|2(0) + 2(0) + 4(2) + 11|}{\sqrt{24}} = \frac{19}{\sqrt{24}} \text{ units}$$

- b** Choose a point on the plane $ax + by + cz + d_1 = 0$, for example, $(0, 0, -\frac{d_1}{c})$.

Using the formula obtained in **15 b** to calculate the distance from this point to the second plane,

$$\therefore d = \frac{|ax_1 + by_1 + cz_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|a(0) + b(0) + c(-\frac{d_1}{c}) + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}} \text{ units}$$

18 The line $x = 2 + t$, $y = -1 + 2t$, $z = -3t$ has direction vector $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$,

and $\begin{bmatrix} 11 \\ -4 \\ 1 \end{bmatrix}$ is a vector normal to the plane $11x - 4y + z = 0$.

But $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \bullet \begin{bmatrix} 11 \\ -4 \\ 1 \end{bmatrix} = 11 - 8 - 3 = 0$

$\therefore$ these vectors are perpendicular and so the line is parallel to the plane.

Choose any point on the line, say $t = 0$, which corresponds to the point $(2, -1, 0)$.

Then the distance $d = \frac{|11x_1 - 4y_1 + z_1|}{\sqrt{11^2 + (-4)^2 + 1^2}} = \frac{|11(2) - 4(-1) + 0|}{\sqrt{138}} = \frac{26}{\sqrt{138}}$ units.

19 Since the planes are parallel to $2x - y + 2z = 5$, they have equation $2x - y + 2z = a$ for some a .

Choose any point on $2x - y + 2z = 5$, for example, $(0, -5, 0)$

Then the distance from this point to the plane $2x - y + 2z = a$ is

$$d = \frac{|2x_1 - y_1 + 2z_1 - a|}{\sqrt{2^2 + (-1)^2 + 2^2}} \quad \therefore \quad 2 = \frac{|2(0) - (-5) + 2(0) - a|}{3}$$

$$\therefore \quad 6 = |5 - a| \quad \text{and so} \quad 5 - a = \pm 6$$

$$\therefore \quad a = 5 \pm 6$$

$$\therefore \quad a = -1 \quad \text{or} \quad a = 11$$

$\therefore$ the planes are $2x - y + 2z = -1$ and $2x - y + 2z = 11$

EXERCISE 17E

1 a $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{l} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$

$$\therefore \sin \phi = \frac{|\mathbf{n} \bullet \mathbf{l}|}{|\mathbf{n}| |\mathbf{l}|}$$

$$= \frac{|4 - 3 + 1|}{\sqrt{3}\sqrt{26}}$$

$$= \frac{2}{\sqrt{78}} \quad \text{and so} \quad \phi \doteq 13.09^\circ$$

b $\mathbf{n} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{l} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

$$\therefore \sin \phi = \frac{|\mathbf{n} \bullet \mathbf{l}|}{|\mathbf{n}| |\mathbf{l}|}$$

$$= \frac{|2 - 3 + 1|}{\sqrt{6}\sqrt{11}}$$

$$= 0 \quad \text{and so} \quad \phi = 0^\circ,$$

i.e., the line and plane are parallel

c $\mathbf{n} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$ and if $x - 4 = 3 - y = 2(z + 1) = t$

i.e., $x = 4 + t$, $y = 3 - t$, $z = -1 + \frac{1}{2}t$ then $\mathbf{l} = \begin{bmatrix} 1 \\ -1 \\ \frac{1}{2} \end{bmatrix}$

$$\therefore \sin \phi = \frac{|\mathbf{n} \bullet \mathbf{l}|}{|\mathbf{n}| |\mathbf{l}|} = \frac{|3 + (-4) + (-\frac{1}{2})|}{\sqrt{26}\sqrt{\frac{9}{4}}} = \frac{|\frac{-3}{2}|}{\frac{3}{2}\sqrt{26}} = \frac{1}{\sqrt{26}} \quad \text{and so} \quad \phi \doteq 11.31^\circ$$

d The plane has normal vector

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & -1 \\ 1 & 1 & -2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -4 & -1 \\ 1 & -2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & 3 \\ -2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix} = 9\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

and the line has direction vector $\mathbf{l} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\therefore \sin \phi = \frac{|\mathbf{n} \bullet \mathbf{l}|}{|\mathbf{n}| |\mathbf{l}|} = \frac{|9 - 5 + 7|}{\sqrt{81 + 25 + 49}\sqrt{1 + 1 + 1}} = \frac{11}{\sqrt{155}\sqrt{3}}$$

$$\therefore \phi \doteq 30.67^\circ$$

2 a $\mathbf{n}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ **b** $\mathbf{n}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$

$$\therefore \cos \theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{|2 - 3 + 2|}{\sqrt{6}\sqrt{14}} = \frac{1}{\sqrt{84}} \quad \therefore \cos \theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{|3 - 1 - 3|}{\sqrt{11}\sqrt{11}} = \frac{1}{11}$$

$$\therefore \theta \doteq 83.7^\circ \quad \therefore \theta \doteq 84.8^\circ$$

c $\mathbf{n}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{n}_2 = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$ $\therefore \cos \theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{|6 - 4 - 1|}{\sqrt{11}\sqrt{21}} = \frac{1}{\sqrt{231}}$

$$\therefore \theta \doteq 86.2^\circ$$

d $\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -1 \\ 2 & -4 & 3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & -1 \\ -4 & 3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & -1 \\ 3 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 1 \\ 2 & -4 \end{vmatrix}$

$$= -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}$

$$= 0\mathbf{i} + 1\mathbf{j} - \mathbf{k}$$

$$\therefore \cos \theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{|0 + 1 - 2|}{\sqrt{1+1+4}\sqrt{0+1+1}} = \frac{1}{\sqrt{6}\sqrt{2}} = \frac{1}{\sqrt{12}}$$

$$\therefore \theta \doteq 73.2^\circ$$

e $\mathbf{n}_1 = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$ and $\mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 0 \\ 2 & 1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}$

$$= -\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

$$\therefore \cos \theta = \frac{|\mathbf{n}_1 \bullet \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{|(3)(-1) + (-4)(-3) + (1)(5)|}{\sqrt{9+16+1}\sqrt{1+9+25}} = \frac{14}{\sqrt{26} \times 35} \therefore \theta \doteq 62.3^\circ$$

EXERCISE 17F

1 a Either (1) no solutions or (2) an infinite number of solutions.

b i They are parallel if

$$\begin{aligned} a_1 &= ka_2 \\ b_1 &= kb_2 \\ \text{and } c_1 &= kc_2 \quad \text{for some } k. \end{aligned}$$

ii They are coincident if

$$\begin{aligned} a_1 &= ka_2 \\ b_1 &= kb_2 \\ c_1 &= kc_2 \\ \text{and } d_1 &= kd_2 \quad \text{for some } k. \end{aligned}$$

c i $\left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 3 & -9 & 2 & 4 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 2 & 8 \\ 0 & 0 & -4 & -20 \end{array} \right] \quad R_2 \rightarrow R_2 - 3R_1$

$$\therefore -4z = -20 \quad \text{and} \quad x - 3y + 2z = 8 \quad \text{i.e.,} \quad x = 3y - 2z + 8$$

$$\therefore z = 5 \quad \text{and if we let } y = t, \quad \text{then } x = 3t - 2(5) + 8 = -2 + 3t$$

$$\therefore \text{the planes meet in the line } x = -2 + 3t, \quad y = t, \quad z = 5, \quad t \text{ in } \mathcal{R}$$

ii $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 1 & -1 & 1 & 3 \end{array} \right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -3 & 1 & 1 \end{array} \right] \quad R_2 \rightarrow 2R_2 - R_1$

$$\therefore -3y + z = 1 \quad \text{and} \quad 2x + y + z = 5$$

$$\begin{aligned} \therefore \text{if we let } y &= t, \quad \text{then } z = 1 + 3y = 1 + 3t \quad \text{and} \quad 2x = 5 - y - z \\ &= 5 - t - (1 + 3t) \\ &= 4 - 4t \end{aligned}$$

$$\text{i.e., } x = 2 - 2t$$

$$\therefore \text{the planes meet in the line } x = 2 - 2t, \quad y = t, \quad z = 1 + 3t, \quad t \text{ in } \mathcal{R}$$

$$\text{iii} \quad \left[\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 3 & 6 & -9 & 18 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 6 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad R_2 \rightarrow R_2 - 3R_1$$

$\therefore$ there are infinitely many solutions, as the planes are coincident.

Let $y = s$ and $z = t$ in $x + 2y - 3z = 6$, $s, t \in \mathcal{R}$

$$\therefore x = 3t - 2s + 6$$

$\therefore x = 3t - 2s + 6$, $y = s$, $z = t$ is the general solution of the plane.

$$\begin{array}{ll} \text{2 a} & \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 2 & 4 & k & 12 \end{array} \right] & \text{If } k = -2, \text{ the two planes are coincident.} \\ & \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 0 & k+2 & 0 \end{array} \right] & \therefore \text{infinitely many solutions.} \\ & R_2 \rightarrow R_2 - 2R_1 & \text{If } k \neq -2, \text{ the two planes meet in a line.} \\ & & \therefore \text{infinitely many solutions.} \end{array}$$

$$\begin{array}{ll} \text{b} & \left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 2 & -2 & 6 & k \end{array} \right] & \text{If } k = 16, \text{ the planes are coincident.} \\ & \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 0 & 0 & 0 & k-16 \end{array} \right] & \therefore \text{infinitely many solutions.} \\ & R_2 \rightarrow R_2 - 2R_1 & \text{If } k \neq 16, \text{ the planes are parallel} \\ & & \text{but not coincident.} \\ & & \therefore \text{no solutions exist.} \end{array}$$

- 3**
- (1) $P_1 = P_2 = P_3$: infinitely many solutions where x , y and z are in terms of two parameters, s and t say, i.e., solution is a plane.
 - (2) $P_1 = P_2$ are coincident and cut by P_3 : infinitely many solutions where x , y and z are in terms of one parameter, t say, i.e., solution is a line.
 - (3) $P_1 = P_2$ with P_3 parallel but not coincident: no solutions exist.
 - (4) P_1 and P_2 are parallel but not coincident, and P_3 cuts both planes: no solutions exist.
 - (5) P_1 , P_2 and P_3 are all parallel but not coincident: no solutions exist.
 - (6) P_1 , P_2 and P_3 meet in a unique point (a, b, c) , so that $x = a$, $y = b$, $z = c$.
 - (7) P_1 , P_2 and P_3 meet in a common line: infinitely many solutions where x , y and z are in terms of one parameter, t , say.
 - (8) P_1 , P_2 and P_3 are such that the line of intersection between any two is parallel to the third plane: no solutions exist.

- 4 a** The system has augmented matrix

$$\begin{array}{ll} & \left[\begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 1 & -1 & 2 & 11 \\ 4 & 1 & -5 & -18 \end{array} \right] & \text{Now } -11z = -4 \quad \therefore z = 4 \\ & \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 0 & -2 & 3 & 16 \\ 0 & -3 & -1 & 2 \end{array} \right] & \text{and } -2y + 3z = 16 \\ & R_2 \rightarrow R_2 - R_1 & \therefore -2y + 12 = 16 \\ & R_3 \rightarrow R_3 - 4R_1 & \therefore -2y = 4 \\ & & \therefore y = -2 \\ & \sim \left[\begin{array}{ccc|c} 1 & 1 & -1 & -5 \\ 0 & -2 & 3 & 16 \\ 0 & 0 & -11 & -44 \end{array} \right] & \text{and } x + y - z = -5 \\ & R_3 \rightarrow 2R_3 - 3R_2 & \therefore x = -5 - (-2) + 4 \\ & & \therefore x = 2 + 4 - 5 \\ & \therefore \text{the planes meet at the unique point } (1, -2, 4) & \therefore x = 1 \end{array}$$

b The system has augmented matrix

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 8 \\ 5 & -2 & 5 & 11 \end{array} \right] \\ & \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 6 \\ 0 & 3 & -5 & 6 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 5R_1 \end{array} \\ & \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -5 & 6 \\ 0 & 0 & 0 & 0 \end{array} \right] R_3 \rightarrow R_3 - R_2 \end{aligned}$$

$\therefore$ the three planes meet in a common line

$$x = \frac{9-t}{3}, \quad y = \frac{5t+6}{3}, \quad z = t, \quad t \text{ in } \mathcal{R}$$

Let $z = t$

$$\begin{aligned} \text{As } 3y - 5z &= 6 \\ 3y &= 5t + 6 \\ \therefore y &= \frac{5t+6}{3} \end{aligned}$$

But $x - y + 2z = 1$

$$\begin{aligned} \therefore x &= 1 - \frac{5t+6}{3} + 2t \\ \therefore x &= \frac{3+5t+6-6t}{3} \\ \therefore x &= \frac{9-t}{3} \end{aligned}$$

c The system has augmented matrix:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 2 & -1 & -1 & 5 \\ 3 & -4 & -1 & 2 \end{array} \right] \\ & \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 0 & -5 & 1 & -11 \\ 0 & -10 & 2 & -22 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \\ & \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 8 \\ 0 & -5 & 1 & -11 \\ 0 & 0 & 0 & 0 \end{array} \right] R_3 \rightarrow R_3 - 2R_2 \end{aligned}$$

$\therefore$ the three planes meet in a common line

$$\therefore x = 3t - 3, \quad y = t, \quad z = 5t - 11, \quad t \text{ in } \mathcal{R}$$

d The system has augmented matrix:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 2 & -2 & 2 & 11 \\ 1 & 3 & -1 & -2 \end{array} \right] \\ & \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 0 & 0 & -5 \\ 0 & 4 & -2 & -10 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \end{aligned}$$

The first two planes are parallel and are cut by the third plane.

$\therefore$ the equations are inconsistent and there are no solutions.

e The system has augmented matrix:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 1 & -1 & 1 & 4 \\ 3 & 3 & -6 & 3 \end{array} \right] \\ & \left[\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & -2 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \end{aligned}$$

There are two coincident planes cut by a third plane.

$\therefore$ infinitely many solutions in a line:

$$x = \frac{t+5}{2}, \quad y = \frac{3t-3}{2}, \quad z = t, \quad t \text{ in } \mathcal{R}$$

Let $z = t$

$$\begin{aligned} \text{Now } -2y + 3z &= 3 \\ \therefore 2y &= 3z - 3 \\ \therefore y &= \frac{3t-3}{2} \end{aligned}$$

and as $x + y - 2z = 1$

$$\begin{aligned} \therefore x &= 1 - y + 2z \\ \therefore x &= 1 - \frac{3t-3}{2} + 2t \\ \therefore x &= \frac{2-3t+3+4t}{2} \\ \therefore x &= \frac{t+5}{2} \end{aligned}$$

f The system has augmented matrix:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & -1 & -1 & 5 \\ 1 & 1 & 1 & 1 \\ 5 & -1 & 2 & 17 \end{array} \right] \\ & \sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 5 \\ 0 & 2 & 2 & -4 \\ 0 & 4 & 7 & -8 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 5R_1 \end{array} \\ & \sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 5 \\ 0 & 2 & 2 & -4 \\ 0 & 0 & 3 & 0 \end{array} \right] \begin{array}{l} R_3 \rightarrow R_3 - 2R_2 \end{array} \end{aligned}$$

Now $3z = 0$
 $\therefore z = 0$
 As $2y + 2z = -4$
 $\therefore 2y = -4$
 $\therefore y = -2$
 and as $x - y - z = 5$
 $\therefore x = 5 + (-2) + 0$
 $\therefore x = 3$

$\therefore$ the planes meet at the unique point $(3, -2, 0)$.

5 The system has augmented matrix:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & -1 & 3 & 1 \\ 2 & -3 & -1 & 3 \\ 3 & -5 & -5 & k \end{array} \right] \\ & \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & -1 & -7 & 1 \\ 0 & -2 & -14 & k-3 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \\ & \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & -1 \\ 0 & -1 & -7 & 1 \\ 0 & 0 & 0 & k-5 \end{array} \right] \begin{array}{l} R_3 \rightarrow R_3 - 2R_2 \end{array} \end{aligned}$$

(1) If $k = 5$, the planes meet in a line
 {as we have a row of zeros}
 Let $z = t$
 Now $-y - 7z = 1$
 $\therefore y = -1 - 7t$
 and $x - y + 3z = 1$
 $\therefore x = 1 + y - 3z$
 $\therefore x = 1 - 1 - 7t - 3t$
 $\therefore x = -10t$
 $\therefore x = -10t, y = -1 - 7t, z = t, t \in \mathcal{R}$

(2) If $k \neq 5$ there are no solutions.

Since the first two planes are not parallel, the line of intersection of any two planes is parallel to the third plane.

6 The augmented matrix is:

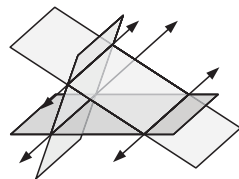
$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 2 & m & -1 \\ 2 & 1 & -1 & 3 \\ m & -2 & 1 & 1 \end{array} \right] \\ & \sim \left[\begin{array}{ccc|c} 1 & 2 & m & -1 \\ 0 & -3 & -1-2m & 5 \\ 0 & -2-2m & 1-m^2 & 1+m \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - mR_1 \end{array} \\ & \sim \left[\begin{array}{ccc|c} 1 & 2 & m & -1 \\ 0 & -3 & -1-2m & 5 \\ 0 & 0 & -m^2-6m-5 & 7m+7 \end{array} \right] \begin{array}{l} R_3 \rightarrow -3R_3 - (-2-2m)R_2 \end{array} \\ & \therefore -(m^2+6m+5)z = 7(m+1) \\ & \text{i.e., } -(m+5)(m+1)z = 7(m+1) \end{aligned}$$

(1) When $m = -5$ we have $0z = -28$.

$\therefore$ the system is inconsistent, and there are no solutions.

The augmented matrix is:
$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 0 & -3 & 9 & 5 \\ 0 & 0 & 0 & -28 \end{array} \right]$$

The line of intersection of any two planes is parallel to the third plane.



$$(2) \text{ When } m = -1, \text{ we have } \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -3 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

i.e., three planes meet in a common line {two coincident planes cut by the third}

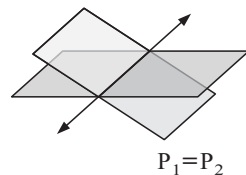
$$\text{Let } z = t. \quad \text{Then } -3y + t = 5, \quad \therefore y = \frac{t-5}{3}$$

$$\text{Also, } x = -1 - 2y + z$$

$$= -1 - 2\left(\frac{t-5}{3}\right) + t$$

$$= -1 + \frac{1}{3}t + \frac{10}{3}$$

$$= \frac{t+7}{3} \quad \text{i.e., } x = \frac{t+7}{3}, \quad y = \frac{t-5}{3}, \quad z = t, \quad t \text{ in } \mathcal{R}$$



- (3) When $m \neq -1$ or -5 , the system has a unique solution,
i.e., the three planes meet in a common point.

7 P_1 meets P_2 where

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + r \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \begin{cases} 2 + 3\lambda + \mu = 3 + 2r + s \\ -1 + \mu = -1 + s \\ \lambda - \mu = 3 - r \end{cases} \quad \text{i.e., } \begin{cases} 3\lambda + \mu = 2r + s + 1 \\ \mu = s \\ \lambda - \mu = 3 - r \end{cases}$$

$$\text{If } \mu = a, \text{ say then } s = a, \quad \begin{cases} 3\lambda + a = 2r + a + 1 \\ \lambda - a = 3 - r \end{cases} \quad \text{i.e., } \begin{cases} r = 3 - \lambda + a \\ \therefore 3\lambda + a = 6 - 2\lambda + 2a + a + 1 \end{cases}$$

$$\text{i.e., } 5\lambda = 2a + 7$$

$$\lambda = \frac{2a+7}{5} \quad \text{and} \quad r = 3 + a - \frac{2a+7}{5}$$

$$\text{i.e., } r = \frac{3a+8}{5}$$

$$\text{i.e., if } \mu = a, \quad \lambda = \frac{2a+7}{5}, \quad r = \frac{3a+8}{5}, \quad s = a \quad \dots (1)$$

P_2 meets P_3 where

$$\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + r \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - u \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

$$\therefore \begin{cases} 3 + 2r + s = 2 + t \\ -1 + s = -1 - t + u \\ 3 - r = 2 - 2u \end{cases} \quad \text{i.e., } \begin{cases} 2r + s + 1 = t \\ s = u - t \\ 2u - r = -1 \end{cases}$$

$$\text{So, if } u = b, \text{ say then } r = 2b + 1 \text{ and } 4b + 2 + b - t + 1 = t$$

$$\therefore 5b + 3 = 2t \quad \text{i.e., } t = \frac{5b+3}{2}$$

$$\text{and } s = u - t = b - \frac{5b+3}{2} = \frac{-3b-3}{2}$$

$$\text{i.e., if } u = b, \quad r = 2b + 1, \quad t = \frac{5b+3}{2}, \quad s = \frac{-3b-3}{2} \quad \dots (2)$$

$$\text{From (1) and (2) } \frac{3a+8}{5} = 2b + 1 \quad \text{and} \quad a = \frac{-3b-3}{2}$$

$$\therefore 3a + 8 = 10b + 5 \quad \text{and} \quad 2a = -3b - 3$$

$$\text{i.e., } \begin{cases} 3a - 10b = -3 \\ 2a + 3b = -3 \end{cases} \quad \text{which has solutions } a = -\frac{39}{29}, \quad b = -\frac{3}{29}$$

$$\text{In (2), } u = -\frac{3}{29}, \quad t = \frac{5(-\frac{3}{29}) + 3}{2} = 1\frac{7}{29} \text{ or } \frac{36}{29}$$

$$\therefore \mathbf{r}_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + \frac{36}{29} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{3}{29} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3\frac{7}{29} \\ -2\frac{10}{29} \\ 2\frac{6}{29} \end{bmatrix} = \begin{bmatrix} \frac{94}{29} \\ -\frac{68}{29} \\ \frac{64}{29} \end{bmatrix}$$

$$\therefore \text{ all 3 planes meet at } \left(\frac{94}{29}, -\frac{68}{29}, \frac{64}{29} \right).$$

REVIEW SET 17A**1 a** The vector equation is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} + t \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

b The parametric equations are

$$x = -6 + 4t, \quad y = 3 - 3t, \quad t \in \mathcal{R}$$

2 The vector equation is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} + t \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad t \in \mathcal{R}$ **3** $(-3, m)$ lies on the line, so $\begin{bmatrix} -3 \\ m \end{bmatrix} = \begin{bmatrix} 18 \\ -2 \end{bmatrix} + \begin{bmatrix} -7t \\ 4t \end{bmatrix}$

$$\therefore -3 = 18 - 7t \quad \text{and} \quad m = -2 + 4t$$

$$\therefore 7t = 21$$

$$\text{i.e., } t = 3 \quad \text{and so} \quad m = -2 + 4(3) = 10$$

$$\text{i.e., } m = 10$$

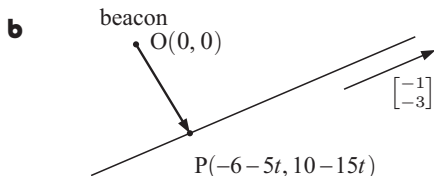
4 a $x(0) = -4$ and $y(0) = 3$, so the initial position is $(-4, 3)$.**b** $x(4) = -4 + 32, \quad y(4) = 3 + 6(4), \quad \text{so at } t = 4, \text{ the position is } (28, 27).$
 $= 28 \quad \quad \quad = 27$ **c** The velocity vector is $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$, so the speed is $\sqrt{8^2 + 6^2} = 10 \text{ m/s}$ **d** $\begin{bmatrix} 8 \\ 6 \end{bmatrix}$ **5** The direction vector is $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ which has length $\sqrt{3^2 + (-1)^2} = \sqrt{10}$ units

$$\therefore 2\sqrt{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ has length } 20. \quad \text{So, the velocity vector is } \begin{bmatrix} 6\sqrt{10} \\ -2\sqrt{10} \end{bmatrix} \text{ or } 2\sqrt{10}(3\mathbf{i} - \mathbf{j})$$

6 a i The yacht is initially at $(-6, 10)$, so its initial position vector is $\begin{bmatrix} -6 \\ 10 \end{bmatrix}$ i.e., $-6\mathbf{i} + 10\mathbf{j}$ **ii** $-\mathbf{i} - 3\mathbf{j}$ has length $\sqrt{(-1)^2 + (-3)^2} = \sqrt{10}$

$$\therefore 5(-\mathbf{i} - 3\mathbf{j}) \text{ has length } 5\sqrt{10}$$

$$\therefore \text{ the direction vector is } -5\mathbf{i} - 15\mathbf{j}$$

iii $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 10 \end{bmatrix} + t \begin{bmatrix} -5 \\ -15 \end{bmatrix}$ i.e., $\begin{bmatrix} x \\ y \end{bmatrix} = -6\mathbf{i} + 10\mathbf{j} + t(-5\mathbf{i} - 15\mathbf{j})$
 $= (-6 - 5t)\mathbf{i} + (10 - 15t)\mathbf{j}$ 

$$\vec{OP} = \begin{bmatrix} -6-5t \\ 10-15t \end{bmatrix} \quad \text{and} \quad \vec{OP} \bullet \begin{bmatrix} -1 \\ -3 \end{bmatrix} = 0$$

$$\therefore -1(-6-5t) - 3(10-15t) = 0$$

$$\therefore 6 + 5t - 30 + 45t = 0$$

$$\therefore 50t = 24$$

$$\therefore t = 0.48 \text{ h}$$

$$(\text{i.e., } 28.8 \text{ min})$$

c When $t = 0.48$, $\vec{OP} = \begin{bmatrix} -6-5(0.48) \\ 10-15(0.48) \end{bmatrix} = \begin{bmatrix} -8.4 \\ 2.8 \end{bmatrix}$

$$\text{and } OP = \sqrt{(-8.4)^2 + (2.8)^2} \doteq 8.85 \text{ km}$$

As the closest distance is 8.85 km and the radius is 8 km, the yacht will miss the reef.

- 7 a** $\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ where $t \geq 0$. When $t = 0$, the time is 2:17 pm
i.e., $x_1(t) = 2 + t$, $y_1(t) = 4 - 3t$
- b** Likewise, $x_2(t) = 11 - (t + 2)$, $y_2(t) = 3 + a(t + 2)$
i.e., $x_2(t) = 9 - t$ $y_2(t) = [3 + 2a] + at$
- c** They meet where $2 + t = 9 - t$ and $4 - 3t = [3 + 2a] + at$
 $\therefore 2t = 7$
 $\therefore t = \frac{7}{2}$ $\therefore$ the time would be 2:17 pm plus $3\frac{1}{2}$ min i.e., 2:20:30 pm

d When $t = \frac{7}{2}$,

$$4 - 3\left(\frac{7}{2}\right) = [3 + 2\left(\frac{7}{2}\right)] + a\left(\frac{7}{2}\right)$$

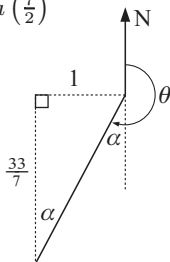
$$\therefore -\frac{13}{2} = 10 + \frac{7a}{2}$$

$$\therefore -13 = 20 + 7a$$

$$\therefore 7a = -33$$

$$\therefore a = -\frac{33}{7}$$

Y18 has velocity vector $\begin{bmatrix} -1 \\ -\frac{33}{7} \end{bmatrix}$



$$\text{with speed} = \sqrt{(-1)^2 + \left(-\frac{33}{7}\right)^2} \\ \div 4.82 \text{ km/min}$$

$$\tan \alpha = \frac{1}{\frac{33}{7}} = \frac{7}{33}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{7}{33}\right) \div 12.0^\circ$$

$\therefore$ the direction is $180^\circ + \alpha^\circ = 192^\circ$
i.e., the torpedo has speed 4.82 km/min
and direction 192° .

- 8 a** Line 1 has direction vector $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and Line 4 has direction vector $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$.
Now $\begin{bmatrix} 5 \\ -2 \end{bmatrix} = -\begin{bmatrix} -5 \\ 2 \end{bmatrix}$, so Lines 1 and 4 are parallel, i.e., $KL \parallel MN$.

- b** $\overrightarrow{KL} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$, $\overrightarrow{NK} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$, $\overrightarrow{MN} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$ $\therefore \overrightarrow{KL} \bullet \overrightarrow{NK} = 20 - 20 = 0$,
and $\overrightarrow{NK} \bullet \overrightarrow{MN} = -20 + 20 = 0$
 $\therefore NK$ is perpendicular to both KL and MN .

- c** Lines 1 and 3 meet at K

$$\therefore \begin{bmatrix} 2 \\ 19 \end{bmatrix} + p \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + r \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 5p - 4r \\ -2p - 10r \end{bmatrix} = \begin{bmatrix} 1 \\ -12 \end{bmatrix}$$

$$\therefore \begin{array}{rcl} 5p - 4r & = & 1 \quad \times 5 \\ 2p + 10r & = & 12 \quad \times 2 \end{array}$$

$$\therefore \begin{array}{rcl} 25p - 20r & = & 5 \\ 4p + 20r & = & 24 \end{array}$$

$$\hline 29p = 29$$

$$\text{Adding, } 29p = 29$$

$$\therefore p = 1$$

$$\text{So, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 19 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$

$\therefore K$ is (7, 17)

Lines 3 and 4 meet at N

$$\therefore \begin{bmatrix} 3 \\ 7 \end{bmatrix} + r \begin{bmatrix} 4 \\ 10 \end{bmatrix} = \begin{bmatrix} 43 \\ -9 \end{bmatrix} + s \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 4r + 5s \\ 10r - 2s \end{bmatrix} = \begin{bmatrix} 40 \\ -16 \end{bmatrix}$$

$$\therefore \begin{array}{rcl} 4r + 5s & = & 40 \quad \times 2 \\ 10r - 2s & = & -16 \quad \times 5 \end{array}$$

$$\therefore \begin{array}{rcl} 8r + 10s & = & 80 \\ 50r - 10s & = & -80 \end{array}$$

$$\hline 58r = 0$$

Lines 2 and 4 meet at M.

$$\therefore \begin{bmatrix} 33 \\ -5 \end{bmatrix} + r \begin{bmatrix} -11 \\ 16 \end{bmatrix} = \begin{bmatrix} 43 \\ -9 \end{bmatrix} + s \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -11r + 5s \\ 16r - 2s \end{bmatrix} = \begin{bmatrix} 10 \\ -4 \end{bmatrix}$$

$$\therefore \begin{array}{rcl} -11r + 5s & = & 10 \quad \times 2 \\ 16r - 2s & = & -4 \quad \times 5 \end{array}$$

$$\therefore \begin{array}{rcl} -22r + 10s & = & 20 \\ 80r - 10s & = & -20 \end{array}$$

$$\hline 58r = 0$$

$$\therefore 58r = 0$$

$$\therefore r = 0$$

$$\text{So, } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 33 \\ -5 \end{bmatrix}$$

$\therefore M$ is (33, -5)

$$\therefore 58r = 0$$

$$\therefore r = 0$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

i.e., N is (3, 7)

d

$$KL = \sqrt{(22-7)^2 + (11-17)^2}$$

$$= \sqrt{225 + 36}$$

$$= \sqrt{261} \text{ units}$$

$$NK = \sqrt{(33-3)^2 + (-5-7)^2}$$

$$= \sqrt{900 + 144}$$

$$= \sqrt{1044} \text{ units}$$

$$KN = \sqrt{(7-3)^2 + (17-7)^2}$$

$$= \sqrt{16 + 100}$$

$$= \sqrt{116}$$

$$\therefore \text{area} = \left(\frac{\sqrt{261} + \sqrt{1044}}{2} \right) \times \sqrt{116} = 261 \text{ units}^2$$

REVIEW SET 17B

- 1 a** We first find the plane containing A, B and C.

The plane has normal

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 8 \\ -2 & 2 & -2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 8 \\ 2 & -2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 8 & 2 \\ -2 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix}$$

$$= -18\mathbf{i} - 12\mathbf{j} + 6\mathbf{k} \quad \text{or} \quad -6[3\mathbf{i} + 2\mathbf{j} - \mathbf{k}]$$

$$\therefore \text{ since A lies on the plane, it has equation } 3x + 2y - z = 3(1) + 2(0) + (-1)(4)$$

$$\text{i.e., } 3x + 2y - z = -1$$

- b** Since the coordinates of D satisfy this equation, all 4 points are coplanar.

The closest point on the plane to E(3, 3, 2) is the foot of the normal from E.

The equation of the normal through E is $x = 3 + 3t$, $y = 3 + 2t$, $z = 2 - t$, and this intersects the plane when

$$3(3 + 3t) + 2(3 + 2t) - (2 - t) = -1$$

$$\therefore 9 + 9t + 6 + 4t - 2 + t = -1$$

$$\therefore 14t = -14$$

$$\therefore t = -1$$

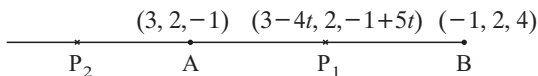
i.e., the nearest point is (0, 1, 3)

2 a $\overrightarrow{AB} = \begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}$ $\therefore$ the line is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 5 \end{bmatrix}$, $t \in \mathcal{R}$

- b** The equation of the plane is

$$-4x + 0y + 5z = -4(-1) + 5(4)$$

$$\text{i.e., } -4x + 5z = 24$$



- c** The distance from a point on the line to A is $d = \sqrt{(-4t)^2 + 0^2 + (5t)^2} = \sqrt{41t^2}$

$$\therefore \text{ since } d = 2\sqrt{41} \text{ units, } \sqrt{41t^2} = 2\sqrt{41}$$

$$\therefore t^2 = 4$$

$$\therefore t = \pm 2 \quad \therefore \text{ the points are } (-5, 2, 9) \text{ and } (11, 2, -11).$$

3 a $\mathbf{n} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ and $\mathbf{l} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$$\therefore \sin \phi = \frac{|2 + (-2) + 2|}{\sqrt{9}\sqrt{6}}$$

$$= \frac{2}{3\sqrt{6}}$$

$$\therefore \phi \doteq 15.79^\circ$$

- b** The planes have normals

$$\mathbf{n}_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{n}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore \cos \theta = \frac{|2 + (-1) + (-4)|}{\sqrt{9}\sqrt{6}}$$

$$= \frac{3}{3\sqrt{6}} \quad \text{i.e., } \frac{1}{\sqrt{6}}$$

$$\therefore \theta \doteq 65.91^\circ$$

$$4 \quad a \quad \vec{AB} = \begin{bmatrix} 0-3 \\ 2-(-1) \\ -1-1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix} \quad \therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ 3 \\ -2 \end{bmatrix}, \quad t \in \mathcal{R}$$

b If P divides BA in the ratio 2 : 5, then $\vec{BP} : \vec{PA} = 2 : 5$

$$\text{If P is } (a, b, c) \text{ then } \begin{bmatrix} a-0 \\ b-2 \\ c-(-1) \end{bmatrix} : \begin{bmatrix} 3-a \\ -1-b \\ 1-c \end{bmatrix} = 2 : 5$$

$$\therefore 5 \begin{bmatrix} a \\ b-2 \\ c+1 \end{bmatrix} = 2 \begin{bmatrix} 3-a \\ -1-b \\ 1-c \end{bmatrix}$$

$$\therefore 5a = 6 - 2a, \quad 5b - 10 = -2 - 2b, \quad 5c + 5 = 2 - 2c$$

$$\therefore 7a = 6, \quad 7b = 8, \quad 7c = -3$$

$$\therefore a = \frac{6}{7}, \quad b = \frac{8}{7}, \quad c = -\frac{3}{7}$$

So, P is $(\frac{6}{7}, \frac{8}{7}, -\frac{3}{7})$

$$5 \quad \text{Given } C(-3, 2, -1) \text{ and } D(0, 1, -4), \quad \vec{CD} = \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix}$$

$\therefore$ the line passing through C and D has parametric equations

$$x = -3 + 3t, \quad y = 2 - t, \quad z = -1 - 3t$$

$$\text{The line meets } 2x - y + z = 3 \text{ when } 2(-3 + 3t) - (2 - t) + (-1 - 3t) = 3$$

$$\therefore -6 + 6t - 2 + t - 1 - 3t = 3$$

$$\therefore 4t = 12$$

$$\therefore t = 3$$

$\therefore$ they meet at $(6, -1, -10)$

$$6 \quad a \quad \frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} \text{ has direction vector } \begin{bmatrix} 3 \\ -16 \\ 7 \end{bmatrix}$$

$$x = 15 + 3t, \quad y = 29 + 8t, \quad z = 5 - 5t \text{ has direction vector } \begin{bmatrix} 3 \\ 8 \\ -5 \end{bmatrix}$$

$\therefore$ the lines are not parallel.

$$\text{If they intersect then } \frac{15+3t-8}{3} = \frac{29+8t+9}{-16} = \frac{5-5t-10}{7}$$

$$\therefore t + \frac{7}{3} = -\frac{1}{2}t - \frac{38}{16} = -\frac{5}{7}t - \frac{5}{7}$$

$$\text{Now } t + \frac{7}{3} = -\frac{1}{2}t - \frac{38}{16} \text{ requires } \frac{3}{2}t = -\frac{19}{8} - \frac{7}{3} = -\frac{113}{24}, \text{ i.e., } t = -\frac{113}{36}$$

$$\text{and } t + \frac{7}{3} = -\frac{5}{7}t - \frac{5}{7} \text{ requires } \frac{12}{7}t = -\frac{5}{7} - \frac{7}{3} = -\frac{64}{21}, \text{ i.e., } t = -\frac{16}{9}$$

Hence the lines do not intersect, and since they are not parallel they are skew.

b If θ is the acute angle between the two lines, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are their direction vectors,

$$\text{then } \cos \theta = \frac{|\mathbf{v}_1 \bullet \mathbf{v}_2|}{|\mathbf{v}_1| |\mathbf{v}_2|} = \frac{\left| \begin{bmatrix} 3 \\ -16 \\ 7 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 8 \\ -5 \end{bmatrix} \right|}{\sqrt{3^2 + (-16)^2 + 7^2} \sqrt{3^2 + 8^2 + (-5)^2}}$$

$$\therefore \cos \theta = \frac{|9 - 128 - 35|}{\sqrt{314} \sqrt{98}} = \frac{154}{\sqrt{30772}} \quad \text{and so } \theta \doteq 28.6^\circ$$

- c** Translate the first line so that it intersects with the second, and find the equation of the plane containing them.

$$\begin{aligned}\text{The plane has normal } \mathbf{n} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} -16 & 7 \\ 8 & -5 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 7 & 3 \\ -5 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 3 & -16 \\ 3 & 8 \end{vmatrix} \\ &= 24\mathbf{i} + 36\mathbf{j} + 72\mathbf{k} \quad \text{or} \quad 12(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})\end{aligned}$$

Choose any point on the second line, for example $(15, 29, 5)$.

$$\begin{aligned}\therefore \text{ the plane has equation } 2x + 3y + 6z &= 2(15) + 3(29) + 6(5) \\ \text{i.e., } 2x + 3y + 6z &= 147\end{aligned}$$

Next choose any point on the first line, for example $(8, -9, 10)$.

Then the shortest distance between the two lines is the distance from this point to the plane.

$$\therefore d = \frac{|2x_1 + 3y_1 + 6z_1 - 147|}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{|2(8) + 3(-9) + 6(10) - 147|}{7} = 14 \text{ units}$$

- 7 a** The distance of $X(-1, 1, 3)$ from $x - 2y - 2z = 8$

$$\text{is } d = \frac{|x_1 - 2y_1 - 2z_1 - 8|}{\sqrt{1^2 + (-2)^2 + (-2)^2}} = \frac{|-1 - 2 - 6 - 8|}{3} = \frac{|-17|}{3} = \frac{17}{3} \text{ units}$$

b Since $2 - x = y - 3 = -\frac{1}{2}z$, $\frac{x-2}{-1} = \frac{y-3}{1} = \frac{z}{-2}$

$$\therefore \text{ the line has direction vector } \mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \text{ and passes through } (2, 3, 0)$$

$\therefore$ if P is a point on the line with coordinates $(2 - t, 3 + t, -2t)$,

$$\text{then } \overrightarrow{QP} = \begin{bmatrix} 3 - t \\ 1 + t \\ -2t - 3 \end{bmatrix}$$

If P is chosen such that $\overrightarrow{QP}$ is perpendicular to the line,
then $\mathbf{u} \bullet \overrightarrow{QP} = 0$

$$\therefore \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \bullet \begin{bmatrix} 3 - t \\ 1 + t \\ -2t - 3 \end{bmatrix} = 0$$

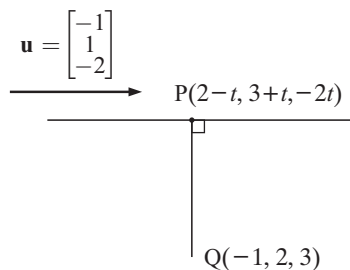
$$\therefore -(3 - t) + 1(1 + t) - 2(-2t - 3) = 0$$

$$\therefore -3 + t + 1 + t + 4t + 6 = 0$$

$$\therefore 6t = -4$$

$$\therefore \text{ meet at } P\left(\frac{8}{3}, \frac{7}{3}, \frac{4}{3}\right).$$

$$\therefore t = -\frac{2}{3}$$



- 8** $P(2, 0, 1)$ $Q(3, 4, -2)$ $R(-1, 3, 2)$

a $\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$

$$\begin{aligned}|\overrightarrow{PQ}| &= \sqrt{1 + 16 + 9} \\ &= \sqrt{26} \text{ units}\end{aligned}$$

and $\overrightarrow{QR} = \begin{bmatrix} -4 \\ -1 \\ 4 \end{bmatrix}$

b Since $\overrightarrow{PQ} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$ and P is at $(2, 0, 1)$,

the line has equation

$$x = 2 + t \quad y = 0 + 4t \quad z = 1 - 3t$$

$$\text{i.e., } x = 2 + t, \quad y = 4t, \quad z = 1 - 3t, \quad t \text{ in } \mathcal{R}$$

$$\mathbf{c} \quad \vec{PQ} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \quad \vec{PR} = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \therefore \vec{PQ} \times \vec{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & -3 \\ -3 & 3 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 4 & -3 \\ 3 & 1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -3 & 1 \\ 1 & -3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 4 \\ -3 & 3 \end{vmatrix} \\ &= 13\mathbf{i} + 8\mathbf{j} + 15\mathbf{k} \end{aligned}$$

$\therefore$ since P lies on the plane, it has equation

$$13x + 8y + 15z = 13(2) + 8(0) + 15(1)$$

$$\text{i.e., } 13x + 8y + 15z = 41$$

$$\therefore \text{ vector equation of plane is } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} + \mu \begin{bmatrix} -4 \\ -1 \\ 4 \end{bmatrix}, \quad \lambda, \mu \in \mathcal{R}$$

- 9 a** Given $A(-1, 3, 2)$ and the plane $2x - y + 2z = 8$,

$$\begin{aligned} \text{the distance from A to the plane is } d &= \frac{|2x_1 - y_1 + 2z_1 - 8|}{\sqrt{2^2 + (-1)^2 + 2^2}} \\ &= \frac{|2(-1) - 3 + 2(2) - 8|}{3} \\ &= \frac{|-9|}{3} \\ &= 3 \text{ units} \end{aligned}$$

- b** The point on the plane nearest A is the foot of the normal to the plane that passes through A.

Since the normal has direction vector $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and passes through $(-1, 3, 2)$,

it has equation $x = -1 + 2t$, $y = 3 - t$, $z = 2 + 2t$, $t \in \mathcal{R}$

and meets the plane when $2(-1 + 2t) - (3 - t) + 2(2 + 2t) = 8$

$$\therefore -2 + 4t - 3 + t + 4 + 4t = 8$$

$$\therefore 9t = 9$$

$$\therefore t = 1$$

$\therefore$ the point is $(1, 2, 4)$.

- c** Call the foot of the perpendicular from A to the line X, so X has coordinates $(7 - 2t, -6 + t, 1 + 5t)$ for some $t \in \mathcal{R}$. Then the shortest distance from A to the line is AX.

$$\text{Now } \vec{AX} = \begin{bmatrix} 8 - 2t \\ t - 9 \\ -1 + 5t \end{bmatrix} \text{ and since the line has direction vector } \mathbf{u} = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix},$$

$$\mathbf{u} \bullet \vec{AX} = 0$$

$$\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} 8 - 2t \\ t - 9 \\ -1 + 5t \end{bmatrix} = 0$$

$$-16 + 4t + t - 9 - 5 + 25t = 0$$

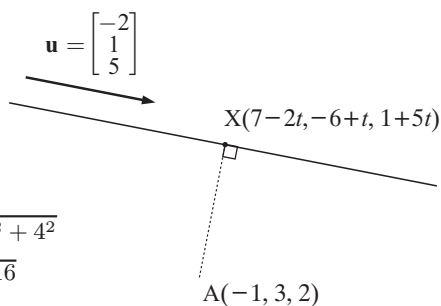
$$\therefore 30t = 30$$

$$\therefore t = 1$$

$$\therefore |AX| = \sqrt{6^2 + (-8)^2 + 4^2}$$

$$= \sqrt{36 + 64 + 16}$$

$$= \sqrt{116} \text{ units}$$



10 Given $A(-1, 0, 2)$, $B(0, -1, 1)$ and $C(1, 2, -1)$

a $\overrightarrow{AB} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$

$$\therefore \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 2 & 2 & -3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & -1 \\ 2 & -3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & 1 \\ -3 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

$\therefore$ since A lies on the plane it has equation $5x + y + 4z = 5(-1) + 0 + 4(2)$
i.e., $5x + y + 4z = 3$

b Since the normal has direction $\begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$ and passes through $(0, 0, 0)$, it has equation

$$x = 0 + 5t, \quad y = 0 + t, \quad z = 0 + 4t$$

i.e., $x = 5t, \quad y = t, \quad z = 4t, \quad t \text{ in } \mathcal{R}$

c The line meets the plane when $5(5t) + t + 4(4t) = 3$
 $\therefore 25t + t + 16t = 3$
 $\therefore 42t = 3$
 $\therefore t = \frac{1}{14}$ i.e., at $(\frac{5}{14}, \frac{1}{14}, \frac{2}{7})$

11 The system has augmented matrix:

$$\begin{bmatrix} 1 & -1 & 1 & 5 \\ 2 & 1 & -1 & -1 \\ 7 & 2 & k & -k \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -11 \\ 0 & 9 & k-7 & -k-35 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 7R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 3 & -3 & -11 \\ 0 & 0 & k+2 & -k-2 \end{bmatrix} \quad R_3 \rightarrow R_3 - 3R_2$$

Thus $(k+2)z = -(k+2)$

If $k \neq -2$ then $z = -1$, and as $3y - 3z = -11$,

$$\text{then } 3y = -14$$

$$\therefore y = -\frac{14}{3}$$

$$\text{and } x - y + z = 5,$$

$$\text{so } x = 5 - \frac{14}{3} + 1 = \frac{4}{3}$$

$\therefore$ we have three planes that meet at the unique point $(\frac{4}{3}, -\frac{14}{3}, -1)$

If $k = -2$, then the 3 planes meet in a common line and hence there are an infinite number of solutions.

In this case, let $z = t, \quad t \text{ in } \mathcal{R}$ Now $3y - 3z = -11$,
 $3y = -11 + 3t$
 $\therefore y = -\frac{11}{3} + t$

and as $x - y + z = 5$

$$\therefore x = 5 + y - z$$

$$\therefore x = 5 - \frac{11}{3} + t - t$$

$$\therefore x = \frac{4}{3}$$

$$\therefore x = \frac{4}{3} \quad y = -\frac{11}{3} + t \quad z = t, \quad t \text{ in } \mathcal{R}$$

REVIEW SET 17C

1 Given: $A(-1, 2, 3)$, $B(2, 0, -1)$ and $C(-3, 2, -4)$

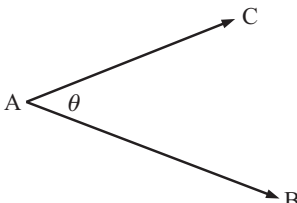
a $\vec{AB} = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}$ $\vec{AC} = \begin{bmatrix} -2 \\ 0 \\ -7 \end{bmatrix}$ $\therefore$ a normal vector to the plane is

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & -7 \\ 3 & -2 & -4 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & -7 \\ -2 & -4 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -2 & -2 \\ -4 & 3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 0 \\ 3 & -2 \end{vmatrix}$$

$$= -14\mathbf{i} - 29\mathbf{j} + 4\mathbf{k}$$

$\therefore$ since B lies on the plane, it has equation $14x + 29y - 4z = 14(2) + 29(0) - 4(-1)$
i.e., $14x + 29y - 4z = 32$

b

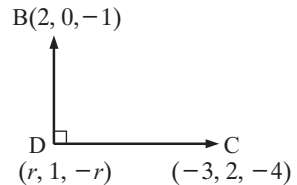


$$\cos \theta = \frac{|\vec{AB} \bullet \vec{AC}|}{|\vec{AB}| |\vec{AC}|}$$

$$= \frac{|3 \times -2 + -2 \times 0 + -4 \times -7|}{\sqrt{9+4+16} \sqrt{4+0+49}}$$

$$= \frac{22}{\sqrt{29 \times 53}} \text{ and so } \theta \doteq 55.9^\circ$$

c



If D is at $[r, 1, -r]$ then $\vec{DB} = \begin{bmatrix} 2-r \\ -1 \\ -1+r \end{bmatrix}$

and $\vec{DC} = \begin{bmatrix} -3-r \\ 1 \\ -4+r \end{bmatrix}$

Now $\angle BDC$ is a right-angle, so $\vec{DB} \bullet \vec{DC} = 0$

$$\therefore (2-r)(-3-r) + (-1) + (-4+r)(-1+r) = 0$$

$$\therefore -6 - 2r + 3r + r^2 - 1 + 4 - 4r - r + r^2 = 0$$

$$\therefore 2r^2 - 4r - 3 = 0$$

$$\therefore r = \frac{4 \pm \sqrt{16+24}}{4}$$

$$\therefore r = \frac{2 \pm \sqrt{10}}{2}$$

2 a $\vec{LM} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ since L lies on the line, it has parametric equations
 $x = 1 + t$, $y = 0 - t$, $z = 1 + t$, t in $\mathcal{R}$

The line meets $x - 2y - 3z = 14$ if

$$(1+t) - 2(-t) - 3(1+t) = 14$$

$$\therefore 1+t+2t-3-3t = 14$$

$$\therefore -2 = 14, \text{ which is absurd.}$$

$\therefore$ the line and plane do not meet (they are parallel).

b Since the line and the plane are parallel we may use any point on the line to determine the shortest distance to the plane. Hence, using L, the distance

$$d = \frac{|x_1 - 2y_1 - 3z_1 - 14|}{\sqrt{1+4+9}} = \frac{|1 - 2(0) - 3(1) - 14|}{\sqrt{14}} = \frac{16}{\sqrt{14}} \text{ units}$$

- 3 a** Given $A(-1, 2, 3)$, $B(1, 0, -1)$ and $C(1, 3, 0)$,

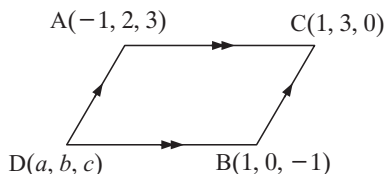
$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

$\therefore$ a normal to the plane containing A, B and C is

$$\therefore \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -2 \\ 2 & 1 & -3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & -2 \\ 1 & -3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -2 & 1 \\ -3 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$\therefore \mathbf{n} = 5\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

b



Suppose D has coordinates (a, b, c)

$$\therefore \text{since } \overrightarrow{AD} = \overrightarrow{CB},$$

$$\begin{bmatrix} a+1 \\ b-2 \\ c-3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}$$

$$\therefore a = -1, \quad b = -1 \quad \text{and} \quad c = 2$$

$$\therefore D \text{ is at } (-1, -1, 2)$$

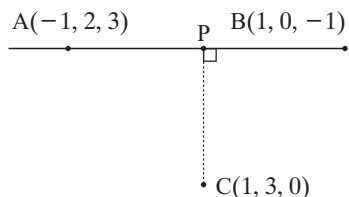
c

From **a**, $\overrightarrow{AB}$ has direction vector $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

$\therefore$ the line through A and B has parametric equations

$$x = 1 + t, \quad y = 0 - t, \quad z = -1 - 2t, \quad t \in \mathcal{R}.$$

$\therefore$ if $P(1 + t, -t, -1 - 2t)$ is the foot of the perpendicular,



$$\text{then } \overrightarrow{CP} = \begin{bmatrix} t \\ -t-3 \\ -1-2t \end{bmatrix}$$

$$\text{and } \overrightarrow{CP} \bullet \overrightarrow{AB} = 0 \quad \therefore \begin{bmatrix} t \\ -t-3 \\ -1-2t \end{bmatrix} \bullet \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} = 0$$

$$\therefore t + t + 3 + 2 + 4t = 0$$

$$\therefore 6t = -5$$

$$\therefore t = -\frac{5}{6}$$

$$\therefore P \text{ is } (1 - \frac{5}{6}, \frac{5}{6}, -1 + \frac{10}{6}) \quad \text{i.e., } (\frac{1}{6}, \frac{5}{6}, \frac{2}{3})$$

- 4** $x - 1 = \frac{y+2}{2} = \frac{z-3}{4}$ has direction vector $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $6x + 7y - 5z = 8$ has $\mathbf{n} = \begin{bmatrix} 6 \\ 7 \\ -5 \end{bmatrix}$.

$$\text{Now } \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \bullet \begin{bmatrix} 6 \\ 7 \\ -5 \end{bmatrix} = 6 + 14 - 20 = 0$$

$\therefore$ since these two vectors are perpendicular, the line is parallel to the plane.

Choose any point on the line, for example, $(1, -2, 3)$.

$$\begin{aligned} \text{Then the distance from the line to the plane is } d &= \frac{|6x_1 + 7y_1 - 5z_1 - 8|}{\sqrt{6^2 + 7^2 + (-5)^2}} \\ &= \frac{|6(1) + 7(-2) - 5(3) - 8|}{\sqrt{110}} \\ &= \frac{31}{\sqrt{110}} \text{ units} \end{aligned}$$

$$5 \quad a \quad \frac{x-3}{2} = \frac{y-4}{1} = \frac{z+1}{-2} \quad \text{has direction vector } \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

while $x = -1 + 3t$, $y = 2 + 2t$, $z = 3 - t$ has direction vector $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

$\therefore$ the lines are not parallel.

If the lines intersect, then $\frac{-1+3t-3}{2} = \frac{2+2t-4}{1} = \frac{3-t+1}{-2}$

i.e., $\frac{3}{2}t - 2 = 2t - 2 = \frac{t}{2} - 2$

Now $t = 0$ satisfies this relation, so the lines intersect at $(-1, 2, 3)$.

b If θ is the acute angle between the lines, then

$$\cos \theta = \frac{|\mathbf{v}_1 \bullet \mathbf{v}_2|}{|\mathbf{v}_1| |\mathbf{v}_2|} = \frac{|2 \times 3 + 1 \times 2 + -2 \times -1|}{\sqrt{9} \sqrt{14}} = \frac{|6 + 2 + 2|}{3\sqrt{14}} = \frac{10}{3\sqrt{14}}$$

$$6 \quad a \quad \overrightarrow{AB} = \begin{bmatrix} -2 \\ 2 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore$ since A lies on the line, it has equations $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

b If C lies on $\overrightarrow{AB}$ and is 2 units from A, then C corresponds to t such that

$$\sqrt{(t)^2 + (-t)^2 + (2t)^2} = 2$$

$$\therefore \sqrt{6t^2} = 2$$

$$\therefore 6t^2 = 4$$

$$\therefore t^2 = \frac{4}{6}$$

$$\therefore t = \pm \frac{2}{\sqrt{6}}$$

$$\therefore C \text{ is } \left(2 + \frac{2}{\sqrt{6}}, -1 - \frac{2}{\sqrt{6}}, 3 + \frac{4}{\sqrt{6}}\right) \quad \text{or} \quad \left(2 - \frac{2}{\sqrt{6}}, -1 + \frac{2}{\sqrt{6}}, 3 - \frac{4}{\sqrt{6}}\right)$$

7 Given $A(-1, 2, 3)$ $B(1, 0, -1)$ and $C(0, -1, 5)$

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \quad \therefore \text{a normal to the plane is}$$

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -4 \\ 1 & -3 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -2 & -4 \\ -3 & 2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -4 & 2 \\ 2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -2 \\ 1 & -3 \end{vmatrix}$$

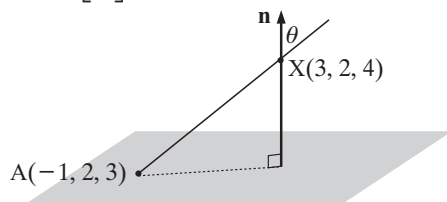
$$= -16\mathbf{i} - 8\mathbf{j} - 4\mathbf{k} \quad \text{or} \quad -4(4\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

$\therefore$ the plane has equation $4x + 2y + z = 4(1) + 2(0) + 1(-1)$

i.e., $4x + 2y + z = 3$

Given the plane has normal $\mathbf{n} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$ and $\overrightarrow{AX} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$,

$$\begin{aligned} \sin \phi &= \frac{|\mathbf{n} \bullet \overrightarrow{AX}|}{|\mathbf{n}| |\overrightarrow{AX}|} \\ &= \frac{|4 \times 4 + 2 \times 0 + 1 \times 1|}{\sqrt{21} \sqrt{17}} \\ &= \frac{17}{\sqrt{21} \sqrt{17}} \quad \text{and so} \quad \phi \doteq 64.1^\circ \end{aligned}$$



- 8 a** All vectors normal to $x - y + z = 6$ have the form $t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix}$, t in $\mathcal{R}$

$$\therefore \text{ if the vector has length 3 units, } \sqrt{t^2 + t^2 + t^2} = 3$$

$$\therefore 3t^2 = 9$$

$$\therefore t^2 = 3$$

$$\therefore t = \pm\sqrt{3}$$

$$\therefore \text{ the vectors are } \begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \\ \sqrt{3} \end{bmatrix} \text{ and } \begin{bmatrix} -\sqrt{3} \\ \sqrt{3} \\ -\sqrt{3} \end{bmatrix}$$

- b** Any vector parallel to $\mathbf{i} + r\mathbf{j} + 3\mathbf{k}$ has the form $t \begin{bmatrix} 1 \\ r \\ 3 \end{bmatrix} = \begin{bmatrix} t \\ rt \\ 3t \end{bmatrix}$, t in $\mathcal{R}$.

$$\text{This is perpendicular to } \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \text{ if } \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} t \\ rt \\ 3t \end{bmatrix} = 0$$

$$\therefore 2t - rt + 6t = 0$$

$$\therefore 8t - rt = 0$$

$$\therefore t(8 - r) = 0$$

$$\therefore t = 0 \text{ or } 8$$

$$\text{But if } t = 0, \text{ the vector has zero length } \therefore r = 8. \quad \therefore \text{ a vector is } \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix},$$

$$\therefore \text{ unit vectors are } \mathbf{u} = \frac{1}{\sqrt{74}}\mathbf{i} + \frac{8}{\sqrt{74}}\mathbf{j} + \frac{3}{\sqrt{74}}\mathbf{k} \quad \text{or} \quad -\frac{1}{\sqrt{74}}\mathbf{i} - \frac{8}{\sqrt{74}}\mathbf{j} - \frac{3}{\sqrt{74}}\mathbf{k}$$

- c** The distance from the plane to A is $d = \frac{|2x_1 - y_1 + 2z_1 - k|}{\sqrt{9}}$

$$\therefore \frac{|2(-1) - (2) + 2(3) - k|}{3} = 3 \quad \therefore |2 - k| = 9$$

$$\therefore 2 - k = 9 \quad \text{or} \quad k - 2 = 9$$

$$\therefore k = -7 \quad \text{or} \quad 11$$

- 9** If A is the origin, AB the x -axis, AD the y -axis, and AP the z -axis, then Q is (4, 0, 7), D is (0, 10, 0) and M is (0, 5, 7).

$$\text{Now } \overrightarrow{DQ} = \begin{bmatrix} 4 - 0 \\ 0 - 10 \\ 7 - 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 7 \end{bmatrix} \text{ and } \overrightarrow{DM} = \begin{bmatrix} 0 - 0 \\ 5 - 10 \\ 7 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$$

$$\text{So, if the required angle is } \theta, \quad \begin{bmatrix} 4 \\ -10 \\ 7 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix} = \sqrt{16 + 100 + 49} \sqrt{0 + 25 + 49} \cos \theta$$

$$\therefore 0 + 50 + 49 = \sqrt{165} \sqrt{74} \cos \theta$$

$$\therefore \frac{99}{\sqrt{165 \times 74}} = \cos \theta \quad \text{and so } \theta \doteq 26.4^\circ$$

$$\begin{aligned} \mathbf{10 \quad a} \quad \overrightarrow{PQ} &= \begin{bmatrix} 4 - -1 \\ 0 - 2 \\ -1 - 3 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} \end{aligned}$$

$$\mathbf{b} \quad \text{For the } x\text{-axis, } \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Now } \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \sqrt{25 + 4 + 16} \sqrt{1 + 0 + 0} \cos \theta$$

$$\therefore 5 + 0 + 0 = \sqrt{45} \cos \theta$$

$$\therefore \cos \theta = \frac{5}{\sqrt{45}} \quad \text{and so } \theta \doteq 41.8^\circ$$

REVIEW SET 17D

$$1 \quad a \quad \vec{AB} = \begin{bmatrix} 2-4 \\ 1-2 \\ 5-(-1) \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix} \quad \text{and} \quad \vec{AC} = \begin{bmatrix} 9-4 \\ 4-2 \\ 1-(-1) \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$$

$$\therefore \vec{AB} \bullet \vec{AC} = (-2)(5) + (-1)(2) + (6)(2) = -10 - 2 + 12 = 0$$

$$\therefore \vec{AB} \perp \vec{AC}$$

$$b \quad \mathbf{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -1 & 6 \\ 5 & 2 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 6 \\ 2 & 2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 6 & -2 \\ 2 & 5 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & -1 \\ 5 & 2 \end{vmatrix}$$

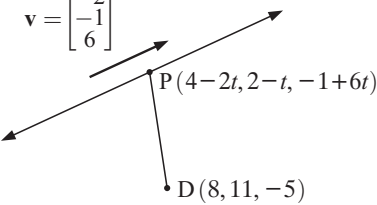
$$= -14\mathbf{i} + 34\mathbf{j} + \mathbf{k}$$

$$\therefore \text{the equation is } -14x + 34y + z = -14(4) + 34(2) + (-1) = 11$$

$$\text{i.e., } 14x - 34y - z = -11$$

$$\text{and distance} = \frac{|14(8) - 34(1) - (-1)|}{\sqrt{14^2 + (-34)^2 + (-1)^2}} = \frac{78}{\sqrt{1353}} \text{ units } (\div 2.12 \text{ units})$$

$$c \quad \text{The equation is } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}, \quad t \in \mathcal{R}$$

$$d \quad \mathbf{v} = \begin{bmatrix} -2 \\ -1 \\ 6 \end{bmatrix}$$


$$\vec{DP} = \begin{bmatrix} 4-2t-8 \\ 2-t-11 \\ -1+6t-(-5) \end{bmatrix} = \begin{bmatrix} -4-2t \\ -9-t \\ 4+6t \end{bmatrix}$$

$$\text{and } \vec{DP} \bullet \mathbf{v} = 0$$

$$\therefore -2(-4-2t) - 1(-9-t) + 6(4+6t) = 0$$

$$\therefore 8+4t+9+t+24+36t = 0$$

$$\therefore 41t = -41$$

$$\therefore t = -1$$

$$\therefore \vec{DP} = \begin{bmatrix} -2 \\ -8 \\ -2 \end{bmatrix} \quad \text{and} \quad |\vec{DP}| = \sqrt{4+64+4}$$

$$= \sqrt{72}$$

$$= 6\sqrt{2} \text{ units}$$

$$2 \quad a \quad l_1 \text{ meets } 2x + y - z = 2 \quad \text{where} \quad 2(3t-4) + (t+2) - (2t-1) = 2$$

$$\therefore 6t - 8 + t + 2 - 2t + 1 = 2$$

$$\therefore 5t = 7$$

$$\therefore t = \frac{7}{5}$$

$$\text{i.e., at } (3(\frac{7}{5}) - 4, \frac{7}{5} + 2, 2(\frac{7}{5}) - 1)$$

$$\text{i.e., } (\frac{1}{5}, \frac{17}{5}, \frac{9}{5})$$

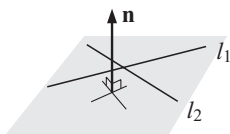
$$b \quad l_1 \text{ meets } l_2 \quad \text{where} \quad 3t - 4 = \frac{t+2-5}{2} = \frac{-(2t-1)-1}{2}$$

$$\text{i.e., } 6t - 8 = t - 3 = -2t$$

$$\therefore 5t = 5 \quad \text{and} \quad 3t = 3$$

$$\text{i.e., } t = 1$$

$$\therefore \text{they meet at } (-1, 3, 1)$$

c


$$\begin{aligned} \mathbf{n} &= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 2 & -2 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -6\mathbf{i} + 8\mathbf{j} + 5\mathbf{k} \\ \therefore \text{equation is } -6x + 8y + 5z &= -6(-1) + 8(3) + 5(1) \\ \text{i.e., } -6x + 8y + 5z &= 35 \\ \text{i.e., } 6x - 8y - 5z &= -35 \end{aligned}$$

3 a We substitute l_1 into the LHS of the plane's equation

$$2(-2t + 2) + (t) + (3t + 1) = -4t + 4 + t + 3t + 1 = 5 \quad \checkmark$$

 $\therefore$ the plane contains the line.

b If $x + ky + z = 3$ contains l_1 then $(-2t + 2) + k(t) + 3t + 1 = 3$

$$\therefore t[-2 + k + 3] + 2 + 1 = 3$$

$$\therefore t[k + 1] = 0$$

$$\therefore k = -1 \quad \text{as } t \in \mathcal{R}$$

c From **a** and **b**, both $2x + y + z = 5$ and $x - y + z = 3$ contain l_1 .

 So, substituting l_1 into plane 3 gives

$$-2(-2t + 2) + pt + 2(3t + 1) = q \quad \text{for all } t \text{ in } \mathcal{R}$$

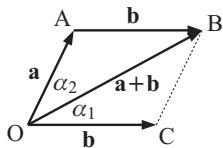
$$\therefore 4t - 4 + pt + 6t + 2 = q \quad \text{for all } t \text{ in } \mathcal{R}$$

$$\therefore [10 + p]t - 2 = q \quad \text{for all } t$$

 This equation has infinitely many solutions for b

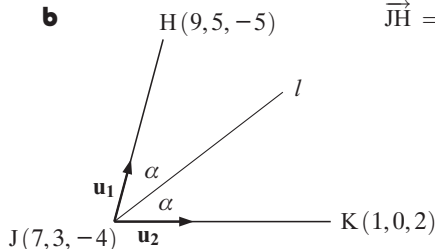
 when $10 + p = 0$ and $-2 = q$ {equating coefficients}

$$\therefore p = -10 \quad \text{and} \quad q = -2$$

4 a

 As $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, OACB is a rhombus.

But the angles of a rhombus are bisected by its diagonals

$$\therefore \alpha_1 = \alpha_2$$

b


$$\vec{JH} = \begin{bmatrix} 9-7 \\ 5-3 \\ -5-(-4) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \quad \vec{JK} = \begin{bmatrix} 1-7 \\ 0-3 \\ 2-(-4) \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$$

$$\therefore \mathbf{u}_1 = \frac{1}{\sqrt{4+4+1}} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$$

$$\text{and } \mathbf{u}_2 = \frac{1}{\sqrt{36+9+36}} \begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$\text{and } \mathbf{u}_1 + \mathbf{u}_2 = \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$\therefore \text{the line's equation is } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ -4 \end{bmatrix} + t \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \quad t \in \mathcal{R}$$

$$\text{c } \overrightarrow{HK} = \begin{bmatrix} 1-9 \\ 0-5 \\ 2-5 \end{bmatrix} = \begin{bmatrix} -8 \\ -5 \\ 7 \end{bmatrix} \quad \therefore \text{ has equation } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + s \begin{bmatrix} -8 \\ -5 \\ 7 \end{bmatrix}$$

and this line meets the first line where:

$$7 = 1 - 8s, \quad 3 + \frac{t}{3} = -5s \quad \text{and} \quad -4 + \frac{t}{3} = 2 + 7s \quad \dots (*)$$

$$\therefore 8s = -6 \quad \text{i.e.,} \quad s = -\frac{3}{4} \quad \text{and so} \quad 3 + \frac{t}{3} = \frac{15}{4} \quad \therefore \frac{t}{3} = \frac{3}{4} \quad \therefore t = \frac{9}{4}$$

$$\begin{array}{ll} \text{In } (*) \quad \text{LHS} = -4 + \frac{t}{3} & \text{RHS} = 2 + 7s \\ & = 2 + 7\left(-\frac{3}{4}\right) \\ & = \frac{8}{4} - \frac{21}{4} \\ & = -\frac{13}{4} \end{array}$$

$$= -\frac{13}{4} \quad \checkmark$$

$$\therefore s = -\frac{3}{4}, \quad t = \frac{9}{4} \quad \text{satisfy all 3 equations}$$

$$\text{So } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} -8 \\ -5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1+6 \\ 0+\frac{15}{4} \\ 2-\frac{21}{4} \end{bmatrix} = \begin{bmatrix} 7 \\ 3\frac{3}{4} \\ -3\frac{1}{4} \end{bmatrix} \quad \therefore M \text{ is } (7, 3\frac{3}{4}, -3\frac{1}{4})$$

$$\text{5 If } A \text{ is } (3, -1, -2) \text{ and } B(5, 3, -4) \text{ then } \overrightarrow{AB} = \begin{bmatrix} 5-3 \\ 3-(-1) \\ -4-(-2) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\therefore \text{ the line has equation } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad t \in \mathcal{R}$$

and it meets $x^2 + y^2 + z^2 = 26$ where

$$(3+t)^2 + (-1+2t)^2 + (-2-t)^2 = 26$$

$$\therefore 9 + 6t + t^2 + 1 - 4t + 4t^2 + 4 + 4t + t^2 - 26 = 0$$

$$\therefore 6t^2 + 6t - 12 = 0$$

$$\therefore t^2 + t - 2 = 0$$

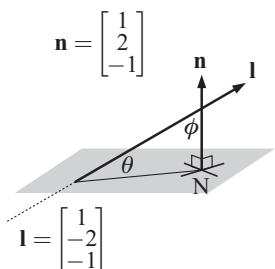
$$\therefore (t+2)(t-1) = 0$$

$$\therefore t = -2 \text{ or } 1$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\therefore \text{ meet at } (1, -5, 0) \text{ and } (4, 1, -3)$$

6



$$\mathbf{n} \bullet \mathbf{l} = |\mathbf{n}| |\mathbf{l}| \cos \phi$$

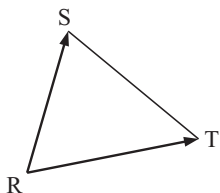
$$\therefore \cos \phi = \frac{\mathbf{n} \bullet \mathbf{l}}{|\mathbf{n}| |\mathbf{l}|}$$

$$\therefore \sin \theta = \frac{|\mathbf{n} \bullet \mathbf{l}|}{|\mathbf{n}| |\mathbf{l}|} \quad \text{as} \quad \cos \phi = \cos \left(\frac{\pi}{2} - \theta \right) = \sin \theta$$

$$= \frac{|2 - 4 + 1|}{\sqrt{4 + 4 + 1} \sqrt{1 + 4 + 1}}$$

$$= \frac{1}{\sqrt{54}}$$

$$\therefore \theta \doteq 7.82^\circ$$

7

$$\overrightarrow{RS} = \overrightarrow{RO} + \overrightarrow{OS} = \overrightarrow{OS} - \overrightarrow{OR}$$

$$\begin{aligned}\therefore \overrightarrow{RS} &= 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} - 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \\ &= 3\mathbf{j} + 3\mathbf{k}\end{aligned}$$

$$\text{Likewise } \overrightarrow{RT} = \overrightarrow{OT} - \overrightarrow{OR} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} - 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} = -\mathbf{i} + 4\mathbf{j}$$

$$\text{Now area} = \frac{1}{2} |\overrightarrow{RS} \times \overrightarrow{RT}|$$

$$\begin{aligned}&= \frac{1}{2} \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 3 \\ -1 & 4 & 0 \end{vmatrix} \right| \\ &= \frac{1}{2} \left| \mathbf{i} \begin{vmatrix} 3 & 3 \\ 4 & 0 \end{vmatrix} + \mathbf{j} \begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 0 & 3 \\ -1 & 4 \end{vmatrix} \right| \\ &= \frac{1}{2} |-12\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}| \\ &= \frac{1}{2} \sqrt{144 + 9 + 9} \\ &= \frac{1}{2} \sqrt{162} \\ &= \frac{1}{2} 9\sqrt{2} \\ &= \frac{9\sqrt{2}}{2} \text{ units}^2\end{aligned}$$

$$\mathbf{8} \quad \mathbf{a} \quad X \text{ is } \left(\frac{4+10}{2}, \frac{4+2}{2}, \frac{-2+0}{2} \right) \text{ i.e., } (7, 3, -1)$$

$$\text{If } D \text{ is } (a, b, c) \quad \overrightarrow{AD} = \begin{bmatrix} a-1 \\ b-3 \\ c-4 \end{bmatrix} = \begin{bmatrix} a-1 \\ b-3 \\ c+4 \end{bmatrix} \quad \text{and} \quad \overrightarrow{BC} = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}$$

$$\begin{aligned}\therefore a-1 &= 6, & b-3 &= -2, & c+4 &= 2 \\ \text{i.e., } a &= 7, & b &= 1, & c &= -2\end{aligned} \quad \therefore D \text{ is } (7, 1, -2)$$

$$\mathbf{b} \quad \overrightarrow{OY} = \overrightarrow{OA} + \overrightarrow{AY} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} + \frac{2}{3} \overrightarrow{AX} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 7-1 \\ 3-3 \\ -1-4 \end{bmatrix}$$

$$\therefore \overrightarrow{OY} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \quad \text{and so } Y \text{ is } (5, 3, -2)$$

$$\mathbf{c} \quad \overrightarrow{BY} = \begin{bmatrix} 5-4 \\ 3-4 \\ -2-(-2) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \overrightarrow{BD} = \begin{bmatrix} 7-4 \\ 1-4 \\ -2-(-2) \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$$

$$\therefore \overrightarrow{BD} = 3\overrightarrow{BY} \quad \text{and so} \quad \overrightarrow{BD} \parallel \overrightarrow{BY} \Rightarrow B, D, Y \text{ are collinear}$$

$$\mathbf{9} \quad \mathbf{a} \quad \mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \\ = -3\mathbf{j} - 3\mathbf{k}$$

$$\mathbf{b} \quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 0 & -3 & -3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & -1 \\ -3 & -3 \end{vmatrix} + \mathbf{j} \begin{vmatrix} -1 & 3 \\ -3 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 3 & 2 \\ 0 & -3 \end{vmatrix} \\ = -9\mathbf{i} + 9\mathbf{j} - 9\mathbf{k} \quad \checkmark$$

and $\mathbf{b}(\mathbf{a} \bullet \mathbf{c}) - \mathbf{c}(\mathbf{a} \bullet \mathbf{b})$

$$\begin{aligned} &= [(3)(2) + (2)(-1) + (-1)(1)] \mathbf{b} - [(3)(1) + (2)(1) + (-1)(-1)] \mathbf{c} \\ &= 3\mathbf{b} - 6\mathbf{c} \\ &= 3(\mathbf{i} + \mathbf{j} - \mathbf{k}) - 6(2\mathbf{i} - \mathbf{j} + \mathbf{k}) \\ &= 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} - 12\mathbf{i} + 6\mathbf{j} - 6\mathbf{k} \\ &= -9\mathbf{i} + 9\mathbf{j} - 9\mathbf{k} \quad \checkmark \end{aligned}$$

10 a

$\mathbf{a} \perp \mathbf{b}$

$$\therefore \mathbf{a} \bullet \mathbf{b} = 0$$

$$\therefore (1)(-t) + (2)(1+t) + (-2)(2t) = 0$$

$$\therefore -t + 2 + 2t - 4t = 0$$

$$\therefore -3t = -2$$

$$\therefore t = \frac{2}{3}$$

b

$\mathbf{a} \parallel \mathbf{b}$

$$\Rightarrow \mathbf{b} = k\mathbf{a} \text{ for some scalar } k$$

$$\therefore \begin{bmatrix} -t \\ 1+t \\ 2t \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$\therefore -t = k, \quad 1+t = 2k, \quad 2t = -2k$$

$$\therefore 1+t = 2(-t)$$

$$\therefore 1 = -3t$$

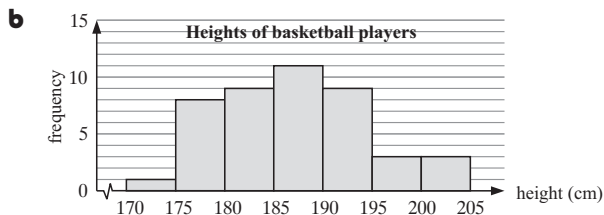
$$\therefore t = -\frac{1}{3}$$

Chapter 18

DESCRIPTIVE STATISTICS

EXERCISE 18A

- 1 a Heights can take any value from 170 cm to 205 cm, including decimal values such as 181.372 cm, i.e., any real number between 170 and 205.



- c The modal class is the class occurring most often. This is '185 -'.

- d The distribution is slightly positively skewed (more of a 'tail' to the right (positive)).

- 2 a The data is continuous numerical. Actual time is continuous and could be measured to the nearest millisecond. After it has been rounded to the nearest minute it becomes discrete numerical data.

b

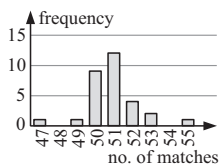
Stem	Leaf
0	3 6 8 8 8 8
1	0 0 0 0 2 2 2 2 4 4 4 4 5 5 5 5 6 6 6 6 7 8 8 8 8 9
2	0 0 0 1 2 4 5 5 5 6 7 7 8
3	1 2 2 2 3 4 5 7 8
4	0 2 5 5 5 6

1 | 2 means 12 minutes

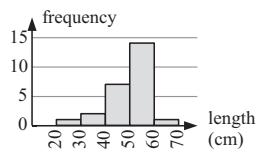
- c The distribution is positive-ly skewed (i.e., skewed to the high end).

- d "The modal travelling time was between 10 and 20 min" if considering classes.
The mode is actually 10.

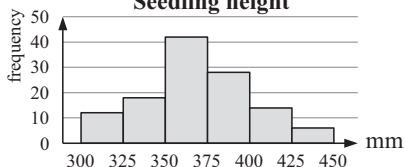
- 3 a The data is discrete numerical, so a column graph should be used.



- b The data is continuous, so a histogram should be used.



- 4 a



- b Number which are ≥ 400 mm is $14 + 6 = 20$ seedlings.

- c % between 349 and 400 = $\frac{42 + 28}{120} \times 100\%$
 $= \frac{70}{120} \times 100\%$
 $\div 58.3\%$

d i

$$\begin{aligned} \text{Number} &= \frac{12 + 18 + 42 + 28}{120} \times 1462 \\ &= \frac{100}{120} \times 1462 \\ &\div 1218 \text{ seedlings} \end{aligned}$$

ii

$$\begin{aligned} \text{Number} &= \frac{28 + 14}{120} \times 1462 \\ &= \frac{42}{120} \times 1462 \\ &\div 512 \text{ seedlings} \end{aligned}$$

EXERCISE 18B.1

1 a i

$$\begin{aligned} \text{mean} &= \frac{2+3+3+3+4+\dots+9+9}{23} \\ &= \frac{129}{23} \\ &\div 5.61 \end{aligned}$$

ii median = 12th score (when in order)
 $= 6$

iii mode = 6 (6 occurs most often)

$$\begin{array}{ll} \text{b i} & \text{mean} = \frac{10+12+12+15+\dots+20+21}{15} \\ & = \frac{245}{15} \\ & \div 16.3 \end{array} \quad \begin{array}{l} \text{i} \quad \text{median} = 8\text{th score (when in order)} \\ \quad = 17 \\ \text{iii} \quad \text{mode} = 18 \end{array}$$

$$\begin{array}{ll} \text{c i} & \text{mean} = \frac{22.4+24.6+21.8+\dots+23.5}{11} \\ & = \frac{273}{11} \\ & \div 24.8 \end{array} \quad \begin{array}{l} \text{ii} \quad \text{median} = 6\text{th score (when in order)} \\ \quad = 24.9 \\ \text{iii} \quad \text{mode} = 23.5 \end{array}$$

$$\begin{array}{ll} \text{2 a} & \text{mean of set A} \qquad \qquad \qquad \text{mean of set B} \\ & = \frac{3+4+4+5+\dots+10}{13} \qquad \qquad \qquad = \frac{3+4+4+5+\dots+15}{13} \\ & = 6.46 \qquad \qquad \qquad = 6.85 \end{array}$$

b median of set A = 7th score = 7 median of set B = 7th score = 7

c The data sets are the same except for the last value, and the last value of set A is less than that of set B. So, the mean of set A is less than that of set B.

d The middle value of both data sets is the same, so the median is the same.

$$\begin{array}{l} \text{3 a} \quad \text{mean} = \frac{23\,000 + 46\,000 + 23\,000 + \dots + 32\,000}{10} = \$29\,300 \\ \text{median} = \text{middle score when in order of size} = \frac{\$23\,000 + \$24\,000}{2} = \$23\,500 \\ \text{mode} = \$23\,000 \end{array}$$

b The mode is unsatisfactory because it is the lowest salary. It does not take the higher values into account.

c The median is too close to the lower end of the distribution and it does not take the higher values into account. So the median is not a satisfactory measure of the middle.

$$\begin{array}{l} \text{4 a} \quad \text{mean} = \frac{3+1+0+0+\dots+1+0+0}{31} = \frac{99}{31} \div 3.19 \\ \text{median} = 16\text{th score (when in order)} = 0 \\ \text{mode} = 0 \quad (\text{most frequently occurring score}) \end{array}$$

b The median is not in the centre. It ignores the high upper values of the distribution.

c The mode is the lowest value. It does not take the higher values into account.

d Yes, 42 and 21. **e** No, as this would ignore actual valid data.

$$\text{5 a} \quad \text{mean} = \frac{43+55+41+37}{4} = \frac{176}{4} = 44 \qquad \text{b} \quad \text{another 44}$$

$$\text{c} \quad \text{new mean} = \frac{43+55+41+37+25}{5} = 40.2$$

d It will increase the new mean to 40.3 as 41 is greater than the old mean of 40.2.

$$\left\{ \frac{5 \times 40.2 + 41}{6} \div 40.3 \right\}$$

$$\text{6} \quad \text{mean} = \frac{\text{total}}{10}, \therefore 11.6 = \frac{\text{total}}{10}, \therefore \text{total} = 11.6 \times 10 = 116$$

$$\text{7} \quad \text{mean} = \frac{\text{total}}{12}, \therefore 262 = \frac{\text{total}}{12}, \therefore \text{total} = 262 \times 12 = 3144 \text{ km}$$

$$\text{8} \quad \text{mean} = \frac{\text{total}}{12}, \therefore 15\,467 = \frac{\text{total}}{12}, \therefore \text{total} = 15\,467 \times 12 = \$185\,604$$

$$9 \quad \frac{5 + 9 + 11 + 12 + 13 + 14 + 17 + x}{8} = 12$$

$$\therefore \frac{81 + x}{8} = 12$$

$$\therefore 81 + x = 96$$

$$\therefore x = 15$$

$$10 \quad \frac{3 + 0 + a + a + 4 + a + 6 + a + 3}{9} = 4$$

$$\therefore \frac{4a + 16}{9} = 4$$

$$\therefore 4a + 16 = 36$$

$$\therefore 4a = 20$$

$$\therefore a = 5$$

$$11 \quad \frac{29 + 36 + 32 + 38 + 35 + 34 + 39 + x}{8} = 35$$

$$\therefore \frac{243 + x}{8} = 35$$

$$\therefore 243 + x = 280$$

$$\therefore x = 37, \text{ so her 8th result was 37}$$

$$12 \quad \text{Total for first 10 measurements} = 10 \times 15.7 = 157$$

$$\text{Total for next 20 measurements} = 20 \times 14.3 = 286$$

$$\therefore \text{mean} = \frac{157 + 286}{30} \div 14.8$$

$$13 \quad \text{Scores were } 5 \ 7 \ 9 \ 9 \ 10 \ a \ b \text{ where } a \leq b \text{ say.}$$

$$\text{mean} = \frac{5 + 7 + 9 + 9 + 10 + a + b}{7} = 8$$

$$\therefore \frac{40 + a + b}{7} = 8$$

$$\therefore 40 + a + b = 56$$

$$\therefore a + b = 16 \quad \{a \leq 12, \ b \leq 12\}$$

Possibilities are:

a	5	6	7	8
b	11	10	9	8

×

×

✓

×

↑ reject as modes are 8 and 9
 ↑ reject as modes are 9 and 10
 ↑ reject as modes are 5 and 9

So, the missing results are 7 and 9.

EXERCISE 18B.2

1 a The mode is 1 (occurs most often).

b The median is the average of the 15th and 16th scores
 $= \frac{1 + 1}{2} = 1$

c

x	f	fx
0	4	0
1	12	12
2	11	22
3	3	9
Σ	30	43

$$\text{mean} = \frac{\Sigma fx}{\Sigma f} = \frac{43}{30} \div 1.43$$

2 a i

x	f	fx
0	5	0
1	8	8
2	13	26
3	8	24
4	6	24
5	3	15
6	3	18
7	2	14
8	1	8
9	0	0
10	0	0
11	1	11
Σ	50	148

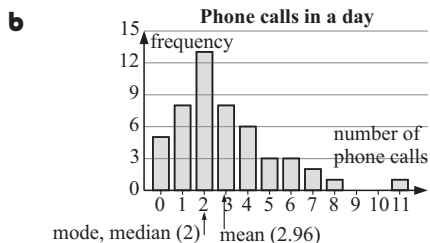
$$\begin{aligned} \text{mean} &= \frac{\Sigma fx}{\Sigma f} \\ &= \frac{148}{50} \\ &= 2.96 \end{aligned}$$

ii median
 = average of 25th and 26th (when in order)

$$= \frac{2 + 2}{2} \quad \left\{ \begin{array}{l} 13 \text{ scores are 1 or 0} \\ 26 \text{ scores are 2, 1 or 0} \end{array} \right\}$$

$$= 2$$

iii mode = 2 {occurs most often}



- c** The distribution is positively skewed.
11 is an outlier.
- d** The mean takes into account the larger numbers of phone calls.
- e** The mean as it best represents all the data.

- 3 a i** mode = 49 {occurs most often}
- ii** median = average of 15th and 16th values (when in order)
- $$= \frac{49 + 49}{2} = 49 \quad \{9 \text{ are } 47 \text{ or } 48 \text{ and the next } 11 \text{ are } 49\}$$

iii

x	f	fx
47	5	235
48	4	192
49	11	539
50	6	300
51	3	153
52	1	52
Σ	30	1471

$$\begin{aligned} \text{mean} &= \frac{\sum fx}{\sum f} \\ &= \frac{1471}{30} \\ &\div 49.0 \end{aligned}$$

- b** No, as they claim the average (mean) is 50 matches/box.

The sample of only 30 is not large enough. The company could have won its case by arguing that a larger sample would have found an average of 50 matches per box.

4 a i

x	f	fx
1	5	5
2	28	56
3	15	45
4	8	32
5	2	10
6	1	6
Σ	59	154

$$\begin{aligned} \text{mean} &= \frac{\sum fx}{\sum f} \\ &= \frac{154}{59} \\ &\div 2.61 \end{aligned}$$

- ii** mode = 2 {occurs most often}

- iii** median = 30th score = 2

- b** This school has more children per family (2.61) than the average Australian family (2.2).

- c** Positive as the higher values are more spread out.

- d** The mean is higher than the mode and median.

5 a i mean = $\frac{53 + 55 + 56 + 60 + \dots + 91}{17} = \frac{1175}{17} \div 69.1$

- ii** median = 9th score (when in order) = 67

- iii** mode = 73 (73 is the only one occurring more than once)

b i mean = $\frac{37 + 40 + 44 + 48 + \dots + 81}{23} = \frac{1347}{23} \div 5.86$

- ii** median = 12th score (when in order) = 5.8 **iii** mode = 6.7 {occurs most often}

6 a Without fertiliser

2	
3	
4	
5	
6	
7	
8	
9	

x	f	fx	cf
2	2	4	2
3	11	33	13
4	19	76	32
5	29	145	61
6	51	306	112
7	25	175	137
8	12	96	149
9	1	9	150

i mean = $\frac{\sum fx}{\sum f} = \frac{845}{150} \div 5.63$

- ii** mode = 6 {occurs most often}

$$\text{iii} \quad \text{median} = \text{average of 75th and 76th scores} = \frac{6+6}{2} = 6$$

b With fertiliser

3	
4	
5	
6	
7	
8	
9	
10	
13	

x	f	fx	cf
3	4	12	4
4	13	52	17
5	11	55	28
6	28	168	56
7	48	336	104
8	27	216	131
9	14	126	145
10	4	40	149
13	1	13	150

$$\text{i} \quad \text{mean} = \frac{\sum fx}{\sum f} = \frac{1018}{150} \div 6.79$$

$$\text{ii} \quad \text{mode} = 7 \quad \{\text{occurs most often}\}$$

$$\text{iii} \quad \text{median} = \text{average of 75th and 76th scores} = \frac{7+7}{2} = 7$$

c The mean best represents the centre for this data.

d Yes, as 6.79 is significantly greater than 5.63.

Note: The total yield of the crop may not have improved as, for example, the number of pods per plant may have decreased when using the fertiliser.

$$7 \quad \text{a} \quad \text{mean selling price} = \frac{146\,400 + 127\,600 + 211\,000 + \dots + 162\,500}{10} = \$163\,770$$

$$\text{median selling price} = \frac{5\text{th} + 6\text{th}}{2} = \frac{146\,400 + 148\,000}{2} = \$147\,200$$

These figures differ by \$16\,570. There are more selling prices at the lower end of the market (i.e., smaller prices).

b i Use the mean as it tends to inflate the average house value of that district.

ii Use the median as you want to buy at the lowest price possible.

$$8 \quad \text{a} \quad \text{mean birth weight} = \frac{75 + 70 + 80 + \dots + 83}{8} = \frac{567}{8} \div 70.9 \text{ grams}$$

$$\text{b} \quad \text{mean after 2 weeks} = \frac{210 + 200 + 200 + \dots + 230}{8} = \frac{1681}{8} \div 210 \text{ grams}$$

$$\text{c} \quad \text{mean increase} \div (210.13 - 70.88) \text{ grams} \div 139 \text{ grams}$$

9 15 of these 10.1, 10.4, 10.7, 10.9, 12 of these

Median = 16th score (when in order) = 10.1 cm

10 a Brand A

x	f	fx
46	1	46
47	1	47
48	2	96
49	7	343
50	10	500
51	20	1020
52	15	780
53	3	159
55	1	55
Σ	60	3046

$$\begin{aligned} \text{mean} &= \frac{\sum fx}{\sum f} \\ &= \frac{3046}{60} \\ &\div 50.8 \end{aligned}$$

Brand B

x	f	fx
48	3	144
49	17	833
50	30	1500
51	7	357
52	2	104
53	1	53
54	1	54
Σ	61	3045

$$\begin{aligned} \text{mean} &= \frac{\sum fx}{\sum f} \\ &= \frac{3045}{61} \\ &\div 49.9 \end{aligned}$$

- b** Based on average contents the CPS should not prosecute either manufacturer. To the nearest match, the average contents of A is 51 and B, 50.

11 Total for first 14 matches = 14×16.5 goals = 231 goals

$$\therefore \text{new average} = \frac{231 + 21 + 24}{16} = \frac{276}{16} = 17.25 \text{ goals/game}$$

12 The measurements are 7, 9, 11, 13, 14, 17, 19, a , b where $a \leq b$

$$\begin{aligned} \text{mean} &= \frac{7 + 9 + 11 + 13 + \dots + a + b}{9} = \frac{90 + a + b}{9} \quad \therefore \frac{90 + a + b}{9} = 12 \\ &\quad \therefore 90 + a + b = 108 \\ &\quad \therefore a + b = 18 \end{aligned}$$

7 9 11 13 14 17 19

If $b \geq 13$, then $a \leq 5$ and the median = 13 ($\times$)

If $b = 12$, then $a = 6$ and the median = 12 ($\checkmark$)

The remaining cases are

a	7	8	9
b	11	10	9
median	11	11	11

So the other two data values are 6 and 12.

13 a i median salary

$$\begin{aligned} &= \frac{10\text{th} + 11\text{th}}{2} \quad \{\text{when in order}\} \\ &= \frac{35\,000 + 28\,000}{2} \\ &= \$31\,500 \end{aligned}$$

ii modal salary

$$= \$28\,000 \quad \{\text{occurs most often}\}$$

iii

x	f	fx
50 000	1	50 000
42 000	3	126 000
35 000	6	210 000
28 000	10	280 000
Σ	20	666 000

$$\begin{aligned} \text{mean} &= \frac{\Sigma fx}{\Sigma f} \\ &= \frac{666\,000}{20} \\ &= \$33\,300 \end{aligned}$$

- b** The mean as it is the highest value.

EXERCISE 18B.3

1

x (midpoint)	f	fx
4.5	2	9
14.5	5	72.5
24.5	7	171.5
34.5	27	931.5
44.5	9	400.5
Σ	50	1585

$$\begin{aligned} \therefore \text{mean result} &= \frac{1585}{50} \\ &= 31.7 \end{aligned}$$

2 a 70 **b** $\div 411\,000$ litres **c**

midpoint (x)	f	fx
2499.5	4	9998
3499.5	4	13 998
4499.5	9	40 495.5
5499.5	14	76 993
6499.5	23	149 488.5
7499.5	16	119 992
Σ	70	410 965

$$\begin{aligned} \text{Approximate mean} &= \frac{\Sigma fx}{\Sigma f} \\ &= \frac{410\,965}{70} \\ &\div 5870 \text{ litres} \end{aligned}$$

3 a $5 + 10 + 25 + 40 + 10 + 15 + 10 + 10 = 125$ people

b

midpoint (x)	frequency (f)	fx
85	5	425
95	10	950
105	25	2625
115	40	4600
125	10	1250
135	15	2025
145	10	1450
155	10	1550
Σ	125	14875

Approximate mean

$$= \frac{\sum fx}{\sum f}$$

$$= \frac{14875}{125}$$

$$\div 119 \text{ people}$$

c $\frac{15}{125}$ scored < 100

i.e., $\frac{3}{25}$ scored < 100

d 20% of 125 people = 25 people and 90 people scored < 130 for the test

$\therefore$ estimate is $130 + \frac{10}{15} \times 10 \div 137$ marks

EXERCISE 18C

1 a Total frequency = $1 + 1 + 0 + 3 + \dots + 0 + 1 = 50$

$\therefore$ the median is the average of the 25th and 26th scores

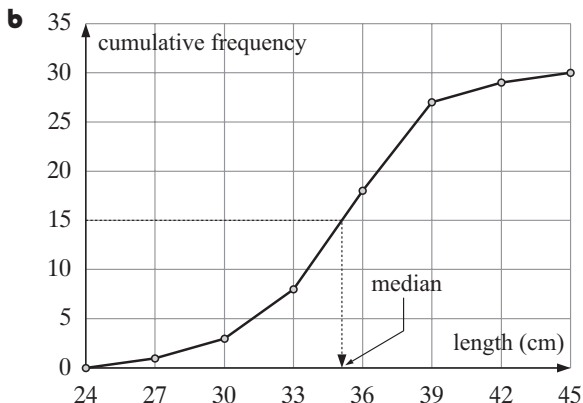
$$\left. \begin{array}{l} 23 \text{ are } 7\frac{1}{2} \text{ or less} \\ 40 \text{ are } 8 \text{ or less} \end{array} \right\} \therefore \text{median} = \frac{8+8}{2} = 8$$

b i $13 + 17 + 7 + 2 + 0 + 1 = 40$ people

ii $1 + 1 + 0 + 3 + 5 + 13 + 17 = 40$ are 8 or less

2 a

Length (x cm)	Freq.	C. freq.
$24 \leq x < 27$	1	1
$27 \leq x < 30$	2	3
$30 \leq x < 33$	5	8
$33 \leq x < 36$	10	18
$36 \leq x < 39$	9	27
$39 \leq x < 42$	2	29
$42 \leq x < 45$	1	30



c median $\div 35$

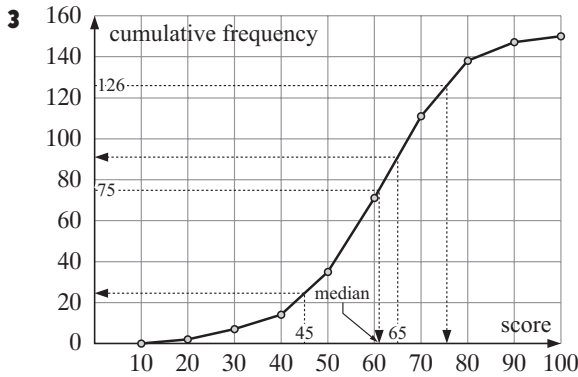
d There are 30 data values. So, the median is the average of the 15th and 16th scores (when in order).

In order they are:

24 27 28 30 31 31 32 32 33 33 33 33 34 34 34 35 35 35 36 etc.

$$\text{median} = \frac{34+35}{2} = 34.5$$

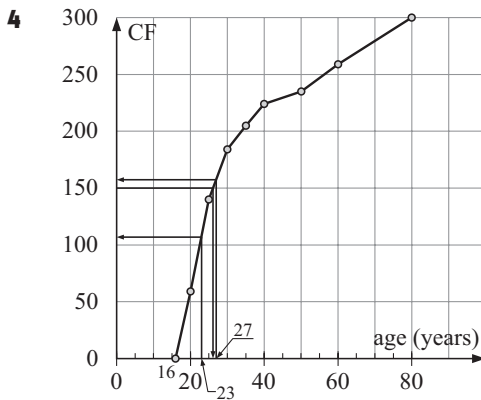
So, the median from the graph is a good approximation.



- a** Total frequency = 150
 $\therefore$ median $\div 61$ {from the graph}
- b** When the score = 65,
 CF $\div 91$ {from the graph}
 i.e., about 91 students scored 65 or less.
- c** $\div 36 + 40 = 76$ students scored between 50 and 70

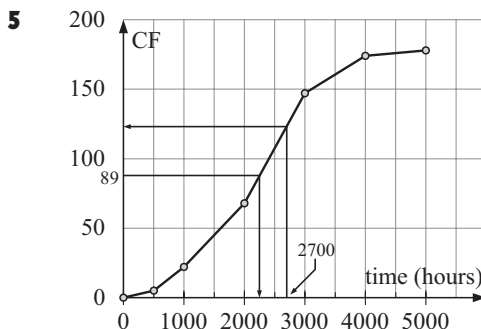
- d** For a pass mark of 45, CF $\div 24.5$
 $\therefore$ 24 or 25 students failed the exam.

- e** 84% of 150 = 126
 So, for a CF of 126, the score value is 76.
 So, the minimum credit mark would be 76.



Total frequency = 300

- a** median $\div 26$ years {from the graph}
- b** when the age = 23, CF $\div 108$
 and $\frac{108}{300} \times 100\% \div 36\%$
- c i** when age is 27, CF $\div 158$
 $\therefore P(\leq 27) = \frac{158}{300} = 0.527$
- ii** when age is 27, CF $\div 158$
 when age is 28, CF $\div 167$
 i.e., 9 died
 $\therefore P(\text{aged } 27) \div \frac{9}{300} \div 0.030$



Total frequency = 178

- a** median $\div 2300$ hours
- b** For a life of 2700 hours
 CF $\div 123$
 and $\frac{123}{178} \times 100\% \div 69\%$
 So, about 69% have a life ≤ 2700 h
- c** For a life of 1500 h, CF $\div 45$
 For a life of 2500 h, CF $\div 107.5$
 i.e., $107.5 - 45 = 62.5$
 $\therefore$ 62 or 63 failed

EXERCISE 18D.1**1 a** 2 3 3 3 4 4 4 5 5 5 5 6 6 6 6 7 7 8 8 8 9 9 ($n = 23$) $\uparrow$ Q_1 $\uparrow$

median

 $\uparrow$ Q_3 **i** median = 6**ii** $Q_1 = 4,$
 $Q_3 = 7$ **iii** range
 $= 9 - 2$
 $= 7$ **iv** IQR
 $= Q_3 - Q_1$
 $= 3$ **b** 10 12 12 14 15 15 16 16 17 18 18 18 18 19 20 21 22 24 ($n = 18$) $\uparrow$ Q_1 $\uparrow$

median

 $\uparrow$ Q_3 **i** median = 17.5**ii** $Q_1 = 15,$
 $Q_3 = 19$ **iii** range
 $= 24 - 10$
 $= 14$ **iv** IQR
 $= Q_3 - Q_1$
 $= 4$ **c** 21.8 22.4 23.5 23.5 24.6 24.9 25.0 25.3 26.1 26.4 29.5 ($n = 11$) $\uparrow$ Q_1 $\uparrow$

median

 $\uparrow$ Q_3 **i** median = 24.9**ii** $Q_1 = 23.5,$
 $Q_3 = 26.1$ **iii** range
 $= 29.5 - 21.8$
 $= 7.7$ **iv** IQR
 $= Q_3 - Q_1$
 $= 2.6$ **2** 0 0 0 0.8 1.4 1.5 1.6 1.9 2.1 2.2 2.7 3.0 3.4 3.6 3.8 3.8 4.5 4.8 5.2 5.2 $\uparrow$

min

 $\uparrow$ Q_1 $\uparrow$

median

 $\uparrow$ Q_3 $\uparrow$

max

a median = $\frac{2.2 + 2.7}{2}$
 $= 2.45$ $Q_1 = 1.45$
 $Q_3 = 3.8$ **b** range = $5.2 - 0$
 $= 5.2$ IQR = $Q_3 - Q_1$
 $= 3.8 - 1.45$
 $= 2.35$ **c i** the median, i.e., 2.45 min **ii** Q_3 , i.e., 3.8 min**iii** The minimum waiting time was 0 minutes and the maximum waiting time was 5.2 minutes.
The waiting time was spread over 5.2 minutes.**3** 3 4 7 9 10 13 14 16 17 18 20 20 23 25 26 29 29 29 31 33 37 38 42 ($n = 23$) $\uparrow$

min

 $\uparrow$ Q_1 $\uparrow$

median

 $\uparrow$ Q_3 $\uparrow$

max

a min = 3**b** max = 42**c** median = 20**d** $Q_1 = 13$ **e** $Q_3 = 29$ **f** range = $42 - 3 = 39$ **g** IQR = $Q_3 - Q_1 = 29 - 13 = 16$ $\downarrow$ min**4** 109 111 113 114 114 118 119 122 122 124 124 126 128 129 129 131 132 ($n = 20$) $\uparrow$

max

 $\uparrow$ Q_1 $\uparrow$

median

 $\uparrow$ Q_3 **a i** median = 124 cm**ii** $Q_3 = 130$ cm,
 $Q_1 = 116$ cm**d** the IQR, i.e., over 14 cm**b i** 124 cm tall**ii** 130 cm tall**c i** range = $138 - 109$
 $= 29$ cm**ii** IQR = $Q_3 - Q_1$
 $= 14$ cm

5 a Without fertiliser - See Exercise 18B.2 solution to question 6.

- i** range = $9 - 2 = 7$ **ii** median = 6
iii lower quartile = 38th score = 5 **iv** upper quartile = 113th score = 7
v interquartile range = $7 - 5 = 2$

b With fertiliser

- i** range = $13 - 3 = 10$ **ii** median = 7
iii lower quartile = 38th score = 6 **iv** upper quartile = 113th score = 8
v interquartile range = $8 - 6 = 2$

EXERCISE 18D.2

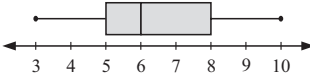
- 1 a i** median = 35 **ii** max. value = 78 **iii** min. value = 13
iv $Q_3 = 53$ **v** $Q_1 = 26$

b i range = $78 - 13 = 65$ **ii** $IQR = Q_3 - Q_1 = 53 - 26 = 27$

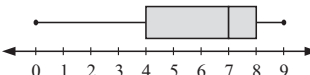
- 2 a** highest mark was 98, lowest mark was 25 **b** the median which is 70
c Q_3 which is 85 **d** $Q_1 = 55$ and $Q_3 = 85$ **e** range = $98 - 25 = 73$
f $IQR = Q_3 - Q_1 = 85 - 55 = 30$

- 3 a i** 3 4 5 5 5 6 6 6 7 7 8 8 9 10
 ↑ ↑ ↑ ↑ ↑
 min Q_1 median Q_3 max

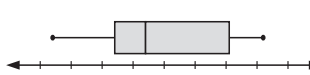
So, min = 3, $Q_1 = 5$, median = 6, $Q_3 = 8$, max = 10

- ii**  **iii** range = $10 - 3 = 7$ **iv** $IQR = Q_3 - Q_1 = 8 - 5 = 3$

- b i** 0 1 2 3 4 5 6 6 7 7 7 8 8 8 8 8 8 9 9
 ↑ ↑ ↑ ↑ ↑
 min Q_1 median Q_3 max
- So, min = 0, $Q_1 = 4$, median = 7, $Q_3 = 8$, max = 9

- ii**  **iii** range = $9 - 0 = 9$ **iv** $IQR = Q_3 - Q_1 = 8 - 4 = 4$

- c i** 117 120 123 126 126 128 130 131 131 131 133 135 135 137 144 147 147
 149 149 151 ↑ ↑ ↑ ↑
 ↑ median Q_3
 max Q_1
- So, min = 117, $Q_1 = 127$, median = 132, $Q_3 = 145.5$, max = 151

- ii**  **iii** range = $151 - 117 = 34$ **iv** $IQR = Q_3 - Q_1 = 145.5 - 127 = 18.5$

d $IQR = Q_3 - Q_1 = 31\frac{1}{2} - 27 = 4\frac{1}{2} \text{ min}$

e For the 40th percentile, $CF = 40\% \text{ of } 80 = 32$

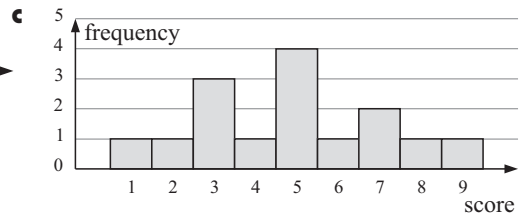
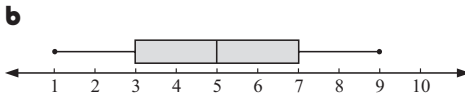
From the similar Δ s, $\frac{x - 27}{12} = \frac{3}{30}$

$\therefore x - 27 = 1.2$

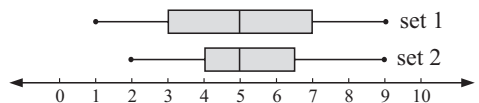
$\therefore x = 28.2$ So about 28 min 10 sec.

EXERCISE 18E.1

1 a $\bar{x} \doteq 4.87$, $\text{Min}_x = 1$, $Q_1 = 3$, $Q_2 = 5$, $Q_3 = 7$, $\text{Max}_x = 9$

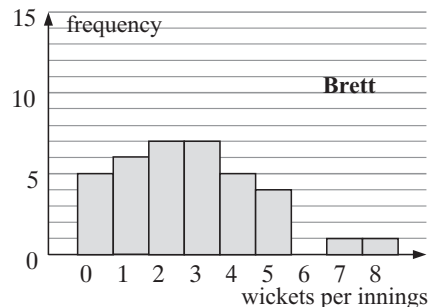
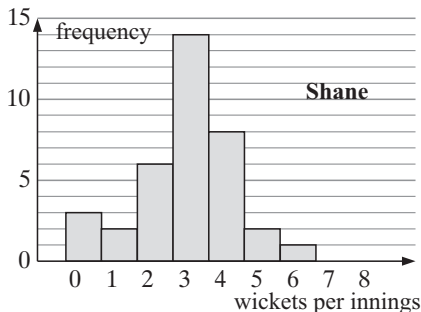


d $\bar{x} \doteq 5.24$, $\text{Min}_x = 2$, $Q_1 = 4$, $Q_2 = 5$, $Q_3 = 6.5$, $\text{Max}_x = 9$



EXERCISE 18E.2

1 a discrete **c**

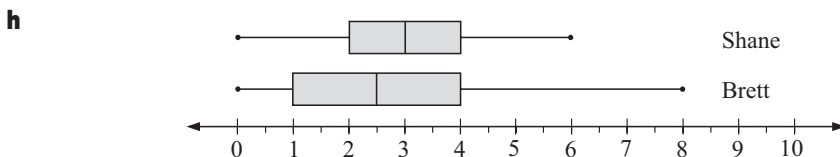


d There are no outliers for Shane. Brett has outliers of 7 and 8 which must not be removed.

e Shane's distribution is reasonably symmetrical. Brett's distribution is positively skewed.

f Shane has a higher mean ($\doteq 2.89$ wickets) compared with Brett ($\doteq 2.67$ wickets). Shane has a higher median (3 wickets) compared with Brett (2.5 wickets). Shane's modal number of wickets is 3 (14 times) compared with Brett, who has a bi-modal distribution of 2 and 3 (7 times each).

g Shane's range is 6 wickets, compared with Brett's range of 8 wickets. Shane's IQR is 2 wickets, compared with Brett's IQR of 3 wickets. Brett's wicket taking shows greater spread or variability.



j Generally, Shane takes more wickets than Brett and is a more consistent bowler.

2 a continuous

c For the 'New type' globes, 191 hours could be considered an outlier. However, it could be a genuine piece of data, so we will include it in the analysis.

d

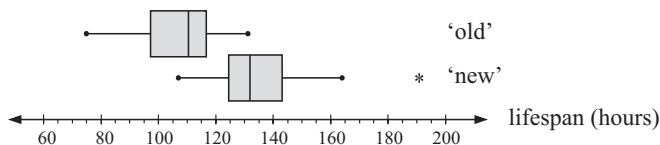
	Old type	New type
Mean	107	134
Median	110.5	132
Range	56	84
IQR	19	18.5

The mean and median are $\div 25\%$ and $\div 19\%$ higher for the 'new type' of globe compared with the 'old type'.

The range is higher for the 'new type' of globe (but has been affected by the 191 hours).

The IQR for each type of globe is almost the same.

e



f For the 'old type' of globe, the data is bunched to the right of the median, hence the distribution is negatively skewed. For the 'new type' of globe, the data is bunched to the left of the median, hence the distribution is positively skewed.

g The manufacturer's claim, that the 'new type' of globe has a 20% longer life than the 'old type' seems to be backed up by the 25% higher mean life and 19.5% higher median life.

EXERCISE 18F

1 a Sally

x	$(x - \bar{x})^2$
23	4
17	64
31	36
25	0
25	0
19	36
28	9
32	49
200	198

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n} \\ &= \frac{200}{8} \\ &= 25\end{aligned}$$

$$\begin{aligned}s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\ &= \sqrt{\frac{198}{8}} \\ &\div 4.97\end{aligned}$$

Joanne

x	$(x - \bar{x})^2$
9	$(21.5)^2$
29	$(1.5)^2$
41	$(10.5)^2$
26	$(4.5)^2$
14	$(16.5)^2$
44	$(13.5)^2$
38	$(7.5)^2$
43	$(12.5)^2$
244	1262

$$\begin{aligned}\bar{x} &= \frac{\sum x}{n} \\ &= \frac{244}{8} \\ &= 30.5\end{aligned}$$

$$\begin{aligned}s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\ &= \sqrt{\frac{1262}{8}} \\ &\div 12.56\end{aligned}$$

b The standard deviation. The lower the value of s , the more consistent.

2 a Glen has mean = $\frac{0 + 10 + 1 + 9 + 11 + 0 + 8 + 5 + 6 + 7}{10} = \frac{57}{10} = 5.7$

Shane has mean = $\frac{4 + 3 + 4 + 1 + 4 + 11 + 7 + 6 + 12 + 5}{10} = \frac{57}{10} = 5.7$

Glen's range = $11 - 0 = 11$ Shane's range = $12 - 1 = 11$

b We suspect Glen's as he has more high and low values.

c Glen

x	$(x - \bar{x})^2$
0	$(5.7)^2$
10	$(4.3)^2$
1	$(4.7)^2$
9	$(3.3)^2$
11	$(5.3)^2$
0	$(5.7)^2$
8	$(2.3)^2$
5	$(0.7)^2$
6	$(0.3)^2$
7	$(1.3)^2$
	154.9

$$\begin{aligned}
 s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\
 &= \sqrt{\frac{154.9}{10}} \\
 &\div 3.9 \\
 &\quad \uparrow \\
 &\text{greater} \\
 &\text{variability}
 \end{aligned}$$

Shane

x	$(x - \bar{x})^2$
4	$(1.7)^2$
3	$(2.7)^2$
4	$(1.7)^2$
1	$(4.7)^2$
4	$(1.7)^2$
11	$(5.3)^2$
7	$(1.3)^2$
6	$(0.3)^2$
12	$(6.3)^2$
5	$(0.7)^2$
	108.1

$$\begin{aligned}
 s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\
 &= \sqrt{\frac{108.1}{10}} \\
 &\div 3.29
 \end{aligned}$$

d The standard deviation as it takes all values into account, not just the lowest and highest.**3 a** We suspect variability in standard deviation since the factors may change every day.**b i** sample mean **ii** sample standard deviation **c** less variability**4 a**

x	$(x - \bar{x})^2$
79	10^2
64	5^2
59	10^2
71	2^2
68	1^2
68	1^2
74	5^2
483	256

$$\begin{aligned}
 \bar{x} &= \frac{\sum x}{n} \\
 &= \frac{483}{7} \\
 &= 69 \\
 s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\
 &= \sqrt{\frac{256}{7}} \\
 &\div 6.05
 \end{aligned}$$

b

x	$(x - \bar{x})^2$
89	10^2
74	5^2
69	10^2
81	2^2
78	1^2
78	1^2
84	5^2
553	256

$$\begin{aligned}
 \bar{x} &= \frac{\sum x}{n} \\
 &= \frac{553}{7} \\
 &= 79 \\
 s &\div 6.05
 \end{aligned}$$

c The distribution has simply shifted by 10 kg. The mean increases by 10 kg and the standard deviation remains the same.**5 a**

x	$(x - \bar{x})^2$
0.8	$(0.21)^2$
1.1	$(0.09)^2$
1.2	$(0.19)^2$
0.9	$(0.11)^2$
1.2	$(0.19)^2$
1.2	$(0.19)^2$
0.9	$(0.11)^2$
0.7	$(0.31)^2$
1.0	$(0.01)^2$
1.1	$(0.09)^2$
10.4	0.289

$$\begin{aligned}
 \bar{x} &= \frac{\sum x}{n} \\
 &= \frac{10.1}{10} \\
 &= 1.01 \text{ kg} \\
 s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\
 &= \sqrt{\frac{0.289}{10}} \\
 &\div 0.17 \text{ kg}
 \end{aligned}$$

b

x	$(x - \bar{x})^2$
1.6	$(0.42)^2$
2.2	$(0.18)^2$
2.4	$(0.38)^2$
1.8	$(0.22)^2$
2.4	$(0.38)^2$
2.4	$(0.38)^2$
1.8	$(0.22)^2$
1.4	$(0.62)^2$
2.0	$(0.02)^2$
2.2	$(0.18)^2$
20.2	1.156

$$\begin{aligned}
 \bar{x} &= \frac{\sum x}{n} \\
 &= \frac{20.2}{10} \\
 &= 2.02 \text{ kg} \\
 s &= \sqrt{\frac{\sum (x - \bar{x})^2}{n}} \\
 &= \sqrt{\frac{1.156}{10}} \\
 &= 0.34 \text{ kg}
 \end{aligned}$$

c Doubling the values doubles the mean and standard deviation.

6 **a** $s = \sqrt{\text{variance}}$ **b** Unbiased estimate of μ is $\bar{x} = 93.8$ kg
 $= \sqrt{45.9}$ kg Unbiased estimate of σ is
 $= 6.77$ kg $\sqrt{\frac{87}{86}} \times 6.77$ kg $\doteq 6.81$ kg

7 **a** Using technology, $\bar{x} \doteq 77.5$ g and $s_n \doteq 7.445$ g

b Unbiased estimate of μ is $\bar{x}$ i.e., 77.5 g
 Unbiased estimate of σ is $\sqrt{\frac{17}{16}} \times 7.445 \doteq 7.67$ g

8 mean $= \frac{1+3+5+7+4+5+p+q}{8} = 5$
 $25 + p + q = 40$
 $\therefore p + q = 15$
 $\therefore q = 15 - p$

and $s = \sqrt{\frac{(-4)^2 + (-2)^2 + 0^2 + 2^2 + (-1)^2 + 0^2 + (p-5)^2 + (q-5)^2}{8}} = \sqrt{5.25}$
 $\therefore \frac{16 + 4 + 4 + 1 + (p-5)^2 + (15-p-5)^2}{8} = 5.25$

$\therefore 25 + p^2 - 10p + 25 + 100 - 20p + p^2 = 42$

$\therefore 2p^2 - 30p + 108 = 0$

$\therefore p^2 - 15p + 54 = 0$

$\therefore (p-6)(p-9) = 0$

$\therefore p = 6$ or 9 and $q = 9$ or 6

But $p < q$ $\therefore p = 6, q = 9$

9 mean $= \frac{3+9+5+5+6+4+6+8+a+b}{10} = 6$

$\therefore \frac{46 + a + b}{10} = 6$

$\therefore 46 + a + b = 60$

$\therefore a + b = 14$

$\therefore b = 14 - a$

and $s = \sqrt{\frac{(-3)^2 + 3^2 + (-1)^2 + (-1)^2 + (-2)^2 + (a-6)^2 + (b-6)^2 + 2^2}{10}} = \sqrt{3.2}$

$\therefore 9 + 9 + 1 + 1 + 4 + 4 + (a-6)^2 + (8-a)^2 = 32$

$\therefore 28 + a^2 - 12a + 36 + 64 - 16a + a^2 = 32$

$\therefore 2a^2 - 28a + 96 = 0$

$\therefore a^2 - 14a + 48 = 0$

$\therefore (a-6)(a-8) = 0$

$\therefore a = 6$ or 8 and $b = 8$ or 6

$\therefore a = 8, b = 6$

10 **a** $\sum_{i=1}^3 (x_i - \bar{x})^2 = (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2$
 $= x_1^2 - 2\bar{x}x_1 + \bar{x}^2 + x_2^2 - 2\bar{x}x_2 + \bar{x}^2 + x_3^2 - 2\bar{x}x_3 + \bar{x}^2$
 $= x_1^2 + x_2^2 + x_3^2 - 2\bar{x}(x_1 + x_2 + x_3) + 3\bar{x}^2$

$$\begin{aligned}
 &= \sum_{i=1}^3 (x_i^2) - 2\bar{x} \sum_{i=1}^3 x_i + 3\bar{x}^2 \quad \text{where} \quad \bar{x} = \frac{\sum_{i=1}^3 x_i}{3} \\
 &= \sum_{i=1}^3 (x_i^2) - 2\bar{x} \times 3\bar{x} + 3\bar{x}^2 \\
 &= \sum_{i=1}^3 (x_i^2) - 6\bar{x}^2 + 3\bar{x}^2 \\
 &= \sum_{i=1}^3 (x_i^2) - 3\bar{x}^2
 \end{aligned}$$

Generalisation is: $\sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{i=1}^n (x_i^2) - n\bar{x}^2$

b Unbiased estimate of μ is $\bar{x} = \frac{\sum_{i=1}^{16} x_i}{16} = \frac{519}{16} \div 32.4 \text{ min}$

$$\begin{aligned}
 \text{and } s &= \sqrt{\frac{\sum_{i=1}^{16} (x_i - \bar{x})^2}{16}} = \sqrt{\frac{\sum_{i=1}^{16} (x_i^2) - 16\bar{x}^2}{16}} = \sqrt{\frac{16\,983 - 16 \times 32.4375^2}{16}} \\
 &\div \sqrt{9.246\dots} \\
 &\div 3.041 \text{ min}
 \end{aligned}$$

$\therefore$ unbiased estimate of $\sigma = \sqrt{\frac{16}{15}} \times 3.041 \div 3.140 \text{ min}$

$\therefore$ unbiased estimate of σ^2 is $\div 9.86 \text{ min}$

11

a

x	f	fx	$f(x - \bar{x})^2$
0	14	0	41.61
1	18	18	9.44
2	13	26	0.99
3	5	15	8.14
4	3	12	15.54
5	2	10	21.46
6	2	12	36.57
7	1	7	27.84
Σ	58	100	161.59

$$\begin{aligned}
 \bar{x} &= \frac{\sum fx}{\sum f} & s &= \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \\
 &= \frac{100}{58} & &\div \sqrt{\frac{161.59}{58}} \\
 &\div 1.72 \text{ children} & &\div 1.67 \text{ children}
 \end{aligned}$$

b Unbiased estimate of μ is $\bar{x} = 1.72 \text{ children}$

Unbiased estimate of σ is $\sqrt{\frac{58}{57}} \times s$
 $\div 1.68 \text{ children}$

12

a

x	f	fx	$f(x - \bar{x})^2$
11	2	22	24.22
12	1	12	6.15
13	4	52	8.76
14	5	70	1.15
15	6	90	1.62
16	4	64	9.24
17	2	34	12.70
18	1	18	12.39
Σ	25	362	76.23

$$\begin{aligned}
 \bar{x} &= \frac{\sum fx}{\sum f} & s &= \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \\
 &= \frac{362}{25} & &= \sqrt{\frac{76.25}{25}} \\
 &\div 14.5 \text{ years} & &\div 1.75 \text{ years}
 \end{aligned}$$

b Unbiased estimate of μ is $\bar{x} = 14.5 \text{ years}$

Unbiased est. of σ is $\sqrt{\frac{25}{24}} \times 1.7464 \dots \div 1.78 \text{ years}$

13**a**

x	f	fx	$f(x - \bar{x})^2$
33	1	33	18.24
35	5	175	25.78
36	7	252	11.31
37	13	481	0.95
38	12	456	6.38
39	8	312	23.92
40	2	80	14.90
Σ	48	1789	101.48

$$\begin{aligned}\bar{x} &= \frac{\sum fx}{\sum f} & s &= \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \\ &= \frac{1789}{48} & &= \sqrt{\frac{101.48}{48}} \\ &\div 37.3 \text{ toothpicks} & &\div 1.45 \text{ toothpicks}\end{aligned}$$

b Unbiased estimate of μ is $\bar{x} \div 37.3$ toothpicksUnbiased est. of σ is $\sqrt{\frac{48}{47}} \times s \div 1.47$ toothpicks**14****a**

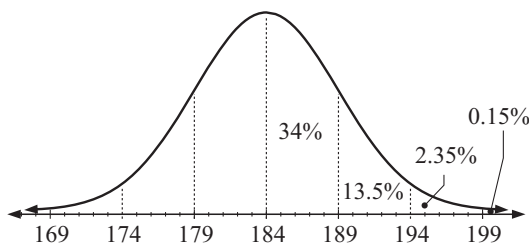
Midpoint (x)	f	fx	$f(x - \bar{x})^2$
40.5	1	40.5	52.85
42.5	1	42.5	27.77
44.5	3	133.5	32.08
46.5	7	325.5	11.29
48.5	11	533.5	5.86
50.5	5	252.5	37.26
52.5	2	105	44.75
Σ	30	1433	211.86

$$\begin{aligned}\bar{x} &= \frac{\sum fx}{\sum f} & s &= \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \\ &= \frac{1433}{30} & &\div \sqrt{\frac{211.86}{30}} \\ &= 47.7666\dots & &\div 2.657\dots \\ &\div 47.8 \text{ cm} & &\div 2.66 \text{ cm}\end{aligned}$$

b Unbiased estimate of μ is $\bar{x} \div 47.8$ cmUnbiased estimate of σ is $\sqrt{\frac{30}{29}} \times 2.657\dots \div 2.70$ cm**15****a**

Midpoint (x)	f	fx	$f(x - \bar{x})^2$
364.995	17	6204.9	10 885.83
374.995	38	14 249.8	8901.23
384.995	47	18 094.8	1322.72
394.995	57	22 514.7	1256.45
404.995	18	7289.9	3886.97
414.995	10	4149.9	6098.43
424.995	10	4250.0	12 037.43
434.995	3	1305.0	5992.93
Σ	200	78 059	50 381.99

$$\begin{aligned}\bar{x} &= \frac{\sum fx}{\sum f} & s &= \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \\ &= \frac{78\,059}{200} & &= \sqrt{\frac{50\,381.99}{200}} \\ &\div \$390.30 & &\div \$15.87\end{aligned}$$

b Unbiased estimate of μ is $\bar{x} \div \$390.30$ Unbiased estimate of σ is $\sqrt{\frac{200}{199}} \times \$15.87\dots \div \$15.91$ **EXERCISE 18G****1****a** $50\% + 34\% = 84\%$

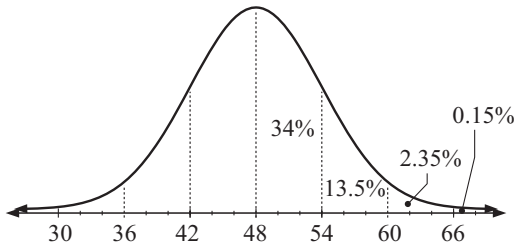
So, 16% are above 189.

b $50\% + 34\% = 84\%$

So, 84% are above 179.

c $2 \times 34\% + 2 \times 13.5\% + 2.35\%$
 $\div 97.4\%$ **d** 0.15%

2



$$100\% - 50\% - 34\% = 16\%$$

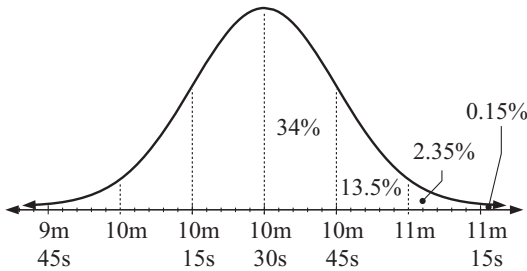
$$\text{and } 20 \times 16\%$$

$$= 20 \times 0.16$$

$$= 3.2$$

On 3 occasions.

3



$$\mathbf{a} \quad 200 \times (50 + 34 + 13.5)\%$$

$$= 200 \times 95.5\%$$

$$\div 195$$

$$\therefore 200 - 195 = 5 \text{ took } > 11 \text{ minutes}$$

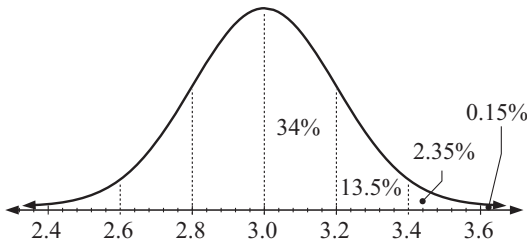
$$\mathbf{b} \quad 200 \times (0.15 + 2.35 + 13.5)\%$$

$$= 200 \times 16\%$$

$$= 32 \text{ lifesavers}$$

$$\mathbf{c} \quad 200 \times 68\% = 136 \text{ lifesavers}$$

4



$$\mathbf{a} \quad 545 \times (50 + 34)\%$$

$$= 545 \times 0.84$$

$$\div 458 \text{ babies}$$

$$\mathbf{b} \quad 545 \times (68 + 13.5)\%$$

$$= 545 \times 81.5\%$$

$$\div 444 \text{ babies}$$

REVIEW SET 18A

1 **a** Diameter of bacteria colonies

0	4 8 9
1	3 5 5 7
2	1 1 5 6 8 8
3	0 1 2 3 4 5 5 6 6 7 7 9
4	0 1 2 7 9

leaf unit: 0.1 cm

b Using technology

$$\mathbf{i} \quad \text{median} = 3.15 \text{ cm}$$

$$\mathbf{ii} \quad \text{range} = 4.9 - 0.4 = 4.5 \text{ cm}$$

c The distribution has a large tail on the left.
So, it is negatively skewed.

2 **a** highest = 97.5 m, lowest = 64.6 m

c

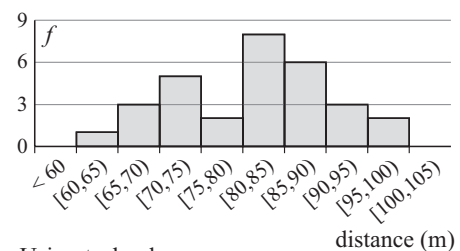
A frequency distribution table for distances thrown by Thabiso		
distance (m)	tally	freq. (f)
60 -		1
65 -		3
70 -		5
75 -		2
80 -		8
85 -		6
90 -		3
95 < 100		2
	Total	30

b The range = $97.5 - 64.5 = 33$

So, if intervals of length 5 are used we need about 7 of them.

We choose 60 -, 65 -, 70 -, 75 -, 80 -, etc.

d i/ii Frequency histogram displaying the distance Thabiso throws a baseball



e Using technology

$$\mathbf{i} \quad \bar{x} \div 81.1 \text{ m}$$

$$\mathbf{ii} \quad \text{median} \div 83.0 \text{ m}$$

$$\begin{aligned} 3 \quad \text{Mean} &= \frac{5+6+8+3+a+b}{6} = 6 & \therefore 22+a+b &= 36 \\ & & \therefore a+b &= 14 \\ & & \therefore b &= 14-a \end{aligned}$$

$$\text{and } s = \sqrt{\frac{(-1)^2 + 0^2 + 2^2 + (-3)^2 + (a-6)^2 + (b-6)^2}{6}} = \sqrt{3}$$

$$\therefore \frac{1+4+9+(a-6)^2+(8-a)^2}{6} = 3$$

$$\therefore 14+a^2-12a+36+64-16a+a^2 = 18$$

$$\therefore 2a^2-28a+96 = 0$$

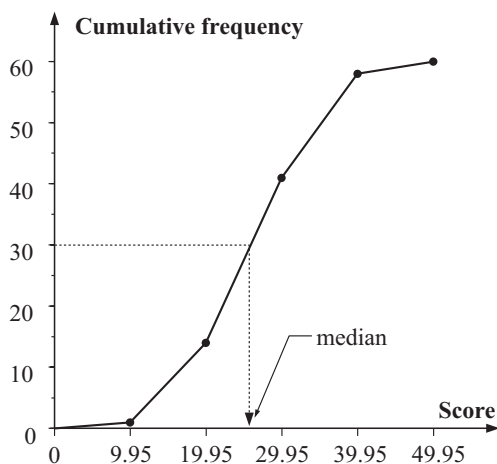
$$\therefore a^2-14a+48 = 0$$

$$\therefore (a-6)(a-8) = 0$$

$$\therefore a = 6 \text{ or } 8 \quad \text{and} \quad b = 8 \text{ or } 6$$

$$\therefore a = 6, b = 8 \quad \text{or} \quad a = 8, b = 6$$

4 a

b median $\doteq$ 25.9 (see graph)

$$\begin{aligned} c \quad \text{IQR} &= Q_3 - Q_1 \\ &= (\text{score for CF of } 45) \\ &\quad - (\text{score for CF of } 15) \\ &\doteq 32 - 20 \doteq 12 \end{aligned}$$

d

	f	midpt x	fx	$(x - \bar{x})^2$
0 – 9.9	1	4.95	4.95	441
10 – 19.9	13	14.95	194.35	121
20 – 29.9	27	24.95	673.65	1
30 – 39.9	17	34.95	594.15	81
40 – 49.9	2	44.95	89.9	361
	60		1557	

$$\begin{aligned} \bar{x} &= \frac{\sum fx}{\sum x} & s &= \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \\ &= \frac{1557}{60} \doteq 26.0 & &= \sqrt{\frac{4140}{60}} \doteq 8.31 \end{aligned}$$

5 a Using technology

	Girls	Boys
shape	pos. skewed	approx. symm.
centre (median)	36.3 sec	34.9 sec
spread (range)	7.7 sec	4.9 sec

b The girls' distribution is positively skewed and the boys' distribution is approximately symmetrical. The median swim times for boys is 1.4 seconds lower than for girls but the range of the girls' swim times is 2.8 seconds higher than for boys. The analysis supports the conjecture that boys generally swim faster than girls with less spread of times.

6 a Reading from the box-plots

	A	B
Min	11	11.2
Q_1	11.6	12
Median	12	12.6
Q_3	12.6	13.2
Max	13	13.8

b i range of A

$$= 13 - 11$$

$$= 2$$

range of B

$$= 13.8 - 11.2$$

$$= 2.6$$

ii IQR of A

$$= 12.6 - 11.6$$

$$= 1.0$$

IQR of B

$$= 13.2 - 12$$

$$= 1.2$$

- c** **i** The members in squad *A* generally ran faster times.
ii The times in squad *B* were more varied.

7 Using technology **a** **i** 101.5 **ii** 98 **iii** 105.5 **b** 7.5 **c** $\bar{x} = 100.2$, $s \doteq 7.59$

8 **a** This question could be done using technology or

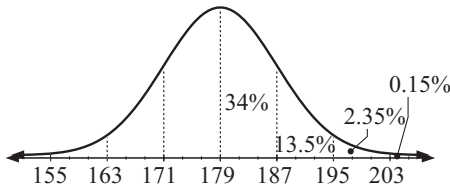
litres (<i>x</i>)	<i>f</i>	<i>f x</i>	$f(x - \bar{x})^2$
17	5	85	1299.27
22	13	286	1607.50
27	17	459	636.72
32	29	928	36.38
37	27	999	406.47
42	18	756	1419.38
47	7	329	1348.58
Σ	116	3842	6754.30

$$\begin{aligned}\bar{x} &= \frac{\sum f x}{\sum f} \\ &= \frac{3842}{116} \\ &\doteq 33.1 \text{ litres}\end{aligned}$$

$$\begin{aligned}s &= \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}} \\ &\doteq \sqrt{\frac{6754.30}{116}} \\ &\doteq 7.63 \text{ litres}\end{aligned}$$

- b** Unbiased estimate of μ is $\bar{x} \doteq 33.1$ litres
 Unbiased estimate of σ is $\sqrt{\frac{116}{115}} \times 7.63 \doteq 7.66$ litres.

9

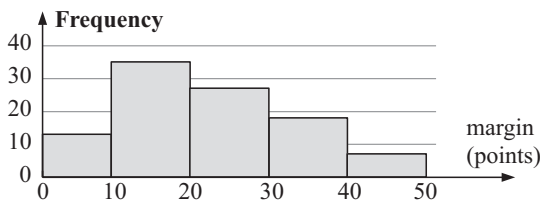


- c** % between 171 cm and 187 cm = 34% + 34% = 68%

- a** % > 195 cm
 = (2.35 + 0.15)%
 = 2.5%
b % between 163 cm and 195 cm
 = 2 × (34% + 13.5%)
 = 95%

REVIEW SET 18B

1



2

Using technology,
 $\bar{x} \doteq 49.6$, $s \doteq 1.60$ does not justify claim.
 A much greater sample is needed.

3 Use technology or

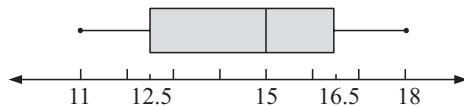
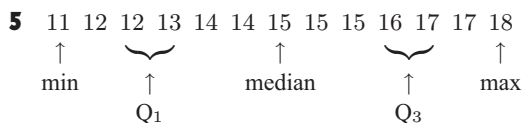
Midpoint (<i>x</i>)	<i>f</i>	<i>f x</i>
274.5	14	3843
324.5	34	11 033
374.5	68	25 466
424.5	72	30 564
474.5	54	25 623
524.5	23	12 063.5
574.5	7	4021.5
Σ	272	112 613.5

$$\begin{aligned}\bar{x} &= \frac{\sum f x}{\sum f} = \frac{112\,613.5}{272} \\ &\doteq 414 \text{ customers}\end{aligned}$$

4 116 118 120 122 127 128 132 135 ($n = 8$)

↑ min ↑ Q₁ ↑ median ↑ Q₃ ↑ max

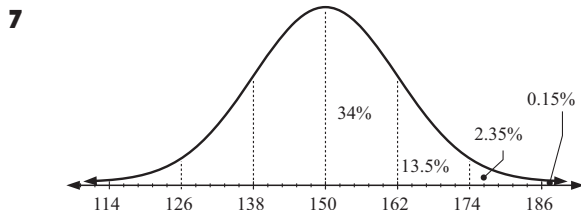
$$\begin{aligned}\text{range} &= 135 - 116 = 19 & Q_1 &= \frac{118 + 120}{2} = 119 & Q_3 &= \frac{128 + 132}{2} = 130 & s &\doteq 6.38 \text{ \{technology\}}\end{aligned}$$



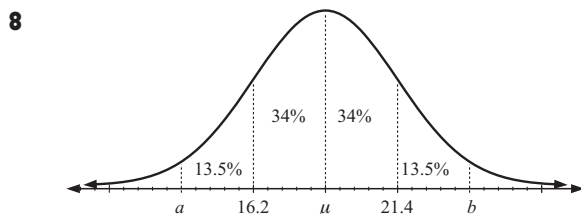
- 6 a** Using technology with x values 74.995, 84.995, 94.995, etc.
 $\bar{x} = \$103.50$ and $s = \$19.40$

- b** Unbiased estimate of μ is $\bar{x} = \$103.50$

Unbiased estimate of σ is $\sqrt{\frac{n}{n-1}} \times \$19.40 \div \sqrt{\frac{215}{214}} \times \$19.40 \div \$19.45$



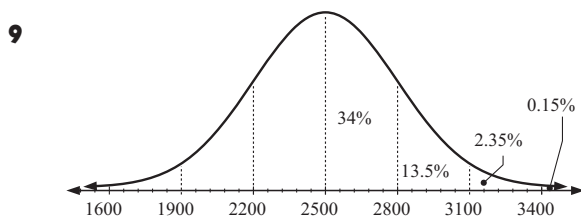
- a** 68% lie here
b $2 \times (34 + 13.5) = 95\%$ lie here
c $2 \times 34 + 13.5 = 81.5\%$ lie here
d 13.5% lie here



a $\mu = \frac{16.2 + 21.4}{2} = \frac{37.6}{2} = 18.8$

$\sigma = 21.4 - 18.8 = 2.6$

- b** The middle 95% of data lies between a and b
 where $a = 16.2 - 2.6 = 13.6$
 and $b = 21.4 + 2.6 = 24.0$
 i.e., between 13.6 and 24.0



- a** $(50 - 34 - 13.5)\% = 2.5\%$
 days are less than 1900
b $(50 + 34)\% = 84\%$
c $(2 \times 34 + 13.5)\% = 81.5\%$

10 a i mean $= \frac{\sum_{i=1}^{30} x_i}{30} = \frac{116.3}{30} \div 3.877 \text{ cm} \div 3.88 \text{ cm}$

ii $s^2 = \frac{\sum_{i=1}^{30} (x_i - \bar{x})^2}{30} = \frac{\sum_{i=1}^{30} (x_i^2) - 30\bar{x}^2}{30}$

$\therefore s^2 = \frac{452.57 - 30 \times 3.877^2}{30} \div 0.0571 \text{ cm}^2$

- b** unbiased estimate of μ is $\bar{x} = 3.88 \text{ cm}$

unbiased estimate of σ is $\sqrt{\frac{n}{n-1}} \times s$
 $\div \sqrt{\frac{30}{29}} \times \sqrt{0.0545}$
 $\div 0.243 \text{ cm}^2$

Chapter 19

PROBABILITY

EXERCISE 19A

1 a P(inside a square)

$$\div \frac{113}{145}$$

$$\div 0.780$$

b P(on a line)

$$\div \frac{32}{145}$$

$$\div 0.221$$

2 Total frequency = $17 + 38 + 19 + 4 = 78$

a P(20 to 39 seconds)

$$= \frac{38}{78}$$

$$\div 0.487$$

b P(> 60 seconds)

$$= \frac{4}{78}$$

$$\div 0.051$$

c P(between 20 and 59 sec (inc))

$$= \frac{38 + 19}{78}$$

$$\div 0.731$$

3

Calls/day	No. of days
0	2
1	7
2	11
3	8
4	7
5	4
6	3
7	0
8	1

a Survey lasted $2 + 7 + 11 + 8 + 7 + 4 + 3 + 0 + 1$
= 43 days

b i P(0 calls)

$$= \frac{2}{43}$$

$$\div 0.047$$

ii P(≥ 5 calls)

$$= \frac{4 + 3 + 0 + 1}{43}$$

$$\div 0.186$$

iii P(< 3 calls)

$$= \frac{2 + 7 + 11}{43}$$

$$\div 0.465$$

4 Total frequency

$$= 37 + 81 + 48 + 17 + 6 + 1$$

$$= 190$$

a P(4 days gap)

$$= \frac{17}{190}$$

$$\div 0.089$$

b P(at least 4 days gap)

$$= \frac{17 + 6 + 1}{190}$$

$$\div 0.126$$

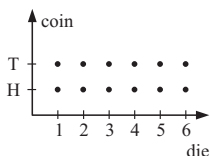
EXERCISE 19B

1 a {A, B, C, D} **b** {BB, BG, GB, GG}

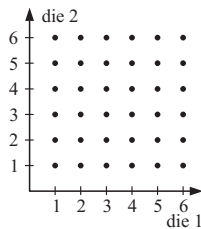
c {ABCD, ABDC, ACBD, ACDB, ADCB, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDAB, DABC, DACB, DBAC, DBCA, DCAB, DCBA}

d {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

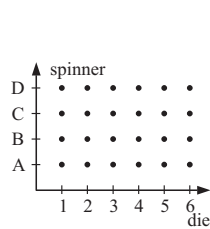
2 a



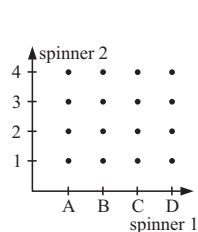
b



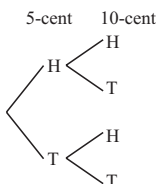
c



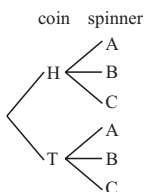
d



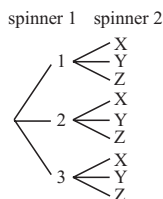
3 a



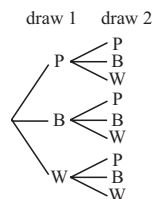
b



c



d



EXERCISE 19C

1 Total number of marbles = $5 + 3 + 7 = 15$

a $P(\text{red})$

$$= \frac{3}{15}$$

$$= \frac{1}{5}$$

b $P(\text{green})$

$$= \frac{5}{15}$$

$$= \frac{1}{3}$$

c $P(\text{blue})$

$$= \frac{7}{15}$$

d $P(\text{not red})$

$$= \frac{5+7}{15}$$

$$= \frac{12}{15} \text{ or } = \frac{4}{5}$$

e $P(\text{neither green nor blue})$

$$= P(\text{red})$$

$$= \frac{1}{5}$$

f $P(\text{green or red}) = \frac{5+3}{15} = \frac{8}{15}$

2 a 8 are brown and so 4 are white.

b i $P(\text{brown})$

$$= \frac{8}{12}$$

$$= \frac{2}{3}$$

ii $P(\text{white})$

$$= \frac{4}{12}$$

$$= \frac{1}{3}$$

3 a $P(\text{multiple of 4})$

$$= P(4, 8, 12, 16, 20, 24, 28, 32, 36)$$

$$= \frac{9}{36}$$

$$= \frac{1}{4}$$

c $P(> 20)$

$$= P(21, 22, 23, 24, \dots, 35, 36)$$

$$= \frac{36-20}{36}$$

$$= \frac{16}{36}$$

$$= \frac{4}{9}$$

e $P(\text{multiple of 13})$

$$= P(13 \text{ or } 26)$$

$$= \frac{2}{36}$$

$$= \frac{1}{18}$$

b $P(\text{between 6 and 9 inclusive})$

$$= P(6, 7, 8 \text{ or } 9)$$

$$= \frac{4}{36}$$

$$= \frac{1}{9}$$

d $P(9)$

$$= \frac{1}{36}$$

f $P(\text{odd multiple of 3})$

$$= P(3, 9, 15, 21, 27, \text{ or } 33)$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

4 a $P(\text{on Tuesday})$

$$= \frac{1}{7}$$

b $P(\text{on a weekend})$

$$= \frac{2}{7}$$

c $P(\text{in July})$

$$= \frac{4 \times 31}{365 \times 3 + 366} \quad \{\text{over a 4 year period}\}$$

d $P(\text{in January or February})$

$$= \frac{4 \times 31 + 3 \times 28 + 1 \times 29}{3 \times 365 + 1 \times 366} \quad \{\text{over a 4 year period}\}$$

$$= \frac{237}{1461}$$

5 Let A denote Antti, K denote Kai and N denote Neda.

Possible orders are: {AKN, ANK, KAN, KNA, NAK, NKA}

a $P(\text{A in middle})$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

b $P(\text{A at left end})$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

c $P(\text{A at right end})$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

d $P(\text{K and N are together}) = \frac{4}{6} = \frac{2}{3}$

6 Let G denote ‘a girl’ and B denote ‘a boy’.

a Possible orders are: {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}

b i	$P(\text{all boys})$	ii	$P(\text{all girls})$	iii	$P(\text{boy, then girl, then girl})$
	$= P(BBB)$		$= P(GGG)$		$= P(BGG)$
	$= \frac{1}{8}$		$= \frac{1}{8}$		$= \frac{1}{8}$

iv	$P(2 \text{ girls and a boy})$	v	$P(\text{girl is eldest})$
	$= P(GGB \text{ or } GBG \text{ or } BGG)$		$= P(GGG \text{ or } GBG \text{ or } GBB \text{ or } GGB)$
	$= \frac{3}{8}$		$= \frac{4}{8}$

vi	$P(\text{at least one boy})$	$= \frac{1}{2}$
	$= \frac{7}{8} \quad \{\text{all except GGG}\}$	

7 a {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}

b i	$P(\text{A sits on one end})$	ii	$P(\text{B sits on one of the two middle seats})$
	$= \frac{12}{24}$		$= \frac{12}{24}$
	$= \frac{1}{2}$		$= \frac{1}{2}$

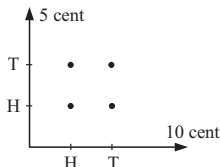
iii	$P(\text{A and B are together})$	iv	$P(\text{A, B and C are together})$
	$= \frac{12}{24}$		$= \frac{12}{24}$
	$= \frac{1}{2}$		$= \frac{1}{2}$

8 $P(\text{hits the bulls-eye}) = \frac{\text{area of bulls-eye}}{\text{area of whole target}} = \frac{\pi \times 20^2}{\pi \times 30^2}$

$$= \frac{400\pi}{900\pi}$$

$$= \frac{4}{9}$$

EXERCISE 19D

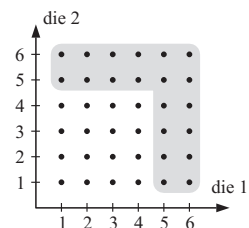
1		a	$P(2 \text{ heads})$	b	$P(2 \text{ tails})$	c	$P(\text{exactly 1 head})$
			$= \frac{1}{4}$		$= \frac{1}{4}$		$= P(\text{HT or TH})$
							$= \frac{2}{4} \text{ or } \frac{1}{2}$
		d	$P(\text{at least one H})$				
			$= P(\text{HT or TH or HH})$				
			$= \frac{3}{4}$				

2 a		b	There are $2 \times 5 = 10$ possible outcomes.
		c i	$P(\text{T and 3})$
			$= \frac{1}{10}$
		ii	$P(\text{H and even})$
			$= P(\text{H2 or H4})$
			$= \frac{2}{10} \text{ i.e., } \frac{1}{5}$
		iii	$P(\text{an odd})$
			$= P(\text{H1, T1, H3, T3, H5, T5})$
			$= \frac{6}{10}$
			$= \frac{3}{5}$
		iv	$P(\text{H or 5})$
			$= \frac{6}{10}$
			$= \frac{3}{5}$
			{those encircled}

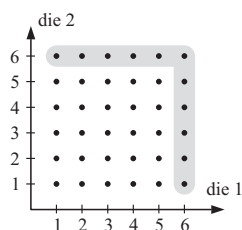
3 a $P(\text{two 3s})$
 $= P((3, 3))$
 $= \frac{1}{36}$

b $P(5 \text{ and a } 6)$
 $= P((5, 6), (6, 5))$
 $= \frac{2}{36}$
 $= \frac{1}{18}$

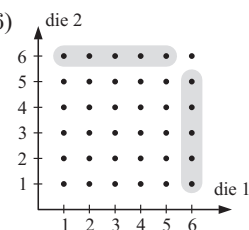
c $P(5 \text{ or a } 6)$
 $= \frac{20}{36}$
 $= \frac{5}{9}$



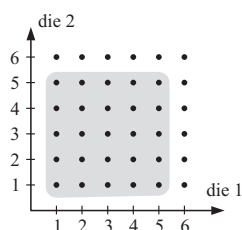
d $P(\text{at least one } 6)$
 $= \frac{11}{36}$



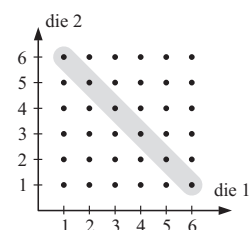
e $P(\text{exactly one } 6)$
 $= \frac{10}{36}$
 $= \frac{5}{18}$



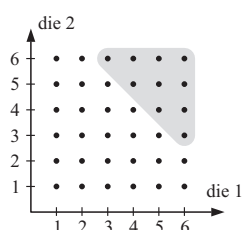
f $P(\text{no sixes})$
 $= \frac{25}{36}$



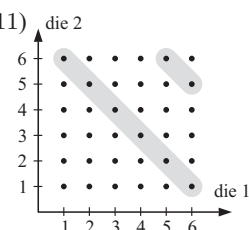
g $P(\text{sum of } 7)$
 $= \frac{6}{36}$
 $= \frac{1}{6}$



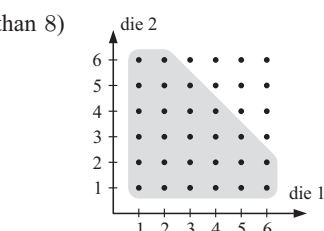
h $P(\text{sum} > 8)$
 $= \frac{10}{36}$
 $= \frac{5}{18}$



i $P(\text{sum of } 7 \text{ or } 11)$
 $= \frac{6+2}{36}$
 $= \frac{2}{9}$



j $P(\text{sum no more than } 8)$
 $= P(\text{sum} \leq 8)$
 $= \frac{26}{36}$
 $= \frac{13}{18}$



EXERCISE 19E.1

1 a $P(\text{rains on any one day})$
 $= \frac{6}{7}$

c $P(\text{rains on 3 successive days})$
 $= P(R \text{ and } R \text{ and } R)$
 $= \frac{6}{7} \times \frac{6}{7} \times \frac{6}{7} \text{ or } \frac{216}{343}$

2 a $P(H, \text{ then } H, \text{ then } H)$
 $= P(H \text{ and } H \text{ and } H)$
 $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
 $= \frac{1}{8}$

b $P(\text{rains on 2 successive days})$
 $= P(R \text{ and } R)$
 $= \frac{6}{7} \times \frac{6}{7}$
 $= \frac{36}{49}$

b $P(T, \text{ then } H, \text{ then } T)$
 $= P(T \text{ and } H \text{ and } T)$
 $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
 $= \frac{1}{8}$

- 3** Let A be the event of photocopier A malfunctioning and
B be the event of photocopier B malfunctioning.

$$\begin{aligned}\mathbf{a} \quad & P(\text{both malfunction}) \\ &= P(A \text{ and } B) \\ &= 0.08 \times 0.12 \\ &= 0.0096\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & P(\text{both work}) \\ &= P(A' \text{ and } B') \\ &= 0.92 \times 0.88 \\ &= 0.8096\end{aligned}$$

4 a $P(\text{they will be happy})$
 $= P(B, \text{ then } G, \text{ then } B, \text{ then } G)$
 $= P(B \text{ and } G \text{ and } B \text{ and } G)$
 $= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
 $= \frac{1}{16}$

b $P(\text{they will be unhappy})$
 $= 1 - P(\text{they will be happy})$
 $= 1 - \frac{1}{16}$
 $= \frac{15}{16}$

- 5** Let J be the event of Jiri hitting the target and
B be the event of Benita hitting the target.

$$\begin{aligned}\mathbf{a} \quad & P(\text{both hit}) \\ &= P(JB) \\ &= 0.7 \times 0.8 \\ &= 0.56\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & P(\text{both miss}) \\ &= P(J'B') \\ &= 0.3 \times 0.2 \\ &= 0.06\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad & P(J \text{ hits and } B \text{ misses}) \\ &= P(JB') \\ &= 0.7 \times 0.2 \\ &= 0.14\end{aligned}$$

d $P(B \text{ hits and } J \text{ misses}) = P(BJ') = 0.8 \times 0.3 = 0.24$

- 6** Let H be the event the archer hits the target.

$$\therefore P(H) = \frac{2}{5}, \quad P(H') = \frac{3}{5}$$

$$\begin{aligned}\mathbf{a} \quad & P(3 \text{ hits}) \\ &= P(HHH) \\ &= \frac{2}{5} \times \frac{2}{5} \times \frac{2}{5} \\ &= \frac{8}{125}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad & P(2 \text{ hits then a miss}) \\ &= P(HHH') \\ &= \frac{2}{5} \times \frac{2}{5} \times \frac{3}{5} \\ &= \frac{12}{125}\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad & P(\text{all misses}) \\ &= P(H'H'H') \\ &= \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \\ &= \frac{27}{125}\end{aligned}$$

EXERCISE 19E.2

1 a $P(\text{both red})$
 $= P(RR)$
 $= \frac{7}{10} \times \frac{6}{9}$
 $= \frac{7}{15}$

b $P(GR)$
 $= \frac{3}{10} \times \frac{7}{9}$
 $= \frac{7}{30}$

c $P(\text{a green and a red})$
 $= P(GR \text{ or } RG)$
 $= \frac{3}{10} \times \frac{7}{9} + \frac{7}{10} \times \frac{3}{9}$
 $= \frac{7}{15}$

2 a $P(\text{all strawberry creams})$
 $= P(SSS)$
 $= \frac{8}{12} \times \frac{7}{11} \times \frac{6}{10}$
 $= \frac{14}{55}$

b $P(\text{none is a strawberry cream})$
 $= P(S'S'S')$
 $= \frac{4}{12} \times \frac{3}{11} \times \frac{2}{10}$
 $= \frac{1}{55}$

3 a $P(\text{wins first prize})$
 $= \frac{3}{100}$

b $P(\text{wins 1st and 2nd})$
 $= P(WW)$
 $= \frac{3}{100} \times \frac{2}{99}$
 $\div 0.000\,606$

c $P(\text{wins all 3})$
 $= P(WWW)$
 $= \frac{3}{100} \times \frac{2}{99} \times \frac{1}{98}$
 $\div 0.000\,006\,18$

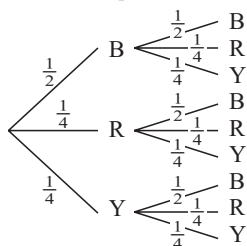
d $P(\text{wins none of them})$
 $= P(W'W'W')$
 $= \frac{97}{100} \times \frac{96}{99} \times \frac{95}{98}$
 $\div 0.912$

4 a P(contains the captain)

$$\begin{aligned}
 &= P(CC'C' \text{ or } C'CC' \text{ or } C'C'C) \\
 &= \frac{1}{7} \times \frac{6}{6} \times \frac{5}{5} + \frac{6}{7} \times \frac{1}{6} \times \frac{5}{5} + \frac{6}{7} \times \frac{5}{6} \times \frac{1}{5} \\
 &= 3 \left(\frac{30}{7 \times 6 \times 5} \right) \\
 &= \frac{3}{7}
 \end{aligned}$$

b P(contains captain and vice captain)

$$\begin{aligned}
 &= P(CVO \text{ or } COV \text{ or } VCO \text{ or } VOC \\
 &\quad \text{or } OCV \text{ or } OVC) \\
 &= \frac{1}{7} \times \frac{1}{6} \times \frac{5}{5} + \frac{1}{7} \times \frac{5}{6} \times \frac{1}{5} + \frac{1}{7} \times \frac{1}{6} \times \frac{5}{5} \\
 &\quad + \frac{1}{7} \times \frac{5}{6} \times \frac{1}{5} + \frac{5}{7} \times \frac{1}{6} \times \frac{1}{5} + \frac{5}{7} \times \frac{1}{6} \times \frac{1}{5} \\
 &= 6 \left(\frac{5}{7 \times 6 \times 5} \right) \\
 &= \frac{1}{7}
 \end{aligned}$$

EXERCISE 19F**1 a** 1st spin 2nd spin**b** P(both black)

$$\begin{aligned}
 &= P(BB) \\
 &= \frac{1}{2} \times \frac{1}{2} \\
 &= \frac{1}{4}
 \end{aligned}$$

c P(both yellow)

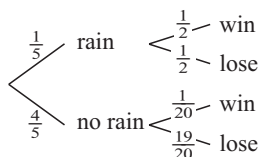
$$\begin{aligned}
 &= P(YY) \\
 &= \frac{1}{4} \times \frac{1}{4} \\
 &= \frac{1}{16}
 \end{aligned}$$

d P(both different)

$$\begin{aligned}
 &= P(BR \text{ or } BY \text{ or } RB \text{ or } RY \text{ or } YB \text{ or } YR) \\
 &= \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4} \\
 &= \frac{4}{8} + \frac{2}{16} \\
 &= \frac{5}{8}
 \end{aligned}$$

e P(B appears on either spin)

$$\begin{aligned}
 &= P(BB \text{ or } BR \text{ or } BY \text{ or } RB \text{ or } YB) \\
 &= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{2} \\
 &= 4 \left(\frac{1}{8} \right) + \frac{1}{4} \\
 &= \frac{3}{4}
 \end{aligned}$$

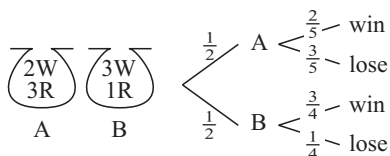
2

P(M wins)

$$\begin{aligned}
 &= P(\text{rain and win or no rain and win}) \\
 &= \frac{1}{5} \times \frac{1}{2} + \frac{4}{5} \times \frac{1}{20} \\
 &= \frac{1}{10} \times \frac{10}{10} + \frac{4}{100} \\
 &= \frac{14}{100} \\
 &= \frac{7}{50}
 \end{aligned}$$

3 P(next is spoiled) = P(from A and spoiled or from B and spoiled)

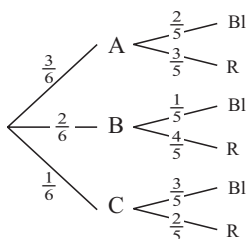
$$\begin{aligned}
 &= 0.4 \times 0.05 + 0.6 \times 0.02 \\
 &= 0.020 + 0.012 \\
 &= 0.032 \quad (3.2\%)
 \end{aligned}$$

4

P(red)

$$\begin{aligned}
 &= P(A \text{ and red or } B \text{ and red}) \\
 &= \frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{1}{4} \\
 &= \frac{3}{10} + \frac{1}{8} \\
 &= \frac{17}{40}
 \end{aligned}$$

5



$$\begin{aligned} \mathbf{a} \quad P(\text{blue}) &= P(\text{A and Bl or B and Bl or C and Bl}) \\ &= \frac{3}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{1}{5} + \frac{1}{6} \times \frac{3}{5} \\ &= \frac{11}{30} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad P(\text{red}) &= 1 - P(\text{blue}) \\ &= 1 - \frac{11}{30} \\ &= \frac{19}{30} \end{aligned}$$

EXERCISE 19G

1



$$\begin{aligned} \mathbf{a} \quad P(\text{different colours}) &= P(\text{PG or GP}) \\ &= \frac{2}{7} \times \frac{5}{7} + \frac{5}{7} \times \frac{2}{7} \\ &= \frac{20}{49} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad P(\text{different colours}) &= P(\text{PG or GP}) \\ &= \frac{2}{7} \times \frac{5}{6} + \frac{5}{7} \times \frac{2}{6} \\ &= \frac{20}{42} \quad \text{i.e.,} \quad \frac{10}{21} \end{aligned}$$

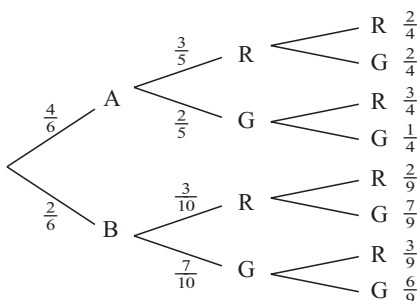
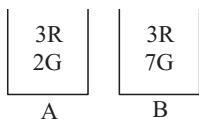
2

$$\begin{aligned} \mathbf{a} \quad P(\text{both odd}) &= P(\text{odd and odd}) \\ &= \frac{3}{5} \times \frac{2}{4} \\ &= \frac{3}{10} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad P(\text{both even}) &= P(\text{even and even}) \\ &= \frac{2}{5} \times \frac{1}{4} \\ &= \frac{1}{10} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad P(\text{one odd and other even}) &= 1 - P(\text{both odd}) - P(\text{both even}) \\ &= 1 - \frac{3}{10} - \frac{1}{10} \\ &= \frac{6}{10} \\ &= \frac{3}{5} \end{aligned}$$

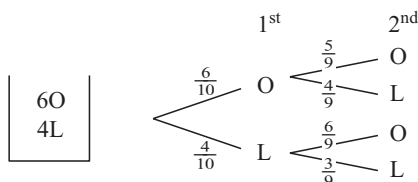
3



$$\begin{aligned} \mathbf{a} \quad P(\text{both green}) &= P(\text{AGG or BGG}) \\ &= \frac{4}{6} \times \frac{2}{5} \times \frac{1}{4} + \frac{2}{6} \times \frac{7}{10} \times \frac{6}{9} \\ &= \frac{1}{15} + \frac{7}{45} \\ &= \frac{10}{45} \\ &= \frac{2}{9} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad P(\text{different in colour}) &= 1 - P(\text{both green}) - P(\text{both red}) \\ &= 1 - \frac{2}{9} - P(\text{ARR or BRR}) \\ &= \frac{7}{9} - \left(\frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} + \frac{2}{6} \times \frac{3}{10} \times \frac{2}{9} \right) \\ &= \frac{7}{9} - \left(\frac{1}{5} + \frac{1}{45} \right) \\ &= \frac{5}{9} \end{aligned}$$

4



$$\begin{aligned} \mathbf{a} \quad P(\text{both O}) &= \frac{6}{10} \times \frac{5}{9} \\ &= \frac{1}{3} \end{aligned}$$

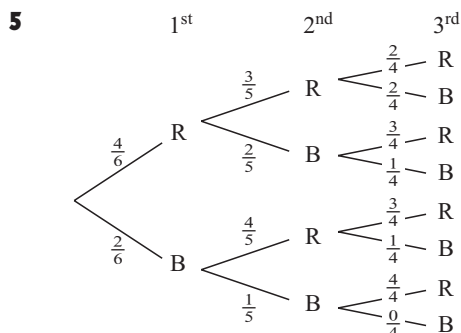
$$\begin{aligned} \mathbf{b} \quad P(\text{both L}) &= \frac{4}{10} \times \frac{3}{9} \\ &= \frac{2}{15} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad P(\text{OL}) &= \frac{6}{10} \times \frac{4}{9} \\ &= \frac{4}{15} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad P(\text{LO}) &= \frac{4}{10} \times \frac{6}{9} \\ &= \frac{4}{15} \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{3} + \frac{2}{15} + \frac{4}{15} + \frac{4}{15} \\
 &= \frac{5}{15} + \frac{2}{15} + \frac{4}{15} + \frac{4}{15} \\
 &= \frac{15}{15} \text{ which is } 1
 \end{aligned}$$

The answer must be 1 as the four categories **a**, **b**, **c**, **d** are all the possibilities that could occur.



a P(all red)

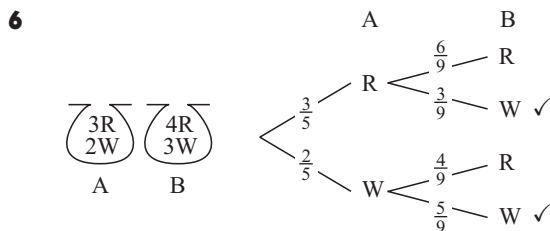
$$\begin{aligned}
 &= P(RRR) \\
 &= \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} \\
 &= \frac{1}{5}
 \end{aligned}$$

b P(only two are red)

$$\begin{aligned}
 &= P(RRB \text{ or } RBR \text{ or } BRR) \\
 &= \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} + \frac{4}{6} \times \frac{2}{5} \times \frac{3}{4} + \frac{2}{6} \times \frac{4}{5} \times \frac{3}{4} \\
 &= 3 \times \left(\frac{24}{6 \times 5 \times 4} \right) \\
 &= \frac{3}{5}
 \end{aligned}$$

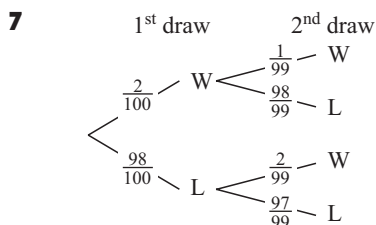
c P(at least two are red)

$$\begin{aligned}
 &= P(\text{all red or only two are red}) \\
 &= \frac{1}{5} + \frac{3}{5} \quad \{\text{from a and b}\} \\
 &= \frac{4}{5}
 \end{aligned}$$



P(marble from B is W) = P(RW or WW) {paths ticked}

$$\begin{aligned}
 &= \frac{3}{5} \times \frac{3}{9} + \frac{2}{5} \times \frac{5}{9} \\
 &= \frac{19}{45}
 \end{aligned}$$



a P(wins both)

$$\begin{aligned}
 &= P(WW) \\
 &= \frac{2}{100} \times \frac{1}{99} \\
 &\div 0.000\ 202
 \end{aligned}$$

b P(wins neither)

$$\begin{aligned}
 &= P(LL) \\
 &= \frac{98}{100} \times \frac{97}{99} \\
 &\div 0.960
 \end{aligned}$$

c P(wins at least one prize)

$$\begin{aligned}
 &= 1 - P(\text{wins neither}) \\
 &= 1 - \frac{98}{100} \times \frac{97}{99} \\
 &\div 0.0398
 \end{aligned}$$

EXERCISE 19H

1 a $(p + q)^4 = p^4 + 4p^3q + 6p^2q^2 + 4pq^3 + q^4$

b

$$\begin{aligned}
 & \uparrow \text{P(3 heads)} = 4p^3q \\
 &= 4 \left(\frac{1}{2} \right)^3 \left(\frac{1}{2} \right) \quad \{\text{as } p = q = \frac{1}{2}\} \\
 &= \frac{1}{4}
 \end{aligned}$$

$$2 \quad a \quad (p+q)^5 = p^5 + 5p^4q + 10p^3q^2 + 10p^2q^3 + 5pq^4 + q^5$$

$$b \quad \begin{aligned} &P(4H \text{ and } 1T) \quad \uparrow \\ &= 5p^4q \\ &= 5\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) \\ &= \frac{5}{32} \end{aligned}$$

$$c \quad \begin{aligned} &P(2H \text{ and } 3T) \\ &= 10p^2q^3 \\ &= 10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 \\ &= \frac{10}{32} \\ &= \frac{5}{16} \end{aligned}$$

$$3 \quad a \quad \left(\frac{2}{3} + \frac{1}{3}\right)^4 = \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + 4\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4$$

$$b \quad P(S) = \frac{2}{3}, \quad P(S') = \frac{1}{3} \quad S' \text{ represents a non-strawberry cream (or an almond centre)}$$

$$i \quad \begin{aligned} &P(\text{all } S) \\ &= \left(\frac{2}{3}\right)^4 \\ &= \frac{16}{81} \end{aligned}$$

$$ii \quad \begin{aligned} &P(\text{two of each}) \\ &= 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 \\ &= \frac{8}{27} \end{aligned}$$

$$iii \quad \begin{aligned} &P(\text{at least 2 strawberry creams}) \\ &= P(\text{all } S \text{ or } 3S, 1T \text{ or } 2S, 2T) \\ &= \left(\frac{2}{3}\right)^4 + 4\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + 6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) \\ &= \frac{16}{81} + \frac{32}{81} + \frac{24}{81} \\ &= \frac{72}{81} \\ &= \frac{8}{9} \end{aligned}$$

$$4 \quad a \quad \left(\frac{3}{4} + \frac{1}{4}\right)^5 = \left(\frac{3}{4}\right)^5 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5$$

$$b \quad i \quad \begin{aligned} &P(2 \text{ 'flat backs'}) \\ &= P(2Fs \text{ and } 3F's) \\ &= 10 \times \left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 \\ &= \frac{135}{512} \end{aligned}$$

$$ii \quad \begin{aligned} &P(\text{at least 3 are 'flat backs'}) \\ &= P(3F, 2F' \text{ or } 4F, 1F' \text{ or } 5F) \\ &= 10\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + 5\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 \\ &= \frac{53}{512} \quad \text{on simplifying} \end{aligned}$$

$$5 \quad X \text{ is Bin } (4, 0.8)$$

$$a \quad \begin{aligned} &P(X = 2) \\ &= \text{binompdf}(4, 0.8, 2) \\ &\doteq 0.154 \end{aligned}$$

$$b \quad \begin{aligned} &P(X \geq 2) \\ &= 1 - P(X \leq 1) \\ &= 1 - \text{binomcdf}(4, 0.8, 1) \\ &\doteq 0.973 \end{aligned}$$

$$6 \quad X \text{ is Bin } (6, 0.05)$$

$$a \quad \begin{aligned} &P(X = 2) \\ &= \text{binompdf}(6, 0.05, 2) \\ &\doteq 0.0305 \end{aligned}$$

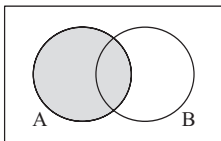
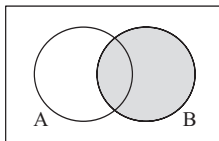
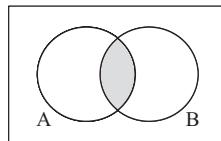
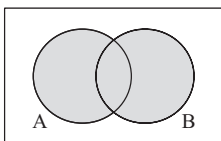
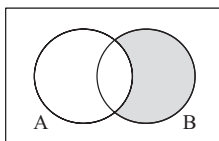
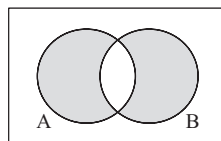
$$b \quad \begin{aligned} &P(X \geq 1) \\ &= 1 - P(X = 0) \\ &= 1 - \text{binompdf}(6, 0.05, 0) \\ &= 0.265 \end{aligned}$$

$$7 \quad X \text{ is Bin } (10, 0.2)$$

$$\begin{aligned} P(\text{Raj passes}) &= P(X \geq 7) \\ &= 1 - P(X \leq 6) \\ &= 1 - \text{binomcdf}(10, 0.2, 6) \\ &\doteq 0.000864 \quad \{\text{i.e., about 9 in 10 000}\} \end{aligned}$$

$$8 \quad P(\text{M wins a game against J}) = \frac{2}{3} \quad \text{i.e.,} \quad P(\text{M wins}) = \frac{2}{3} \quad P(\text{J wins}) = \frac{1}{3}$$

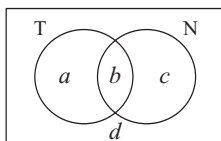
$$\begin{aligned} &P(\text{J wins a set 6 games to 4}) \\ &= P(\underbrace{\text{J wins 5 of the first 9 games}}_{\text{this is Bin } (9, \frac{1}{3})} \text{ and J wins the 10th game}) \\ &= \text{Binompdf}\left(9, \frac{1}{3}, 5\right) \times \frac{1}{3} \\ &\doteq 0.0341 \end{aligned}$$

EXERCISE 19I**1 a****b****c****d****e****f****2**

- a** Total number in the class = $3 + 4 + 5 + 17 = 29$
b Number who study both = 17 {the intersection}
c Number who study at least one = $5 + 17 + 4 = 26$ {the union}
d Number who study only Chem. = 5

3

- a** Total number in the survey = $37 + 9 + 15 + 4 = 65$
b Number who liked both = 9 {the intersection}
c Number who liked neither = 4
d Number who liked exactly one = $37 + 15 = 52$

4

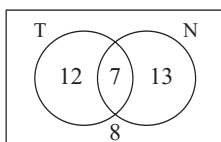
T represents those playing tennis
 N represents those playing netball

$$\begin{aligned}\therefore a + b + c + d &= 40 \\ a + b &= 19 \\ b + c &= 20 \\ d &= 8\end{aligned}$$

$$\text{So, } a + b + c = 32$$

$$\begin{aligned}\therefore 19 + c &= 32 \quad \text{and} \quad a + 20 = 32 \\ \therefore c &= 13 \quad \text{and} \quad a = 12\end{aligned}$$

$$\begin{aligned}\text{Hence, } 12 + b + 13 + 8 &= 40 \\ \therefore b &= 7\end{aligned}$$

**a** P(plays tennis)

$$\begin{aligned}&= \frac{12 + 7}{40} \\ &= \frac{19}{40}\end{aligned}$$

b P(does not play netball)

$$\begin{aligned}&= \frac{12 + 8}{40} \\ &= \frac{1}{2}\end{aligned}$$

c P(plays at least one)

$$\begin{aligned}&= \frac{12 + 7 + 13}{40} \\ &= \frac{32}{40} \\ &= \frac{4}{5}\end{aligned}$$

d P(plays one and only one)

$$\begin{aligned}&= \frac{12 + 13}{40} \\ &= \frac{25}{40} \\ &= \frac{5}{8}\end{aligned}$$

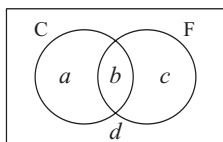
e P(plays netball, but not tennis)

$$= \frac{13}{40}$$

f P(plays tennis given plays netball)

$$\begin{aligned}&= \frac{7}{7 + 13} \\ &= \frac{7}{20}\end{aligned}$$

5



C represents men who gave chocolates

F represents men who gave flowers

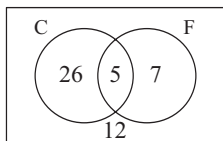
$$\therefore a + b + c + d = 50$$

$$a + b = 31$$

$$b + c = 12$$

$$b = 5$$

$$\text{Thus } c = 7, a = 26 \text{ and } 26 + 5 + 7 + d = 50 \therefore d = 12$$



a $P(C \text{ or } F)$

$$= \frac{26+5+7}{50}$$

$$= \frac{38}{50} \text{ i.e., } \frac{19}{25}$$

b $P(C \text{ but not } F)$

$$= \frac{26}{50}$$

$$= \frac{13}{25}$$

c $P(\text{neither } C \text{ nor } F)$

$$= \frac{12}{50}$$

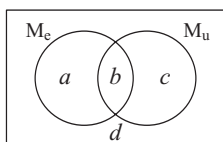
$$= \frac{6}{25}$$

d $P(F \text{ given that } C')$

$$= \frac{7}{7+12}$$

$$= \frac{7}{19}$$

6



$$a + b + c + d = 30$$

$$a + b = 24$$

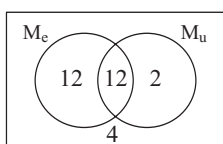
$$b = 12$$

$$a + b + c = 26$$

$$\therefore 26 + d = 30 \text{ i.e., } d = 4$$

$$24 + c = 26 \text{ i.e., } c = 2$$

$$\text{and } a + 12 + 2 = 26 \therefore a = 12$$



a $P(\text{Mu})$

$$= \frac{14}{30}$$

$$= \frac{7}{15}$$

b $P(\text{Mu, but not Me})$

$$= \frac{2}{30}$$

$$= \frac{1}{15}$$

c $P(\text{neither Mu nor Me})$

$$= \frac{4}{30}$$

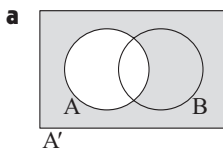
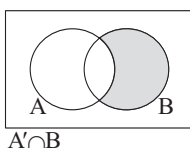
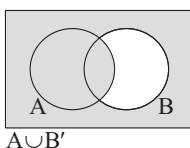
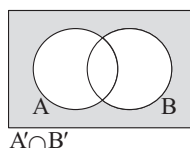
$$= \frac{2}{15}$$

d $P(\text{Me given Mu})$

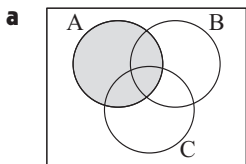
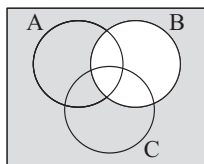
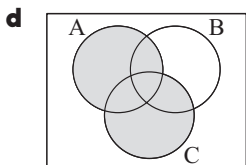
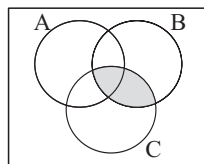
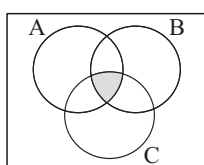
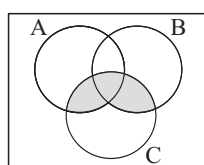
$$= \frac{12}{14}$$

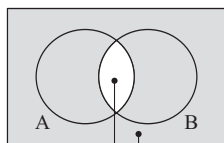
$$= \frac{6}{7}$$

7

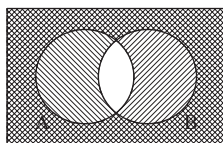

 A'
b

 $A \cap B$
c

 $A \cup B'$
d

 $A' \cap B'$

8


b

c

e

f


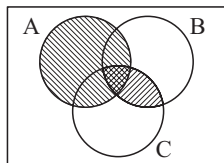
9 a

$A \cap B$ ——— $(A \cap B)'$ is shaded


 A'
 B'

So $A' \cup B'$ is the region containing either type of shading.

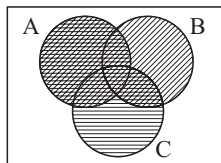
Thus, as the regions are the same, $(A \cap B)' = A' \cup B'$ is verified.

b

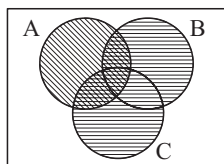
$A \cup (B \cap C)$ consists of the shaded region

As the two regions are identical

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ is verified.



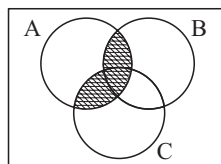
$(A \cup B) \cap (A \cup C)$ consists of the 'double shaded' region.

c

$A \cap (B \cup C)$ consists of the double shaded region



As the regions are identical, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ is verified.



$(A \cap B) \cup (A \cap C)$ consists of the region shaded. (all forms and)

10**a** $A = \{7, 14, 21, 28, 35, \dots, 98\}$ $B = \{5, 10, 15, 20, 25, \dots, 95\}$ **i** as $98 = 7 \times 14$, $n(A) = 14$ **ii** as $95 = 5 \times 19$, $n(B) = 19$ **iii** $A \cap B = \{35, 70\}$ $\therefore n(A \cap B) = 2$ **iv** $A \cup B = \{5, 7, 10, 14, 15, 20, 21, 25, 28, 30, 35, 40, 42, 45, 49, 50, 55, 56, 60, 63, 65, 70, 75, 77, 80, 84, 85, 90, 91, 95, 98\}$ $\therefore n(A \cup B) = 31$ **b** $n(A) + n(B) - n(A \cap B) = 14 + 19 - 2$ $= 31$ $= n(A \cup B) \quad \checkmark$ **11****a i**

$$\begin{aligned} P(B) &= \frac{n(B)}{n(S)} \\ &= \frac{b+c}{a+b+c+d} \end{aligned}$$

ii

$$\begin{aligned} P(A \text{ and } B) &= \frac{n(A \cap B)}{n(S)} \\ &= \frac{b}{a+b+c+d} \end{aligned}$$

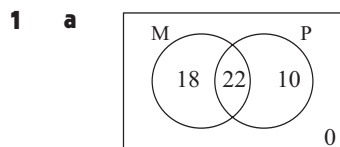
iii

$$\begin{aligned} P(A \text{ or } B) &= \frac{n(A \cup B)}{n(S)} \\ &= \frac{a+b+c}{a+b+c+d} \end{aligned}$$

iv

$$\begin{aligned} P(A) + P(B) - P(A \text{ and } B) &= \frac{a+b+b+c-b}{a+b+c+d} \\ &= \frac{a+b+c}{a+b+c+d} \end{aligned}$$

$$\mathbf{b} \quad P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \quad \{\text{using iii and iv}\}$$

EXERCISE 19J


So 22 study both.

b i $P(M \text{ but not } P)$ **ii** $P(P \text{ given } M)$

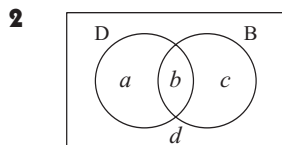
$$= \frac{18}{50}$$

$$= \frac{9}{25}$$

$$= \frac{22}{18 + 22}$$

$$= \frac{22}{40}$$

$$= \frac{11}{20}$$

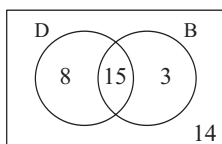


$$a + b + c + d = 40 \quad \dots (1) \quad \therefore d = 14 \quad \{\text{using (1) and (4)}\}$$

$$a + b = 23 \quad \dots (2) \quad 23 + c = 26 \quad \text{and} \quad a + 18 = 26$$

$$b + c = 18 \quad \dots (3) \quad \therefore c = 3 \quad \text{and} \quad a = 8$$

$$a + b + c = 26 \quad \dots (4) \quad \text{Thus} \quad b = 18 - c = 15$$



a $P(D \text{ and } B)$ **b** $P(\text{neither } D \text{ nor } B)$

$$= \frac{15}{40}$$

$$= \frac{3}{8}$$

$$= \frac{14}{40}$$

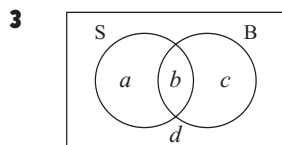
$$= \frac{7}{20}$$

c $P(D, \text{ but not } B)$ **d** $P(B \text{ given } D)$

$$= \frac{8}{40}$$

$$= \frac{1}{5}$$

$$= \frac{15}{23}$$

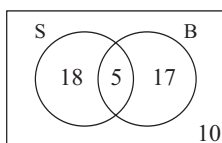


$$a + b + c + d = 50 \quad \therefore c = 17, \quad a = 18$$

$$a + b = 23 \quad \text{and} \quad 18 + 5 + 17 + d = 50$$

$$b + c = 22 \quad \therefore d = 10$$

$$b = 5$$



a $P(\text{not } B)$ **b** $P(B \text{ or } S)$ **c** $P(\text{neither } B \text{ nor } S)$

$$= P(B')$$

$$= \frac{28}{50}$$

$$= \frac{14}{25}$$

$$= \frac{18 + 5 + 17}{50}$$

$$= \frac{40}{50}$$

$$= \frac{4}{5}$$

$$= \frac{10}{50}$$

$$= \frac{1}{5}$$

d $P(B, \text{ given } S)$

$$= \frac{5}{18 + 5}$$

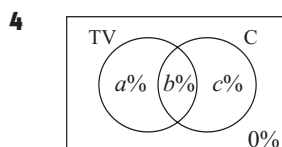
$$= \frac{5}{23}$$

e $P(S, \text{ given } B')$

$$= \frac{18}{18 + 10}$$

$$= \frac{18}{28}$$

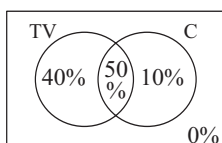
$$= \frac{9}{14}$$



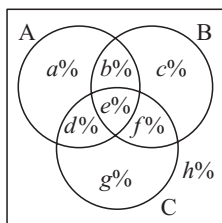
$$a + b + c = 100$$

$$a + b = 90 \quad \therefore c = 10 \quad \text{and} \quad a = 40$$

$$b + c = 60 \quad \therefore b = 50$$



$$P(\text{TV, given } C) = \frac{50}{50 + 10} = \frac{5}{6}$$

5

$$a + b + c + d + e + f + g + h = 100$$

$$a + b + d + e = 20$$

$$b + c + e + f = 16$$

$$d + e + f + g = 14$$

$$b + e = 8$$

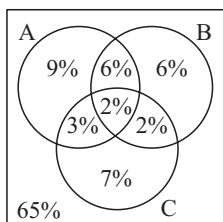
$$d + e = 5$$

$$e + f = 4$$

$$e = 2$$

i.e., $e = 2$, $f = 2$, $d = 3$, $b = 6$,

$$\begin{cases} a + 6 + 3 + 2 = 20 \\ 6 + c + 2 + 2 = 16 \\ 3 + 2 + 2 + g = 14 \end{cases} \quad \therefore \begin{cases} a = 9 \\ c = 6 \\ g = 7 \end{cases}$$

**a** P (none)

$$= \frac{65}{100} \\ = \frac{13}{20}$$

b P (at least one)

$$= 1 - \text{P (none)} \\ = 1 - \frac{13}{20} \\ = \frac{7}{20}$$

c P (exactly one)

$$= \frac{9 + 6 + 7}{100} \\ = \frac{22}{100} \\ = \frac{11}{50}$$

d P (A or B)

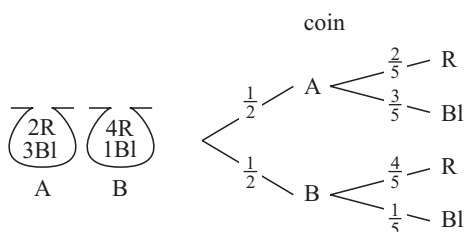
$$= \frac{9 + 6 + 6 + 3 + 2 + 2}{100} \\ = \frac{28}{100} \\ = \frac{7}{25}$$

e P (A, given at least one)

$$= \frac{9 + 6 + 2 + 3}{35} \\ = \frac{20}{35} \\ = \frac{4}{7}$$

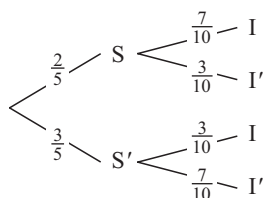
f P (C, given A or B or both)

$$= \frac{3 + 2 + 2}{9 + 6 + 6 + 3 + 2 + 2} \\ = \frac{7}{28} \\ = \frac{1}{4}$$

6

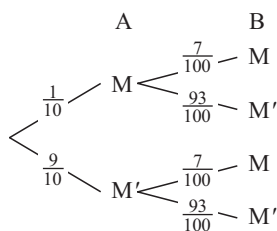
$$\text{a} \quad \text{P(R)} = \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{4}{5} \\ = \frac{3}{5}$$

$$\text{b} \quad \text{P(B|R)} = \frac{\text{P(B} \cap \text{R)}}{\text{P(R)}} \\ = \frac{\frac{1}{2} \times \frac{4}{5}}{\frac{3}{5}} \\ = \frac{2}{3}$$

7

$$\text{a} \quad \text{P(I)} = \frac{2}{5} \times \frac{7}{10} + \frac{3}{5} \times \frac{3}{10} \\ = \frac{23}{50} \quad (\text{or } 0.46)$$

$$\text{b} \quad \text{P(S|I)} = \frac{\text{P(S} \cap \text{I)}}{\text{P(I)}} \\ = \frac{\frac{2}{5} \times \frac{7}{10}}{\frac{23}{50}} \\ = \frac{14}{23}$$

8

P(B | at least one malfunctions)

$$= \frac{\text{P(B} \cap \text{at least one malfunctions)}}{\text{P(at least one malfunctions)}}$$

$$= \frac{\frac{1}{10} \times \frac{7}{100} + \frac{9}{10} \times \frac{7}{100}}{\frac{1}{10} \times \frac{7}{100} + \frac{1}{10} \times \frac{93}{100} + \frac{9}{10} \times \frac{7}{100}} \\ = \frac{7 + 63}{7 + 93 + 63} \\ = \frac{70}{163}$$

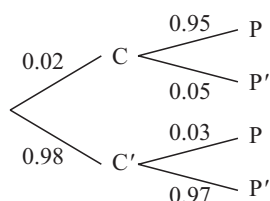
9 $P(B) = 0.5$, $P(G) = 0.6$, $P(G | B) = 0.9$

a $P(\text{both eat})$
 $= P(B \cap G)$
 $= P(G | B) \times P(B) \quad \left\{ \text{as } P(G | B) = \frac{P(G \cap B)}{P(B)} \right\}$
 $= 0.9 \times 0.5$
 $= 0.45$

b $P(B | G)$
 $= \frac{P(B \cap G)}{P(G)}$
 $= \frac{0.45}{0.6}$
 $= 0.75$

c $P(\text{at least one eats})$
 $= P(B \cup G)$
 $= P(B) + P(G) - P(B \cap G)$
 $= 0.5 + 0.6 - 0.45$
 $= 0.65$

10



a $P(P)$
 $= 0.02 \times 0.95 + 0.98 \times 0.03$
 $= 0.0484$

b $P(C | P)$
 $= \frac{P(C \cap P)}{P(P)}$
 $= \frac{0.02 \times 0.95}{0.0484}$
 $\div 0.3926$

11 The coins are H, H T, T and H, T.

Any one of these 6 faces could be seen uppermost, $\therefore P(\text{falls H}) = \frac{3}{6} = \frac{1}{2}$

Now $P(\text{HH coin} | \text{falls H}) = \frac{P(\text{HH coin} \cap \text{falls H})}{P(\text{falls H})}$
 $= \frac{P(\text{HH})}{P(\text{falls H})}$
 $= \frac{\frac{1}{3}}{\frac{1}{2}}$
 $= \frac{2}{3}$

EXERCISE 19K

1 $P(R \cap S)$ Also, $P(R) \times P(S)$
 $= P(R) + P(S) - P(R \cup S) \quad = 0.4 \times 0.5$
 $= 0.4 + 0.5 - 0.7 \quad = 0.2$
 $= 0.2$

So, $P(R \cap S) = P(R) \times P(S) \Rightarrow R$ and S are independent events.

2 a $P(A \cap B)$
 $= P(A) + P(B) - P(A \cup B)$
 $= \frac{2}{5} + \frac{1}{3} - \frac{1}{2}$
 $= \frac{7}{30}$

b $P(B | A)$
 $= \frac{P(B \cap A)}{P(A)}$
 $= \frac{\frac{7}{30}}{\frac{2}{5}}$
 $= \frac{7}{12}$

c $P(A | B)$
 $= \frac{P(A \cap B)}{P(B)}$
 $= \frac{\frac{7}{30}}{\frac{1}{3}}$
 $= \frac{7}{10}$

A and B are not independent as $P(A \cap B) = \frac{7}{30}$ whereas $P(A) \times P(B) = \frac{2}{15}$ i.e., $P(A \cap B) \neq P(A) \times P(B)$

3 a As X and Y are independent

$$P(X \cap Y) = P(X) \times P(Y)$$

$$= 0.5 \times 0.7$$

$$= 0.35$$

$$\text{i.e., } P(\text{both X and Y}) = 0.35$$

c P(neither X nor Y)

$$= 0.15$$

$$\text{e } P(X|Y) = \frac{P(X \cap Y)}{P(Y)} = \frac{0.35}{0.70} = \frac{1}{2}$$

b $P(X \text{ or } Y)$

$$= P(X \cup Y)$$

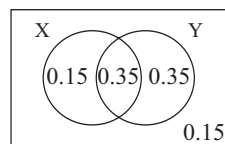
$$= P(X) + P(Y) - P(X \cap Y)$$

$$= 0.5 + 0.7 - 0.35$$

$$= 0.85$$

d $P(X \text{ but not } Y)$

$$= 0.15$$

**4** $P(\text{at least one solves it})$

$$= 1 - P(\text{no-one solves it})$$

$$= 1 - P(A' \text{ and } B' \text{ and } C')$$

$$= 1 - \frac{2}{5} \times \frac{1}{3} \times \frac{1}{2}$$

$$= 1 - \frac{1}{15}$$

$$= \frac{14}{15}$$

5 a $P(\text{at least one 6})$

$$= 1 - P(\text{no 6s})$$

$$= 1 - P(6' \text{ and } 6' \text{ and } 6')$$

$$= 1 - \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$$

$$= 1 - \frac{125}{216}$$

$$= \frac{91}{216}$$

b $P(\text{at least one 6 in } n \text{ throws})$

$$= 1 - \left(\frac{5}{6}\right)^n$$

$$\text{So we want } 1 - \left(\frac{5}{6}\right)^n > 0.99$$

$$\therefore -\left(\frac{5}{6}\right)^n > -0.01$$

$$\therefore \left(\frac{5}{6}\right)^n < 0.01$$

$$\therefore n \log\left(\frac{5}{6}\right) < \log(0.01)$$

$$\therefore n > \frac{\log(0.01)}{\log\left(\frac{5}{6}\right)}$$

$$\{\text{as } \log\left(\frac{5}{6}\right) < 0\}$$

$$\therefore n > 25.2585\dots$$

$$\text{i.e., } n = 26$$

6 A and B are independent $\Rightarrow P(A \cap B) = P(A)P(B)$ (1)Now $P(A \cap B')$

$$= P(A \cup B) - P(B) \quad \{\text{see diagram}\}$$

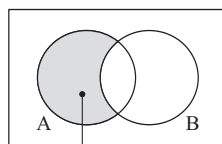
$$= P(A) + P(B) - P(A \cap B) - P(B)$$

$$= P(A) - P(A \cap B)$$

$$= P(A) - P(A)P(B) \quad \{\text{from (1)}\}$$

$$= P(A)[1 - P(B)]$$

$$= P(A) \times P(B')$$

 $A \cap B'$ $\therefore$ A and B' are also independent.**EXERCISE 19L****1** The total number of different committees is ${}_{11}C_4$. (Note: $C_4^{11} = {}_{11}C_4$)The number of ways of X and Y being on a committee is ${}_2C_2 \times {}_9C_2$.

$$\therefore P(X \text{ and } Y \text{ are on the committee}) = \frac{{}_2C_2 \times {}_9C_2}{{}_{11}C_4}$$

$$\div 0.109$$

2 AIDS and SAID are 2 of the $4!$ different orderings.

$$\therefore P(\text{AIDS or SAID}) = \frac{2}{4!} = \frac{1}{12}$$

3 There are ${}_{12}C_7$ different teams that can be selected.

$$\therefore P(\text{captain and vice captain are chosen}) = \frac{{}_2C_2 \times {}_{10}C_5}{{}_{12}C_7} \doteq 0.318$$

4 $P(\text{none of the golfers was killed}) = \frac{{}_3C_0 \times {}_{19}C_4}{{}_{22}C_4} \doteq 0.530$

5

5	4	3	2	1
---	---	---	---	---

 i.e., there are $5!$ different possible seating arrangements.

a

2	4	3	2	1
---	---	---	---	---

 there are $2 \times 4!$ seating arrangements if K and U sit at the ends

$$\therefore P(\text{K and U sit at the ends}) = \frac{2 \times 4!}{5!} = \frac{2}{5}$$

b K and U can sit together in $2!$ ways. They as a pair plus the other three people can then be ordered in $4!$ ways.

$$\therefore P(\text{sit together}) = \frac{2! \times 4!}{5!} = \frac{2}{5}$$

6 There are ${}_{16}C_5$ different committees possible.

a $P(\text{all men}) = \frac{{}_9C_5 \times {}_7C_0}{{}_{16}C_5} = 0.0288$

b $P(\text{at least 3 men}) = P(3 \text{ men or } 4 \text{ men or } 5 \text{ men})$

$$= \frac{{}_9C_3 \times {}_7C_2 + {}_9C_4 \times {}_7C_1 + {}_9C_5 \times {}_7C_0}{{}_{16}C_5}$$

$$\doteq 0.635$$

c $P(\text{at least one of each sex}) = 1 - P(\text{no men or no women})$

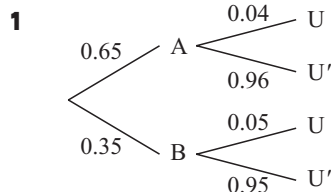
$$= 1 - \frac{{}_9C_0 \times {}_7C_5 + {}_9C_5 \times {}_7C_0}{{}_{16}C_5}$$

$$\doteq 0.966$$

7 If there are no restrictions there are $6!$ different orderings possible. A, B and C can be ordered in $3!$ ways. This triple together with the 3 others can be ordered in $4!$ ways.

$$\therefore P(\text{A, B, C together}) = \frac{3!4!}{6!} = \frac{1}{5}$$

EXERCISE 19M



a $P(\text{underfilled})$

$$= P(\text{A and U or B and U})$$

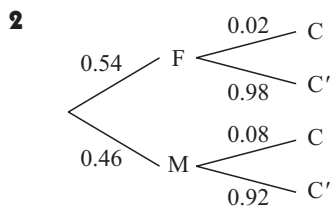
$$= 0.65 \times 0.04 + 0.35 \times 0.05$$

$$= 0.0435$$

b $P(\text{A|U}) = \frac{P(\text{A} \cap \text{U})}{P(\text{U})}$

$$= \frac{0.65 \times 0.04}{0.0435}$$

$$\doteq 0.598$$



a $P(\text{M|C}) = \frac{P(\text{C|M}) \times P(\text{M})}{P(\text{C|M}) \times P(\text{M}) + P(\text{C|F}) \times P(\text{F})}$

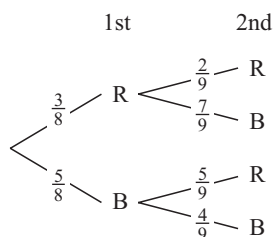
$$= \frac{0.08 \times 0.46}{0.08 \times 0.46 + 0.02 \times 0.54}$$

$$\doteq 0.773$$

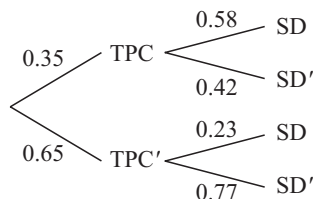
b $P(\text{F|C'}) = \frac{P(\text{C'|F}) \times P(\text{F})}{P(\text{C'|F}) \times P(\text{F}) + P(\text{C'|M}) \times P(\text{M})}$

$$= \frac{0.98 \times 0.54}{0.98 \times 0.54 + 0.92 \times 0.46}$$

$$\doteq 0.556$$

3

$$\begin{aligned}
 & P(BB \mid RR \text{ or } BB) \\
 &= \frac{P(BB \cap (RR \text{ or } BB))}{P(RR \text{ or } BB)} \\
 &= \frac{P(BB)}{P(RR \text{ or } BB)} \\
 &= \frac{\frac{5}{8} \times \frac{4}{9}}{\frac{3}{8} \times \frac{2}{9} + \frac{5}{8} \times \frac{4}{9}} \\
 &= \frac{20}{26} \\
 &= \frac{10}{13}
 \end{aligned}$$

4

$$\begin{aligned}
 & P(TPC' \mid SD) \\
 &= \frac{P(TPC' \cap SD)}{P(SD)} \\
 &= \frac{0.65 \times 0.23}{0.35 \times 0.58 + 0.65 \times 0.23} \\
 &\div 0.424
 \end{aligned}$$

5

a $P(A) = P(A \text{ and in } C_1 \text{ or } A \text{ and in } C_2 \text{ or } A \text{ and in } C_3)$

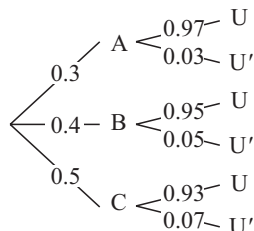
$$= P((A \cap C_1) \cup (A \cap C_2) \cup (A \cap C_3))$$

$$= P(A \cap C_1) + P(A \cap C_2) + P(A \cap C_3)$$

as $A \cap C_1$, $A \cap C_2$, $A \cap C_3$ are disjoint

$$= P(C_1) P(A \mid C_1) + P(C_2) P(A \mid C_2) + P(C_3) P(A \mid C_3)$$

$$\{\text{as } P(X \mid Y) = \frac{P(X \cap Y)}{P(Y)}, \text{ then } P(X \cap Y) = P(Y) P(X \mid Y)\}$$

b

i $P(U) = P(A \cap U \text{ or } B \cap U \text{ or } C \cap U)$

$$= 0.3 \times 0.97 + 0.4 \times 0.95 + 0.3 \times 0.93$$

$$= 0.95$$

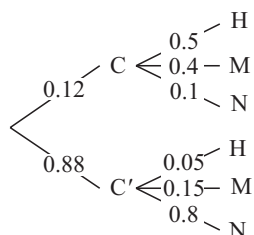
ii $P(A \mid U) = \frac{P(A \cap U)}{P(U)} = \frac{0.3 \times 0.97}{0.95} \div 0.306$

iii $P(A \text{ or } C \mid U') = \frac{P((A \cup C) \cap U')}{P(U')}$

$$= \frac{P((A \cap U') \cup (C \cap U'))}{P(U')}$$

$$= \frac{0.3 \times 0.03 + 0.3 \times 0.07}{0.05}$$

$$= 0.6$$

6

a $P(H)$

$$= P(C \cap H \text{ or } C' \cap H)$$

$$= 0.12 \times 0.5 + 0.88 \times 0.05$$

$$= 0.104$$

$$\begin{aligned} \mathbf{b} \quad P(C|M) &= \frac{P(C \cap M)}{P(M)} \\ &= \frac{0.12 \times 0.4}{0.12 \times 0.4 + 0.88 \times 0.15} \\ &\div 0.267 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad P(C|N) &= \frac{P(C \cap N)}{P(N)} \\ &= \frac{0.12 \times 0.1}{0.12 \times 0.1 + 0.88 \times 0.8} \\ &\div 0.0168 \end{aligned}$$

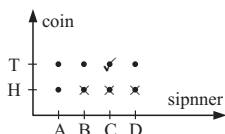
REVIEW SET 19A

- 1** ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA

- a** There are 24 possible orderings.
 $\therefore P(A \text{ is next to } C)$
 $= \frac{12}{24} \quad \{12 \text{ have } A \text{ next to } C\}$
 $= \frac{1}{2}$

- b** $P(\text{exactly one person between } A \text{ and } C)$
 $= \frac{8}{24} \quad \{8 \text{ have one person between } A \text{ and } C\}$
 $= \frac{1}{3}$

2

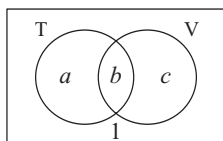


- a** Consonants are B, C and D
 $\therefore P(H \text{ and a consonant})$
 $= \frac{3}{8} \quad \{\text{those with } \times\}$

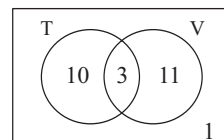
- b** $P(T \text{ and } C)$
 $= \frac{1}{8} \quad \{\text{those with a } \checkmark\}$

- c** $P(T \text{ or vowel})$
 $= P(T \text{ or } A)$
 $= P(T) + P(A) - P(T \text{ and } A)$
 $= \frac{4}{8} + \frac{2}{8} - \frac{1}{8}$
 $= \frac{5}{8}$

3



- $a + b + c = 24$
 $a + b = 13$
 $b + c = 14$
 Also $b = 13 - a$
 $= 3$
- $\therefore 13 + c = 24 \quad \text{and} \quad a + 14 = 24$
 $\therefore c = 11 \quad \text{and} \quad a = 10$

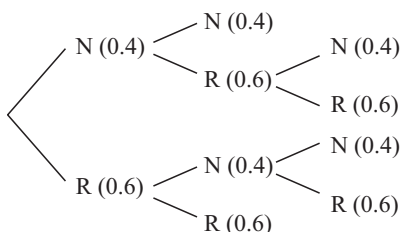


- a** $P(T \text{ and } V)$
 $= \frac{3}{25}$

- b** $P(\text{at least one})$
 $= 1 - P(\text{neither})$
 $= 1 - \frac{1}{25}$
 $= \frac{24}{25}$

- c** $P(V | T')$
 $= \frac{11}{11 + 1}$
 $= \frac{11}{12}$

4



- $P(\text{Niklas wins})$
 $= (0.4)(0.4) + (0.4)(0.6)(0.4) + (0.6)(0.4)(0.4)$
 $= 0.352$

5 $P(M) = \frac{3}{5}, P(W) = \frac{2}{3}$

a $P(M \text{ and } W)$
 $= \frac{3}{5} \times \frac{2}{3}$ {assuming
 $= \frac{2}{5}$ independence}

b $P(\text{at least one})$
 $= P(M \text{ or } W)$
 $= P(M) + P(W) - P(M \text{ and } W)$
 $= \frac{3}{5} + \frac{2}{3} - \frac{2}{5}$
 $= \frac{13}{15}$

c $P(M' \text{ and } W)$
 $= (1 - \frac{3}{5}) \times \frac{2}{3}$
 $= \frac{2}{5} \times \frac{2}{3}$
 $= \frac{4}{15}$

6 A repetition of 5 independent events $\therefore$ binomial model applies.

$P(M) = \frac{4}{5}, P(M') = \frac{1}{5}$

a $P(M \text{ wins 3 games})$
 $= C_3^5 (\frac{4}{5})^3 (\frac{1}{5})^2$
 $\div 0.205$

b $P(M \text{ wins 4 or 5 games})$
 $= C_4^5 (\frac{4}{5})^4 (\frac{1}{5})^1 + C_5^5 (\frac{4}{5})^5$
 $\div 0.737$

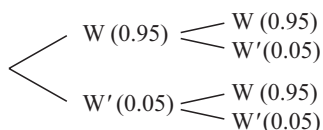
7 a $P(\text{wins first 3 prizes})$

$= P(WWW)$
 $= \frac{4}{500} \times \frac{3}{499} \times \frac{2}{498}$
 $\div 1.93 \times 10^{-7}$

b $P(\text{wins at least one of the 3 prizes})$

$= 1 - P(\text{wins none of them})$
 $= 1 - P(W'W'W')$
 $= 1 - \frac{496}{500} \times \frac{495}{499} \times \frac{494}{498}$
 $\div 0.0239$

8



$P(\text{works on at least one day})$

$= 0.95 \times 0.95 + 0.95 \times 0.05 + 0.05 \times 0.95$
 $= 0.9975$

9 a $P(\text{all doctors}) = \frac{{}^6C_5 \times {}^4C_0}{{}^{10}C_5} \div 0.0238$

b $P(\text{at least 2 doctors}) = 1 - P(1 \text{ doctor}) = 1 - \frac{{}^6C_1 \times {}^4C_4}{{}^{10}C_5} \div 0.976$

10 If there are no restrictions, there are 6! different orderings.

a

3	3	2	2	1	1
B	G	B	G	B	G

or

3	3	2	2	1	1
G	B	G	B	G	B

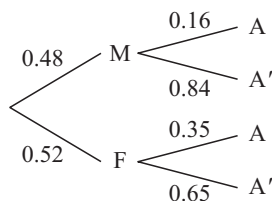
i.e., 36 + 36 consist
of alternate sexes.

$\therefore P(\text{alternate sexes}) = \frac{72}{6!} = \frac{1}{10}$

b The girls as a group can be ordered in 3! ways. This group plus the 3 boys can be ordered in 4! ways.

$\therefore P(\text{girls are together}) = \frac{3! \times 4!}{6!} = \frac{1}{5}$

11

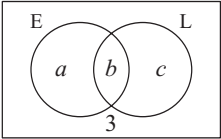


a $P(A) = P(M \cap A \text{ or } F \cap A)$
 $= 0.48 \times 0.16 + 0.52 \times 0.35$
 $= 0.2588 \quad (\div 0.259)$

b $P(F|A) = \frac{P(F \cap A)}{P(A)}$
 $= \frac{0.52 \times 0.35}{0.2588}$
 $\div 0.703$

REVIEW SET 19B

- 1 BBBB, BBBG, BBGB, BGBB, GBBB, BBGG, BGBG, BGGB, GGBB, GBBG, GBGB, BGGG, GBGG, GGBG, GGGG
- $$\begin{aligned}
 P(2B \text{ and } 2G) &= \frac{6}{16} \leftarrow 6 \text{ have } 2B \text{ and } 2G \\
 &= \frac{3}{8}
 \end{aligned}$$

- 2
- 
- $$\begin{aligned}
 a + b + c &= 37 \\
 a + b &= 22 & \therefore 22 + c = 37 \text{ and } a + 25 = 37 \\
 b + c &= 25 & \therefore c = 15 \text{ and } a = 12
 \end{aligned}$$
- Hence, $b = 22 - a = 10$

- a $P(E \text{ and } L)$

$$= \frac{10}{40}$$

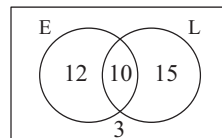
$$= \frac{1}{4}$$

- c $P(E | L) = \frac{10}{15+10} = \frac{10}{25} = \frac{2}{5}$

- b $P(\text{at least one})$

$$= \frac{12+10+15}{40}$$

$$= \frac{37}{40}$$



- 3 a $P(\text{both blue})$

$$= P(BB)$$

$$= \frac{5}{12} \times \frac{4}{11}$$

$$= \frac{5}{33}$$

- b $P(\text{both same colour})$

$$= P(BB \text{ or } RR \text{ or } YY)$$

$$= \frac{5}{12} \times \frac{4}{11} + \frac{3}{12} \times \frac{2}{11} + \frac{4}{12} \times \frac{3}{11}$$

$$= \frac{19}{66}$$

- c $P(\text{at least one } R)$

$$= 1 - P(\text{no reds})$$

$$= 1 - P(R'R')$$

$$= 1 - \frac{9}{12} \times \frac{8}{11}$$

$$= 1 - \frac{6}{11}$$

$$= \frac{5}{11}$$

- d $P(\text{exactly one } Y)$

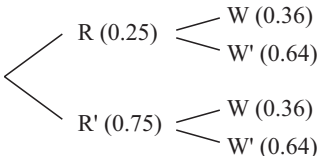
$$= P(YY' \text{ or } Y'Y)$$

$$= \frac{4}{12} \times \frac{8}{11} + \frac{8}{12} \times \frac{4}{11}$$

$$= \frac{16}{33}$$

- 4 a Two events are independent if the occurrence of one does not influence the occurrence of the other. For A and B independent, $P(A) \times P(B) = P(A \text{ and } B)$

- b Two events, A and B, are disjoint if they have no common outcomes, i.e., $P(A \text{ and } B) = 0$ and so $P(A \text{ or } B) = P(A) + P(B)$

- 5
- 

- a $P(W \text{ and } R)$

$$= 0.25 \times 0.36$$

$$= 0.09$$

- b $P(W \text{ or } R)$

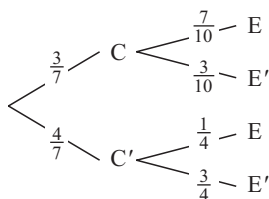
$$= P(W) + P(R) - P(W \text{ and } R)$$

$$= 0.36 + 0.25 - 0.09$$

$$= 0.52$$

$$\begin{aligned}
 \text{or } P(W \text{ or } R) &= 1 - P(W'R') \\
 &= 1 - 0.64 \times 0.75 \\
 &= 0.52
 \end{aligned}$$

- 6 $P(A) = 0.1, P(B) = 0.2, P(C) = 0.3 \therefore P(\text{group solves it}) = P(\text{at least one solves it})$
- $$\begin{aligned}
 &= 1 - P(\text{no-one solves it}) \\
 &= 1 - P(A' \text{ and } B' \text{ and } C') \\
 &= 1 - (0.9 \times 0.8 \times 0.7) \\
 &= 0.496
 \end{aligned}$$

7

$$\begin{aligned} \mathbf{a} \quad P(E) &= \frac{3}{7} \times \frac{7}{10} + \frac{4}{7} \times \frac{1}{4} \\ &= \frac{3}{10} + \frac{1}{7} \\ &= \frac{31}{70} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad P(C|E) &= \frac{P(C \text{ and } E)}{P(E)} \\ &= \frac{\frac{3}{7} \times \frac{7}{10}}{\frac{31}{70}} \\ &= \frac{21}{31} \end{aligned}$$

8

$$\begin{aligned} \mathbf{a} \quad \left(\frac{3}{5} + \frac{2}{5}\right)^4 &= \underbrace{\left(\frac{3}{5}\right)^4}_{4B} + 4 \underbrace{\left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)}_{\substack{3B \\ 1B'}} + 6 \underbrace{\left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^2}_{\substack{2B \\ 2B'}} + 4 \underbrace{\left(\frac{3}{5}\right) \left(\frac{2}{5}\right)^3}_{\substack{1B \\ 3B'}} + \underbrace{\left(\frac{2}{5}\right)^4}_{4B'} \end{aligned}$$

$$\begin{aligned} P(B) &= \frac{12}{20} \\ &= \frac{3}{5} \end{aligned}$$

$$\therefore P(B') = \frac{2}{5}$$

b i P(2 Blue inks)

$$= P(2B \text{ and } 2B')$$

$$= 6 \left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^2$$

$$= \frac{6 \times 9 \times 4}{5^4}$$

$$= \frac{144}{625}$$

ii P(at most 2 Blue inks)

$$= P(2B \text{ and } 2B' \text{ or } 1B \text{ and } 3B' \text{ or } 4B')$$

$$= 6 \left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^2 + 4 \left(\frac{3}{5}\right) \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4$$

$$= \frac{6 \times 9 \times 4 + 4 \times 3 \times 8 + 16}{625}$$

$$= \frac{328}{625}$$

9

$$\mathbf{a} \quad P(X \text{ wins}) = \frac{3}{5} \quad P(Y \text{ wins}) = \frac{2}{5}$$

$$\text{Probability generator is } \left(\frac{3}{5} + \frac{2}{5}\right)^6$$

$$= \left(\frac{3}{5}\right)^6 + 6 \left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right) + 15 \left(\frac{3}{5}\right)^4 \left(\frac{2}{5}\right)^2 + 20 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^3 + 15 \left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^4$$

↑

X wins 6

↑

X wins 5

Y wins 1

↑

X wins 4

Y wins 2

↑

X wins 3

Y wins 3

↑

X wins 2

Y wins 4

$$+ 6 \left(\frac{3}{5}\right) \left(\frac{2}{5}\right)^5 + \left(\frac{3}{5}\right)^6$$

↑

X wins 1

Y wins 5

↑

X wins 0

Y wins 6

b i P(Y wins 3)

$$= 20(0.6)^3(0.4)^3$$

$$\div 0.276$$

ii P(Y wins at least 5)

$$= 6(0.6)^1(0.4)^5 + (0.4)^6$$

$$\div 0.0410$$

10 X = the number of goals scored X is Bin (5, 0.8)**a** P(3 goals then 2 misses)

$$= P(GGGG'G')$$

$$= (0.8)^3 \times (0.2)^2$$

$$\div 0.0205$$

b P(3 goals and 2 misses)

$$= P(X = 3)$$

$$= \text{binompdf}(5, 0.8, 3)$$

$$\div 0.205$$

Chapter 20

INTRODUCTION TO CALCULUS

EXERCISE 20A.1

- 1 a Average speed from Taillem Bend to Nhill

$$\begin{aligned} &= \frac{\text{distance travelled}}{\text{time taken}} \\ &= \frac{(324 - 98) \text{ km}}{\left(\frac{204 - 63}{60}\right) \text{ h}} \\ &= \frac{226 \times 60}{141} \\ &= 96.2 \text{ kmph} \end{aligned}$$

- b Average speed from Horsham to Melbourne

$$\begin{aligned} &= \frac{729 - 431 \text{ km}}{\left(\frac{534 - 261}{60}\right) \text{ h}} \\ &= \frac{298 \times 60}{273} \\ &= 65.5 \text{ kmph} \end{aligned}$$

- 2 a 800 m to the newsagency

b $m = \frac{y\text{-step}}{x\text{-step}} = \frac{500 - 0}{4 - 0} = 125$

c Average speed $= \frac{(500 - 0) \text{ m}}{(4 - 0) \text{ min}} = 125 \text{ m/minute}$

- d The *slope* represents the *average walking speed*.

- e Paul stayed between the 12th and 20th minutes i.e., 8 minutes.

f Average speed $= \frac{(800 - 0) \text{ m}}{(32 - 20) \text{ min}}$ {return journey is from (20, 800) to (32, 0)}

$$= 66.7 \text{ m/minute}$$

- g Total distance travelled $= 1600 \text{ m} = 1.6 \text{ km}$ (to the shop and return)

3 Average water loss $= \frac{\text{amount of water lost}}{\text{time taken}}$

$$\begin{aligned} &= \frac{53.8 - 48.2}{11} \\ &= 0.509 \text{ million kL/day} \\ &= 509\,000 \text{ kL/day} \end{aligned}$$

- 4 a First quarter: Used 106.8 kL Number of days $= 31 + 28 + 31$

$$\begin{aligned} \therefore \text{rate} &= \frac{106.8}{90} \\ &= 1.19 \text{ kL/day} \end{aligned}$$

- b First six months: Used 106.8

$$\begin{array}{r} + 79.4 \\ \hline 186.2 \end{array} \text{ kL}$$

Number of days $= 90 + 30 + 31 + 30$

$$= 181 \text{ days}$$

$$\therefore \text{rate} = \frac{186.2}{181} = 1.03 \text{ kL/day}$$

- c The whole year: Used $= 106.8 + 79.4 + 81.8 + 115.8$
- $$= 383.8 \text{ kL}$$

$$\begin{aligned} \therefore \text{rate} &= \frac{383.8}{365} \text{ kg} \\ &= 1.05 \text{ kL/day} \end{aligned}$$

EXERCISE 20A.2

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad \text{In the first 4 seconds:} \quad \text{average speed} &= \frac{\text{distance travelled}}{\text{time taken}} \\
 &= \frac{(2.4 - 2) \text{ m}}{(4 - 0) \text{ sec}} \\
 &= 0.1 \text{ m/s}
 \end{aligned}$$

$$\mathbf{b} \quad \text{In the last 4 seconds:} \quad \text{average speed} = \frac{(6 - 2.4) \text{ m}}{(4 - 0) \text{ sec}} = 0.9 \text{ m/s}$$

$$\mathbf{c} \quad \text{In the 8 second period:} \quad \text{average speed} = \frac{(6 - 2) \text{ m}}{(8 - 0) \text{ sec}} = 0.5 \text{ m/s}$$

- 2 a i** Initially there are 35 beetles. After a 10 g dose there are 3 beetles.

$$\begin{aligned}
 \text{rate of change} &= \frac{\text{change in population}}{\text{change in dose}} \\
 &= \frac{(35 - 3) \text{ beetles}}{(0 - 10) \text{ gm}} \\
 &= -3.2 \text{ beetles/gm}
 \end{aligned}$$

i.e., rate of decrease = 3.2 beetles/gram

$$\begin{aligned}
 \mathbf{ii} \quad \text{rate of change} &= \frac{(26 - 8) \text{ beetles}}{(4 - 8) \text{ beetles}} \\
 &= -4.5 \text{ beetles/gm}
 \end{aligned}$$

i.e., rate of decrease = 4.5 beetles/gram

- b** A dose of 0 to 1 g has little or no effect.

A dose of 1 to 8 g has good or considerable effect (rapid decrease).

A dose of 8 to 14 g has an effect, but less rapid than the 1 to 8 rate.

EXERCISE 20B.1

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad \text{The tangent passes through (2, 3) and (0, 1)} \quad \therefore \text{ slope of tangent} &\div \frac{(3 - 1) \text{ m}}{(2 - 0) \text{ s}} \\
 &\div \frac{2}{2} \text{ m/s} \\
 &\div 1 \text{ m/s}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{The tangent passes through (3.5, 6) and (2.2, 2)} \quad \therefore \text{ slope of tangent} &\div \frac{(6 - 2) \text{ km}}{(3 - 2.2) \text{ h}} \\
 &\div \frac{4}{1.3} \text{ km/h} \\
 &\div 3 \text{ km/h}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \text{The tangent passes through (20, 2700) and (40, 3700)} \quad \therefore \text{ slope of tangent} \\
 &\div \frac{(3.7 - 2.7) \text{ thousands of \$}}{(40 - 20) \text{ items}} \\
 &\div 0.05 \text{ thousands of \$/item} \\
 &\div \$50/\text{item}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \text{The tangent passes through (0, 35) and (7, 0)} \quad \therefore \text{ slope of tangent} &\div \frac{(35 - 0) \text{ bats}}{(0 - 7) \text{ weeks}} \\
 &\div -5 \text{ bats/week}
 \end{aligned}$$

i.e., the population is decreasing at the rate of 5 bats/week after 5 weeks.

2 a Originally (when $x = 0$) the volume was 8000 L.

b After 1 hour ($x = 1$) the volume was 3000 L.

c The tangent at (0, 8) passes through (1, 0.5)

$$\begin{aligned}\therefore \text{ slope of tangent} &= \frac{(8 - 0.5) \text{ L}}{(0 - 1) \text{ h}} \\ &= -7.5 \quad \text{i.e., loses 7500 L/hour}\end{aligned}$$

d After 1 hour, $x = 1$ and the tangent at (1, 3) passes through (0, 6)

$$\begin{aligned}\therefore \text{ slope of tangent} &= \frac{(6 - 3) \text{ L}}{(0 - 1) \text{ h}} \\ &= -3 \text{ L/h} \quad \text{i.e., the rate of loss is 3000 L/hour}\end{aligned}$$

EXERCISE 20B.2

1 Let M $(1.5 + h, (1.5 + h)^2)$ be a point on $y = x^2$ which is close to F $(1.5, 1.5^2)$.

$$\begin{aligned}\therefore \text{ slope of MF} &= \frac{y\text{-step}}{x\text{-step}} \\ &= \frac{(1.5 + h)^2 - (1.5)^2}{(1.5 + h) - 1.5} \\ &= \frac{2.25 + h^2 + 3h - 2.25}{h} \\ &= \frac{h^2 + 3h}{h} \\ &= h + 3 \quad \{\text{as } h \neq 0\}\end{aligned}$$

As M approaches F, h approaches 0, $\therefore h + 3$ approaches 3, $\therefore$ slope of the tangent = 3

2 a

$$\begin{aligned}(x + h)^3 &= (x + h)^2(x + h) \\ &= (x^2 + 2xh + h^2)(x + h) \\ &= x^3 + 2x^2h + h^2x + hx^2 + 2xh^2 + h^3 \\ &= x^3 + 3x^2h + 3xh^2 + h^3\end{aligned}$$

b

$$\begin{aligned}(1 + h)^3 &= 1^3 + 3(1)^2h + 3(1)h^2 + h^3 \\ &= 1 + 3h + 3h^2 + h^3\end{aligned}$$

c M $(1 + h, (1 + h)^3)$ is the point on $y = x^3$ which is close to F(1, 1).

d slope of chord MF

$$\begin{aligned}&= \frac{y\text{-step}}{x\text{-step}} \\ &= \frac{(1 + h)^3 - 1}{(1 + h) - 1} \\ &= \frac{1 + 3h + 3h^2 + h^3 - 1}{h} \\ &= \frac{3h + 3h^2 + h^3}{h} \\ &= 3 + 3h + h^2 \quad \{\text{as } h \neq 0\}\end{aligned}$$

e As M approaches F, h approaches 0

$\therefore$ slope MF approaches 3

$\therefore$ slope of tangent = 3

3 b from **2 a** $(2 + h)^3 = 2^3 + 3(2)^2h + 3(2)h^2 + h^3$

$$= 8 + 12h + 6h^2 + h^3$$

c M $(2 + h, (2 + h)^3)$ is the point on $y = x^3$ which is close to F(2, 8).

d slope of chord MF

$$\begin{aligned}
&= \frac{y\text{-step}}{x\text{-step}} \\
&= \frac{(2+h)^3 - 2^3}{2+h-2} \\
&= \frac{8+12h+6h^2+h^3-8}{h} \\
&= \frac{12h+6h^2+h^3}{h} \\
&= 12+6h+h^2 \quad \{\text{as } h \neq 0\}
\end{aligned}$$

e as M approaches F, h approaches 0,

slope approaches 12

 $\therefore$ slope of tangent = 12

4 a $\frac{1}{x+h} - \frac{1}{x} = \frac{x(1) - 1(x+h)}{(x+h)x}$

$$\begin{aligned}
&= \frac{x-x-h}{x(x+h)} \\
&= -\frac{h}{x(x+h)}
\end{aligned}$$

b M is $\left(2+h, \frac{1}{2+h}\right)$

$\therefore$ y -coordinate is $\frac{1}{2+h}$

$\therefore$ slope of MF = $\frac{y\text{-step}}{x\text{-step}}$

$$\begin{aligned}
&= \frac{\frac{1}{2+h} - \frac{1}{2}}{(2+h)-2} \\
&= \frac{-h}{2(2+h)} \quad \{\text{using a with } x=2\} \\
&= \frac{-h}{2h(2+h)} \\
&= \frac{-1}{2(2+h)} \quad \{\text{as } h \neq 0\}
\end{aligned}$$

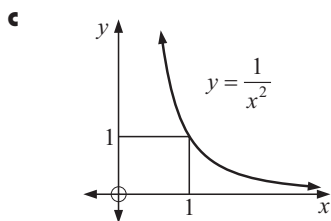
c F is the point where $x = 2$. Now as h approaches 0, M approaches F, and the slope MF approximates the slope of the tangent at M.But as h approaches 0, the slope MF approaches $-\frac{1}{4}$, $\therefore$ slope of tangent = $-\frac{1}{4}$ **d** At the point where $x = 3$, $y = \frac{1}{3}$.Also, the point with x -coordinate $3+h$ has y -coordinate $\frac{1}{3+h}$.Using the method in **b**, the slope of the line between these points is $-\frac{1}{3(3+h)}$, $h \neq 0$.As h approaches 0, the slope of the line approximates the slope of the tangent when $x = 3$, and this is $-\frac{1}{9}$.

5 a $\frac{1}{(x+h)^2} - \frac{1}{x^2} = \frac{x^2 - (x+h)^2}{x^2(x+h)^2}$

$$\begin{aligned}
&= \frac{x^2 - x^2 - 2xh - h^2}{x^2(x+h)^2} \\
&= \frac{-2xh - h^2}{x^2(x+h)^2}
\end{aligned}$$

b When $x = 2$,

$$\frac{1}{(2+h)^2} - \frac{1}{4} = \frac{-4h - h^2}{4(2+h)^2}$$



d Let M be close to $F(2, \frac{1}{2^2})$ i.e., $(2 + h, \frac{1}{(2 + h)^2})$

$$\begin{aligned} \therefore \text{slope MF} &= \frac{y\text{-step}}{x\text{-step}} \\ &= \frac{\frac{1}{(2 + h)^2} - \frac{1}{2^2}}{(2 + h) - 2} \\ &= \frac{\frac{-4h - h^2}{4(2 + h)^2}}{h} \quad \{\text{using b}\} \\ &= \frac{-4h - h^2}{4h(2 + h)^2} \\ &= \frac{-4 - h}{4(2 + h)^2} \quad \{\text{as } h \neq 0\} \end{aligned}$$

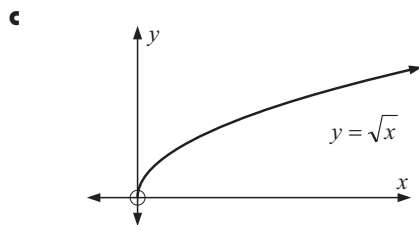
as h approaches 0, slope MF approaches $\frac{-4}{16}$ i.e., $-\frac{1}{4}$ $\therefore$ slope of tangent $= -\frac{1}{4}$

6 a

$$\begin{aligned} &\frac{\sqrt{x + h} - \sqrt{x}}{h} \\ &= \frac{\sqrt{x + h} - \sqrt{x}}{h} \left(\frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}} \right) \\ &= \frac{(x + h) - x}{h(\sqrt{x + h} + \sqrt{x})} \\ &= \frac{h}{h(\sqrt{x + h} + \sqrt{x})} \\ &= \frac{1}{\sqrt{x + h} + \sqrt{x}} \quad h \neq 0 \end{aligned}$$

b When $x = 9$,

$$\frac{\sqrt{9 + h} - 3}{h} = \frac{1}{\sqrt{9 + h} + 3} \quad \dots (*)$$



d Slope MF $= \frac{y\text{-step}}{x\text{-step}}$

$$\begin{aligned} &= \frac{\sqrt{9 + h} - \sqrt{9}}{(9 + h) - 9} \\ &= \frac{\sqrt{9 + h} - 3}{h} \\ &= \frac{1}{\sqrt{9 + h} + 3} \quad \{\text{using } *\} \end{aligned}$$

$\therefore$ as h approaches 0, slope MF approaches $\frac{1}{6}$, $\therefore$ slope of tangent $= \frac{1}{6}$

- e** $N = \sqrt{t}$ for $t \geq 4$ is the same graph as $y = \sqrt{x}$ for $x \geq 4$,
 $\therefore$ rate after 9 days is the same as the slope of the tangent in **d**, i.e., $\frac{1}{6}$
 $\therefore$ the population is increasing at a rate of $\frac{1}{6}$ (thousand insects/day) $= 167$ insects/day

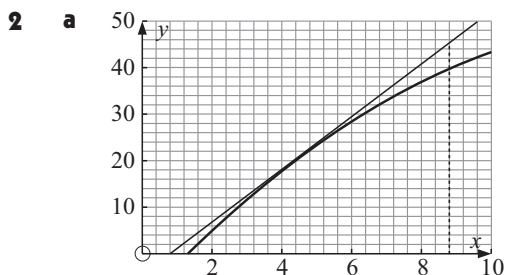
REVIEW SET 20

- 1 a** When $t = 1$, $d = 40$ and when $t = 4$, $d = 70$

$$\begin{aligned} \therefore \text{average speed on } 1 \leq t \leq 4 \text{ is } & \frac{\text{distance travelled}}{\text{time taken}} \\ &= \frac{70 - 40}{4 - 1} \\ &= \frac{30}{3} \\ &= 10 \text{ m/s} \end{aligned}$$

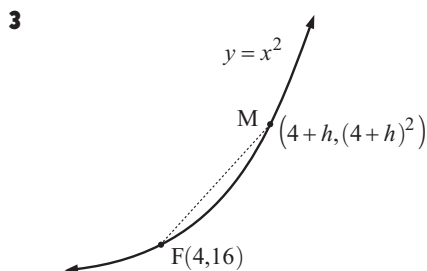
- b** When $t = 1$, $d = 40$ and when $t = 10$, $d = 81$

$$\therefore \text{average speed} = \frac{81 - 40}{10 - 1} = \frac{41}{9} \div 4.6 \text{ m/s}$$



b slope = $\frac{y\text{-step}}{x\text{-step}}$

$$\begin{aligned} &= \frac{46 - 0}{8.8 - 0.8} \\ &= \frac{46}{8} \\ &= 5.8 \end{aligned}$$



Let $M(4+h, (4+h)^2)$ be a point close to $F(4, 16)$.

$$\begin{aligned} \therefore \text{slope of MF} &= \frac{(4+h)^2 - 16}{4+h-4} \\ &= \frac{16+8h+h^2-16}{h} \\ &= \frac{8h+h^2}{h} \\ &= 8+h \quad (\text{as } h \neq 0) \end{aligned}$$

Now as M approaches F , $h \rightarrow 0$ and $8+h$ approaches 8. $\therefore$ the tangent at F has slope 8.

4 a $f(x+h)$

$$\begin{aligned} &= (2(x+h)+3)^2 \\ &= ((2x+2h)+3)^2 \\ &= (2x+2h)^2 + 2(2x+2h)3 + 3^2 \\ &= 4x^2 + 8xh + 4h^2 + 12x + 12h + 9 \end{aligned}$$

b $\frac{f(x+h)-f(x)}{h}$

$$\begin{aligned} &= \frac{\cancel{4x^2} + 8xh + 4h^2 + \cancel{12x} + 12h + \cancel{9} - \cancel{4x^2} - \cancel{12x} - \cancel{9}}{h} \\ &= \frac{8xh + 4h^2 + 12h}{h} \\ &= 8x + 4h + 12 \quad \{\text{as } h \neq 0\} \end{aligned}$$

c $\frac{f(x+h)-f(x)}{h}$ is the slope of the secant (chord) AB.

d i As $h \rightarrow 0$, $\frac{f(x+h)-f(x)}{h} = 8x + 4h + 12$ approaches $8x + 12$.

ii As $h \rightarrow 0$, $\frac{f(1+h)-f(1)}{h} = 8 + 4h + 12$ approaches 20.

e i $8x + 12$ gives us the slope of the tangent at a point with x -coordinate x .

ii 20 is the slope of the tangent at $x = 1$.

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad \text{Average speed on } 2 \leq t \leq 5 \text{ is } & \frac{s(5) - s(2)}{5 - 2} \text{ m/s} \\
 &= \frac{(25 + 20) - (4 + 8)}{3} \text{ m/s} \\
 &= \frac{33}{3} \text{ m/s} \\
 &= 11 \text{ m/s}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \text{Average speed on } 2 \leq t \leq 2 + h \text{ is } & \frac{s(2 + h) - s(2)}{2 + h - 2} \text{ m/s} \\
 &= \frac{(2 + h)^2 + 4(2 + h) - 12}{h} \text{ m/s} \\
 &= \frac{4 + 4h + h^2 + 8 + 4h - 12}{h} \text{ m/s} \\
 &= \frac{8h + h^2}{h} \text{ m/s} \\
 &= (8 + h) \text{ m/s}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \text{as } h \rightarrow 0, \quad 8 + h &\rightarrow 8 \\
 \therefore \frac{s(2 + h) - s(2)}{h} &\rightarrow 8 \text{ m/s}
 \end{aligned}$$

8 m/s is the instantaneous velocity at $t = 2$ seconds.

Chapter 21

DIFFERENTIAL CALCULUS

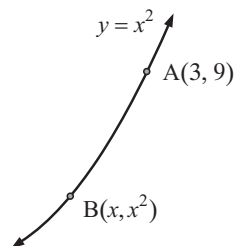
EXERCISE 21A

$$\begin{aligned} 1 \quad a \quad \text{Slope AB} &= \frac{x^2 - 9}{x - 3} \\ &= \frac{(x+3)(\cancel{x-3})}{(\cancel{x-3})} \\ &= x+3 \quad (\text{provided } x \neq 3) \end{aligned}$$

$$\therefore \lim_{x \rightarrow 3} (\text{slope AB}) = 6 \quad \dots\dots (1)$$

But, as $B \rightarrow A$, i.e., $x \rightarrow 3$ and $\lim_{x \rightarrow 3} (\text{slope AB}) = \text{slope of tangent at A}$

$\therefore$ slope of tangent = 6

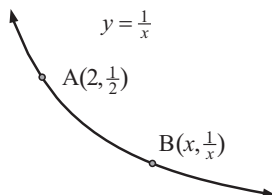


$$\begin{aligned} b \quad \text{Slope AB} &= \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} \\ &= \frac{2 - x}{2x(x - 2)} \\ &= \frac{-1(\cancel{x-2})}{2x(\cancel{x-2})} \\ &= -\frac{1}{2x} \quad (\text{provided } x \neq 2) \end{aligned}$$

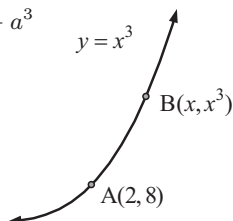
$$\therefore \lim_{x \rightarrow 2} (\text{slope AB}) = -\frac{1}{4}$$

But, as $B \rightarrow A$, i.e., $x \rightarrow 2$ and $\lim_{x \rightarrow 2} (\text{slope AB}) = \text{slope of tangent at A}$

$\therefore$ slope of tangent = $-\frac{1}{4}$

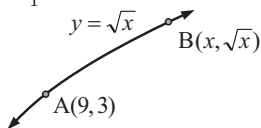


$$\begin{aligned} 2 \quad a \quad (x-a)(x^2+ax+a^2) &= x^3 - a^3 \\ &= x^3 - a^3 \\ &= x^3 - a^3 \end{aligned} \quad b \quad \begin{aligned} \text{Slope AB} &= \frac{x^3 - 8}{x - 2} \\ &= \frac{(x-2)(x^2+2x+4)}{(x-2)} \\ &= x^2 + 2x + 4 \quad (\text{provided } x \neq 2) \\ \therefore \lim_{x \rightarrow 2} (\text{slope of AB}) &= 2^2 + 4 + 4 = 12 \end{aligned}$$



But, as $B \rightarrow A$, i.e., $x \rightarrow 2$,
 $\lim_{x \rightarrow 2} (\text{slope AB}) = \text{slope of tangent at A}$
 $\therefore$ slope of tangent = 12

$$\begin{aligned} 3 \quad a \quad \frac{\sqrt{x} - \sqrt{a}}{x - a} &= \frac{1}{\sqrt{x} + \sqrt{a}} \\ &= \frac{1}{\sqrt{x} + \sqrt{a}} \end{aligned}$$



$$\begin{aligned} b \quad \text{Slope AB} &= \frac{\sqrt{x} - 3}{x - 9} = \frac{1}{\sqrt{x} + 3} \\ &\quad \{\text{from a, provided } x \neq 9\} \end{aligned}$$

$$\therefore \lim_{x \rightarrow 9} (\text{slope AB}) = \frac{1}{6}$$

But, as $B \rightarrow A$, i.e., $x \rightarrow 9$,
 $\lim_{x \rightarrow 9} (\text{slope AB}) = \text{slope of tangent at A}$
 $\therefore$ slope of tangent = $\frac{1}{6}$

EXERCISE 21B

1 a $f(x) = 1 - x^2$

$$\therefore f(2) = 1 - 2^2 = -3$$

$$\begin{aligned} f'(2) &= \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(1 - x^2) - (-3)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{4 - x^2}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{-(x+2)(\cancel{x-2})}{\cancel{x-2}} \\ &= \lim_{x \rightarrow 2} -(x+2) \quad \{\text{as } x \neq 2\} \\ &= -4 \end{aligned}$$

c $f(x) = 5 - 2x^2$ at $x = 3$

$$\therefore f(3) = 5 - 2(3)^2 = -13$$

$$\begin{aligned} f'(3) &= \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{(5 - 2x^2) - (-13)}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{18 - 2x^2}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{-2(x^2 - 9)}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{-2(x+3)(\cancel{x-3})}{\cancel{x-3}} \\ &= \lim_{x \rightarrow 3} -2(x+3) \quad \{\text{as } x \neq 3\} \\ &= -2(6) \\ &= -12 \end{aligned}$$

2 a $f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$

$$\begin{aligned} &= \lim_{x \rightarrow 2} \frac{\frac{4}{x} - \frac{4}{2}}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{8 - 4x}{2x(x - 2)} \\ &= \lim_{x \rightarrow 2} \frac{-4(\cancel{x-2})}{2x(\cancel{x-2})} \\ &= \lim_{x \rightarrow 2} \frac{-4}{2x} \quad \{\text{as } x \neq 2\} \\ &= -\frac{4}{4} \\ &= -1 \end{aligned}$$

b $f(x) = 2x^2 + 5x$ at $x = -1$

$$f(-1) = 2(-1)^2 + 5(-1) = -3$$

$$\begin{aligned} f'(-1) &= \lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x - (-1)} \\ &= \lim_{x \rightarrow -1} \frac{(2x^2 + 5x) - (-3)}{x + 1} \\ &= \lim_{x \rightarrow -1} \frac{2x^2 + 5x + 3}{x + 1} \\ &= \lim_{x \rightarrow -1} \frac{(2x+3)(\cancel{x+1})}{\cancel{x+1}} \\ &= \lim_{x \rightarrow -1} 2x + 3 \quad \{\text{as } x \neq -1\} \\ &= 1 \end{aligned}$$

d $f(x) = 3x + 5$ at $x = -2$

$$\therefore f(-2) = 3(-2) + 5 = -1$$

$$\begin{aligned} f'(-2) &= \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x - (-2)} \quad \text{where} \\ &= \lim_{x \rightarrow -2} \frac{(3x + 5) - (-1)}{x + 2} \\ &= \lim_{x \rightarrow -2} \frac{3x + 6}{x + 2} \\ &= \lim_{x \rightarrow -2} \frac{3(\cancel{x+2})}{\cancel{x+2}} \\ &= \lim_{x \rightarrow -2} 3 \quad \{\text{as } x \neq -2\} \\ &= 3 \end{aligned}$$

b $f'(-2) = \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x - (-2)}$

$$\begin{aligned} &= \lim_{x \rightarrow -2} \frac{-\frac{3}{x} - \frac{3}{2}}{(x + 2)} \\ &= \lim_{x \rightarrow -2} \frac{-6 - 3x}{2x(x + 2)} \\ &= \lim_{x \rightarrow -2} \frac{-3(\cancel{x+2})}{2x(\cancel{x+2})} \\ &= \lim_{x \rightarrow -2} \frac{-3}{2x} \quad \{\text{as } x \neq -2\} \\ &= \frac{-3}{-4} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad f'(4) &= \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} \\
 &= \lim_{x \rightarrow 4} \frac{\frac{1}{x^2} - \frac{1}{16}}{x - 4} \\
 &= \lim_{x \rightarrow 4} \frac{16 - x^2}{16x^2(x - 4)} \\
 &= \lim_{x \rightarrow 4} \frac{-(x+4)(\cancel{x-4})}{16x^2(\cancel{x-4})} \\
 &= \lim_{x \rightarrow 4} \frac{-(x+4)}{16x^2} \quad \{\text{as } x \neq 4\} \\
 &= \frac{-8}{256} \\
 &= -\frac{1}{32}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad f'(2) &= \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} \\
 &= \lim_{x \rightarrow 2} \frac{\frac{4x}{x-3} - \left(-\frac{8}{1}\right)}{x - 2} \\
 &= \lim_{x \rightarrow 2} \frac{4x + 8(x-3)}{1(x-2)(x-3)} \\
 &= \lim_{x \rightarrow 2} \frac{12(\cancel{x-2})}{(\cancel{x-2})(x-3)} \\
 &= \lim_{x \rightarrow 2} \frac{12}{x-3} \quad \{\text{as } x \neq 2\} \\
 &= \frac{12}{-1} \\
 &= -12
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad f'(5) &= \lim_{x \rightarrow 5} \frac{f(x) - f(5)}{(x-5)} \\
 &= \lim_{x \rightarrow 5} \frac{\frac{4x+1}{x-2} - \frac{7}{1}}{x-5} \\
 &= \lim_{x \rightarrow 5} \frac{4x+1-7(x-2)}{(x-2)(x-5)} \\
 &= \lim_{x \rightarrow 5} \frac{4x+1-7x+14}{(x-2)(x-5)} \\
 &= \lim_{x \rightarrow 5} \frac{-3x+15}{(x-2)(x-5)} \\
 &= \lim_{x \rightarrow 5} \frac{-3(\cancel{x-5})}{(x-2)(\cancel{x-5})} \\
 &= \lim_{x \rightarrow 5} \frac{-3}{x-2} \quad \{\text{as } x \neq 5\} \\
 &= -\frac{3}{3} \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad f'(-4) &= \lim_{x \rightarrow -4} \frac{f(x) - f(-4)}{x - (-4)} \\
 &= \lim_{x \rightarrow -4} \frac{\frac{3x}{x^2+1} - \left(-\frac{12}{17}\right)}{x+4} \\
 &= \lim_{x \rightarrow -4} \frac{51x+12(x^2+1)}{17(x^2+1)(x+4)} \\
 &= \lim_{x \rightarrow -4} \frac{12x^2+51x+12}{17(x+4)(x^2+1)} \\
 &= \lim_{x \rightarrow -4} \frac{(\cancel{x+4})(12x+3)}{17(x^2+1)(\cancel{x+4})} \\
 &= \lim_{x \rightarrow -4} \frac{12x+3}{17(x^2+1)} \quad \{x \neq -4\} \\
 &= -\frac{45}{17 \times 17} \\
 &= -\frac{45}{289}
 \end{aligned}$$

$$3 \quad \text{a} \quad f(x) = \sqrt{x} \quad \text{and} \quad f(4) = \sqrt{4} = 2$$

$$\begin{aligned}
 f'(4) &= \lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} \\
 &= \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} \\
 &= \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{(\sqrt{x} + 2)(\cancel{\sqrt{x} - 2})} \\
 &= \lim_{x \rightarrow 4} \frac{1}{(\sqrt{x} + 2)} \quad \{\text{as } x \neq 4\} \\
 &= \frac{1}{2+2} \\
 &= \frac{1}{4}
 \end{aligned}$$

$$\text{b} \quad f(x) = \sqrt{x} \quad \text{and} \quad f\left(\frac{1}{4}\right) = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\begin{aligned}
 f'\left(\frac{1}{4}\right) &= \lim_{x \rightarrow \frac{1}{4}} \frac{f(x) - f\left(\frac{1}{4}\right)}{x - \frac{1}{4}} \\
 &= \lim_{x \rightarrow \frac{1}{4}} \frac{\sqrt{x} - \frac{1}{2}}{x - \frac{1}{4}} \\
 &= \lim_{x \rightarrow \frac{1}{4}} \frac{(\sqrt{x} - \frac{1}{2})}{(\sqrt{x} + \frac{1}{2})(\cancel{\sqrt{x} - \frac{1}{2}})} \\
 &= \lim_{x \rightarrow \frac{1}{4}} \frac{1}{\sqrt{x} + \frac{1}{2}} \quad \{\text{as } x \neq \frac{1}{4}\} \\
 &= \frac{1}{\frac{1}{2} + \frac{1}{2}} \\
 &= 1
 \end{aligned}$$

$$\text{c} \quad f(x) = \frac{2}{\sqrt{x}} \quad \text{and} \quad f(9) = \frac{2}{3}$$

$$\begin{aligned} f'(9) &= \lim_{x \rightarrow 9} \frac{f(x) - f(9)}{x - 9} \\ &= \lim_{x \rightarrow 9} \frac{\frac{2}{\sqrt{x}} - \frac{2}{3}}{x - 9} \\ &= \lim_{x \rightarrow 9} \frac{2(3 - \sqrt{x})}{3\sqrt{x}(x - 9)} \\ &= \lim_{x \rightarrow 9} \frac{-2(\sqrt{x} - 3)}{3\sqrt{x}(\sqrt{x} + 3)(\sqrt{x} - 3)} \\ &= \lim_{x \rightarrow 9} \frac{-2}{3\sqrt{x}(\sqrt{x} + 3)} \quad \{x \neq 9\} \\ &= \frac{-2}{3(3)(6)} \\ &= -\frac{1}{27} \end{aligned}$$

$$\text{d} \quad f(x) = \sqrt{x-6} \quad \text{and} \quad f(10) = 2$$

$$\begin{aligned} f'(10) &= \lim_{x \rightarrow 10} \frac{f(x) - f(10)}{x - 10} \\ &= \lim_{x \rightarrow 10} \frac{\sqrt{x-6} - 2}{x - 10} \\ &= \lim_{x \rightarrow 10} \frac{(\sqrt{x-6} - 2)(\sqrt{x-6} + 2)}{(x - 10)(\sqrt{x-6} + 2)} \\ &= \lim_{x \rightarrow 10} \frac{x - 6 - 4}{(x - 10)(\sqrt{x-6} + 2)} \\ &= \lim_{x \rightarrow 10} \frac{1}{(\cancel{x-10})(\sqrt{x-6} + 2)} \\ &= \lim_{x \rightarrow 10} \frac{1}{\sqrt{x-6} + 2} \quad \{\text{as } x \neq 10\} \\ &= \frac{1}{2 + 2} \\ &= \frac{1}{4} \end{aligned}$$

$$4 \quad \text{a} \quad f(x) = x^2 + 3x - 4 \quad \text{at} \quad x = 3$$

$$\begin{aligned} f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \quad \text{where} \quad f(3) = 3^2 + 3(3) - 4 = 14 \\ &= \lim_{h \rightarrow 0} \frac{[(3+h)^2 + 3(3+h) - 4] - 14}{h} \\ &= \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 + 9 + 3h - 4 - 14}{h} \\ &= \lim_{h \rightarrow 0} \frac{9h + h^2}{h} \\ &= \lim_{h \rightarrow 0} 9 + h \quad \{\text{as } h \neq 0\} \\ &= 9 \end{aligned}$$

$$\text{b} \quad f(x) = 5 - 2x - 3x^2 \quad \text{at} \quad x = -2$$

$$\begin{aligned} f'(-2) &= \lim_{h \rightarrow 0} \frac{f(-2+h) - f(-2)}{h} \quad \text{where} \quad f(-2) = 5 - 2(-2) - 3(4) = -3 \\ &= \lim_{h \rightarrow 0} \frac{[5 - 2(-2+h) - 3(-2+h)^2] - (-3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{5 + 4 - 2h - 12 + 12h - 3h^2 + 3}{h} \\ &= \lim_{h \rightarrow 0} \frac{10h - 3h^2}{h} \\ &= \lim_{h \rightarrow 0} 10 - 3h \quad \{\text{as } h \neq 0\} \\ &= 10 \end{aligned}$$

$$\mathbf{c} \quad f(x) = \frac{1}{2x-1}$$

$$\therefore f(-2) = \frac{1}{2(-2)-1} = -\frac{1}{5}$$

$$\begin{aligned} f'(-2) &= \lim_{h \rightarrow 0} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{2(-2+h)-1} - \left(-\frac{1}{5}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{2h-5} + \frac{1}{5}}{h} \\ &= \lim_{h \rightarrow 0} \frac{5 + 1(2h-5)}{5h(2h-5)} \\ &= \lim_{h \rightarrow 0} \frac{2h}{5h(2h-5)} \\ &= \lim_{h \rightarrow 0} \frac{2}{5(2h-5)} \quad \{\text{as } h \neq 0\} \\ &= -\frac{2}{25} \end{aligned}$$

$$\mathbf{d} \quad f(x) = \frac{1}{x^2}$$

$$\therefore f(3) = \frac{1}{3^2} = \frac{1}{9}$$

$$\begin{aligned} f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{(3+h)^2} - \frac{1}{9}}{h} \\ &= \lim_{h \rightarrow 0} \frac{9 - (3+h)^2}{9h(3+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{9 - 9 - 6h - h^2}{9h(3+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{-\mathcal{K}(6+h)}{9\mathcal{K}(3+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{-(6+h)}{9(3+h)^2} \quad \{\text{as } h \neq 0\} \\ &= \frac{-6}{81} \\ &= -\frac{2}{27} \end{aligned}$$

$$\mathbf{e} \quad f(x) = \sqrt{x}$$

$$\therefore f(4) = \sqrt{4} = 2$$

$$\begin{aligned} f'(4) &= \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{4+h} - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{4+h} - 2}{h} \left(\frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \right) \\ &= \lim_{h \rightarrow 0} \frac{4+h-4}{h(\sqrt{4+h}+2)} \\ &= \lim_{h \rightarrow 0} \frac{1\mathcal{K}}{\mathcal{K}(\sqrt{4+h}+2)} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{4+h}+2} \quad \{\text{as } h \neq 0\} \\ &= \frac{1}{4} \end{aligned}$$

$$\mathbf{f} \quad f(x) = \frac{1}{\sqrt{x}}$$

$$\therefore f(1) = \frac{1}{\sqrt{1}} = 1$$

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{1+h}} - \frac{1}{1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 - \sqrt{1+h}}{h\sqrt{1+h}} \\ &= \lim_{h \rightarrow 0} \frac{(1 - \sqrt{1+h})}{h\sqrt{1+h}} \left(\frac{1 + \sqrt{1+h}}{1 + \sqrt{1+h}} \right) \\ &= \lim_{h \rightarrow 0} \frac{1 - (1+h)}{h(\sqrt{1+h})(1 + \sqrt{1+h})} \\ &= \lim_{h \rightarrow 0} \frac{-\mathcal{K}^1}{\mathcal{K}\sqrt{1+h}(1 + \sqrt{1+h})} \\ &= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{1+h}(1 + \sqrt{1+h})} \quad \{h \neq 0\} \\ &= \frac{-1}{1(1+1)} \\ &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad f(x) = x^3, \quad \therefore f'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \quad \text{where } f(2) = 2^3 = 8 \\
 &= \lim_{h \rightarrow 0} \frac{(2+h)^3 - 8}{h} \\
 &= \lim_{h \rightarrow 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\
 &= \lim_{h \rightarrow 0} \frac{12h + 6h^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} 12 + 6h + h^2 \quad \{\text{as } h \neq 0\} \\
 &= 12
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad f(x) = x^4, \quad \therefore f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \quad \text{where } f(3) = 3^4 = 81 \\
 &= \lim_{h \rightarrow 0} \frac{(3+h)^4 - 3^4}{h} \\
 &= \lim_{h \rightarrow 0} \frac{81 + 108h + 54h^2 + 12h^3 + h^4 - 81}{h} \\
 &= \lim_{h \rightarrow 0} \frac{108h + 54h^2 + 12h^3 + h^4}{h} \\
 &= \lim_{h \rightarrow 0} 108 + 54h + 12h^2 + h^3 \quad \{\text{as } h \neq 0\} \\
 &= 108
 \end{aligned}$$

EXERCISE 21C

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad f(x) &= x \\
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h) - x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h}{h} \\
 &= \lim_{h \rightarrow 0} 1 \quad \{\text{as } h \neq 0\} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad f(x) &= 5 \\
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{5 - 5}{h} \\
 &= \lim_{h \rightarrow 0} \frac{0}{h} \\
 &= \lim_{h \rightarrow 0} 0 \quad \{\text{as } h \neq 0\} \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad f(x) = x^3, \quad \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
 &= \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 \quad \{\text{as } h \neq 0\} \\
 &= 3x^2
 \end{aligned}$$

$$\mathbf{d} \quad f(x) = x^4$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^4 - x^4}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\ &= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\ &= \lim_{h \rightarrow 0} 4x^3 + 6x^2h + 4xh^2 + h^3 \quad (\text{as } h \neq 0) \\ &= 4x^3 \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad f(x) = 2x + 5$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2(x+h) + 5) - (2x + 5)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2x + 2h + 5 - 2x - 5}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h}{h} \\ &= \lim_{h \rightarrow 0} 2 \quad \{\text{as } h \neq 0\} \\ &= 2 \end{aligned}$$

$$\mathbf{b} \quad f(x) = x^2 - 3x$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 - 3(x+h)] - [x^2 - 3x]}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2 - 3h}{h} \\ &= \lim_{h \rightarrow 0} 2x + h - 3 \quad \{\text{as } h \neq 0\} \\ &= 2x - 3 \end{aligned}$$

$$\mathbf{c} \quad f(x) = x^3 - 2x^2 + 3$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 2(x+h)^2 + 3] - [x^3 - 2x^2 + 3]}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 2x^2 - 4xh - 2h^2 + 3) - (x^3 - 2x^2 + 3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 4xh - 2h^2}{h} \\ &= \lim_{h \rightarrow 0} 3x^2 + 3xh + h^2 - 4x - 2h \quad \{\text{as } h \neq 0\} \\ &= 3x^2 - 4x \end{aligned}$$

$$\mathbf{3} \quad \mathbf{a} \quad f(x) = \frac{1}{x+2},$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)+2} - \frac{1}{x+2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+2) - (x+h+2)}{h(x+h+2)(x+2)} \\ &= \lim_{h \rightarrow 0} \frac{-\cancel{h}^1}{\cancel{h}(x+2)(x+h+2)} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(x+2)(x+h+2)} \quad \{\text{as } h \neq 0\} \\ &= \frac{-1}{(x+2)^2} \end{aligned}$$

$$\mathbf{b} \quad f(x) = \frac{1}{2x-1},$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{2(x+h)-1} - \frac{1}{2x-1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2x-1) - [2(x+h)-1]}{h[2(x+h)-1](2x-1)} \\ &= \lim_{h \rightarrow 0} \frac{2x-1-2x-2h+1}{h[2(x+h)-1](2x-1)} \\ &= \lim_{h \rightarrow 0} \frac{-2\cancel{h}}{\cancel{h}[2(x+h)-1](2x-1)} \\ &= \lim_{h \rightarrow 0} \frac{-2}{[2(x+h)-1](2x-1)} \quad \{\text{as } h \neq 0\} \\ &= \frac{-2}{(2x-1)^2} \end{aligned}$$

$$\mathbf{c} \quad f(x) = \frac{1}{x^2},$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{[x + (x+h)][x - (x+h)]}{hx^2(x+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{(2x+h)(-\cancel{h})}{\cancel{h}x^2(x+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{-(2x+h)}{x^2(x+h)^2} \quad \{\text{as } h \neq 0\} \\ &= \frac{-2x}{x^4} \\ &= -\frac{2}{x^3} \end{aligned}$$

$$\mathbf{d} \quad f(x) = \frac{1}{x^3},$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^3} - \frac{1}{x^3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^3 - (x+h)^3}{hx^3(x+h)^3} \\ &= \lim_{h \rightarrow 0} \frac{x^3 - x^3 - 3x^2h - 3xh^2 - h^3}{hx^3(x+h)^3} \\ &= \lim_{h \rightarrow 0} \frac{-3x^2h - 3xh^2 - h^3}{hx^3(x+h)^3} \\ &= \lim_{h \rightarrow 0} \frac{-3x^2 - 3xh - h^2}{x^3(x+h)^3} \quad \{\text{as } h \neq 0\} \\ &= \frac{-3x^2}{x^3 \times x^3} \\ &= -\frac{3}{x^4} \end{aligned}$$

4 a $f(x) = \sqrt{x+2},$

$$\begin{aligned}\therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \\&= \lim_{h \rightarrow 0} \frac{(\sqrt{x+2+h} - \sqrt{x+2})(\sqrt{x+2+h} + \sqrt{x+2})}{h(\sqrt{x+2+h} + \sqrt{x+2})} \\&= \lim_{h \rightarrow 0} \frac{(x+2+h) - (x+2)}{h(\sqrt{x+2+h} + \sqrt{x+2})} \\&= \lim_{h \rightarrow 0} \frac{\cancel{h}^1}{\cancel{h}(\sqrt{x+2+h} + \sqrt{x+2})} \\&= \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+2+h} + \sqrt{x+2}} \\&= \frac{1}{\sqrt{x+2} + \sqrt{x+2}} \\&= \frac{1}{2\sqrt{x+2}}\end{aligned}$$

b $f(x) = \sqrt{2x+1}$

$$\begin{aligned}\therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \\&= \lim_{h \rightarrow 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \left(\frac{\sqrt{2(x+h)+1} + \sqrt{2x+1}}{\sqrt{2(x+h)+1} + \sqrt{2x+1}} \right) \\&= \lim_{h \rightarrow 0} \frac{[2(x+h)+1] - [2x+1]}{h(\sqrt{2(x+h)+1} + \sqrt{2x+1})} \\&= \lim_{h \rightarrow 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{2(x+h)+1} + \sqrt{2x+1})} \quad \{\text{as } h \neq 0\} \\&= \lim_{h \rightarrow 0} \frac{2}{(\sqrt{2(x+h)+1} + \sqrt{2x+1})} \\&= \frac{2}{2\sqrt{2x+1}} \\&= \frac{1}{\sqrt{2x+1}}\end{aligned}$$

b $f(x) = \frac{1}{\sqrt{x}},$

$$\begin{aligned}\therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \\&= \lim_{h \rightarrow 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h} \times \sqrt{x}}}{h\sqrt{x+h} \times \sqrt{x}} \\&= \lim_{h \rightarrow 0} \frac{(\sqrt{x} - \sqrt{x+h})}{h\sqrt{x+h} \times \sqrt{x}} \left(\frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} \right) \\&= \lim_{h \rightarrow 0} \frac{x - (x+h)}{h\sqrt{x+h} \times \sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\&= \lim_{h \rightarrow 0} \frac{-\cancel{h}^1}{\cancel{h}\sqrt{x+h} \times \sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\&= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{x+h} \times \sqrt{x}(\sqrt{x} + \sqrt{x+h})} \\&= \frac{-1}{\sqrt{x} \times \sqrt{x} \times 2\sqrt{x}} \\&= \frac{-1}{2x\sqrt{x}}\end{aligned}$$

5

Function	Derivative	Function	Derivative	Function	Derivative
x^1	$1x^0 = 1$	x^{-1}	$-1x^{-2}$	$x^{\frac{1}{2}}$	$\frac{1}{2}x^{-\frac{1}{2}}$
x^2	$2x^1 = 2x$	x^{-2}	$-2x^{-3}$	$x^{-\frac{1}{2}}$	$-\frac{1}{2}x^{-\frac{3}{2}}$
x^3	$3x^2$	x^{-3}	$-3x^{-4}$		
x^4	$4x^3$				

$$\text{If } f(x) = x^n, \quad f'(x) = nx^{n-1}$$

6 In order to prove this rule, we need the *binomial expansion* discussed in **Chapter 9**.Now if $f(x) = x^n$,

$$\begin{aligned}
 \text{then } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\left[\binom{n}{0}x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + \binom{n}{n-1}xh^{n-1} + \binom{n}{n}h^n \right] - x^n}{h} \\
 &\quad \{\text{for } n \in \mathbb{Z}^+ \text{ using the Binomial expansion}\} \\
 &= \lim_{h \rightarrow 0} \frac{[x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + \binom{n}{n-1}xh^{n-1} + h^n] - x^n}{h} \\
 &= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \dots + \binom{n}{n-1}xh^{n-1} + h^n}{h} \\
 &= \lim_{h \rightarrow 0} [nx^{n-1} + \binom{n}{2}x^{n-2}h + \dots + \binom{n}{n-1}xh^{n-2} + h^{n-1}] \\
 &= nx^{n-1} \quad \text{for } n \in \mathbb{Z}^+ \text{ as required}
 \end{aligned}$$

EXERCISE 21D

1 a $f(x) = x^3$,

$\therefore f'(x) = 3x^2$

b $f(x) = 2x^3$,

$\therefore f'(x) = 3 \times 2x^2 = 6x^2$

c $f(x) = 7x^2$,

$\therefore f'(x) = 2 \times 7x = 14x$

d $f(x) = x^2 + x$,

$\therefore f'(x) = 2x + 1$

e $f(x) = 4 - 2x^2$,

$\therefore f'(x) = 0 - 2 \times 2x = -4x$

f $f(x) = x^2 + 3x - 5$,

$\therefore f'(x) = 2x + 3 - 0 = 2x + 3$

g $f(x) = x^3 + 3x^2 + 4x - 1$

$\therefore f'(x) = 3x^2 + 2(3x) + 4 - 0 = 3x^2 + 6x + 4$

h $f(x) = 5x^4 - 6x^2$

$\therefore f'(x) = 4(5x^3) - 2(6x) = 20x^3 - 12x$

i $f(x) = \frac{3x-6}{x} = 3 - 6x^{-1}$

$\therefore f'(x) = 0 - (-1) \times 6x^{-2} = \frac{6}{x^2}$

j $f(x) = \frac{2x-3}{x^2} = \frac{2x}{x^2} - \frac{3}{x^2} = 2x^{-1} - 3x^{-2}$

$\therefore f'(x) = -2x^{-2} + 6x^{-3} = \frac{-2}{x^2} + \frac{6}{x^3}$

k $f(x) = \frac{x^3+5}{x} = x^2 + 5x^{-1}$

$\therefore f'(x) = 2x - 5x^{-2} = 2x - \frac{5}{x^2}$

l $f(x) = \frac{x^3+x-3}{x} = x^2 + 1 - 3x^{-1}$

$\therefore f'(x) = 2x + 0 + 3x^{-2} = 2x + \frac{3}{x^2}$

$$\begin{array}{ll} \text{m} & f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \\ & \therefore f'(x) = -\frac{1}{2}x^{-\frac{3}{2}} = \frac{-1}{2x\sqrt{x}} \end{array} \quad \begin{array}{l} \text{n} \quad f(x) = (2x-1)^2 = 4x^2 - 4x + 1 \\ \therefore f'(x) = 8x - 4 \end{array}$$

$$\begin{array}{l} \text{o} \quad f(x) = (x+2)^3 \\ \quad = x^3 + 3x^2(2) + 3x(2^2) + 2^3 \\ \quad = x^3 + 6x^2 + 12x + 8 \end{array} \quad \therefore f'(x) = 3x^2 + 12x + 12$$

$$\begin{array}{lll} \text{2 a} & y = 2x^3 - 7x^2 - 1 & \text{b} \quad y = \pi x^2 \\ & \therefore \frac{dy}{dx} = 6x^2 - 14x & \therefore \frac{dy}{dx} = 2\pi x \end{array} \quad \text{c} \quad y = \frac{1}{5x^2} = \frac{1}{5}x^{-2}$$

$$\therefore \frac{dy}{dx} = -\frac{2}{5}x^{-3} = \frac{-2}{5x^3}$$

$$\begin{array}{lll} \text{d} & y = 100x & \text{e} \quad y = 10(x+1) \\ & \therefore \frac{dy}{dx} = 100 & \quad = 10x + 10 \\ & & \therefore \frac{dy}{dx} = 10 \end{array} \quad \text{f} \quad y = 4\pi x^3$$

$$\therefore \frac{dy}{dx} = 12\pi x^2$$

$$\begin{array}{lll} \text{3 a} & \frac{d}{dx}(6x+2) & \text{b} \quad \frac{d}{dx}(x\sqrt{x}) \\ & = 6 & = \frac{d}{dx}(x^{\frac{3}{2}}) \\ & & = \frac{3}{2}x^{\frac{1}{2}} \\ & & = \frac{3}{2}\sqrt{x} \end{array} \quad \text{c} \quad \frac{d}{dx}(5-x)^2$$

$$= \frac{d}{dx}(25 - 10x + x^2)$$

$$= -10 + 2x$$

$$= 2x - 10$$

$$\begin{array}{lll} \text{d} & \frac{d}{dx}\left(\frac{6x^2 - 9x^4}{3x}\right) & \text{e} \quad \frac{d}{dx}\left(4x - \frac{1}{4x}\right) \\ & = \frac{d}{dx}(2x - 3x^3) & = \frac{d}{dx}\left(4x - \frac{1}{4}x^{-1}\right) \\ & = 2 - 9x^2 & = 4 + \frac{1}{4}x^{-2} \\ & & = 4 + \frac{1}{4x^2} \end{array} \quad \text{f} \quad \frac{d}{dx}(x(x+1)(2x-5))$$

$$= \frac{d}{dx}(x(2x^2 - 3x - 5))$$

$$= \frac{d}{dx}(2x^3 - 3x^2 - 5x)$$

$$= 6x^2 - 6x - 5$$

$$\begin{array}{ll} \text{4 a} & y = x^2 \quad \text{at } x = 2 \\ & f(x) = x^2 \\ & \therefore f'(x) = 2x \\ & \text{and so } f'(2) = 2(2) \\ & \therefore \text{tangent has slope of } 4 \end{array} \quad \text{b} \quad y = \frac{8}{x^2} \quad \text{at } x = 9$$

$$f(x) = 8x^{-2}$$

$$\therefore f'(x) = -16x^{-3} = -\frac{16}{x^3}$$

$$\text{and so } f'(9) = -\frac{16}{729}$$

$$\therefore \text{tangent has slope of } -\frac{16}{729}$$

$$\begin{array}{ll} \text{c} & y = 2x^2 - 3x + 7 \quad \text{at } x = -1 \\ & f(x) = 2x^2 - 3x + 7 \\ & \therefore f'(x) = 4x - 3 + 0 \\ & \text{and so } f'(-1) = 4(-1) - 3 \\ & \quad = -7 \\ & \therefore \text{tangent has slope of } -7 \end{array} \quad \text{d} \quad y = \frac{2x^2 - 5}{x} \quad \text{at } x = 2$$

$$f(x) = 2x - 5x^{-1}$$

$$\therefore f'(x) = 2 + 5x^{-2}$$

$$= 2 + \frac{5}{x^2}$$

$$\text{and so } f'(2) = 2 + \frac{5}{4} = \frac{13}{4}$$

$$\therefore \text{tangent has slope of } \frac{13}{4}$$

$$\text{e } y = \frac{x^2 - 4}{x^2} \text{ at } x = 4 \qquad \text{f } y = \frac{x^3 - 4x - 8}{x^2} \text{ at } x = -1$$

$$\begin{aligned} f(x) &= 1 - 4x^{-2} \\ \therefore f'(x) &= 0 + 8x^{-3} \\ &= \frac{8}{x^3} \end{aligned}$$

$$\text{and so } f'(4) = \frac{8}{4^3} = \frac{1}{8}$$

$\therefore$ tangent has slope of $\frac{1}{8}$

$$\begin{aligned} f(x) &= x - 4x^{-1} - 8x^{-2} \\ \therefore f'(x) &= 1 + 4x^{-2} + 16x^{-3} \\ &= 1 + \frac{4}{x^2} + \frac{16}{x^3} \end{aligned}$$

$$\text{and so } f'(-1) = 1 + 4 - 16 = -11$$

$\therefore$ tangent has slope of -11

$$\begin{aligned} \text{5 a } f(x) &= 4\sqrt{x} + x = 4x^{\frac{1}{2}} + x \\ \therefore f'(x) &= 4\left(\frac{1}{2}\right)x^{-\frac{1}{2}} + 1 \\ &= \frac{2}{\sqrt{x}} + 1 \end{aligned}$$

$$\begin{aligned} \text{b } f(x) &= \sqrt[3]{x} = x^{\frac{1}{3}} \\ \therefore f'(x) &= \frac{1}{3}x^{-\frac{2}{3}} \\ &= \frac{1}{3\sqrt[3]{x^2}} \end{aligned}$$

$$\begin{aligned} \text{c } f(x) &= -\frac{2}{\sqrt{x}} = -2x^{-\frac{1}{2}} \\ \therefore f'(x) &= -2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} \\ &= x^{-\frac{3}{2}} \\ &= \frac{1}{x\sqrt{x}} \end{aligned}$$

$$\begin{aligned} \text{d } f(x) &= 2x - \sqrt{x} = 2x - x^{\frac{1}{2}} \\ \therefore f'(x) &= 2 - \frac{1}{2}x^{-\frac{1}{2}} \\ &= 2 - \frac{1}{2\sqrt{x}} \end{aligned}$$

$$\begin{aligned} \text{e } f(x) &= \frac{4}{\sqrt{x}} - 5 = 4x^{-\frac{1}{2}} - 5 \\ \therefore f'(x) &= 4\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} - 5 \\ &= -2x^{-\frac{3}{2}} \text{ or } \frac{-2}{x\sqrt{x}} \end{aligned}$$

$$\begin{aligned} \text{f } f(x) &= 3x^2 - x\sqrt{x} = 3x^2 - x^{\frac{3}{2}} \\ \therefore f'(x) &= 6x - \frac{3}{2}x^{\frac{1}{2}} \\ &= 6x - \frac{3}{2}\sqrt{x} \end{aligned}$$

$$\begin{aligned} \text{g } f(x) &= \frac{5}{x^2\sqrt{x}} = 5x^{-\frac{5}{2}} \\ \therefore f'(x) &= 5\left(-\frac{5}{2}\right)x^{-\frac{7}{2}} \\ &= -\frac{25}{2}x^{-\frac{7}{2}} \\ &= \frac{-25}{2x^3\sqrt{x}} \end{aligned}$$

$$\begin{aligned} \text{h } f(x) &= 2x - \frac{3}{x\sqrt{x}} = 2x - 3x^{-\frac{3}{2}} \\ \therefore f'(x) &= 2 - 3\left(-\frac{3}{2}\right)x^{-\frac{5}{2}} \\ &= 2 + \frac{9}{2}x^{-\frac{5}{2}} \\ &= 2 + \frac{9}{2x^2\sqrt{x}} \end{aligned}$$

$$\text{6 a } y = 4x - \frac{3}{x} = 4x - 3x^{-1} \quad \therefore \frac{dy}{dx} = 4 + 3x^{-2} = 4 + \frac{3}{x^2}$$

$\frac{dy}{dx}$ is the slope function of $y = 4x - \frac{3}{x}$ from which the slope at any point can be found.

$$\text{b } S = 2t^2 + 4t \text{ m} \quad \therefore \frac{dS}{dt} = 4t + 4 \text{ ms}^{-1}$$

$\frac{dS}{dt}$ is the instantaneous rate of change in position at time t , i.e., it is the velocity function.

$$\text{c } C = 1785 + 3x + 0.002x^2 \text{ dollars.}$$

$$\frac{dC}{dx} = 3 + 2(0.002)x = 3 + 0.004x \text{ dollars/toaster}$$

$\frac{dC}{dx}$ is the instantaneous rate of change in cost as the number of toasters changes.

EXERCISE 21E.1

- 1** **a** $f(x) = x^2$, $g(x) = 2x + 7$,
 $\therefore f(g(x)) = f(2x + 7) = (2x + 7)^2$
- b** $f(x) = 2x + 7$, $g(x) = x^2$,
 $f(g(x)) = f(x^2) = 2x^2 + 7$
- c** $f(x) = \sqrt{x}$, $g(x) = 3 - 4x$,
 $f(g(x)) = f(3 - 4x) = \sqrt{3 - 4x}$
- d** $f(x) = 3 - 4x$, $g(x) = \sqrt{x}$,
 $f(g(x)) = f(\sqrt{x}) = 3 - 4\sqrt{x}$
- e** $f(x) = \frac{2}{x}$, $g(x) = x^2 + 3$,
 $f(g(x)) = f(x^2 + 3) = \frac{2}{x^2 + 3}$
- f** $f(x) = x^2 + 3$, $g(x) = \frac{2}{x}$,
 $f(g(x)) = f\left(\frac{2}{x}\right) = \left(\frac{2}{x}\right)^2 + 3 = \frac{4}{x^2} + 3$
- 2** **a** $f(g(x)) = (3x + 10)^3 \therefore f(x) = x^3$, $g(x) = 3x + 10$
- b** $f(g(x)) = \frac{1}{2x + 4} \therefore f(x) = \frac{1}{x}$, $g(x) = 2x + 4$
- c** $f(g(x)) = \sqrt{x^2 - 3x} \therefore f(x) = \sqrt{x}$, $g(x) = x^2 - 3x$
- d** $f(g(x)) = \frac{10}{(3x - x^2)^3} \therefore f(x) = \frac{10}{x^3}$, $g(x) = 3x - x^2$

EXERCISE 21E.2

- 1** **a** $\frac{1}{(2x - 1)^2}$
 $= (2x - 1)^{-2}$
 $= u^{-2}$,
 where $u = 2x - 1$
- b** $\sqrt{x^2 - 3x}$
 $= (x^2 - 3x)^{\frac{1}{2}}$
 $= u^{\frac{1}{2}}$,
 where $u = x^2 - 3x$
- c** $\frac{2}{\sqrt{2 - x^2}}$
 $= 2(2 - x^2)^{-\frac{1}{2}}$
 $= 2u^{-\frac{1}{2}}$,
 where $u = 2 - x^2$
- d** $\sqrt[3]{x^3 - x^2}$
 $= (x^3 - x^2)^{\frac{1}{3}}$
 $= u^{\frac{1}{3}}$,
 where $u = x^3 - x^2$
- e** $\frac{4}{(3 - x)^3}$
 $= 4(3 - x)^{-3}$
 $= 4u^{-3}$,
 where $u = 3 - x$
- f** $\frac{10}{x^2 - 3}$
 $= 10(x^2 - 3)^{-1}$
 $= 10u^{-1}$,
 where $u = x^2 - 3$
- 2** **a** $y = (4x - 5)^2$
 $\therefore y = u^2$ where $u = 4x - 5$
 Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 2u(4)$
 $= 8u$
 $= 8(4x - 5)$
- b** $y = \frac{1}{5 - 2x} = u^{-1}$ where $u = 5 - 2x$
 Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = -u^{-2}(-2)$
 $= \frac{2}{u^2}$
 $= \frac{2}{(5 - 2x)^2}$
- c** $y = \sqrt{3x - x^2}$
 $\therefore y = u^{\frac{1}{2}}$ where $u = 3x - x^2$
 Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}}(3 - 2x)$
 $= \frac{(3 - 2x)}{2\sqrt{u}}$
 $= \frac{3 - 2x}{2\sqrt{3x - x^2}}$
- d** $y = (1 - 3x)^4$
 $\therefore y = u^4$ where $u = 1 - 3x$
 Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = 4u^3(-3)$
 $= -12u^3$
 $= -12(1 - 3x)^3$

$$\begin{aligned} \mathbf{e} \quad y &= 6(5-x)^3 \\ \therefore y &= 6u^3 \text{ where } u = 5-x \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = 18u^2(-1) \\ &= -18u^2 \\ &= -18(5-x)^2 \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad y &= \sqrt[3]{2x^3 - x^2} \\ \therefore y &= u^{\frac{1}{3}} \text{ where } u = 2x^3 - x^2 \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = \frac{1}{3}u^{-\frac{2}{3}}(6x^2 - 3x) \\ &= \frac{6x^2 - 3x}{3\sqrt[3]{(2x^3 - x^2)^2}} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad y &= \frac{6}{(5x-4)^2} \\ \therefore y &= 6u^{-2} \text{ where } u = 5x-4 \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = -12u^{-3}(5) \\ &= -\frac{60}{u^3} \\ &= \frac{-60}{(5x-4)^3} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad y &= \frac{4}{3x-x^2} \\ \therefore y &= 4u^{-1} \text{ where } u = 3x-x^2 \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = -4u^{-2}(3-2x) \\ &= \frac{-4(3-2x)}{u^2} \\ &= \frac{-4(3-2x)}{(3x-x^2)^2} \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad y &= 2\left(x^2 - \frac{2}{x}\right)^3 \\ \therefore y &= 2u^3 \text{ where } u = x^2 - 2x^{-1} \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = 6u^2(2x + 2x^{-2}) \\ &= 6\left(2x + \frac{2}{x^2}\right)\left(x^2 - \frac{2}{x}\right)^2 \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad y &= \sqrt{1-x^2} \text{ at } x = \frac{1}{2} \\ \therefore y &= \sqrt{u} \text{ where } u = 1-x^2 \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}}(-2x) \\ &= \frac{-x}{\sqrt{u}} \\ &= \frac{-x}{\sqrt{1-x^2}} \\ \text{at } x = \frac{1}{2}, \quad \frac{dy}{dx} &= \frac{-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} = -\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right) \\ \therefore \text{ slope of tangent} &= -\frac{1}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= (3x+2)^6 \text{ at } x = -1 \\ \therefore y &= u^6 \text{ where } u = 3x+2 \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = 6u^5(3) \\ &= 18u^5 \\ &= 18(3x+2)^5 \\ \text{at } x = -1, \quad \frac{dy}{dx} &= 18(-1)^5 \\ \therefore \text{ slope of tangent} &= -18 \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad y &= \frac{1}{(2x-1)^4} \text{ at } x = 1 \\ \therefore y &= u^{-4} \text{ where } u = 2x-1 \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = -4u^{-5}(2) \\ &= \frac{-8}{u^5} \\ &= \frac{-8}{(2x-1)^5} \\ \text{at } x = 1, \quad \frac{dy}{dx} &= \frac{-8}{1^5} \\ \therefore \text{ slope of tangent} &= -8 \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad y &= 6 \times \sqrt[3]{1-2x} \text{ at } x = 0 \\ \therefore y &= 6u^{\frac{1}{3}} \text{ where } u = 1-2x \\ \text{Now } \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} = 6\left(\frac{1}{3}\right)u^{-\frac{2}{3}}(-2) \\ &= 2u^{-\frac{2}{3}}(-2) \\ &= \frac{-4}{\sqrt[3]{u^2}} \\ &= \frac{-4}{\sqrt[3]{(1-2x)^2}} \\ \text{at } x = 0, \quad \frac{dy}{dx} &= \frac{-4}{\sqrt[3]{1^2}} \\ \therefore \text{ slope of tangent} &= -4 \end{aligned}$$

$$\mathbf{e} \quad y = \frac{4}{x + 2\sqrt{x}} \quad \text{at } x = 4$$

$$\therefore y = 4u^{-1} \quad \text{where } u = x + 2x^{\frac{1}{2}}$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

$$= -4u^{-2}(1 + x^{-\frac{1}{2}})$$

$$= -\frac{4}{u^2} \left(1 + \frac{1}{\sqrt{x}}\right)$$

$$= \frac{-4}{(x + 2\sqrt{x})^2} \left(1 + \frac{1}{\sqrt{x}}\right)$$

$$\text{at } x = 4, \quad \frac{dy}{dx} = \frac{-4}{(4 + 4)^2} \left(1 + \frac{1}{2}\right)$$

$$= -\frac{6}{64}$$

$$\therefore \text{ slope of tangent} = -\frac{3}{32}$$

$$\mathbf{f} \quad y = \left(x + \frac{1}{x}\right)^3 \quad \text{at } x = 1$$

$$\therefore y = u^3 \quad \text{where } u = x + x^{-1}$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

$$= 3u^2(1 - u^{-2})$$

$$= 3 \left(x + \frac{1}{x}\right)^2 \left(1 - \frac{1}{x^2}\right)$$

$$\text{at } x = 1, \quad \frac{dy}{dx} = 3(1 + 1)^2(1 - 1)$$

$$\therefore \text{ slope of tangent} = 0$$

$$\mathbf{4} \quad \mathbf{a} \quad y = x^3 \quad \therefore \quad \frac{dy}{dx} = 3x^2$$

$$x = y^{\frac{1}{3}} \quad \therefore \quad \frac{dx}{dy} = \frac{1}{3}y^{-\frac{2}{3}}$$

$$\frac{dy}{dx} \frac{dx}{dy} = 3x^2 \left(\frac{1}{3}\right) y^{-\frac{2}{3}}$$

$$= x^2(y)^{-\frac{2}{3}}$$

$$= x^2(x^3)^{-\frac{2}{3}} \quad \{\text{substituting } y = x^3\}$$

$$= x^2(x^{-2})$$

$$= x^0$$

$$= 1 \quad \text{as required.}$$

$$\mathbf{b} \quad \text{We know that } \frac{dy}{du} \frac{du}{dx} = \frac{dy}{dx} \quad \{\text{chain rule}\}$$

$$\therefore \frac{dy}{dx} \frac{dx}{dy} = \frac{dy}{dy} = 1$$

EXERCISE 21F.1

$$\mathbf{1} \quad \mathbf{a} \quad y = x^2(2x - 1) \quad \text{is the product of } u = x^2 \quad \text{and } v = 2x - 1$$

$$\therefore u' = 2x \quad \text{and } v' = 2$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = 2x(2x - 1) + x^2(2)$$

$$= 4x^2 - 2x + 2x^2$$

$$= 6x^2 - 2x$$

$$\mathbf{b} \quad y = 4x(2x + 1)^3 \quad \text{is the product of } u = 4x \quad \text{and } v = (2x + 1)^3$$

$$\therefore u' = 4 \quad \text{and } v' = 6(2x + 1)^2$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = 4(2x + 1)^3 + 24x(2x + 1)^2$$

$$= [4(2x + 1) + 24x](2x + 1)^2$$

$$= [32x + 4](2x + 1)^2$$

c $y = x^2\sqrt{3-x}$ is the product of $u = x^2$ and $v = (3-x)^{\frac{1}{2}}$
 $\therefore u' = 2x$ and $v' = -\frac{1}{2}(3-x)^{-\frac{1}{2}}$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$$\begin{aligned}\therefore \frac{dy}{dx} &= 2x(3-x)^{\frac{1}{2}} + x^2 \left[-\frac{1}{2}(3-x)^{-\frac{1}{2}} \right] \\ &= 2x\sqrt{3-x} - \frac{x^2}{2\sqrt{3-x}}\end{aligned}$$

d $y = \sqrt{x}(x-3)^2$ is the product of $y = x^{\frac{1}{2}}$ and $v = (x-3)^2$
 $\therefore u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(x-3)^1$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x-3)^2 + 2\sqrt{x}(x-3)$$

e $y = 5x^2(3x^2-1)^2$ is the product of $u = 5x^2$ and $v = (3x^2-1)^2$
 $\therefore u' = 10x$ and $v' = 12x(3x^2-1)$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$$\begin{aligned}\therefore \frac{dy}{dx} &= 10x(3x^2-1)^2 + 5x^2(12x)(3x^2-1) \\ &= 10x(3x^2-1)^2 + 60x^3(3x^2-1)\end{aligned}$$

f $y = \sqrt{x}(x-x^2)^3$ is the product of $u = x^{\frac{1}{2}}$ and $v = (x-x^2)^3$
 $\therefore u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 3(1-2x)(x-x^2)^2$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3 + 3\sqrt{x}(1-2x)(x-x^2)^2$$

2 a $y = x^4(1-2x)^2$ is the product of $u = x^4$ and $v = (1-2x)^2$
 $u' = 4x^3$ and $v' = 2(1-2x)(-2)$
 $= -4(1-2x)$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore \frac{dy}{dx} = 4x^3(1-2x)^2 - 4x^4(1-2x)$$

at $x = -1$, $\frac{dy}{dx} = -4(3^2) - 4(3) = -48$ $\therefore$ slope of tangent = -48

b $y = \sqrt{x}(x^2-x+1)^2$ is the product of $u = x^{\frac{1}{2}}$ and $v = (x^2-x+1)^2$
 $\therefore u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(x^2-x+1)(2x-1)$

Now $\frac{dy}{dx} = u'v + uv'$ {product rule}

$$\therefore \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x^2-x+1)^2 + 2\sqrt{x}(2x-1)(x^2-x+1)$$

at $x = 4$, $\frac{dy}{dx} = \frac{1}{4}(13)^2 + 4(7)(13) = 406\frac{1}{4}$

$\therefore$ slope of tangent = $406\frac{1}{4}$

$$\begin{aligned} \text{c } y &= x\sqrt{1-2x} \text{ is the product of } u = x \text{ and } v = (1-2x)^{\frac{1}{2}} \\ \therefore u' &= 1 \text{ and } v' = \frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2) \\ &= -(1-2x)^{-\frac{1}{2}} \end{aligned}$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = \sqrt{1-2x} - \frac{x}{\sqrt{1-2x}}$$

$$\text{at } x = -4, \quad \frac{dy}{dx} = \sqrt{9} - \frac{(-4)}{\sqrt{9}} = 3 + \frac{4}{3} = \frac{13}{3}$$

$$\therefore \text{ slope of tangent } = \frac{13}{3}$$

$$\begin{aligned} \text{d } y &= x^3\sqrt{5-x^2} \text{ is the product of } u = x^3 \text{ and } v = (5-x^2)^{\frac{1}{2}} \\ \therefore u' &= 3x^2 \text{ and } v' = \frac{1}{2}(5-x^2)^{-\frac{1}{2}}(-2x) \\ &= -x(5-x^2)^{-\frac{1}{2}} \end{aligned}$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\therefore \frac{dy}{dx} = 3x^2\sqrt{5-x^2} - \frac{x}{\sqrt{5-x^2}}$$

$$\text{at } x = 1, \quad \frac{dy}{dx} = 3(1)^2\sqrt{4} - \frac{1}{\sqrt{4}} = 6 - \frac{1}{2} = \frac{11}{2} \quad \therefore \text{ slope of tangent } = \frac{11}{2}$$

$$\begin{aligned} \text{3 } y &= \sqrt{x}(3-x)^2 \text{ is the product of } u = x^{\frac{1}{2}} \text{ and } v = (3-x)^2 \\ \therefore u' &= \frac{1}{2}x^{-\frac{1}{2}} \text{ and } v' = 2(3-x)^1(-1) = -2(3-x) \end{aligned}$$

$$\text{Now } \frac{dy}{dx} = u'v + uv' \quad \{\text{product rule}\}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{1}{2\sqrt{x}}(3-x)^2 - 2\sqrt{x}(3-x) \\ &= \frac{(3-x)^2 - 4x(3-x)}{2\sqrt{x}} \\ &= \frac{(3-x)[(3-x) - 4x]}{2\sqrt{x}} \\ &= \frac{(3-x)(3-5x)}{2\sqrt{x}} \text{ as required} \end{aligned}$$

Tangents are horizontal when their slopes are 0.

$$\begin{aligned} \therefore \frac{dy}{dx} &= 0 \text{ when } (3-x)(3-5x) = 0 \\ &\text{i.e., } 3-x = 0 \text{ or } 3-5x = 0 \\ &\text{i.e., } x = 3 \text{ or } x = \frac{3}{5} \end{aligned}$$

EXERCISE 21F.2

$$\text{1 a } y = \frac{1+3x}{2-x} \text{ is a quotient where } u = 1+3x \text{ and } v = 2-x$$

$$\therefore u' = 3 \text{ and } v' = -1$$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$$

$$\therefore \frac{dy}{dx} = \frac{3(2-x) - (-1)(1+3x)}{(2-x)^2} = \frac{7}{(2-x)^2}$$

b $y = \frac{x^2}{2x+1}$ is a quotient where $u = x^2$ and $v = 2x+1$

$$\therefore u' = 2x \text{ and } v' = 2$$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore \frac{dy}{dx} = \frac{2x(2x+1) - x^2(2)}{(2x+1)^2} = \frac{2x^2 + 2x}{(2x+1)^2}$$

c $y = \frac{x}{x^2-3}$ is a quotient where $u = x$ and $v = x^2-3$

$$\therefore u' = 1 \text{ and } v' = 2x$$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore \frac{dy}{dx} = \frac{1(x^2-3) - x(2x)}{(x^2-3)^2} = \frac{-3-x^2}{(x^2-3)^2}$$

d $y = \frac{\sqrt{x}}{1-2x}$ is a quotient where $u = x^{\frac{1}{2}}$ and $v = 1-2x$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \text{ and } v' = -2$$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore \frac{dy}{dx} = \frac{\frac{1-2x}{2\sqrt{x}} - (-2)\sqrt{x}}{(1-2x)^2} = \frac{1-2x+4x}{2\sqrt{x}(1-2x)^2} = \frac{1+2x}{2\sqrt{x}(1-2x)^2}$$

e $y = \frac{x^2-3}{3x-x^2}$ is a quotient where $u = x^2-3$ and $v = 3x-x^2$

$$\therefore u' = 2x \text{ and } v' = 3-2x$$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{2x(3x-x^2) - (x^2-3)(3-2x)}{(3x-x^2)^2} = \frac{6x^2-2x^3-3x^2+2x^3+9-6x}{(3x-x^2)^2} \\ &= \frac{3x^2-6x+9}{(3x-x^2)^2} \end{aligned}$$

f $y = \frac{x}{\sqrt{1-3x}}$ is a quotient where $u = x$ and $v = (1-3x)^{\frac{1}{2}}$

$$\therefore u' = 1 \text{ and } v' = -\frac{3}{2}(1-3x)^{-\frac{1}{2}}$$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore \frac{dy}{dx} = \frac{\sqrt{1-3x} - x\left(\frac{-3}{2\sqrt{1-3x}}\right)}{(1-3x)} = \frac{2(1-3x) + 3x}{2(1-3x)^{\frac{3}{2}}} = \frac{2-3x}{2(1-3x)^{\frac{3}{2}}}$$

2 a $y = \frac{x}{1-2x}$ is a quotient where $u = x$ and $v = 1-2x$

$$\therefore u' = 1 \text{ and } v' = -2 \quad \text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} \text{ {quotient rule}}$$

$$\therefore \frac{dy}{dx} = \frac{1(1-2x) - x(-2)}{(1-2x)^2} = \frac{1}{(1-2x)^2}$$

$$\text{at } x = 1 \quad \frac{dy}{dx} = \frac{1}{(1-2)^2} = \frac{1}{(-1)^2} = 1 \quad \therefore \text{slope of tangent} = 1$$

b $y = \frac{x^3}{x^2 + 1}$ is a quotient where $u = x^3$ and $v = x^2 + 1$

$$\therefore u' = 3x^2 \text{ and } v' = 2x$$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{3x^2(x^2 + 1) - x^3(2x)}{(x^2 + 1)^2} = \frac{x^4 + 3x^2}{(x^2 + 1)^2}$$

$$\therefore \text{ at } x = -1 \quad \frac{dy}{dx} = \frac{1 + 3}{(1 + 1)^2} = \frac{4}{4} = 1$$

and so, the slope of tangent = 1

c $y = \frac{\sqrt{x}}{2x + 1}$ is a quotient where $u = x^{\frac{1}{2}}$ and $v = 2x + 1$

$$u' = \frac{1}{2}x^{-\frac{1}{2}} \text{ and } v' = 2$$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{\frac{1}{2\sqrt{x}}(2x + 1) - \sqrt{x}(2)}{(2x + 1)^2}$$

$$\text{at } x = 4 \quad \frac{dy}{dx} = \frac{\frac{9}{4} - 4}{81} = \frac{(\frac{9}{4} - 4)}{81} \times \frac{4}{4} = \frac{9 - 16}{324}$$

$$\therefore \text{ slope of tangent} = -\frac{7}{324}$$

d $y = \frac{x^2}{\sqrt{x^2 + 5}}$ is a quotient where $u = x^2$ and $v = (x^2 + 5)^{\frac{1}{2}}$

$$\begin{aligned} \therefore u' &= 2x \text{ and } v' = \frac{1}{2}(x^2 + 5)^{-\frac{1}{2}}(2x) \\ &= x(x^2 + 5)^{-\frac{1}{2}} \end{aligned}$$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{2x\sqrt{x^2 + 5} - x^2 \left(\frac{x}{\sqrt{x^2 + 5}} \right)}{(x^2 + 5)}$$

$$\text{at } x = -2 \quad \frac{dy}{dx} = \frac{-4(3) - \left(\frac{-8}{3}\right)}{9} = \frac{(-12 + \frac{8}{3})}{9} \times \frac{3}{3} = \frac{-36 + 8}{27}$$

$$\therefore \text{ slope of tangent} = -\frac{28}{27}$$

3 a $y = \frac{2\sqrt{x}}{1 - x}$ is a quotient where $u = 2x^{\frac{1}{2}}$ and $v = 1 - x$

$$\therefore u' = 2x^{-\frac{1}{2}} \text{ and } v' = -1$$

$$\text{Now } \frac{dy}{dx} = \frac{u'v - uv'}{v^2} \quad \{\text{quotient rule}\}$$

$$\therefore \frac{dy}{dx} = \left(\frac{\frac{1}{\sqrt{x}}(1 - x) - 2\sqrt{x}(-1)}{(1 - x)^2} \right) \frac{\sqrt{x}}{\sqrt{x}} = \frac{(1 - x) + 2x}{\sqrt{x}(1 - x)^2} = \frac{x + 1}{\sqrt{x}(1 - x)^2} \text{ as required.}$$

i $\frac{dy}{dx} = 0$ when $x + 1 = 0$ i.e., $x = -1$. However $\frac{dy}{dx}$ is not defined for $x \leq 0$ because of the $\sqrt{x}$ term. Hence $\frac{dy}{dx}$ never equals 0.

ii $\frac{dy}{dx}$ is undefined when $x \leq 0$ and when $x = 1$

b $y = \frac{x^2 - 3x + 1}{x + 2}$ is a quotient where $y = x^2 - 3x + 1$ and $v = x + 2$

$\therefore u' = 2x - 3$ and $v' = 1$

Now $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{(2x - 3)(x + 2) - (x^2 - 3x + 1)(1)}{(x + 2)^2} = \frac{2x^2 + 4x - 3x - 6 - x^2 + 3x - 1}{(x + 2)^2} \\ &= \frac{x^2 + 4x - 7}{(x + 2)^2} \text{ as required.}\end{aligned}$$

i $\frac{dy}{dx} = 0$ when $x^2 + 4x - 7 = 0$ i.e., $x = \frac{-4 \pm \sqrt{44}}{2} = -2 \pm \sqrt{11}$

ii $\frac{dy}{dx}$ is undefined when $(x + 2)^2 = 0$ i.e., $x = -2$

EXERCISE 21G

1 a $y = x - 2x^2 + 3$ at $x = 2$

Since when $x = 2$, $y = 2 - 2(2)^2 + 3 = -3$, the point of contact is $(2, -3)$.

Now $\frac{dy}{dx} = 1 - 4x$, $\therefore$ at $x = 2$, $\frac{dy}{dx} = 1 - 8 = -7$

$\therefore$ the tangent has equation $\frac{y - (-3)}{x - 2} = -7$ i.e., $y + 3 = -7(x - 2)$
i.e., $y = -7x + 14 - 3$
i.e., $y = -7x + 11$

b $y = \sqrt{x} + 1 = x^{\frac{1}{2}} + 1$ at $x = 4$

Since when $x = 4$, $y = \sqrt{4} + 1 = 3$, the point of contact is $(4, 3)$.

Now $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$, $\therefore$ at $x = 4$, $\frac{dy}{dx} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$\therefore$ the tangent has equation $\frac{y - 3}{x - 4} = \frac{1}{4}$ i.e., $4y - 12 = x - 4$
i.e., $4y = x + 8$

c $y = x^3 - 5x$ at $x = 1$

Since when $x = 1$, $y = 1^3 - 5(1) = -4$ the point of contact is $(1, -4)$.

Now $\frac{dy}{dx} = 3x^2 - 5$ $\therefore$ at $x = 1$, $\frac{dy}{dx} = 3 - 5 = -2$

$\therefore$ tangent has equation $\frac{y - (-4)}{x - 1} = -2$ i.e., $y + 4 = -2x + 2$
i.e., $y = -2x - 2$

d $y = \frac{4}{\sqrt{x}}$ at $(1, 4)$. Now $y = \frac{4}{\sqrt{x}} = 4x^{-\frac{1}{2}}$

$\therefore \frac{dy}{dx} = -2x^{-\frac{3}{2}}$

at $x = 1$, $\frac{dy}{dx} = -2 \left(1^{-\frac{3}{2}}\right) = -2$

$\therefore$ the tangent has equation $\frac{y - 4}{x - 1} = -2$ i.e., $y - 4 = -2x + 2$
i.e., $y = -2x + 6$

2 a $y = x^2$ at $(3, 9)$

$$\frac{dy}{dx} = 2x \quad \therefore \quad \text{at } x = 3, \quad \frac{dy}{dx} = 2(3) = 6 = \frac{6}{1}$$

$\therefore$ the normal at $(3, 9)$ has slope $-\frac{1}{6}$, so the equation of the normal is

$$\frac{y-9}{x-3} = -\frac{1}{6} \quad \text{i.e., } 6y - 54 = -x + 3 \quad \text{i.e., } 6y = -x + 57$$

b $y = x^3 - 5x + 2$ at $x = -2$

Since when $x = -2$, $y = (-2)^3 - 5(-2) + 2 = 4$

and so the point of contact is $(-2, 4)$

Now $\frac{dy}{dx} = 3x^2 - 5 \quad \therefore \quad \text{at } x = -2, \quad \frac{dy}{dx} = 3(-2)^2 - 5 = 7$

$\therefore$ the normal at $(-2, 4)$ has slope $-\frac{1}{7}$, so the equation of the normal is

$$\frac{y-4}{x-(-2)} = -\frac{1}{7} \quad \text{i.e., } 7y - 28 = -(x+2) \quad \text{i.e., } 7y = -x + 26$$

c $y = \frac{5}{\sqrt{x}} - \sqrt{x}$ at $(1, 4)$

Now $y = 5x^{-\frac{1}{2}} - x^{\frac{1}{2}} \quad \therefore \quad \frac{dy}{dx} = -\frac{5}{2}x^{-\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore$ at $x = 1$, $\frac{dy}{dx} = -\frac{5}{2}\left(1^{-\frac{3}{2}}\right) - \frac{1}{2}\left(1^{-\frac{1}{2}}\right) = -\frac{5}{2} - \frac{1}{2} = -3$

$\therefore$ the normal at $(1, 4)$ has slope $\frac{1}{3}$, so the equation of the normal is

$$\frac{y-4}{x-1} = \frac{1}{3} \quad \text{i.e., } 3y - 12 = x - 1$$

$$\text{i.e., } 3y = x + 11$$

d $y = 8\sqrt{x} - \frac{1}{x^2}$ at $x = 1$

Since when $x = 1$, $y = 8\sqrt{1} - \frac{1}{1^2} = 7$ the point of contact is $(1, 7)$

Now $y = 8\sqrt{x} - \frac{1}{x^2} = 8x^{\frac{1}{2}} - x^{-2}$

$\therefore \quad \frac{dy}{dx} = 4x^{-\frac{1}{2}} + 2x^{-3}$

$\therefore$ at $x = 1$, $\frac{dy}{dx} = 4 + 2 = 6$

$\therefore$ the normal at $(1, 7)$ has slope $-\frac{1}{6}$, so the equation of the normal is

$$\frac{y-7}{x-1} = -\frac{1}{6} \quad \text{i.e., } 6y - 42 = -x + 1$$

$$\text{i.e., } 6y = -x + 43$$

3 a $y = 2x^3 + 3x^2 - 12x + 1 \quad \therefore \quad \frac{dy}{dx} = 6x^2 + 6x - 12$

Horizontal tangents have slope $= 0$ so $6x^2 + 6x - 12 = 0$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$$\therefore \quad x = -2 \quad \text{or} \quad x = 1$$

Now at $x = -2$, $y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 1 = 21$

and at $x = 1$, $y = 2(1)^3 + 3(1)^2 - 12(1) + 1 = -6$

i.e., points of contact are $(-2, 21)$ and $(1, -6)$

$\therefore$ tangents are $y = -6$ and $y = 21$

b Now $y = \frac{2x+1}{\sqrt{x}} = 2\sqrt{x} + \frac{1}{\sqrt{x}} = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{\sqrt{x}} - \frac{1}{2x\sqrt{x}}$$

Horizontal tangents have slope = 0

$$\therefore \frac{1}{\sqrt{x}} - \frac{1}{2x\sqrt{x}} = 0$$

$$\therefore \frac{2x-1}{2x\sqrt{x}} = 0$$

$$\therefore 2x-1 = 0$$

$$\therefore x = \frac{1}{2}$$

Now at $x = \frac{1}{2}$, $y = \frac{2(\frac{1}{2})+1}{\sqrt{\frac{1}{2}}} = \frac{2}{\frac{1}{2}\sqrt{2}} = 2\sqrt{2}$

$\therefore$ the only horizontal tangent touches at $(\frac{1}{2}, 2\sqrt{2})$

c Now $y = 2x^3 + kx^2 - 3 \quad \therefore \frac{dy}{dx} = 6x^2 + 2kx$

and so when $x = 2$, $\frac{dy}{dx} = 4 \quad \therefore 6(2)^2 + 2k(2) = 4$

$$\therefore 24 + 4k = 4$$

$$\therefore 4k = -20$$

$$\therefore k = -5$$

d Now $y = 1 - 3x + 12x^2 - 8x^3 \quad \therefore \frac{dy}{dx} = -3 + 24x - 24x^2$

when $x = 1$, $\frac{dy}{dx} = -3 + 24 - 24 = -3$

i.e., the tangent at (1, 2) has slope -3

$$\therefore -3 + 24x - 24x^2 = -3 \quad \text{for all points } x \text{ where the tangent has slope } = -3$$

$$\therefore 24x^2 - 24x = 0$$

$$\therefore 24x(x-1) = 0$$

i.e., when $x = 0$ or $x = 1$

$\therefore$ the other x -value for which the tangent of the curve has slope -3 is $x = 0$ and when $x = 0$, $y = 1 - 0 + 0 - 0 = 1$

i.e., the tangent to the curve at (0, 1) is parallel to the tangent at (1, 2)

This tangent has equation $\frac{y-1}{x-0} = -3$ i.e., $y = -3x + 1$

4 a Now $y = x^2 + ax + b \quad \therefore \frac{dy}{dx} = 2x + a$

$$\therefore \text{ at } x = 1, \frac{dy}{dx} = 2 + a$$

$\therefore$ the slope of the tangent to the curve at $x = 1$ will be $2 + a$

However the equation of the tangent is $y + 2x = 6$ i.e., $y = -2x + 6$

and so the slope of the tangent is -2 . $\therefore 2 + a = -2$

$$a = -4$$

$$\therefore y = x^2 - 4x + b$$

Also, at $x = 1$, the tangent line contacts the curve

$$\text{i.e., } 1^2 - 4(1) + b = -2(1) + 6$$

$$1 - 4 + b = 4$$

$$b = 7$$

$$\therefore a = -4, b = 7$$

b Now $y = a\sqrt{x} + \frac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}}$$

$$\therefore \text{ at } x = 4, \frac{dy}{dx} = \frac{a}{2}\left(4^{-\frac{1}{2}}\right) - \frac{b}{2}\left(4^{-\frac{3}{2}}\right) = \frac{a}{2}\left(\frac{1}{2}\right) - \frac{b}{2}\left(\frac{1}{8}\right) = \frac{a}{4} - \frac{b}{16}$$

$$\therefore \text{ the slope of the tangent to the curve at } x = 4 \text{ will be } \frac{a}{4} - \frac{b}{16} = \frac{4a - b}{16}$$

However the equation of the *normal* is $4x + y = 22$, i.e., $y = -4x + 22$

$\therefore$ the normal has slope -4

$$\therefore \text{ the tangent has slope } \frac{1}{4} \quad \text{and so, } \frac{4a - b}{16} = \frac{1}{4} \quad \therefore \quad 4a - b = 4$$

$$\text{i.e., } b = 4a - 4 \dots\dots (1)$$

Also, at $x = 4$ the normal line intersects the curve.

$$\text{i.e., } a\sqrt{4} + \frac{b}{\sqrt{4}} = -4(4) + 22$$

$$\therefore 2a + \frac{b}{2} = 6$$

$$\text{Consequently, } 2a + \frac{4a - 4}{2} = 6 \quad \text{using (1)}$$

$$\therefore 2a + 2a - 2 = 6$$

$$\therefore 4a = 8$$

$$\therefore a = 2 \quad \text{and so } b = 4(2) - 4 = 4 \quad \text{from (1)}$$

5 a $y = \sqrt{2x + 1}$ when $x = 4$, $y = \sqrt{2(4) + 1} = 3$ $\therefore$ the point of contact is $(4, 3)$

$$\text{Since } y = (2x + 1)^{\frac{1}{2}} \quad \text{then } \frac{dy}{dx} = \frac{1}{2}(2x + 1)^{-\frac{1}{2}}(2) = \frac{1}{\sqrt{2x + 1}}$$

$$\therefore \text{ at } x = 4, \frac{dy}{dx} = \frac{1}{\sqrt{2(4) + 1}} = \frac{1}{3}$$

$$\text{so the tangent has equation } \frac{y - 3}{x - 4} = \frac{1}{3} \quad \text{i.e., } 3y = x + 5$$

b $y = \frac{1}{2 - x} = (2 - x)^{-1}$ at $x = -1$, $y = \frac{1}{2 - (-1)} = \frac{1}{3}$

$$\therefore \text{ the point of contact is } (-1, \frac{1}{3})$$

$$\text{Now } \frac{dy}{dx} = -1(2 - x)^{-2}(-1) = \frac{1}{(2 - x)^2}$$

$$\therefore \text{ at } x = -1, \frac{dy}{dx} = \frac{1}{(2 - (-1))^2} = \frac{1}{9}$$

$$\therefore \text{ the tangent has equation } \frac{y - \frac{1}{3}}{x - (-1)} = \frac{1}{9} \quad \text{i.e., } 9y - 3 = x + 1 \quad \text{i.e., } 9y = x + 4$$

c $f(x) = \frac{x}{1 - 3x}$ at $(-1, -\frac{1}{4})$

$$f(x) \text{ is a quotient where } u = x \quad \text{and } v = 1 - 3x \quad \therefore u' = 1 \quad \text{and } v' = -3$$

Now $f'(x) = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore f'(x) = \frac{1(1-3x) - x(-3)}{(1-3x)^2} = \frac{1}{(1-3x)^2}$$

$$\therefore f'(-1) = \frac{1}{(1-3(-1))^2} = \frac{1}{16}$$

So the tangent has equation $\frac{y - (-\frac{1}{4})}{x - (-1)} = \frac{1}{16}$ i.e., $16y + 4 = x + 1$
i.e., $16y = x - 3$

d $f(x) = \frac{x^2}{1-x}$ at $(2, -4)$

$f(x)$ is a quotient where $u = x^2$ and $v = 1 - x$

$u' = 2x$ and $v' = -1$

Now $f'(x) = \frac{u'v - uv'}{v^2}$ {quotient rule}

$$\therefore f'(x) = \frac{2x(1-x) - x^2(-1)}{(1-x)^2} = \frac{2x - 2x^2 + x^2}{(1-x)^2} = \frac{2x - x^2}{(1-x)^2}$$

$$\therefore f'(2) = \frac{2(2) - 2^2}{(1-2)^2} = \frac{4-4}{1} = 0$$

As the tangent has slope 0, i.e., is horizontal, it has equation $y = c$ and as the contact point is $(2, -4)$, the tangent has equation $y = -4$

6 a $y = \frac{1}{(x^2 + 1)^2}$ at $(1, \frac{1}{4})$

As $y = (x^2 + 1)^{-2}$,

$$\frac{dy}{dx} = -2(x^2 + 1)^{-3}(2x) = \frac{-4x}{(x^2 + 1)^3}$$

$$\therefore \text{ at } x = 1, \frac{dy}{dx} = \frac{-4}{(1+1)^3} = \frac{-4}{8} = -\frac{1}{2}$$

$\therefore$ normal at $(1, \frac{1}{4})$ has slope 2.

So the equation of the normal is $\frac{y - \frac{1}{4}}{x - 1} = 2$ i.e., $y - \frac{1}{4} = 2x - 2$

$$2x - y = \frac{7}{4}$$

$$\text{i.e., } y = 2x - \frac{7}{4}$$

b $y = \frac{1}{\sqrt{3-2x}}$ at $x = -3$, $y = \frac{1}{\sqrt{3-2(-3)}} = \frac{1}{3}$

$\therefore$ the point of contact is $(-3, \frac{1}{3})$

Now $y = (3 - 2x)^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}(3 - 2x)^{-\frac{3}{2}}(-2) = (3 - 2x)^{-\frac{3}{2}}$$

$$\therefore \text{ at } x = -3, \frac{dy}{dx} = (3 - 2(-3))^{-\frac{3}{2}} = 9^{-\frac{3}{2}} = 3^{-3} = \frac{1}{27}$$

$\therefore$ normal at $(-3, \frac{1}{3})$ has slope -27 ,

So the equation of the normal is $\frac{y - \frac{1}{3}}{x - (-3)} = -27$ i.e., $y - \frac{1}{3} = -27(x + 3)$

$$\text{i.e., } y = -27x - \frac{242}{3}$$

c $f(x) = \sqrt{x}(1-x)^2$

since $f(4) = \sqrt{4}(1-4)^2 = 18$, the point of contact is $(4, 18)$

Now $f(x)$ is a product where $u = x^{\frac{1}{2}}$ and $v = (1-x)^2$

$$\therefore u' = \frac{1}{2}x^{-\frac{1}{2}} \quad \text{and} \quad v' = 2(1-x)(-1) = -2(1-x)$$

Now $f'(x) = u'v + uv'$ {product rule}

$$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}}(1-x)^2 - x^{\frac{1}{2}}2(1-x)$$

$$\therefore f'(4) = \frac{1}{2\sqrt{4}}(1-4)^2 - \sqrt{4}(2)(1-4) = \frac{1}{4}(9) - 2(2)(-3) = \frac{57}{4}$$

$\therefore$ the normal at $(4, 18)$ has slope $-\frac{4}{57}$.

So the equation of the normal is $\frac{y-18}{x-4} = -\frac{4}{57}$ i.e., $57(y-18) = -4(x-4)$
i.e., $57y = -4x + 1042$

d $f(x) = \frac{x^2-1}{2x+3}$

Since $f(-1) = \frac{(-1)^2-1}{2(-1)+3} = \frac{0}{1} = 0$ the point of contact is $(-1, 0)$

Now $f(x)$ is a quotient where $u = x^2-1$ and $v = 2x+3$

$$\therefore u' = 2x \quad \text{and} \quad v' = 2$$

$$\text{Now } f'(x) = \frac{u'v - uv'}{v^2} = \frac{2x(2x+3) - (x^2-1)(2)}{(2x+3)^2}$$

$$\therefore f'(-1) = \frac{2(-1)(-2+3) - ((-1)^2-1)(2)}{(2(-1)+3)^2} = \frac{-2(-1) - (0)(2)}{(1)^2} = -2$$

$\therefore$ the normal at $(-1, 0)$ has slope $\frac{1}{2}$.

So the equation of the normal is $\frac{y-0}{x-(-1)} = \frac{1}{2}$ i.e., $2y = x+1$

7 The tangent has equation $3x+y=5$, i.e., $y=-3x+5$

$\therefore$ tangent has slope -3 (1)

Also, at $x=-1$, $y=-3(-1)+5=8$

$\therefore$ the tangent contacts the curve at $(-1, 8)$ (2)

$$\text{Now } y = a(1-bx)^{\frac{1}{2}}, \quad \therefore \frac{dy}{dx} = \frac{1}{2}a(1-bx)^{-\frac{1}{2}}(-b)$$

$$\therefore -3 = \frac{1}{2}a(1+b)^{-\frac{1}{2}}(-b) \quad \text{using (1)}$$

$$6 = \frac{ab}{\sqrt{1+b}} \quad \text{..... (3)}$$

and $(-1, 8)$ must lie on the curve $y = a\sqrt{1-bx}$

$$\text{and so } 8 = a\sqrt{1+b} \quad \text{..... (4)}$$

$$\therefore \frac{6\sqrt{1+b}}{b} = \frac{8}{\sqrt{1+b}} \quad \{\text{equating } a\text{'s in (3) and (4)}\}$$

$$\therefore 6(1+b) = 8b$$

$$\therefore 6+6b = 8b$$

$$\therefore 6 = 2b$$

$$\therefore b = 3 \quad \text{and} \quad a = \frac{8}{\sqrt{4}} = 4$$

8 a $y = x^3$ at $x = 2$

Since when $x = 2$, $y = 2^3 = 8$ the point of contact is $(2, 8)$

Now $\frac{dy}{dx} = 3x^2$ and so at $x = 2$, $\frac{dy}{dx} = 3(2)^2 = 12$

$\therefore$ tangent at $(2, 8)$ has slope 12 and its equation is $\frac{y-8}{x-2} = 12$ i.e., $y - 8 = 12x - 24$
i.e., $y = 12x - 16$

$\therefore$ the tangent meets the curve where $12x - 16 = x^3$
i.e., $x^3 - 12x + 16 = 0$

Because the tangent touches the curve at $x = 2$, there must be a repeated solution at this point. $\therefore (x - 2)^2$ must be a factor of this cubic

$$\therefore (x - 2)^2(x + 4) = 0$$

$\therefore$ tangent meets curve again when $x = -4$

and when $x = -4$, $y = (-4)^3 = -64$

$\therefore$ tangent meets curve again at $(-4, -64)$

b $y = -x^3 + 2x^2 + 1$ at $x = -1$

Since when $x = -1$, $y = -(-1)^3 + 2(-1)^2 + 1 = 4$

and so the point of contact is $(-1, 4)$

Now $\frac{dy}{dx} = -3x^2 + 4x$ $\therefore$ at $x = -1$, $\frac{dy}{dx} = -3(-1)^2 + 4(-1) = -7$

$\therefore$ tangent at $(-1, 4)$ has slope -7

$\therefore$ its equation is $\frac{y-4}{x-(-1)} = -7$ $\therefore y - 4 = -7(x + 1)$
i.e., $y = -7x - 3$

Now the tangent meets the curve where $-7x - 3 = -x^3 + 2x^2 + 1$
i.e., $x^3 - 2x^2 - 7x - 4 = 0$

Because the tangent touches the curve at $x = -1$, there must be a repeated solution at this point. $\therefore (x + 1)^2$ must be a factor of this cubic

$$\therefore (x + 1)^2(x - 4) = 0$$

$\therefore$ tangent meets curve again when $x = 4$

and when $x = 4$, $y = -(4)^3 + 2(4)^2 + 1 = -64 + 32 + 1 = -31$

$\therefore$ tangent meets curve again at $(4, -31)$

c $y = x^3 + \frac{4}{x}$ at $x = 1$

Since when $x = 1$, $y = 1^3 + \frac{4}{1} = 5$ and so the point of contact is $(1, 5)$

Now $\frac{dy}{dx} = 3x^2 - \frac{4}{x^2}$ and at $x = 1$, $\frac{dy}{dx} = 3 - 4 = -1$

$\therefore$ tangent at $(1, 5)$ has slope -1 and therefore its equation is

$$\frac{y-5}{x-1} = -1 \quad \text{i.e., } y - 5 = -x + 1$$

$$\text{i.e., } y = -x + 6$$

Now the tangent meets the curve where $-x + 6 = x^3 + \frac{4}{x}$

$$\therefore x^3 + x - 6 + \frac{4}{x} = 0$$

$$\therefore x^4 + x^2 - 6x + 4 = 0$$

Using a graphics calculator, this quartic has a graph which touches the x -axis at $x = 1$ and has no other x -intercepts. i.e., the tangent *never* meets the curve again.

9 a $y = x^2 - x + 9$ at $x = a$

Since when $x = a$, $y = a^2 - a + 9$ the point of contact is $(a, a^2 - a + 9)$

Now $\frac{dy}{dx} = 2x - 1$ $\therefore$ at $x = a$, $\frac{dy}{dx} = 2a - 1$

$\therefore$ the slope of the tangent at $(a, a^2 - a + 9)$ is $2a - 1$

$\therefore$ the equation of the tangent is $\frac{y - (a^2 - a + 9)}{x - a} = 2a - 1$

i.e., $y - (a^2 - a + 9) = (2a - 1)(x - a)$

i.e., $y = (2a - 1)x - 2a^2 + a + a^2 - a + 9$

i.e., $y = (2a - 1)x - a^2 + 9$

But this tangent passes through $(0, 0)$ $\therefore 0 = a^2 - 9$

$\therefore (a + 3)(a - 3) = 0$

$\therefore a = \pm 3$

$\therefore$ tangents are: at $a = 3$: $y - (9 - 3 + 9) = 5(x - 3)$

$\therefore y = 5x$, with contact at $(3, 15)$

at $a = -3$: $y - (9 + 3 + 9) = -7(x + 3)$

$\therefore y = -7x$, with contact at $(-3, 21)$

points of contact are $(3, 15)$ and $(-3, 21)$

b Let (a, a^3) lie on $y = x^3$

Now $\frac{dy}{dx} = 3x^2$, $\therefore$ at $x = a$, $\frac{dy}{dx} = 3a^2$

$\therefore$ the slope of the tangent at (a, a^3) is $3a^2$

$\therefore$ the equation of the tangent is $\frac{y - a^3}{x - a} = 3a^2$ i.e., $y - a^3 = (3a^2)(x - a)$

But this tangent passes through $(-2, 0)$ $\therefore a - a^3 = 3a^2(-2 - a)$

$\therefore -a^3 = -6a^2 - 3a^3$

$\therefore 2a^3 + 6a^2 = 0$

$\therefore 2a^2(a + 3) = 0$

$\therefore a = 0$ or -3

If $a = 0$, tangent equation is $y = 0$, with contact point $(0, 0)$

If $a = -3$, tangent equation is $y - (-27) = 27(x + 3)$

i.e., $y = 27x + 54$, with contact point $(-3, -27)$

c Let $(a, \sqrt{a})$ lie on $y = \sqrt{x}$.

Now $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ $\therefore$ at $x = a$, $\frac{dy}{dx} = \frac{1}{2\sqrt{a}}$

$\therefore$ the slope of the tangent at $(a, \sqrt{a})$ is $\frac{1}{2\sqrt{a}}$ and so the slope of the normal is $-2\sqrt{a}$

$\therefore$ the normal has equation $\frac{y - \sqrt{a}}{x - a} = -2\sqrt{a}$ i.e., $y - \sqrt{a} = -2\sqrt{a}(x - a)$

But this normal passes through $(4, 0)$ $\therefore 0 - \sqrt{a} = -2\sqrt{a}(4 - a)$

$\therefore 2\sqrt{a}(4 - a) - \sqrt{a} = 0$

$\therefore \sqrt{a}(8 - 2a - 1) = 0$

$\therefore \sqrt{a}(7 - 2a) = 0$

$\therefore a = 0$ or $\frac{7}{2}$

When $a = 0$, normal equation is $y = 0$, and contact point is $(0, 0)$.

$$\text{When } a = \frac{7}{2}, \quad y - \sqrt{\frac{7}{2}} = -2\sqrt{\frac{7}{2}} \left(x - \frac{7}{2}\right)$$

$$\text{i.e., } \sqrt{2}y - \sqrt{7} = -2\sqrt{7} \left(x - \frac{7}{2}\right)$$

$$\text{i.e., } \sqrt{2}y + 2\sqrt{7}x = 7\sqrt{7} + \sqrt{7}$$

$$\text{i.e., } \sqrt{2}y + 2\sqrt{7}x = 8\sqrt{7}$$

$$\text{i.e., } y = -\sqrt{14}x + 4\sqrt{14} \quad \text{with contact point } \left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$$

EXERCISE 21H

1 a $f(x) = 3x^2 - 6x + 2$

$$\therefore f'(x) = 6x - 6$$

$$\therefore f''(x) = 6$$

b $f(x) = 2x^3 - 3x^2 - x + 5$

$$\therefore f'(x) = 6x^2 - 6x - 1 + 0$$

$$\therefore f''(x) = 12x - 6$$

c $f(x) = \frac{2}{\sqrt{x}} - 1 = 2x^{-\frac{1}{2}} - 1$

$$\therefore f'(x) = -x^{-\frac{3}{2}}$$

$$\therefore f''(x) = \frac{3}{2}x^{-\frac{5}{2}}$$

d $f(x) = \frac{2-3x}{x^2} = 2x^{-2} - 3x^{-1}$

$$\therefore f'(x) = -4x^{-3} + 3x^{-2}$$

$$\therefore f''(x) = 12x^{-4} - 6x^{-3}$$

$$= \frac{12-6x}{x^4}$$

e $f(x) = (1-2x)^3$

$$\therefore f'(x) = 3(1-2x)^2(-2)$$

$$= -6(1-2x)^2 \quad \therefore f''(x) = -12(1-2x)^1(-2) = 24(1-2x)$$

f $f(x) = \frac{x+2}{2x-1}$ is a quotient where $u = x+2$ and $v = 2x-1$

$$\therefore u' = 1 \quad \text{and} \quad v' = 2$$

$$\therefore f'(x) = \frac{1(2x-1) - 2(x+2)}{(2x-1)^2} \quad \{\text{quotient rule}\}$$

$$= \frac{-5}{(2x-1)^2}$$

$$= -5(2x-1)^{-2}$$

$$\text{and } f''(x) = 10(2x-1)^{-3}(2) = \frac{20}{(2x-1)^3}$$

2 a $y = x - x^3$

$$\therefore \frac{dy}{dx} = 1 - 3x^2$$

$$\therefore \frac{d^2y}{dx^2} = -6x$$

b $y = x^2 - \frac{5}{x^2}$

$$= x^2 - 5x^{-2}$$

$$\therefore \frac{dy}{dx} = 2x + 10x^{-3}$$

$$\therefore \frac{d^2y}{dx^2} = 2 - 30x^{-4} = 2 - \frac{30}{x^4}$$

c $y = 2 - \frac{3}{\sqrt{x}}$

$$= 2 - 3x^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{2}x^{-\frac{3}{2}}$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{9}{4}x^{-\frac{5}{2}}$$

d $y = \frac{4-x}{x} = 4x^{-1} - 1$

$$\therefore \frac{dy}{dx} = -4x^{-2}$$

$$\therefore \frac{d^2y}{dx^2} = 8x^{-3} = \frac{8}{x^3}$$

e $y = (x^2 - 3x)^3$

$$\therefore \frac{dy}{dx} = 3(x^2 - 3x)^2(2x - 3) = (6x - 9)(x^2 - 3x)^2$$

which is a product where $u = 6x - 9$ and $v = (x^2 - 3x)^2$

$$\therefore u' = 6 \quad \text{and} \quad v' = 2(x^2 - 3x)^1(2x - 3)$$

$$\begin{aligned} \therefore \frac{d^2y}{dx^2} &= 6(x^2 - 3x)^2 + (6x - 9)(2)(x^2 - 3x)(2x - 3) \\ &= 6(x^2 - 3x) [(x^2 - 3x) + (2x - 3)^2] \\ &= 6(x^2 - 3x)(x^2 - 3x + 4x^2 - 12x + 9) \\ &= 6(x^2 - 3x)(5x^2 - 15x + 9) \end{aligned}$$

$$\mathbf{f} \quad y = x^2 - x + \frac{1}{1-x} = x^2 - x + (1-x)^{-1}$$

$$\therefore \frac{dy}{dx} = 2x - 1 + (-1)(1-x)^{-2}(-1) = 2x - 1 + (1-x)^{-2}$$

$$\therefore \frac{d^2y}{dx^2} = 2 - 2(1-x)^{-3}(-1) = 2 + \frac{2}{(1-x)^3}$$

$$\mathbf{3} \quad \mathbf{a} \quad f(x) = 2x^3 - 6x^2 + 5x + 1$$

$$\therefore f'(x) = 6x^2 - 12x + 5$$

$$\therefore f''(x) = 12x - 12 \quad \text{and so} \quad f''(x) = 0 \quad \text{when} \quad 12x - 12 = 0$$

$$\therefore 12x = 12$$

$$\text{i.e., when } x = 1$$

$$\mathbf{b} \quad f(x) = \frac{x}{x^2 + 2} \quad \text{is a quotient where } u = x \quad \text{and} \quad v = x^2 + 2$$

$$\therefore u' = 1, \quad v' = 2x$$

$$\therefore f'(x) = \frac{1(x^2 + 2) - 2x^2}{(x^2 + 2)^2} \quad \{\text{using the quotient rule}\}$$

$$= \frac{2 - x^2}{(x^2 + 2)^2} \quad \text{which is another quotient}$$

$$\text{with } u = 2 - x^2 \quad \text{and} \quad v = (x^2 + 2)^2$$

$$\therefore u' = -2x, \quad v' = 2(x^2 + 2)(2x)$$

$$f''(x) = \frac{-2x(x^2 + 2)^2 - 4x(x^2 + 2)(2 - x^2)}{(x^2 + 2)^4}$$

$$= \frac{-2x(x^2 + 2)[x^2 + 2 + 2(2 - x^2)]}{(x^2 + 2)^4}$$

$$= \frac{-2x[-x^2 + 6]}{(x^2 + 2)^3}$$

$$= \frac{2x[x^2 - 6]}{(x^2 + 2)^3} \quad \text{and so} \quad f''(x) = 0 \quad \text{when} \quad 2x[x^2 - 6] = 0$$

$$\text{i.e., } x = 0 \quad \text{or} \quad x^2 - 6 = 0$$

$$\text{i.e., when } x = 0 \quad \text{or} \quad x = \pm\sqrt{6}$$

REVIEW SET 21A

$$\mathbf{1} \quad y = -2x^2. \quad \text{Since when } x = -1, \quad y = -2(-1)^2 = -2, \quad \text{the point of contact is } (-1, -2).$$

$$\text{Now } \frac{dy}{dx} = -4x$$

$$\therefore \text{ at } x = -1, \quad \frac{dy}{dx} = -4(-1) = 4$$

$$\therefore \text{ tangent has equation } \frac{y - (-2)}{x + 1} = 4 \quad \text{i.e., } y + 2 = 4x + 4$$

$$\text{i.e., } y = 4x + 2$$

$$\mathbf{2} \quad \mathbf{a} \quad y = 3x^2 - x^4$$

$$\therefore \frac{dy}{dx} = 6x - 4x^3$$

$$\mathbf{b} \quad y = \frac{x^3 - x}{x^2} = x - x^{-1}$$

$$\therefore \frac{dy}{dx} = 1 + x^{-2} = 1 + \frac{1}{x^2}$$

$$\begin{aligned}
 \mathbf{3} \quad f(x) &= x^2 + 2x, \quad \therefore f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 + 2(x+h)] - [x^2 + 2x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 2h}{h} \\
 &= \lim_{h \rightarrow 0} (2x + 2 + h) \quad \{\text{as } h \neq 0\} \\
 &= 2x + 2 \quad \text{Checking: } f(x) = x^2 + 2x, \quad \therefore f'(x) = 2x + 2 \quad \checkmark
 \end{aligned}$$

$$\mathbf{4} \quad y = \frac{1-2x}{x^2}. \quad \text{Since when } x = 1, \quad y = \frac{1-2(1)}{1^2} = -1 \quad \text{the point of contact is } (1, -1)$$

$$\text{Since } y = \frac{1}{x^2} - \frac{2}{x} \quad \text{then } \frac{dy}{dx} = -2x^{-3} + 2x^{-2} = -\frac{2}{x^3} + \frac{2}{x^2}$$

$$\therefore \text{ at } x = 1, \quad \frac{dy}{dx} = -2 + 2 = 0$$

i.e., the tangent is a horizontal line. $\therefore$ the normal is a vertical line, of the form $x = k$.

As the normal passes through $(1, -1)$ its equation must be $x = 1$

$$\mathbf{5} \quad y = 2x^3 + 4x - 1 \quad \text{at } (1, 5)$$

$$\text{Now } \frac{dy}{dx} = 6x^2 + 4 \quad \therefore \text{ at } x = 1, \quad \frac{dy}{dx} = 6(1)^2 + 4 = 10$$

$$\therefore \text{ the tangent has equation } \frac{y-5}{x-1} = 10 \quad \text{i.e., } y = 10x - 5$$

Now the tangent meets the curve again where $10x - 5 = 2x^3 + 4x - 1$

$$2x^3 - 6x + 4 = 0$$

$$x^3 - 3x + 2 = 0$$

We know that $(x-1)^2$ is a factor since the line is tangent to the curve (i.e., touches) at $x = 1$

Consequently, $x^3 - 3x + 2 = (x-1)^2(x+2) = 0$ {since the constant term is 2}

Thus $x = -2$ is the other solution and when $x = -2$, $y = 2(-2)^3 + 4(-2) - 1 = -25$

$\therefore$ tangent meets curve again at $(-2, -25)$

$$\mathbf{6} \quad y = \frac{ax+b}{\sqrt{x}} = a\sqrt{x} + \frac{b}{\sqrt{x}} = ax^{\frac{1}{2}} + bx^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{a}{2}x^{-\frac{1}{2}} - \frac{b}{2}x^{-\frac{3}{2}} = \frac{a}{2\sqrt{x}} - \frac{b}{2x\sqrt{x}}$$

The equation of the tangent at $x = 1$ is $2x - y = 1$

i.e., $y = 2x - 1$ $\therefore$ the slope is 2

$$\therefore \text{ at } x = 1, \quad \frac{dy}{dx} = \frac{a}{2} - \frac{b}{2} = 2 \quad \therefore a - b = 4$$

$$\therefore a = b + 4 \quad \dots\dots (1)$$

Also, at $x = 1$ the tangent touches the curve i.e., $\frac{a(1)+b}{\sqrt{1}} = 2(1) - 1$

i.e., $a + b = 1$

$\therefore b + 4 + b = 1$ {using (1)}

$$\therefore 2b = -3$$

$$\therefore b = -\frac{3}{2}$$

$$\therefore a = 4 - \frac{3}{2} = \frac{5}{2}$$

$$\text{i.e., } a = \frac{5}{2}, \quad b = -\frac{3}{2}$$

7 $y = 4(ax + 1)^{-2}$

Since when $x = 0$, $y = 4(0 + 1)^{-2} = 4$, the point of contact is $(0, 4)$

Now $\frac{dy}{dx} = -8(ax + 1)^{-3}(a) = \frac{-8a}{(ax + 1)^3}$ $\therefore$ at $x = 0$, $\frac{dy}{dx} = -8a$

$\therefore$ the tangent has equation $\frac{y - 4}{x - 0} = -8a$ i.e., $y - 4 = -8ax$

and this passes through $(1, 0)$ $\therefore 0 - 4 = -8a(1)$ $\therefore a = \frac{1}{2}$

8 $y = \frac{1}{\sqrt{x}}$ at $x = 4$

Since when $x = 4$ $y = \frac{1}{\sqrt{4}} = \frac{1}{2}$ the point of contact is $(4, \frac{1}{2})$

Now $\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$ $\therefore$ at $x = 4$, $\frac{dy}{dx} = -\frac{1}{2}\left(4^{-\frac{3}{2}}\right) = -\frac{1}{2}\left(\frac{1}{8}\right) = -\frac{1}{16}$

$\therefore$ the normal at $(4, \frac{1}{2})$ has slope 16.

So the equation is $\frac{y - \frac{1}{2}}{x - 4} = 16$ i.e., $y - \frac{1}{2} = 16x - 64$ i.e., $y = 16x - \frac{127}{2}$

9 a $M = (t^2 + 3)^4$
 $\frac{dM}{dt} = 4(t^2 + 3)^3(2t)$
 $\frac{dM}{dt} = 8t(t^2 + 3)^3$

b $A = \frac{\sqrt{t+5}}{t^2}$ is a quotient with
 $u = (t + 5)^{\frac{1}{2}}$ and $v = t^2$
 $\therefore u' = \frac{1}{2}(t + 5)^{-\frac{1}{2}}, v' = 2t$
 $\frac{dA}{dt} = \frac{\left(\frac{1}{2\sqrt{t+5}}\right)t^2 - \sqrt{t+5}(2t)}{t^4}$
 $= \frac{t^2 - 2(2t)(t+5)}{2t^4\sqrt{t+5}}$
 $= \frac{t - 4(t+5)}{2t^3\sqrt{t+5}}$
 $= \frac{-3t - 20}{2t^3\sqrt{t+5}}$

10 a $y = \frac{4}{\sqrt{x}} - 3x = 4x^{-\frac{1}{2}} - 3x$
 $\therefore \frac{dy}{dt} = -2x^{-\frac{3}{2}} - 3 = \frac{-2}{x\sqrt{x}} - 3$

b $y = \left(x - \frac{1}{x}\right)^4 = (x - x^{-1})^4$
 $\therefore \frac{dy}{dx} = 4(x - x^{-1})^3(1 + x^{-2})$
 $= 4\left(x - \frac{1}{x}\right)^3\left(1 + \frac{1}{x^2}\right)$

c $y = \sqrt{x^2 - 3x} = (x^2 - 3x)^{\frac{1}{2}}$
 $\therefore \frac{dy}{dx} = \frac{1}{2}(x^2 - 3x)^{-\frac{1}{2}}(2x - 3)$
 $= \frac{2x - 3}{2\sqrt{x^2 - 3x}}$

REVIEW SET 21B

1 a $y = 5x - 3x^{-1}$
 $\frac{dy}{dx} = 5 + 3x^{-2} = 5 + \frac{3}{x^2}$

b $y = (3x^2 + x)^4$
 $\frac{dy}{dx} = 4(3x^2 + x)^3(6x + 1)$

c $y = (x^2 + 1)(1 - x^2)^3$ is a product with $u = x^2 + 1$ and $v = (1 - x^2)^3$
 $\therefore u' = 2x$ and $v' = 3(1 - x^2)^2(-2x)$
 $\frac{dy}{dx} = 2x(1 - x^2)^3 - 6x(x^2 + 1)(1 - x^2)^2$ {using the product rule}

$$2 \quad y = x^3 - 3x^2 - 9x + 2, \quad \therefore \frac{dy}{dx} = 3x^2 - 6x - 9$$

$$\text{horizontal tangents occur when } \frac{dy}{dx} = 0 \quad \text{i.e., when } 3x^2 - 6x - 9 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$\therefore x = 3 \quad \text{or} \quad x = -1$$

when $x = 3$, horizontal tangent has equation $y = -25$

when $x = -1$, horizontal tangent has equation $y = 7$

$$3 \quad y = \frac{x+1}{x^2-2} \quad \text{and since when } x = 1, \quad y = \frac{1+1}{1^2-2} = -2, \quad \text{the point of contact is } (1, -2)$$

$$\text{Now } y = \frac{x+1}{x^2-2} \text{ is a quotient with } u = x+1 \text{ and } v = x^2-2$$

$$\therefore u' = 1 \quad \text{and} \quad v' = 2x$$

$$\frac{dy}{dx} = \frac{1(x^2-2) - (x+1)(2x)}{(x^2-2)^2} \quad \{\text{using the quotient rule}\}$$

$$\therefore \text{ at } x = 1, \quad \frac{dy}{dx} = \frac{(1^2-2) - (1+1)2(1)}{(1^2-2)^2} = \frac{(-1) - 4}{1} = -5$$

$$\therefore \text{ the normal at } (1, -2) \text{ has slope } \frac{1}{5}$$

$$\therefore \text{ the equation of the normal is } \frac{y - (-2)}{x - 1} = \frac{1}{5} \quad \text{i.e., } 5y = x - 11$$

$$4 \quad \mathbf{a} \quad f(x) = \frac{(x+3)^3}{\sqrt{x}} = (x+3)^3 x^{-\frac{1}{2}} \quad \text{which is a product with } u = (x+3)^3 \quad \text{and} \quad v = x^{-\frac{1}{2}}$$

$$\therefore u' = 3(x+3)^2 \quad \text{and} \quad v' = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$\therefore f'(x) = \frac{3(x+3)^2}{\sqrt{x}} - \frac{(x+3)^3}{2x\sqrt{x}} \quad \{\text{using the product rule}\}$$

$$\mathbf{b} \quad f(x) = x^4 \sqrt{x^2+3} \quad \text{which is a product with } u = x^4 \quad \text{and} \quad v = (x^2+3)^{\frac{1}{2}}$$

$$\therefore u' = 4x^3 \quad \text{and} \quad v' = \frac{1}{2}(x^2+3)^{-\frac{1}{2}}(2x) = x(x^2+3)^{-\frac{1}{2}}$$

$$\therefore f'(x) = 4x^3 \sqrt{x^2+3} + \frac{x^5}{\sqrt{x^2+3}} \quad \{\text{using the product rule}\}$$

$$5 \quad \mathbf{a} \quad f(x) = 3x^2 - \frac{1}{x} = 3x^2 - x^{-1} \quad \mathbf{b} \quad f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$\therefore f'(x) = 6x + x^{-2}$$

$$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\text{and } f''(x) = 6 - 2x^{-3} = 6 - \frac{2}{x^3}$$

$$\therefore f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$$

$$6 \quad y = x^2 \sqrt{1-x} \quad \text{at } x = -3$$

$$\text{Since when } x = -3, \quad y = (-3)^2 \sqrt{1 - (-3)} = 9\sqrt{4} = 18,$$

the point of contact is $(-3, 18)$

$$\text{Also, } y = x^2 \sqrt{1-x} \text{ is a product, with } u = x^2 \quad \text{and} \quad v = (1-x)^{\frac{1}{2}}$$

$$\therefore u' = 2x \quad \text{and} \quad v' = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$$

$$\therefore \frac{dy}{dx} = 2x(1-x)^{\frac{1}{2}} - x^2 \left(\frac{1}{2}(1-x)^{-\frac{1}{2}} \right)$$

$$\therefore \text{ at } x = -3, \quad \frac{dy}{dx} = 2(-3)(1-(-3))^{\frac{1}{2}} - (-3)^2 \left(\frac{1}{2} \right) (1-(-3))^{-\frac{1}{2}} = -6(2) - 9\left(\frac{1}{2}\right)^2 = -\frac{57}{4}$$

$$\therefore \text{ the tangent at } (-3, 18) \text{ has equation } \frac{y-18}{x-(-3)} = -\frac{57}{4} \quad \text{i.e., } 4y-72 = -57x-171$$

$$\text{i.e., } 4y = -57x-99$$

$$\text{Now when } x = 0, \quad y = -\frac{99}{4} \quad \text{and when } y = 0, \quad x = -\frac{99}{57}$$

$$\therefore \text{ area of } \triangle OAB = \frac{1}{2} \left(\frac{99}{4} \right) \left(\frac{99}{57} \right) \div 21.5 \text{ units}^2$$

$$\mathbf{7} \quad y = x^3 + ax + b \quad \therefore \quad \frac{dy}{dx} = 3x^2 + a \quad \text{and so at } x = 1, \quad \frac{dy}{dx} = 3 + a$$

Now the equation of the tangent at $x = 1$ is $y = 2x$

$$\therefore \text{ the slope is } 2 \quad \therefore \quad 3 + a = 2 \quad \text{and so } a = -1$$

Also, at $x = 1$ the tangent touches the curve.

$$\text{i.e., } x^3 + ax + b = 2x \quad \text{when } x = 1$$

$$\therefore (1)^3 + (-1)(1) + b = 2(1)$$

$$\therefore 1 - 1 + b = 2$$

$$\therefore b = 2 \quad \text{and so } a = -1.$$

$$\mathbf{8} \quad y = x^3 + ax^2 - 4x + 3 \quad \therefore \quad \frac{dy}{dx} = 3x^2 + 2ax - 4$$

The tangent at $x = 1$ is parallel to $y = 3x$, so at $x = 1$, $\frac{dy}{dx} = 3$

$$\text{i.e., } 3 = 3(1)^2 + 2a(1) - 4$$

$$\therefore 2a = 4$$

$$\therefore a = 2$$

$$\text{Since when } x = 1, \quad y = 1^3 + 2(1)^2 - 4(1) + 3 = 2$$

The contact point is $(1, 2)$ and since the slope is 3, the tangent at $(1, 2)$ has equation

$$\frac{y-2}{x-1} = 3 \quad \text{i.e., } y-2 = 3x-3$$

$$\text{i.e., } y = 3x-1$$

$$\text{The tangent meets the curve where } x^3 + 2x^2 - 4x + 3 = 3x - 1$$

$$\text{i.e., } x^3 + 2x^2 - 7x + 4 = 0$$

Since the line touches the curve (i.e., is tangent to it) at $x = 1$, $(x-1)^2$ must be a factor.

Consequently, $x^3 + 2x^2 - 7x + 4 = (x-1)^2(x+4) = 0$ {since the constant term is 4}

$$\therefore \text{ the curve cuts the tangent at } x = -4, \quad y = (-4)^3 + 2(-4)^2 - 4(-4) + 3 = -13$$

$$\text{i.e., at } (-4, -13)$$

$$\mathbf{9} \quad f(x) = 2x^3 + Ax + B \quad \therefore \quad f'(x) = 6x^2 + A$$

Now as the slope at $(-2, 33)$ is 10,

$$\therefore f'(-2) = 10$$

$$\therefore 10 = 6(-2)^2 + A$$

$$\therefore A = -14$$

$$\therefore f(x) = 2x^3 - 14x + B$$

and as $(-2, 33)$ lies on the curve

$$f(-2) = 33$$

$$\therefore 2(-2)^3 - 14(-2) + B = 33$$

$$\therefore -16 + 28 + B = 33$$

$$\therefore B = 49 - 28$$

$$\therefore B = 21$$

10 So, P_n is: “If $y = x^n$ where $n \in \mathbb{Z}^+$, then $\frac{dy}{dx} = nx^{n-1}$.”

Proof: (By the Principle of Mathematical Induction)

(1) If $p = 1$, then $y = x$. This has slope 1, so $\frac{dy}{dx} = 1 = 1x^0$. $\therefore P_1$ is true.

(2) If P_k is true, then $y = x^k$ implies that $\frac{dy}{dx} = kx^{k-1}$

If $y = x^{k+1} = x^k x$,

$$\begin{aligned}\text{then } \frac{dy}{dx} &= \frac{d}{dx}(x^k)x + x^k \frac{d}{dx}(x) \quad \{\text{Product Rule}\} \\ &= kx^{k-1}x + x^k \times 1 \\ &= kx^k + x^k \\ &= (k+1)x^k\end{aligned}$$

Hence P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for $n \in \mathbb{Z}^+$. {Principle of Mathematical Induction}

REVIEW SET 21C

1 a $y = x^3\sqrt{1-x^2}$ is a product where $y = x^3$ and $v = (1-x^2)^{\frac{1}{2}}$

$$\therefore u' = 3x^2 \quad \text{and} \quad v' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = -x(1-x^2)^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = 3x^2\sqrt{1-x^2} - \frac{x^4}{\sqrt{1-x^2}} \quad \{\text{using the product rule}\}$$

b $y = \frac{x^2-3x}{\sqrt{x+1}}$ is a quotient where $u = x^2-3x$ and $v = (x+1)^{\frac{1}{2}}$

$$u' = 2x-3 \quad \text{and} \quad v' = \frac{1}{2}(x+1)^{-\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \frac{(2x-3)(x+1)^{\frac{1}{2}} - \frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1} \quad \{\text{using the quotient rule}\}$$

2 $y = \frac{x+1}{x^2-2}$. Since when $x = 1$, $y = \frac{1+1}{1^2-2} = -2$, the point of contact is $(1, -2)$

$y = \frac{x+1}{x^2-2}$ is a quotient with $u = x+1$ and $v = x^2-2$

$$\therefore u' = 1 \quad \text{and} \quad v' = 2x$$

$$\therefore \frac{dy}{dx} = \frac{1(x^2-2) - (x+1)2x}{(x^2-2)^2} \quad \{\text{using the quotient rule}\}$$

$$\therefore \text{ at } x = 1, \quad \frac{dy}{dx} = \frac{(1-2) - 2(1+1)}{(1-2)^2} = \frac{-1-4}{1} = -5$$

$\therefore$ the normal at $(1, -2)$ has slope $\frac{1}{5}$.

$$\begin{aligned}\text{So the normal has equation: } \frac{y - (-2)}{x - 1} &= \frac{1}{5} \quad \text{i.e., } 5y + 10 = x - 1 \\ &\text{i.e., } 5y = x - 11\end{aligned}$$

3 $f(x) = 2x^4 - 4x^3 - 9x^2 + 4x + 7$

$$\therefore f'(x) = 8x^3 - 12x^2 - 18x + 4$$

$$\therefore f''(x) = 24x^2 - 24x - 18, \quad \therefore f''(x) = 0 \quad \text{where } 24x^2 - 24x - 18 = 0$$

$$4x^2 - 4x - 3 = 0$$

$$(2x+1)(2x-3) = 0$$

$$\therefore x = -\frac{1}{2} \quad \text{or} \quad x = \frac{3}{2}$$

4 $f(x) = \frac{3x}{1+x}$ at $(2, 2)$

$f(x)$ is the product of u and v where $u = 3x$ and $v = 1 + x \quad \therefore \quad u' = 3 \quad \text{and} \quad v' = 1$

$\therefore$ by the quotient rule $f'(x) = \frac{3(1+x) - 1(3x)}{(1+x)^2} = \frac{3}{(1+x)^2} \quad \therefore \quad f'(2) = \frac{3}{9} = \frac{1}{3}$

$\therefore$ the normal at $(2, 2)$ has slope -3

So the equation of the normal is $\frac{y-2}{x-2} = -3 \quad \therefore \quad y-2 = -3(x-2)$
 i.e., $y-2 = -3x+6$
 i.e., $y = -3x+8$

$\therefore$ when $x = 0$, $y = 8$ and when $y = 0$, $x = \frac{8}{3}$

$\therefore$ B and C are at $(0, 8)$ and $(\frac{8}{3}, 0)$, and the distance BC

$$= \sqrt{\left(0 - \frac{8}{3}\right)^2 + (8-0)^2} = \sqrt{\frac{64}{9} + 64} = \sqrt{\frac{640}{9}} = \frac{8\sqrt{10}}{3} \text{ units}$$

5 a $y = 3x^4 - \frac{2}{x} = 3x^4 - 2x^{-1}$

$$\therefore \frac{dy}{dx} = 12x^3 + 2x^{-2}$$

$$\therefore \frac{d^2y}{dx^2} = 36x^2 - 4x^{-3} = 36x^2 - \frac{4}{x^3}$$

b $y = x^3 - x + \frac{1}{\sqrt{x}} = x^3 - x + x^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = 3x^2 - 1 - \frac{1}{2}x^{-\frac{3}{2}}$$

$$\therefore \frac{d^2y}{dx^2} = 6x + \frac{3}{4}x^{-\frac{5}{2}}$$

6 $y = \frac{x}{\sqrt{1-x}}$ at $x = -3$.

Since when $x = -3$, $y = \frac{-3}{\sqrt{1-(-3)}} = -\frac{3}{2}$ the point of contact is $(-3, -\frac{3}{2})$

Now $y = \frac{x}{\sqrt{1-x}}$ is a quotient with $u = x$ and $v = (1-x)^{\frac{1}{2}}$

$$\therefore u' = 1, \quad v' = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$$

$$\therefore \frac{dy}{dx} = \frac{(1)\sqrt{1-x} - x(\frac{1}{2})(1-x)^{-\frac{1}{2}}(-1)}{1-x} \quad \{\text{quotient rule}\}$$

$$\text{at } x = -3, \quad \frac{dy}{dx} = \frac{\sqrt{4} - (-3)(\frac{1}{2})(4)^{-\frac{1}{2}}(-1)}{4} = \frac{2 - 3(\frac{1}{2})(\frac{1}{2})}{4} = \frac{2 - \frac{3}{4}}{4} = \frac{5}{16}$$

$$\therefore \text{ the tangent at } (-3, -\frac{3}{2}) \text{ has equation } \frac{y - (-\frac{3}{2})}{x - (-3)} = \frac{5}{16} \quad \text{i.e., } 16(y + \frac{3}{2}) = 5(x + 3)$$

$$16y + 24 = 5x + 15$$

$$\text{i.e., } 5x - 16y = 9$$

$$\therefore b = -16 \quad \text{and} \quad a = 9$$

7 $f(x) = 3x^3 + Ax^2 + B$

As the point $(-2, 14)$ lies on the curve, $14 = -24 + 4A + B$

$$4A + B = 38 \quad \dots\dots (1)$$

Now $f'(x) = 9x^2 + 2Ax$ and as $f'(-2) = 0$

$$36 - 4A = 0$$

$$\therefore A = 9$$

$$\therefore 4(9) + B = 38 \quad \{\text{using (1)}\}$$

$$\therefore B = 38 - 36 = 2$$

That is, $A = 9$ and $B = 2$

$$\therefore f'(x) = 9x^2 + 18x$$

$$\therefore f''(x) = 18x + 18 \quad \text{and so} \quad f''(-2) = -36 + 18 = -18$$

$$8 \quad y = \frac{a}{(x+2)^2} = a(x+2)^{-2}$$

The slope of the line AB is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 4}{0 - 2} = \frac{4}{-2} = -2$

$\therefore$ the equation of the tangent is $\frac{y - 8}{x - 0} = -2$ i.e., $y = -2x + 8$

$$\frac{dy}{dx} = -2(a)(x+2)^{-3} = -2$$

$$\therefore \frac{a}{(x+2)^3} = 1 \quad \therefore a = (x+2)^3 \quad \dots (1)$$

The line AB meets the curve where

$$-2x + 8 = \frac{a}{(x+2)^2}$$

$$\therefore -2x + 8 = \frac{(x+2)^3}{(x+2)^2} \quad \{\text{using (1)}\}$$

$$\therefore -2x + 8 = x + 2$$

$$\therefore -3x = -6$$

$$\therefore x = 2$$

$$\text{and so } a = (2+2)^3 = 64$$

9 The curves $y = \sqrt{3x+1}$ and $y = \sqrt{5x-x^2}$ meet when $\sqrt{(3x+1)} = \sqrt{5x-x^2}$

Squaring both sides, $3x+1 = 5x-x^2$

$$\therefore x^2 - 2x + 1 = 0$$

$$\therefore (x-1)^2 = 0$$

$$\therefore x = 1$$

and when $x = 1$, $y = \sqrt{3+1} = 2$ $\therefore$ the curves meet at $(1, 2)$

Now for $y = \sqrt{3x+1} = (3x+1)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(3x+1)^{-\frac{1}{2}}(3)$$

$$\therefore \text{ at } (1, 2) \quad \frac{dy}{dx} = \frac{3}{2(3+1)^{\frac{1}{2}}} = \frac{3}{4}$$

and for $y = \sqrt{5x-x^2} = (5x-x^2)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(5x-x^2)^{-\frac{1}{2}}(5-2x) = \frac{5-2x}{2\sqrt{5x-x^2}}$$

$$\therefore \text{ at } (1, 2) \quad \frac{dy}{dx} = \frac{5-2}{2\sqrt{5-1}} = \frac{3}{4}$$

i.e., the curves have the same slope of $\frac{3}{4}$ at their point of intersection.

Now the equation of the common tangent at $(1, 2)$ is $\frac{y-2}{x-1} = \frac{3}{4}$ i.e., $4(y-2) = 3(x-1)$

$$4y - 8 = 3x - 3$$

$$\text{i.e., } 4y = 3x + 5$$

10 a $y = \frac{1}{1-x} = (1-x)^{-1}$

$$\therefore \frac{dy}{dx} = -(-1)(1-x)^{-2} = \frac{1}{(1-x)^2}$$

b P_n is: “If $y = \frac{1}{1-x}$ then $\frac{d^n y}{dx^n} = \frac{n!}{(1-x)^{n+1}}$ for $n \in \mathbb{Z}^+$.”

Proof: (By the Principle of Mathematical Induction)

(1) Using **a**, $\frac{d^1 y}{dx^1} = \frac{dy}{dx} = \frac{1}{(1-x)^2} = \frac{1!}{(1-x)^{(1+1)}}$

$\therefore P_1$ is true.

(2) If P_k is true, then $\frac{d^k y}{dx^k} = \frac{k!}{(1-x)^{k+1}}$

$$= k!(1-x)^{-(k+1)}$$

$$\therefore \frac{d^{k+1} y}{dx^{k+1}} = -k!(k+1)(-1)(1-x)^{-(k+1)-1}$$

$$= \frac{(k+1)!}{(1-x)^{(k+1)+1}}$$

Hence P_{k+1} is true whenever P_k is true.

$\therefore$ since P_1 is true, P_n is true for $n \in \mathbb{Z}^+$. {Principle of Mathematical Induction}

Chapter 22

APPLICATIONS OF DIFFERENTIAL CALCULUS

EXERCISE 22A

1 $P(t) = 2t^2 - 12t + 118$ thousand dollars, $t \geq 0$

a $P(0) = \$118\,000$ is the current annual profit

b $\frac{dP}{dt} = 4t - 12$ thousand dollars/year **c** $\frac{dP}{dt}$ is the rate of change in profit with time

d i Profit decreases when $\frac{dP}{dt} \leq 0$ i.e., $4t - 12 \leq 0$
 $\therefore 4t \leq 12$
i.e., $t \leq 3$

But $t \geq 0 \therefore 0 \leq t \leq 3$ years

ii Profit increases when $\frac{dP}{dt} \geq 0$ i.e., $t \geq 3$ years

e The profit function is a quadratic with $a > 0 \therefore$ shape is 

So, a minimum profit occurs when $\frac{dP}{dt} = 0$ i.e., at $t = 3$ years

and $P(3) = 18 - 36 + 118 = 100$ thousand dollars i.e., \$100 000.

f i When $t = 4$, $\frac{dP}{dt} = 4$ thousand dollars per year.

So, the profit is increasing at \$4000/year after 4 years.

ii When $t = 10$, $\frac{dP}{dt} = 28$ thousand dollars per year.

So, the profit is increasing at \$28 000/year after 10 years.

iii When $t = 25$, $\frac{dP}{dt} = 88$ thousand dollars per year.

So, the profit is increasing at \$88 000/year after 25 years.

2 $V = 200(50 - t)^2 \text{ m}^3$

a average rate on $0 \leq t \leq 5$

$$= \frac{V(5) - V(0)}{5 - 0}$$

$$= \frac{200(45)^2 - 200(50)^2}{5}$$

$$= -19\,000 \text{ m}^3/\text{min}$$

i.e., leaving at 19 000 m³/min

b $V'(t) = 400(50 - t)^1 \times (-1)$

$$\therefore V'(5) = 400 \times 45 \times -1$$

$$= -18\,000 \text{ m}^3/\text{min}$$

i.e., leaving at 18 000 m³/min

3 $s(t) = 1.2 + 28.1t - 4.9t^2$ metres

a When released, $t = 0$ and $s(0) = 1.2 \text{ m}$ $\therefore$ it is released 1.2 m above the ground.

b $s'(t) = 28.1 - 9.8t \text{ m/s}$ is the instantaneous velocity of the ball at the time t seconds after release.

c When $s'(t) = 0$, $28.1 - 9.8t = 0 \therefore t = \frac{28.1}{9.8} \div 2.87 \text{ sec.}$

So, after 2.87 sec the ball has reached its maximum height.

d $s(2.867) = 1.2 + 28.1 \times 2.867 - 4.9 \times 2.867^2 \div 41.5 \text{ m}$

So, the maximum height reached is about 41.5 m.

$$\text{e} \quad \text{i} \quad s'(0) = 28.1 \text{ m/s} \quad \text{ii} \quad s'(2) = 28.1 - 19.6 = 8.5 \text{ m/s} \quad \text{iii} \quad s'(5) = 28.1 - 49 = -20.9 \text{ m/s}$$

If $s'(t) \geq 0$, the ball is travelling upwards.

If $s'(t) \leq 0$, the ball is travelling downwards.

$$\text{f} \quad s(t) = 0 \quad \text{when} \quad 1.2 + 28.1t - 4.9t^2 = 0$$

$$\text{i.e.,} \quad 4.9t^2 - 28.1t - 1.2 = 0$$

$$\therefore t = \frac{28.1 \pm \sqrt{28.1^2 - 4(4.9)(-1.2)}}{9.8}$$

$$\div -0.0424 \quad \text{or} \quad 5.777$$

$\therefore$ hits the ground after 5.78 sec.

$$\text{g} \quad \frac{d^2s}{dt^2} = -9.8 \text{ m/s}^2 \quad \text{and is the constant rate of change in} \quad \frac{ds}{dt}$$

i.e., the instantaneous acceleration is constant at -9.8 m/s^2 for the entire motion.

$$\text{4 a} \quad s(t) = bt - 4.9t^2$$

$$s'(t) = b - 9.8t$$

$$\therefore s'(0) = b$$

i.e., the initial velocity is $b \text{ m/s}$

$$\text{b} \quad \text{since} \quad s(14.2) = 0$$

$$b(14.2) - 4.9(14.2)^2 = 0$$

$$\therefore 14.2[b - 4.9 \times 14.2] = 0$$

$$\therefore b = 4.9 \times 14.2$$

$$\therefore b \div 69.6$$

$\therefore$ the initial velocity is 69.6 m/s

EXERCISE 22B

$$\text{1} \quad Q = 100 - 10\sqrt{t}, \quad t \geq 0$$

$$\text{a} \quad \text{i} \quad \text{At } t = 0, \quad Q = 100 \text{ units}$$

$$\text{ii} \quad \text{At } t = 25, \quad Q = 50 \text{ units}$$

$$\text{iii} \quad \text{At } t = 100, \quad Q = 0 \text{ units}$$

$$\text{b} \quad \frac{dQ}{dt} = -5t^{-\frac{1}{2}} = -\frac{5}{\sqrt{t}}$$

$$\text{i} \quad \text{At } t = 25, \quad \frac{dQ}{dt} = -1 \text{ units/year}$$

i.e., decreasing at 1 unit/year

$$\text{ii} \quad \text{At } t = 50, \quad \frac{dQ}{dt} = -\frac{5}{\sqrt{50}}$$

$$= -\frac{1}{\sqrt{2}} \text{ units/year}$$

i.e., decreasing at $\frac{1}{\sqrt{2}}$ units/year

$$\text{c} \quad \frac{dQ}{dt} = -\frac{5}{\sqrt{t}}$$

$$\therefore \frac{dQ}{dt} < 0, \quad \text{for all } t > 0, \quad \therefore \frac{dQ}{dt} \text{ is decreasing for all } t > 0$$

$$\text{2} \quad H = 20 - \frac{9}{t+5} \text{ m}, \quad t \geq 0$$

$$\text{a} \quad \text{At planting, } t = 0 \quad \therefore H(0) = 20 - \frac{9}{0+5} = 18.2 \text{ m}$$

$$\text{b} \quad \text{i} \quad \text{At } t = 4, \quad H(4) = 20 - \frac{9}{4+5}$$

$$= 19 \text{ m}$$

$$\text{ii} \quad \text{At } t = 8, \quad H(8) = 20 - \frac{9}{8+5}$$

$$\div 19.3 \text{ m}$$

$$\text{iii} \quad \text{At } t = 12, \quad H(12) = 20 - \frac{9}{12+5}$$

$$\div 19.5 \text{ m}$$

$$\text{c} \quad \text{Now} \quad \frac{dH}{dt} = 9(t+5)^{-2} = \frac{9}{(t+5)^2}$$

$$\text{i} \quad \text{When } t = 0,$$

$$\frac{dH}{dt} = \frac{9}{25} = 0.36 \text{ m/yr}$$

$$\text{ii} \quad \text{When } t = 5,$$

$$\frac{dH}{dt} = \frac{9}{100} = 0.09 \text{ m/yr}$$

$$\text{iii} \quad \text{When } t = 10,$$

$$\frac{dH}{dt} = \frac{9}{225} = 0.04 \text{ m/yr}$$

d Now $\frac{dH}{dt} = \frac{9}{(t+5)^2}$, $\therefore$ as $(t+5)^2 > 0$ for all $t \geq 0$, $\frac{dH}{dt} > 0$ for all $t \geq 0$
i.e., the height of the tree is always increasing, which means that the tree is always growing.

3 a $C(v) = 200v + 10\,000v^{-1}$ dollars **i** At $v = 20$ kmph, $C = \$4500$
ii At $v = 40$ kmph, $C = \$8250$

b $\frac{dC}{dv} = 200 - 10\,000v^{-2} = 200 - \frac{10\,000}{v^2}$

i At $v = 10$ kmph, $\frac{dC}{dv} = \$100$ per kmph

ii At $v = 30$ kmph, $\frac{dC}{dv} = \$188.89$ per kmph

Cost is a minimum when $\frac{dC}{dv} = 0$, i.e., $200 - \frac{10\,000}{v^2} = 0$

$\therefore 200 = \frac{10\,000}{v^2}$

$\therefore v^2 = 50$

$\therefore v = \pm\sqrt{50}$

i.e., $v = \sqrt{50}$ kmph

$$\mathbf{4} \quad y = \frac{1}{10}x(x-2)(x-3) = \frac{1}{10}(x^3 - 5x^2 + 6x)$$

a When $y = 0$, $x = 0, 2$ or 3
 $\therefore$ the lake is between 2 and 3 km from the shoreline.

b $\frac{dy}{dx} = \frac{1}{10}(3x^2 - 10x + 6)$ When $x = \frac{1}{2}$, $\frac{dy}{dx} = \frac{7}{40} \therefore$ land is sloping upwards.

When $x = \frac{3}{2}$, $\frac{dy}{dx} = -\frac{9}{40} \therefore$ land is sloping downwards.

The deepest point of the lake occurs when the slope of the land is 0, i.e., $\frac{dy}{dx} = 0$
 $\therefore \frac{1}{10}(3x^2 - 10x + 6) = 0$
 i.e., $3x^2 - 10x + 6 = 0 \quad \therefore x = \frac{10 \pm \sqrt{100 - 72}}{6} = \frac{5 \pm \sqrt{7}}{3}$

but it must be the value between 2 and 3 km, i.e., $x = \frac{5 + \sqrt{7}}{3} \div 2.549$ km

$$\begin{aligned} \therefore \text{ the depth at this point is } & y(2.549) \div \frac{1}{10}(2.549)(0.549)(-0.451) \\ & \div -0.06311 \text{ km} \\ \text{i.e., } & \div 63.11 \text{ m below sea level.} \end{aligned}$$

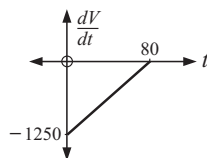
5 a $V = 50\,000 \left(1 - \frac{t}{80}\right)^2, \quad 0 \leq t \leq 80$

$$\frac{dV}{dt} = 2 \times 50\,000 \left(1 - \frac{t}{80}\right)^1 \times \left(-\frac{1}{80}\right) = -1250 \left(1 - \frac{t}{80}\right)$$

b Outflow was fastest when $t = 0$ (when the tap was first opened).

As $\frac{dV}{dt} = -1250 + \frac{1250}{80}t$, then $\frac{d^2V}{dt^2} = \frac{1250}{80} = \frac{125}{8}$

and since $\frac{d^2V}{dt^2}$ is constant and positive, it shows that $\frac{dV}{dt}$ is constantly increasing i.e., the outflow is decreasing at a constant rate.



- 6 a** $\frac{dP}{dt} = aP \left(1 - \frac{P}{b}\right) - \left(\frac{c}{100}\right)P$ and when $\frac{dP}{dt} = 0$, the rate of change of population is zero, $\therefore$ the population is not changing and is stable.

- b** If $a = 0.06$, $b = 24\,000$, $c = 5$ then

$$\begin{aligned}\frac{dP}{dt} &= 0.06P \left(1 - \frac{P}{24\,000}\right) - \frac{5}{100}P \\ &= 0.06P - 0.05P - \frac{0.06P^2}{24\,000} \\ &= P \left(0.01 - \frac{P}{400\,000}\right)\end{aligned}$$

Now for a stable population

$$\begin{aligned}\frac{dP}{dt} &= 0 \\ \therefore P &= 0 \quad \text{or} \quad \frac{P}{400\,000} = 0.01 \\ \text{i.e., } P &= 0 \quad \text{or} \quad 4000 \\ \text{i.e., the stable population is } &4000 \text{ fish}\end{aligned}$$

- c** If the harvest rate is 4%, then $\frac{dP}{dt} = 0.06P \left(1 - \frac{P}{24\,000}\right) - \frac{4}{100}P$
- $$= P \left(0.02 - \frac{0.06P}{24\,000}\right)$$

For stable population $\frac{dP}{dt} = 0 \quad \therefore 0 = P \left(0.02 - \frac{0.06P}{24\,000}\right)$

$$\therefore P = 0 \quad \text{or} \quad \frac{0.06P}{24\,000} = 0.02$$

$$\therefore P = 0 \quad \text{or} \quad \frac{0.02 \times 24\,000}{0.06}$$

$$\therefore P = 0 \quad \text{or} \quad 8000$$

i.e., the stable population is 8000 fish

- 7 a** $C(x) = 0.0003x^3 + 0.02x^2 + 4x + 2250$

$$\therefore C'(x) = 0.0009x^2 + 0.04x + 4 \quad (\text{dollars/pair})$$

- b** $C'(220) = 0.0009(220)^2 + 0.04(220) + 4 = \56.36 per pair

This estimates the cost of making the 221st pair of jeans if 220 pairs are currently being made.

- c** $C(221) - C(220) = \$7348.98 - \$7292.40 = \$56.58$

This is the actual cost to make the extra pair of jeans (221 instead of 220).

- d** $C''(x) = 0.0018x + 0.04$

$$C''(x) = 0 \quad \text{when} \quad 0.0018x + 0.04 = 0 \quad \text{i.e., } x = -\frac{0.04}{0.0018} \div -22.2 \quad \text{but } x \geq 0$$

$\therefore$ when the rate of change is a minimum it is out of the bounds of our model (we cannot make < 0 jeans!)

EXERCISE 22C.1

- 1 a** $s(t) = t^2 + 3t - 2 \quad t \geq 0$

$$\begin{aligned}\text{Average velocity} &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\ &= \frac{s(3) - s(1)}{3 - 2} \\ &= \frac{16 - 2}{2} \\ &= 7 \text{ ms}^{-1}\end{aligned}$$

$$\begin{aligned}\text{b Average velocity} &= \frac{s(t_2) - s(t_1)}{t_2 - t_1} \\ &= \frac{s(1+h) - s(1)}{(1+h) - 1} \\ &= \frac{(1+h)^2 + 3(1+h) - 2 - 2}{h} \\ &= \frac{2h + h^2 + 3h}{h} \\ &= (5+h) \text{ ms}^{-1}\end{aligned}$$

$$\text{c} \quad \lim_{h \rightarrow 0} \frac{s(1+h) - s(1)}{h}$$

$$= \lim_{h \rightarrow 0} 5 + h$$

$$= 5 \text{ ms}^{-1}$$

This is the *instantaneous velocity* at $t = 1$ second.

d Average velocity

$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$= \frac{s(t+h) - s(t)}{(t+h) - t}$$

$$= \frac{[(t+h)^2 + 3(t+h) - 2] - [t^2 + 3t - 2]}{h}$$

$$= \frac{2ht + h^2 + 3h}{h}$$

$$= (2t + 3 + h) \text{ ms}^{-1}$$

$$\begin{aligned} \text{Now } \lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h} &= \lim_{h \rightarrow 0} (2t + 3 + h) \\ &= (2t + 3) \text{ ms}^{-1} \end{aligned}$$

This is the *instantaneous velocity* at t seconds.

2 a $s(t) = 5 - 2t^2 \text{ cm}$

$$\text{Average velocity} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$= \frac{s(5) - s(2)}{5 - 2}$$

$$= \frac{(-45) - (-3)}{3}$$

$$= -14 \text{ cm s}^{-1}$$

b Average velocity $= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$

$$= \frac{s(2+h) - s(2)}{(2+h) - 2}$$

$$= \frac{5 - 2(2+h)^2 + 3}{h}$$

$$= \frac{-8h - 2h^2}{h}$$

$$= (-8 - 2h) \text{ cm s}^{-1}$$

$$\text{c} \quad \lim_{h \rightarrow 0} \frac{s(2+h) - s(2)}{h} = \lim_{h \rightarrow 0} (-8 - 2h)$$

$$= -8 \text{ cm s}^{-1}$$

This is the *instantaneous velocity* when $t = 2$ seconds.

d $\lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h}$

$$= \lim_{h \rightarrow 0} \frac{[5 - 2(t+h)^2] - [5 - 2t^2]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-4th - 2h^2}{h}$$

$$= \lim_{h \rightarrow 0} (-4t - 2h)$$

$$= -4t \text{ cm s}^{-1}$$

$-4t \text{ cm s}^{-1}$ is the *instantaneous velocity* at t seconds.

3 $v(t) = 2\sqrt{t} + 3 \text{ cm s}^{-1}, \quad t \geq 0$

a Average acceleration

$$= \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

$$= \frac{v(4) - v(1)}{4 - 1}$$

$$= \frac{7 - 5}{3}$$

$$= \frac{2}{3} \text{ cm s}^{-2}$$

b Average acceleration

$$= \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

$$= \frac{v(1+h) - v(1)}{(1+h) - 1}$$

$$= \frac{[2\sqrt{1+h} + 3] - [2\sqrt{1} + 3]}{h}$$

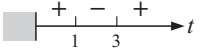
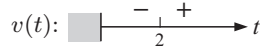
$$= \frac{2\sqrt{1+h} - 2}{h} \text{ cm s}^{-2}$$

$$\begin{aligned}
 \text{c } \lim_{h \rightarrow 0} \frac{v(1+h) - v(1)}{(1+h) - 1} &= \lim_{h \rightarrow 0} \frac{[2\sqrt{1+h} + 3] - [2 + 3]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2[\sqrt{1+h} - 1]}{h} \times \frac{\sqrt{1+h} + 1}{\sqrt{1+h} + 1} \\
 &= \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{1+h} + 1)} \\
 &= \frac{2}{2} \\
 &= 1 \text{ cm s}^{-2} \quad \text{This is the instantaneous acceleration when } t = 1 \text{ second.}
 \end{aligned}$$

$$\begin{aligned}
 \text{d } \lim_{h \rightarrow 0} \frac{v(t+h) - v(t)}{h} &= \lim_{h \rightarrow 0} \frac{2\sqrt{t+h} - 2\sqrt{t}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2(\sqrt{t+h} - \sqrt{t})}{h} \times \frac{\sqrt{t+h} + \sqrt{t}}{\sqrt{t+h} + \sqrt{t}} \\
 &= \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{t+h} + \sqrt{t})} \\
 &= \frac{2}{2\sqrt{t}} \\
 &= \frac{1}{\sqrt{t}} \text{ cm s}^{-2} \quad \text{This is the instantaneous acceleration at } t \text{ seconds.}
 \end{aligned}$$

- 4 a This is the *instantaneous velocity* at $t = 4$ seconds.
 b This is the *instantaneous acceleration* at $t = 4$ seconds.

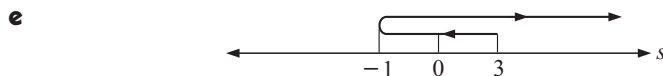
EXERCISE 22C.2

1 a $s(t) = t^2 - 4t + 3 \text{ cm}, t \geq 0 \quad \therefore v(t) = 2t - 4 \text{ cm s}^{-1}$ $s(t)$: 
 and $a(t) = 2 \text{ cm s}^{-2}$ $v(t)$: 

b When $t = 0$, $s(0) = 3 \text{ cm}$
 $v(0) = -4 \text{ cm s}^{-1}$
 $a(0) = 2 \text{ cm s}^{-2} \quad \therefore$ the object is 3 cm right of O and is moving to the left with a velocity of 4 cm s^{-1} and slowing down, its acceleration being 2 cm s^{-2} to the right.

c When $t = 2$, $s(2) = -1 \text{ cm}$
 $v(2) = 0 \text{ cm s}^{-1}$
 $a(2) = 2 \text{ cm s}^{-2} \quad \therefore$ the object is 1 cm left of O, momentarily at rest, but with acceleration 2 cm s^{-2} to the right.

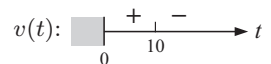
- d The object reverses direction when $v(t) = 0$ i.e., at $t = 2$ seconds.
 At $t = 2$, the particle is 1 cm left of O.



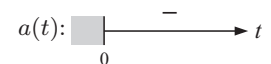
- f Speed decreases when $v(t)$ and $a(t)$ have opposite signs, i.e., when $0 \leq t \leq 2$.

2 $s(t) = 98t - 4.9t^2 \text{ m } t \geq 0$

a $v(t) = 98 - 9.8t \text{ ms}^{-1}$
 $a(t) = -9.8 \text{ ms}^{-2}$



b When $t = 0$, $s(0) = 0 \text{ m}$, $v(0) = 98 \text{ ms}^{-1}$



c When $t = 5$, $s(5) = 367.5 \text{ m}$
 $v(5) = 49 \text{ ms}^{-1}$
 $a(5) = -9.8 \text{ ms}^{-2}$

The stone is 367.5 m above the ground, travelling upwards at 49 ms^{-1} and slowing down.

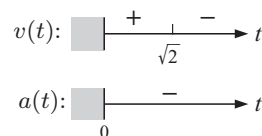
When $t = 12$, $s(12) = 470.4 \text{ m}$
 $v(12) = -19.6 \text{ ms}^{-1}$
 $a(12) = -9.8 \text{ ms}^{-2}$

The stone is 470.4 m above the ground and travelling downwards at 19.6 ms^{-1} , increasing in speed.

- d** Maximum height is reached when $v(t) = 0 \text{ ms}^{-1}$ $\therefore$ the maximum height is
 $\therefore 98 - 9.8t = 0$ $s(10) = 9.8(10) - 4.9(100)$
 $9.8t = 98$ $= 980 - 490$
 $t = 10 \text{ seconds}$ $= 490 \text{ m}$

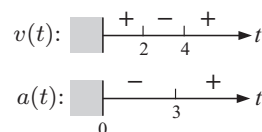
- e** The stone is at ground level when $s(t) = 0$ i.e., when $98t - 4.9t^2 = 0$
 $\therefore 4.9t(20 - t) = 0$
 $\therefore t = 0$ or 20 seconds
 i.e., it hits the ground after 20 seconds .

- 3 a** $s(t) = 12t - 2t^3 - 1 \text{ cm}, t \geq 0$
 $\therefore v(t) = 12 - 6t^2 \text{ cm s}^{-1}$
 and $a(t) = -12t \text{ cm s}^{-2}$

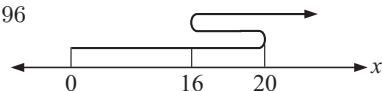


- b** When $t = 0$, $s(0) = -1 \text{ cm}$
 $v(0) = 12 \text{ cm s}^{-1}$
 $a(0) = 0 \text{ cm s}^{-2}$
 The particle is 1 cm left of O , moving right at 12 cm s^{-1} with constant speed.
- c** The particle reverses direction when $v(t) = 0$ i.e., at $t = \sqrt{2}$ seconds.
 When $t = \sqrt{2}$, $s(\sqrt{2}) = 12\sqrt{2} - 2(2\sqrt{2}) - 1$
 $= 8\sqrt{2} - 1$ i.e., the particle is $(8\sqrt{2} - 1) \text{ cm}$ to the right of O .
- d i** From the sign diagrams in **a**, the speed increases for $t \geq \sqrt{2}$ seconds.
ii The velocity of the particle never increases $\{a(t) \leq 0\}$.

- 4 a** $x(t) = t^3 - 9t^2 + 24t \text{ m}, t \geq 0$
 $v(t) = 3t^2 - 18t + 24$ and $a(t) = 6t - 18$
 $= 3(t^2 - 6t + 8)$ $= 6(t - 3) \text{ ms}^{-2}$
 $= 3(t - 4)(t - 2) \text{ ms}^{-1}$



- b** Reverses direction when $v(t) = 0$, i.e., at $t = 2$ seconds and $t = 4$ seconds.
 $x(2) = 8 - 36 + 48 \text{ m}$ and $x(4) = 64 - 144 + 96$
 $= 20 \text{ m}$ $= 16 \text{ m}$



- c i** The speed decreases when $v(t)$ and $a(t)$ have the same sign, i.e., when $0 \leq t \leq 2$ and $3 \leq t \leq 4$.
ii The velocity decreases when $a(t) < 0$, i.e., when $0 \leq t \leq 3$.
- d** When $t = 5$, $s(5) = 5^3 - 9.5^2 + 24.5$ $\therefore$ distance travelled $= 20 + 4 + 4 \text{ m}$
 $= 125 - 225 + 120$ $= 28 \text{ m}$
 $= 20 \text{ m}$

- 5 a** Let the equation be $s(t) = at^2 + bt + c$
 $\therefore v(t) = 2at + b$
 and $a(t) = 2a = g$ {gravitational acceleration}
 $\therefore a = \frac{1}{2}g$

Also $v(t) = gt + b$

But, when $t = 0$, $v(0) = g \times 0 + b$

$\therefore v(0) = b$ i.e., initial velocity is b

$\therefore v(t) = v(0) + gt$ as required

b Now when $t = 0$, $s(0) = 0$

$$\therefore a \times 0^2 + b \times 0 + c = 0$$

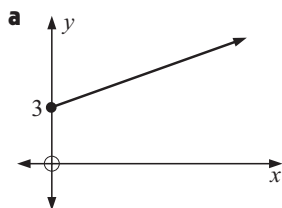
$$\therefore c = 0$$

$$\text{and so } s(t) = \left(\frac{1}{2}g\right)t^2 + v(0)t$$

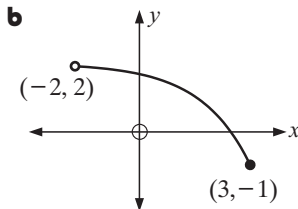
$$\text{i.e., } s(t) = v(0) \times t + \frac{1}{2}gt^2 \text{ as required}$$

EXERCISE 22D.1

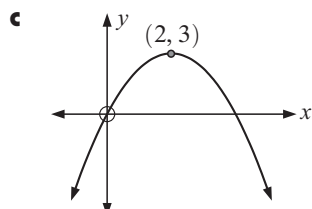
1



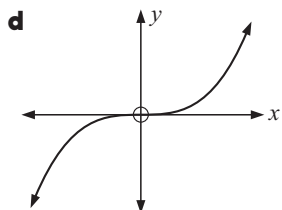
- i** $x \geq 0$
ii never



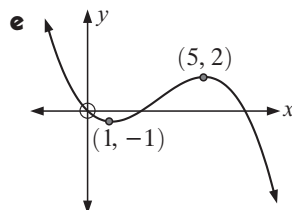
- i** never
ii $-2 < x \leq 3$



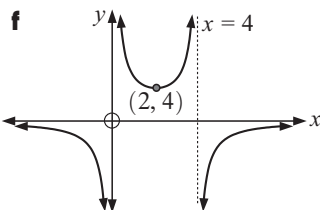
- i** $x \leq 2$
ii $x \geq 2$



- i** all real x
ii never



- i** $1 \leq x \leq 5$
ii $x \leq 1$, $x \geq 5$



- i** $2 \leq x < 4$, $x > 4$
ii $x < 0$, $0 < x \leq 2$

EXERCISE 22D.2

1

a $f(x) = x^2$, $f'(x) = 2x$

Sign $f'(x)$: $\xleftarrow{-} \underset{0}{\mid} \xrightarrow{+}$

increasing when $x \geq 0$,
 decreasing when $x \leq 0$

b $f(x) = -x^3$, $f'(x) = -3x^2$

Sign $f'(x)$: $\xleftarrow{-} \underset{0}{\mid} \xrightarrow{-}$

decreasing for all x

c $f(x) = 2x^2 + 3x - 4$, $f'(x) = 4x + 3$

Sign $f'(x)$: $\xleftarrow{-} \underset{-\frac{3}{4}}{\mid} \xrightarrow{+}$

increasing when $x \geq -\frac{3}{4}$,
 decreasing when $x \leq -\frac{3}{4}$

e $f(x) = \frac{2}{\sqrt{x}}$, $f'(x) = -x^{-\frac{3}{2}}$
 $= 2x^{-\frac{1}{2}}$ $= \frac{-1}{x\sqrt{x}}$

Sign $f'(x)$: $\xleftarrow{-} \underset{0}{\mid} \xrightarrow{-}$

$f(x)$ is only defined for $x > 0$
 decreasing when $x > 0$, never increasing

d $f(x) = \sqrt{x} = x^{\frac{1}{2}}$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

Sign $f'(x)$: $\xleftarrow{-} \underset{0}{\mid} \xrightarrow{+}$

only defined when $x \geq 0$,
 increasing when $x \geq 0$, never decreasing

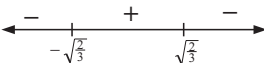
f $f(x) = x^3 - 6x^2$, $f'(x) = 3x^2 - 12x$
 $= 3x(x - 4)$

Sign $f'(x)$: $\xleftarrow{+} \underset{0}{\mid} \xrightarrow{-} \underset{4}{\mid} \xrightarrow{+}$

increasing when $x \leq 0$ or $x \geq 4$,
 decreasing when $0 \leq x \leq 4$

g $f(x) = -2x^3 + 4x$

$$\begin{aligned} f'(x) &= -6x^2 + 4 \\ &= -2(3x^2 - 2) \end{aligned}$$

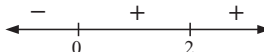
Sign $f'(x)$: 

increasing for $-\sqrt{\frac{2}{3}} \leq x \leq \sqrt{\frac{2}{3}}$,

decreasing for $x \leq -\sqrt{\frac{2}{3}}$ or $x \geq \sqrt{\frac{2}{3}}$

i $f(x) = 3x^4 - 16x^3 + 24x^2 - 2$,

$$\begin{aligned} f'(x) &= 12x^3 + 48x^2 + 48x \\ &= 12x(x^2 - 4x + 4) \\ &= 12x(x - 2)^2 \end{aligned}$$

Sign $f'(x)$: 

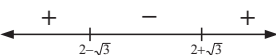
increasing when $x \geq 0$,

decreasing when $x \leq 0$

k $f(x) = x^3 - 6x^2 + 3x - 1$,

$$\begin{aligned} f(x) &= 3x^2 - 12x + 3 \\ &= 3(x^2 - 4x + 1) \end{aligned}$$

$$x = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$$

Sign $f'(x)$: 

increasing when $x \leq 2 - \sqrt{3}$

or $x \geq 2 + \sqrt{3}$,

decreasing when $2 - \sqrt{3} \leq x \leq 2 + \sqrt{3}$

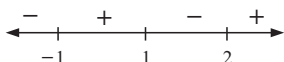
m $y = 3x^4 - 8x^3 - 6x^2 + 24x + 11$

$$\begin{aligned} \frac{dy}{dx} &= 12x^3 - 24x^2 - 12x + 24 \\ &= 12(x^3 - 2x^2 - x + 2) \end{aligned}$$

Using technology, a root is -1 ,

$$\begin{aligned} \therefore \frac{dy}{dx} &= 12(x + 1)(x^2 - 3x + 2) \\ &= 12(x + 1)(x - 1)(x - 2) \end{aligned}$$

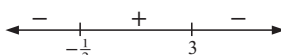
Sign diagram of $\frac{dy}{dx}$:



So $f(x)$ is increasing for $-1 \leq x \leq 1$ and $x \geq 2$, and decreasing for $x \leq -1$ and $1 \leq x \leq 2$.

h $f(x) = -4x^3 + 15x^2 + 18x + 3$

$$\begin{aligned} f'(x) &= -12x^2 + 30x + 18 \\ &= -6(2x^2 - 5x - 3) \\ &= -6(2x + 1)(x - 3) \end{aligned}$$

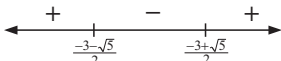
Sign $f'(x)$: 

increasing when $-\frac{1}{2} \leq x \leq 3$,

decreasing when $x \leq -\frac{1}{2}$ and $x \geq 3$

j $f(x) = 2x^3 + 9x^2 + 6x - 7$,

$$\begin{aligned} f'(x) &= 6x^2 + 18x + 6 \\ &= 6(x^2 + 3x + 1) \end{aligned}$$

Sign $f'(x)$: 

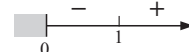
$$x = \frac{-3 \pm \sqrt{9 - 4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$$

increasing for $x \leq \frac{-3 - \sqrt{5}}{2}$ or $x \geq \frac{-3 + \sqrt{5}}{2}$,

decreasing for $\frac{-3 - \sqrt{5}}{2} \leq x \leq \frac{-3 + \sqrt{5}}{2}$

l $f(x) = x - 2\sqrt{x}$, $f'(x) = 1 - x^{-\frac{1}{2}}$

$$\begin{aligned} &= x - 2x^{\frac{1}{2}} &= 1 - \frac{1}{\sqrt{x}} \\ & &= \frac{\sqrt{x} - 1}{\sqrt{x}} \end{aligned}$$

Sign $f'(x)$: 

increasing when $x \geq 1$,

decreasing when $0 \leq x \leq 1$

n $y = x^4 - 4x^3 + 2x^2 + 4x + 1$,

$$\begin{aligned} \frac{dy}{dx} &= 4x^3 - 12x^2 + 4x + 4 \\ &= 4(x^3 - 3x^2 + x + 1) \end{aligned}$$

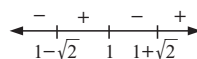
Using technology, a root is 1 ,

$$\begin{aligned} \therefore \frac{dy}{dx} &= 4(x - 1)(x^2 - 2x - 1) \\ &= 0 \end{aligned}$$

when $x = 1$ or $x = \frac{2 \pm \sqrt{8}}{2}$

i.e., $x = 1$ or $x = 1 \pm \sqrt{2}$

Sign diagram of $\frac{dy}{dx}$:



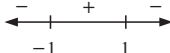
$\therefore$ increasing for $1 - \sqrt{2} \leq x \leq 1$

and $x \geq 1 + \sqrt{2}$, decreasing for $x \leq 1 - \sqrt{2}$ and $1 \leq x \leq 1 + \sqrt{2}$

2 a i $f(x) = \frac{4x}{x^2 + 1},$

let $u = 4x, \quad u' = 4,$
 $v = x^2 + 1, \quad v' = 2x$

$$\begin{aligned}\therefore f'(x) &= \frac{4(x^2 + 1) - 4x \times 2x}{(x^2 + 1)^2} \\ &= \frac{4x^2 + 4 - 8x^2}{(x^2 + 1)^2} \\ &= \frac{4 - 4x^2}{(x^2 + 1)^2} \\ &= \frac{-4(x^2 - 1)}{(x^2 + 1)^2} \\ &= \frac{-4(x + 1)(x - 1)}{(x^2 + 1)^2}\end{aligned}$$

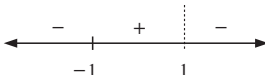
Sign diagram of $\frac{dy}{dx}$: 

ii $\therefore$ increasing for $-1 \leq x \leq 1,$
 decreasing for $x \leq -1$ and $x \geq 1$

b i $f(x) = \frac{4x}{(x - 1)^2},$

let $u = 4x, \quad u' = 4,$
 $v = (x - 1)^2, \quad v' = 2(x - 1)^1$

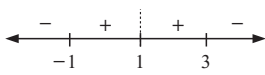
$$\begin{aligned}\therefore f'(x) &= \frac{4(x - 1)^2 - 8x(x - 1)}{(x - 1)^4} \\ &= \frac{4(x - 1)((x - 1) - 2x)}{(x - 1)^4} \\ &= \frac{4(-1 - x)}{(x - 1)^3} \\ &= \frac{-4(x + 1)}{(x - 1)^3}\end{aligned}$$

Sign diagram of $\frac{dy}{dx}$: 

ii $\therefore$ increasing for $-1 \leq x < 1,$
 decreasing for $x \leq -1$ and $x > 1$

c i $f(x) = \frac{-x^2 + 4x - 7}{x - 1},$ let $u = -x^2 + 4x - 7, \quad u' = -2x + 4, \quad v = x - 1, \quad v' = 1$

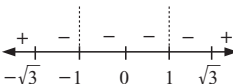
$$\begin{aligned}\therefore f'(x) &= \frac{(-2x + 4)(x - 1) - (-x^2 + 4x - 7)(1)}{(x - 1)^2} \\ &= \frac{-2x^2 + 6x - 4 + x^2 - 4x + 7}{(x - 1)^2} \\ &= \frac{-x^2 + 2x + 3}{(x - 1)^2} \\ &= \frac{-(x^2 - 2x - 3)}{(x - 1)^2} \\ &= \frac{-(x - 3)(x + 1)}{(x - 1)^2}\end{aligned}$$

Sign diagram of $\frac{dy}{dx}$: 

ii $\therefore f(x)$ is increasing for $-1 \leq x < 1$ and $1 < x \leq 3,$ and
 decreasing for $x \leq -1$ and $x \geq 3$

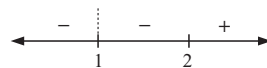
3 a $f(x) = \frac{x^3}{x^2 - 1},$ let $u = x^3, \quad u' = 3x^2, \quad v = x^2 - 1, \quad v' = 2x$

$$\begin{aligned}\therefore f'(x) &= \frac{3x^2(x^2 - 1) - x^3 \times 2x}{(x^2 - 1)^2} \\ &= \frac{3x^4 - 3x^2 - 2x^4}{(x^2 - 1)^2} \\ &= \frac{x^2(x^2 - 3)}{(x^2 - 1)^2} \\ &= \frac{x^2(x + \sqrt{3})(x - \sqrt{3})}{(x^2 - 1)^2}\end{aligned}$$

Sign diagram of $\frac{dy}{dx}$: 

$\therefore f(x)$ is increasing for $x \geq \sqrt{3}$ and $x \leq -\sqrt{3},$ and
 decreasing for $-\sqrt{3} \leq x < -1, \quad -1 < x \leq 0, \quad 0 \leq x < 1$ and $1 < x \leq \sqrt{3}.$

$$\begin{aligned}
 \text{b } f(x) &= x^2 + \frac{4}{x-1}, & f'(x) &= 2x - 4(x-1)^{-2} \times 1 \\
 &= x^2 + 4(x-1)^{-1} & &= 2x - \frac{4}{(x-1)^2} \\
 & & &= \frac{2x(x-1)^2 - 4}{(x-1)^2} \\
 & & &= \frac{2x(x^2 - 2x + 1) - 4}{(x-1)^2} \\
 & & &= \frac{2x^3 - 4x^2 + 2x - 4}{(x-1)^2} \\
 & & &= \frac{(x-2)(2x^2 + 2)}{(x-1)^2}
 \end{aligned}$$

 Sign of $\frac{dy}{dx}$:


$\therefore f(x)$ is increasing for $x \geq 2$, and decreasing for $1 < x \leq 2$ and $x < 1$.

EXERCISE 22D.3

1 a A is a local minimum, B is a local maximum, C is a horizontal inflection.

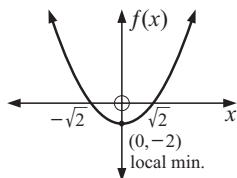
b $f'(x)$ has sign diagram:

c i $f(x)$ is increasing for $x \leq -2$ and $x \geq 3$ ii $f(x)$ is decreasing for $-2 \leq x \leq 3$.

2 a $f(x) = x^2 - 2 \therefore f'(x) = 2x$

with sign diagram:

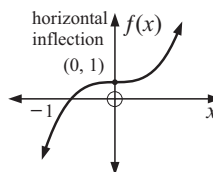
local minimum at $(0, -2)$



b $f(x) = x^3 + 1 \therefore f'(x) = 3x^2$

with sign diagram:

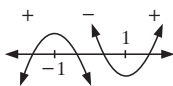
horizontal inflection at $(0, 1)$



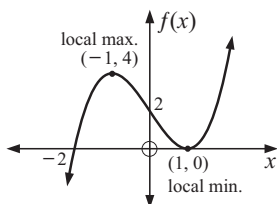
c $f(x) = x^3 - 3x + 2$

$$\begin{aligned}
 \therefore f'(x) &= 3x^2 - 3 \\
 &= 3(x^2 - 1) \\
 &= 3(x+1)(x-1)
 \end{aligned}$$

with sign diagram:



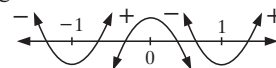
local maximum at $(-1, 4)$,
local minimum at $(1, 0)$



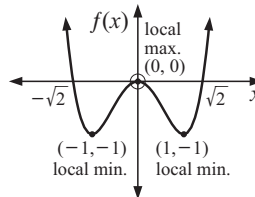
d $f(x) = x^4 - 2x^2$

$$\begin{aligned}
 \therefore f'(x) &= 4x^3 - 4x \\
 &= 4x(x^2 - 1) \\
 &= 4x(x+1)(x-1)
 \end{aligned}$$

with sign diagram:



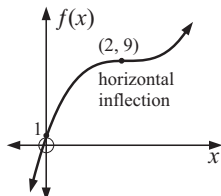
local minima at $(-1, -1)$ and $(1, -1)$,
local maximum at $(0, 0)$



e $f(x) = x^3 - 6x^2 + 12x + 1$
 $\therefore f'(x) = 3x^2 - 12x + 12$
 $= 3(x^2 - 4x + 4)$
 $= 3(x - 2)^2$

with sign diagram:

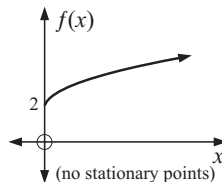
$\therefore$ horizontal inflection at (2, 9)



f $f(x) = \sqrt{x} + 2$
 $\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$
 $= \frac{1}{2\sqrt{x}} \neq 0$

with sign diagram:

$\therefore$ no stationary points.

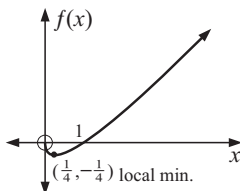


g $f(x) = x - \sqrt{x}$
 $\therefore f'(x) = 1 - \frac{1}{2}x^{-\frac{1}{2}}$
 $= 1 - \frac{1}{2\sqrt{x}}$
 $= \frac{2\sqrt{x} - 1}{2\sqrt{x}}$

with sign diagram:

$f(x)$ is defined for all $x \geq 0$

local minimum at $(\frac{1}{4}, -\frac{1}{4})$

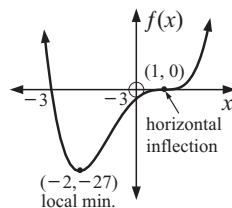


h $f(x) = x^4 - 6x^2 + 8x - 3$
 $\therefore f'(x) = 4x^3 - 12x + 8$
 $= 4(x^3 - 3x + 2)$
 $= 4(x - 1)(x^2 + x - 2)$
 $= 4(x - 1)(x + 2)(x - 1)$

with sign diagram:

local minimum at $(-2, -27)$,

horizontal inflection at (1, 0)

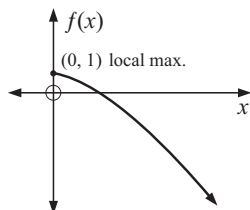


i $f(x) = 1 - x\sqrt{x} = 1 - x^{\frac{3}{2}}$
 $\therefore f'(x) = -\frac{3}{2}x^{\frac{1}{2}}$
 $= -\frac{3\sqrt{x}}{2}$

with sign diagram:

$f(x)$ is only defined when $x \geq 0$

$\therefore$ local maximum at (0, 1)

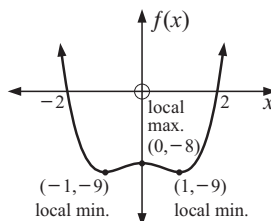


j $f(x) = x^4 - 2x^2 - 8$
 $\therefore f'(x) = 4x^3 - 4x$
 $= 4x(x^2 - 1)$
 $= 4x(x + 1)(x - 1)$

with sign diagram:

local minima at $(-1, -9)$ and $(1, -9)$,

local maximum at $(0, -8)$



$$3 \quad f(x) = ax^2 + bx + c, \quad a \neq 0$$

$$f'(x) = 2ax + b \quad \text{and } f(x) \text{ has a stationary point when } f'(x) = 0 \quad \text{i.e., } x = -\frac{b}{2a}$$

There is a local maximum when $a < 0$  and there is a local minimum when $a > 0$ .

$$4 \quad f(x) = 2x^3 + ax^2 - 24x + 1, \quad \therefore f'(x) = 6x^2 + 2ax - 24$$

$$\text{But } f'(-4) = 0, \quad \therefore 96 - 8a - 24 = 0$$

$$\therefore 72 = 8a \quad \text{and so } a = 9.$$

$$5 \quad \mathbf{a} \quad f(x) = x^3 + ax + b, \quad \therefore f'(x) = 3x^2 + a$$

$$\text{Now } 3x^2 + a = 0 \quad \text{when } x = -2$$

$$\therefore 12 + a = 0$$

$$\text{i.e., } a = -12$$

Now $(-2, 3)$ is a point on the curve.

$$\therefore f(x) = x^3 - 12x + b$$

$$\therefore 3 = -8 + 24 + b$$

$$\therefore b = -13$$

$$\mathbf{b} \quad \therefore f(x) = x^3 - 12x - 13 \quad \text{and } f'(x) = 3x^2 - 12$$

$$f'(x) = 3(x+2)(x-2)$$

where $f'(x)$ has sign diagram:



$\therefore$ there is a local maximum at $(-2, 3)$ and a local minimum at $(2, -29)$

$$6 \quad \text{Let the cubic polynomial be } P(x) = ax^3 + bx^2 + cx + d \quad \dots\dots (1)$$

$$\therefore P'(x) = 3ax^2 + 2bx + c \quad \dots\dots (2)$$

$$(0, 2) \text{ lies on } P(x) \quad \therefore P(0) = 2 \quad \text{and so } a(0) + b(0) + c(0) + d = 2$$

$$\text{i.e., } d = 2$$

$$\text{Now the tangent is } y = 9x + 2, \quad \therefore \text{slope at } (0, 2) \text{ is } 9 \quad \therefore P'(0) = 9$$

$$\therefore 3a(0) + 2b(0) + c = 9$$

$$\therefore c = 9$$

and a stationary point at $(-1, -7)$ means that

$$P'(-1) = 0$$

$$\therefore 3a(-1)^2 + 2b(-1) + c = 0 \quad \{\text{using (2)}\}$$

$$\therefore 3a - 2b + c = 0$$

$$\text{but since } c = 9$$

$$3a - 2b = -9 \quad \dots\dots (3)$$

$$\text{Now } (-1, -7) \text{ also lies on } P(x) \quad \therefore a(-1)^3 + b(-1)^2 + c(-1) + d = -7$$

$$\therefore -a + b - 9 + 2 = -7$$

$$\therefore a - b = 0 \quad \dots\dots (4)$$

$$\therefore a = b$$

$$\therefore 3a - 2a = -9 \quad \{\text{in (3)}\}$$

$$\therefore a = -9$$

$$\therefore a = b = -9$$

$$\therefore P(x) = -9x^3 - 9x^2 + 9x + 2$$

$$7 \quad \mathbf{a} \quad f(x) = x^3 - 12x - 2, \quad \text{for } -3 \leq x \leq 5$$

$$\therefore f'(x) = 3x^2 - 12$$

$$= 3(x+2)(x-2)$$

which is 0 when $x = -2$ or 2

x	-3	-2	2	5
$f(x)$	7	14	-18	63

$\therefore$ maximum value is 63 when $x = 5$, and minimum value is -18, when $x = 2$.

b $f(x) = 4 - 3x^2 + x^3$, for $-2 \leq x \leq 3$

$$\therefore f'(x) = -6x + 3x^2 \\ = 3x(x - 2)$$

which is 0 when $x = 0$ or 2

x	-2	0	2	3
$f(x)$	-16	4	0	4

$\therefore$ maximum value is 4 when $x = 0$ or $x = 3$, minimum value is -16, when $x = -2$.

8 $C(x) = 0.0007x^3 - 0.1796x^2 + 14.663x + 160$

$$C'(x) = 0.0021x^2 - 0.3592x + 14.663$$

$$C'(x) = 0 \quad \text{when} \quad 0.0021x^2 - 0.3592x + 14.663 = 0$$

Using technology, $x \doteq 103.74$ or $x \doteq 67.30$

x	50	67.30	103.74	150
$C(x)$	531.65	546.73	529.80	680.95

$\therefore$ the maximum hourly cost is \$680.95 when 150 hinges are made. The minimum hourly cost is \$529.80 when 104 hinges are made.

EXERCISE 22E.1

1 a $y = \frac{2x}{x^2 - 4} = \frac{2x}{(x+2)(x-2)}$

VAs are $x+2=0$ and $x-2=0$

i.e., $x = -2$ and $x = 2$

HA is $y = 0$ {as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

b $y = \frac{1-x}{(x+2)^2}$

VA is $x+2=0$ i.e., $x = -2$

HA is $y = 0$ {as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

c $y = \frac{3x+2}{x^2+1}$ has no VAs {as $x^2+1=0$ has no real solutions}
and a HA of $y = 0$ {as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

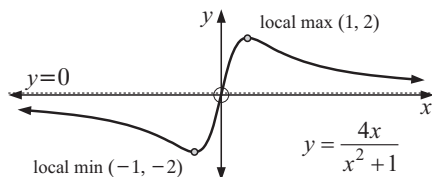
2 a i $f(x) = \frac{4x}{x^2+1}$
has no VAs {as $x^2+1=0$
has no real solutions}
and a HA of $y = 0$
{as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

iii when $x = 0$, $f(0) = 0$
 $\therefore$ y -intercept is 0

when $y = 0$, $\frac{4x}{x^2+1} = 0$

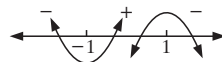
$\therefore x = 0$ and so the x -intercept is 0

iv



ii $f'(x) = \frac{4(x^2+1) - 4x(2x)}{(x^2+1)^2}$
 $= \frac{4x^2+4-8x^2}{(x^2+1)^2}$
 $= \frac{4-4x^2}{(x^2+1)^2}$
 $= \frac{4(1+x)(1-x)}{(x^2+1)^2}$

and has a sign diagram of



$\therefore$ local min. is $\left(-1, \frac{4(-1)}{1+1}\right)$
i.e., $(-1, -2)$

and local max is $\left(1, \frac{4(1)}{1+1}\right)$
i.e., $(1, 2)$

$$\mathbf{b} \quad f(x) = \frac{4x}{x^2 - 4x - 5} = \frac{4x}{(x-5)(x+1)}$$

- i** Vertical asymptotes occur when $(x-5)(x+1) = 0$,
i.e., at $x = -1$ and $x = 5$.

Horizontal asymptote is $y = 0$
{as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

- iii** When $x = 0$, $y = \frac{0}{-5} = 0$

$\therefore$ y -intercept is 0

When $y = 0$, $\frac{4x}{x^2 - 4x - 5} = 0$

$\therefore x = 0$

$\therefore$ x -intercept is 0

c i $f(x) = \frac{4x}{(x-1)^2}$

has VA $x - 1 = 0$ i.e., $x = 1$
and a HA of $y = 0$
{as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

- iii** when $x = 0$, $y = \frac{0}{1} = 0$

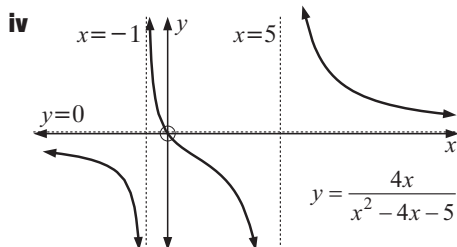
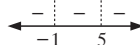
$\therefore$ y -intercept is 0

when $y = 0$, $\frac{4x}{(x-1)^2} = 0$

$\therefore$ x -intercept is 0

$$\begin{aligned} \mathbf{ii} \quad f'(x) &= \frac{4(x^2 - 4x - 5) - 4x(2x - 4)}{(x^2 - 4x - 5)^2} \\ &= \frac{4x^2 - 16x - 20 - 8x^2 + 16x}{(x^2 - 4x - 5)^2} \\ &= \frac{-4x^2 - 20}{(x-5)^2(x+1)^2} \end{aligned}$$

Since $-4x^2 - 20$ is always negative we have no turning points, and sign diagram is:

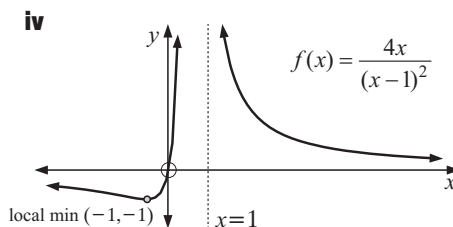


$$\begin{aligned} \mathbf{ii} \quad f'(x) &= \frac{4(x-1)^2 - 4x \times 2(x-1)1}{(x-1)^4} \\ &= \frac{4(x-1)[x-1-2x]}{(x-1)^4} \\ &= \frac{4(-x-1)}{(x-1)^3} \\ &= \frac{-4(x+1)}{(x-1)^3} \end{aligned}$$

which has sign diagram



local min. at $\left(-1, \frac{4(-1)}{(-2)^3}\right)$ i.e., at $(-1, -1)$



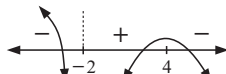
d $f(x) = \frac{3x-3}{(x+2)^2}$

i Vertical asymptotes occur when $(x+2)^2 = 0$, i.e., at $x = -2$.

HA is $y = 0$ {as $|x| \rightarrow \infty$, $y \rightarrow 0$ }

ii
$$\begin{aligned} f'(x) &= \frac{3(x+2)^2 - 2(x+2)(3x-3)}{(x+2)^4} \\ &= \frac{3(x+2) - 2(3x-3)}{(x+2)^3} \\ &= \frac{3x+6-6x+6}{(x+2)^3} \\ &= \frac{12-3x}{(x+2)^3} \\ &= \frac{-3(x-4)}{(x+2)^3} \end{aligned}$$

Sign diagram
of $f'(x)$ is:



$\therefore$ there is a local maximum at $(4, \frac{1}{4})$.

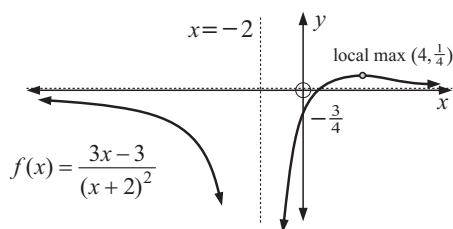
iii when $x = 0$, $y = \frac{-3}{2^2} = -\frac{3}{4}$

$\therefore$ y -intercept is $-\frac{3}{4}$

when $y = 0$, $3x - 3 = 0$
 $\therefore x = 1$

$\therefore$ x -intercept is 1

iv



EXERCISE 22E.2

1 a $y = \frac{2x^2 - x + 2}{x^2 - 1} = \frac{2x^2 - x + 2}{(x+1)(x-1)}$
has VA when $x+1 = 0$, $x-1 = 0$
i.e., $x = -1$, $x = 1$

and as $y = \frac{2 - \frac{1}{x} + \frac{2}{x^2}}{1 - \frac{1}{x^2}}$

HA is $y = 2$
{as $|x| \rightarrow \infty$, $y \rightarrow \frac{2}{1}\}$

c $y = \frac{3x^2 - x + 2}{(x+2)^2} = \frac{3x^2 - x + 2}{x^2 + 4x + 4} = \frac{3 - \frac{1}{x} + \frac{2}{x^2}}{1 + \frac{4}{x} + \frac{4}{x^2}}$

has VA $x+2 = 0$ i.e., $x = -2$ had HA $y = 3$ {as $|x| \rightarrow \infty$, $y \rightarrow \frac{3}{1}\}$

b $y = \frac{-x^2 + 2x - 1}{x^2 + x + 1}$
has a VA when $x^2 + x + 1 = 0$
But $\Delta = 1^2 - 4(1)(1) < 0 \therefore$ no real solutions $\therefore$ no VA's exist

and as $y = \frac{-1 + \frac{2}{x} - \frac{1}{x^2}}{1 + \frac{1}{x} + \frac{1}{x^2}}$

HA is $y = -1$ {as $|x| \rightarrow \infty$, $y = \frac{-1}{1}\}$

$$\mathbf{2 \quad a \quad i} \quad y = \frac{x^2 - x}{x^2 - x - 6} = \frac{x(x-1)}{(x-3)(x+2)}$$

has VA when $x - 3 = 0$, $x + 2 = 0$

i.e., $x = 3$, $x = -2$

and since $y = \frac{1 - \frac{1}{x}}{1 - \frac{1}{x} - \frac{6}{x^2}}$

the HA is $y = 1$

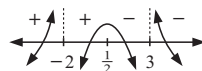
{as $|x| \rightarrow \infty$, $y \rightarrow \frac{1}{1}$ }

$$\begin{aligned} \mathbf{ii} \quad f'(x) &= \frac{(2x-1)(x^2-x-6) - (x^2-x)(2x-1)}{(x^2-x-6)^2} \\ &= \frac{(2x-1)(x^2-x-6-x^2+x)}{(x^2-x-6)^2} \\ &= \frac{-6(2x-1)}{(x^2-x-6)^2} \end{aligned}$$

Turning points are when

$f'(x) = 0$, i.e., when $x = \frac{1}{2}$.

Sign diagram of $f'(x)$ is:



$\therefore$ there is a local maximum at $(\frac{1}{2}, \frac{1}{25})$.

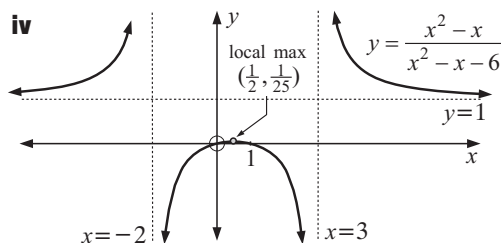
$$\mathbf{iii} \quad \text{When } x = 0, \quad y = \frac{0}{-6} = 0$$

$\therefore$ y -intercept is 0

When $y = 0$, $x(x-1) = 0$

$\therefore$ $x = 0$ or 1

$\therefore$ x -intercepts are 0 and 1



$$\mathbf{b \quad i} \quad y = \frac{x^2 - 1}{x^2 + 1} = \frac{(x+1)(x-1)}{x^2 + 1}$$

has no VA as $x^2 + 1 = 0$ has no real solutions and since

$$y = \frac{1 - \frac{1}{x^2}}{1 + \frac{1}{x^2}} \quad \text{the HA is } y = 1$$

{as $|x| \rightarrow \infty$, $y \rightarrow \frac{1}{1}$ }

$$\begin{aligned} \mathbf{ii} \quad f'(x) &= \frac{2x(x^2+1) - (x^2-1)2x}{(x^2+1)^2} \\ &= \frac{2x[x^2+1-x^2+1]}{(x^2+1)^2} \\ &= \frac{4x}{(x^2+1)^2} \end{aligned}$$

and has sign diagram



$\therefore$ a local minimum of $(0, -1)$.

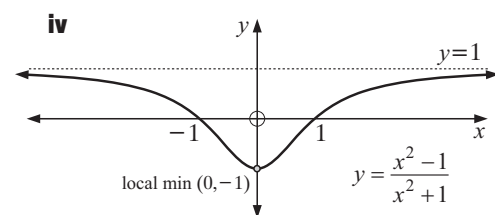
$$\mathbf{iii} \quad \text{When } x = 0, \quad y = \frac{-1}{1} = -1$$

$\therefore$ y -intercept is -1

When $y = 0$, $(x+1)(x-1) = 0$

$\therefore$ $x = \pm 1$

$\therefore$ x -intercepts are ± 1



$$\mathbf{c \quad i} \quad y = \frac{x^2 - 5x + 4}{x^2 + 5x + 4} = \frac{(x-1)(x-4)}{(x+1)(x+4)}$$

has VAs of $x + 1 = 0$ and $x + 4 = 0$
i.e., $x = -1$, $x = -4$

and as $y = \frac{1 - \frac{5}{x} + \frac{4}{x^2}}{1 + \frac{5}{x} + \frac{4}{x^2}}$ the HA is $y = 1$ {as $|x| \rightarrow \infty$, $y \rightarrow \frac{1}{1}$ }

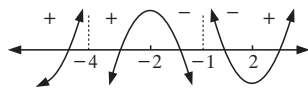
$$\begin{aligned} \mathbf{ii} \quad \frac{dy}{dx} &= \frac{(2x-5)(x^2+5x+4) - (x^2-5x+4)(2x+5)}{(x+1)^2(x+4)^2} \\ &= \frac{[2x^3+10x^2+8x-5x^2-25x-20] - [2x^3-10x^2+8x+5x^2-25x+20]}{(x+1)^2(x+4)^2} \end{aligned}$$

$$= \frac{10x^2 - 40}{(x+1)^2(x+4)^2}$$

$$= \frac{10(x+2)(x-2)}{(x+1)^2(x+4)^2} \quad \text{which has sign diagram}$$

$$\therefore \text{ local max at } \left(-2, \frac{4+10+4}{4-10+4}\right) \quad \text{i.e., at } (-2, -9)$$

$$\text{local min at } \left(2, \frac{4-10+4}{4+10+4}\right) \quad \text{i.e., at } \left(2, -\frac{1}{9}\right)$$

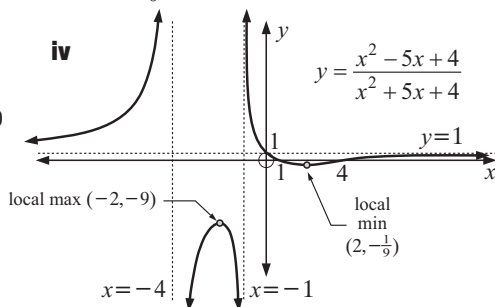


iii When $x = 0$, $y = \frac{4}{4} = 1$
 $\therefore$ the y -intercept is 1

When $y = 0$, $(x-1)(x-4) = 0$

$\therefore x = 1$ or 4

$\therefore x$ -intercepts are 1, 4



d i $y = \frac{x^2 - 6x + 5}{(x+1)^2} = \frac{(x-1)(x-5)}{(x+1)^2}$

has VA $x+1=0$ i.e., $x=-1$ and HA $y=1$

{as $y = \frac{1 - \frac{6}{x} + \frac{5}{x^2}}{1 + \frac{2}{x} + \frac{1}{x^2}} \rightarrow \frac{1}{1} \quad \text{as } |x| \rightarrow \infty$ }

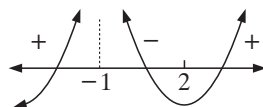
ii $\frac{dy}{dx} = \frac{(2x-6)(x+1)^2 - (x^2-6x+5)2(x+1)^1}{(x+1)^4}$

$$= \frac{(x+1)[2x^2 - 4x - 6 - 2x^2 + 12x - 10]}{(x+1)^4}$$

$$= \frac{8x - 16}{(x+1)^3}$$

$$= \frac{8(x-2)}{(x+1)^3}$$

and has sign diagram



$\therefore$ local minimum at $\left(2, \frac{4-12+5}{3^2}\right)$ i.e., $\left(2, -\frac{1}{3}\right)$

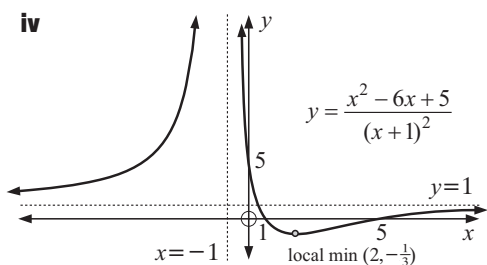
iii When $x = 0$, $y = \frac{5}{1} = 5$

$\therefore y$ -intercept is 5

When $y = 0$, $(x-1)(x-5) = 0$

$\therefore x = 1$ or 5

$\therefore x$ -intercepts are 1, 5



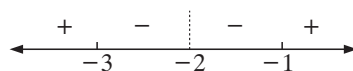
EXERCISE 22E.3

1 a i $y = \frac{x^2 + 4x + 5}{x+2} = \frac{(x+2)^2 + 1}{x+2} = x+2 + \frac{1}{x+2}$

$\therefore$ there is a vertical asymptote of $x = -2$ {as $x \rightarrow -2$, $|y| \rightarrow \infty$ } and an oblique asymptote of $y = x+2$ {as $|x| \rightarrow \infty$, $y \rightarrow x+2$ }

$$\begin{aligned}
 \text{ii } \frac{dy}{dx} &= 1 - \frac{1}{(x+2)^2} \\
 &= \frac{(x+2)^2 - 1}{(x+2)^2} \\
 &= \frac{x^2 + 4x + 3}{(x+2)^2} \\
 &= \frac{(x+1)(x+3)}{(x+2)^2}
 \end{aligned}$$

which has sign diagram



$\therefore$ there is a local maximum at $(-3, -2)$
and a local minimum at $(-1, 2)$

iii When $y = 0$, $x^2 + 4x + 5 = 0$

$$\Delta = 4^2 - 4 \times 5 = -4$$

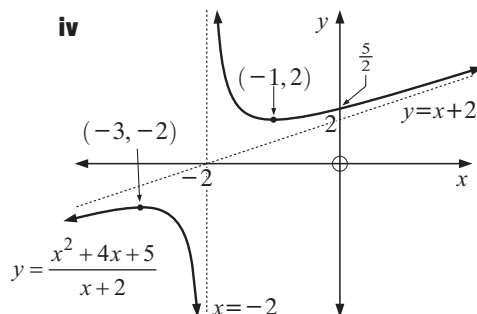
$\therefore$ there are no real roots.

So there are no x -intercepts.

When $x = 0$, $y = \frac{5}{2}$

so the y -intercept is $\frac{5}{2}$

iv



b i $y = \frac{x^2 + 3x}{x + 1} = \frac{(x+1)(x+2) - 2}{x + 1} = x + 2 - \frac{2}{x + 1}$

$\therefore$ there is a vertical asymptote of $x = -1$ {as $x \rightarrow -1$, $|y| \rightarrow \infty$ }

and an oblique asymptote of $y = x + 2$ {as $|x| \rightarrow \infty$, $y \rightarrow x + 2$ }

ii $\frac{dy}{dx} = 1 + \frac{2}{(x+1)^2}$

$$= \frac{(x+1)^2 + 2}{(x+1)^2}$$

$$= \frac{x^2 + 2x + 3}{(x+1)^2}$$

Now $x^2 + 2x + 3$ has $\Delta = 2^2 - 4 \times 3 = -8$

So $\frac{dy}{dx}$ is never zero.

$\therefore$ there are no turning points.

iii When $y = 0$, $x^2 + 3x = 0$

$$\therefore x(x+3) = 0$$

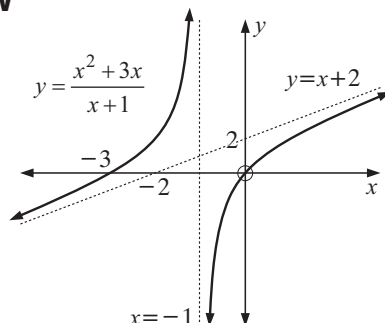
$$\therefore x = -3 \text{ or } 0$$

i.e., there are x -intercepts -3 and 0

When $x = 0$, $y = 0$

so the y -intercept is 0

iv



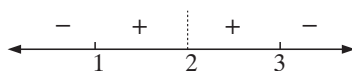
c i $y = -2x + 1 - \frac{2}{x-2}$

has a vertical asymptote of $x = 2$ {as $x \rightarrow 2$, $|y| \rightarrow \infty$ }

and an oblique asymptote of $y = -2x + 1$ {as $|x| \rightarrow \infty$, $y \rightarrow -2x + 1$ }

$$\begin{aligned}
 \text{ii } \frac{dy}{dx} &= -2 + \frac{2}{(x-2)^2} \\
 &= \frac{-2(x-2)^2 + 2}{(x-2)^2} \\
 &= \frac{-2x^2 + 8x - 6}{(x-2)^2} \\
 &= \frac{-2(x^2 - 4x + 3)}{(x-2)^2} \\
 &= \frac{-2(x-3)(x-1)}{(x-2)^2}
 \end{aligned}$$

which has sign diagram



$\therefore$ there is a local minimum at $(1, 1)$
and a local maximum at $(3, -7)$

$$\text{iii When } y = 0, \quad -2x + 1 - \frac{2}{x-2} = 0$$

$$\therefore \frac{(-2x+1)(x-2) - 2}{x-2} = 0$$

$$\therefore \frac{-2x^2 + 4x + x - 2 - 2}{x-2} = 0$$

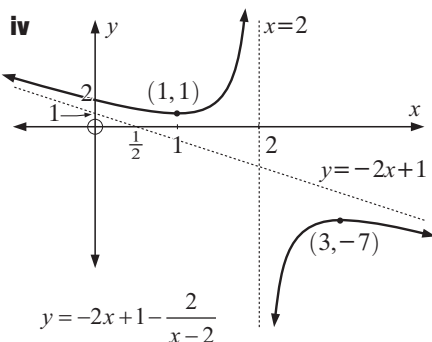
$$\therefore \frac{-2x^2 + 5x - 4}{x-2} = 0$$

$$\therefore -2x^2 + 5x - 4 = 0$$

$$\therefore \Delta = 5^2 - 4(-2)(-4) < 0$$

$\therefore$ there are no real roots, so there are no x -intercepts.

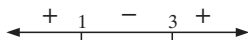
When $x = 0$, $y = -2(0) + 1 - \frac{2}{0-2} = 2$ so the y -intercept is 2.



EXERCISE 22F

- 1 a** $f(x) = x^2 + 3$
 $\therefore f'(x) = 2x$ and $f''(x) = 2$
 Since $f''(x) \neq 0$,
 no points of inflection exist.

c $f(x) = x^3 - 6x^2 + 9x + 1$
 $\therefore f'(x) = 3x^2 - 12x + 9$
 $= 3(x^2 - 4x + 3)$
 $= 3(x-3)(x-1)$

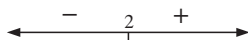


and $f''(x) = 6x - 12$
 $= 6(x-2)$

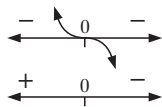
Now $f''(x) = 0$ when $x = 2$

and $f'(2) \neq 0$

$\therefore$ there is a non-horizontal inflection at $(2, 3)$

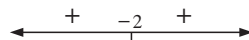


- b** $f(x) = 2 - x^3$
 $\therefore f'(x) = -3x^2$
 and $f''(x) = -6x$
 Now $f''(x) = 0$ when $x = 0$
 and $f'(0) = 0$



$\therefore$ there is a horizontal inflection at $(0, 2)$

d $f(x) = x^3 + 6x^2 + 12x + 5$
 $\therefore f'(x) = 3x^2 + 12x + 12$
 $= 3(x^2 + 4x + 4)$
 $= 3(x+2)^2$

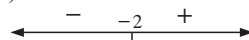


and $f''(x) = 6x + 12$
 $= 6(x+2)$

Now $f''(x) = 0$ when $x = -2$

and $f'(-2) = 0$

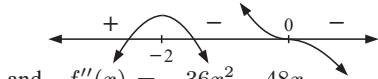
$\therefore$ there is a horizontal inflection at $(-2, -3)$



e $f(x) = -3x^4 - 8x^3 + 2$

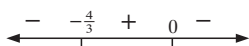
$$\therefore f'(x) = -12x^3 - 24x^2$$

$$= -12x^2(x + 2)$$



and $f''(x) = -36x^2 - 48x$

$$= -12x(3x + 4)$$



$\therefore$ horizontal inflection at $(0, 2)$,

non-horizontal inflection at $(-\frac{4}{3}, \frac{310}{27})$

f $f(x) = 3 - \frac{1}{\sqrt{x}} = 3 - x^{-\frac{1}{2}}$

$$\therefore f'(x) = \frac{1}{2}x^{-\frac{3}{2}}$$

and $f''(x) = -\frac{3}{4}x^{-\frac{5}{2}}$

$$= \frac{-3}{4x^2\sqrt{x}}$$

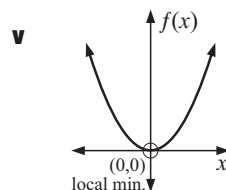
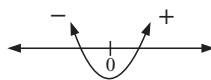
$$f''(x) \neq 0 \text{ for all } x$$

$\therefore$ no points of inflection.

2 a $f(x) = x^2$

$$\therefore f'(x) = 2x \text{ which has sign diagram:}$$

and $f''(x) = 2$



i There is a local minimum at $(0, 0)$.

ii There are no points of inflection as $f''(x) \neq 0$.

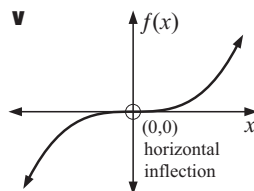
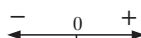
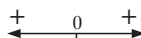
iii $f(x)$ is increasing when $x \geq 0$, and decreasing when $x \leq 0$.

iv $f(x)$ is concave up for all x as $f''(x) > 0$ for all x .

b $f(x) = x^3$

$$\therefore f'(x) = 3x^2 \text{ which has sign diagram:}$$

and $f''(x) = 6x$ which has sign diagram:



i A horizontal inflection at $(0, 0)$. $\{f'(x) = 0\}$

ii see **i**

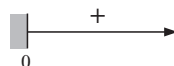
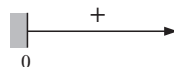
iii $f(x)$ is increasing for all x .

iv $f(x)$ is concave up $x \geq 0$, and concave down $x \leq 0$

c $f(x) = \sqrt{x}$

$$\therefore f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \text{ which has sign diagram:}$$

and $f''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = \frac{-1}{4x\sqrt{x}}$ which has sign diagram:

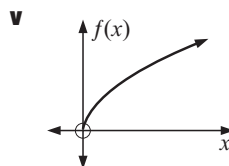


i There are no stationary points as $f'(x) \neq 0$.

ii There are no points of inflection as $f''(x) \neq 0$.

iii $f(x)$ is increasing for all $x \geq 0$.

iv $f(x)$ is concave down for all $x \geq 0$ as $f''(x) < 0$ for all $x > 0$.



d $f(x) = x^3 - 3x^2 - 24x + 1$

$$\therefore f'(x) = 3x^2 - 6x - 24$$

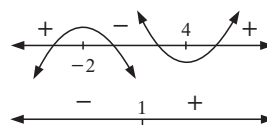
$$= 3(x^2 - 2x - 8)$$

$$= 3(x - 4)(x + 2) \text{ which has sign diagram:}$$

and $f''(x) = 6x - 6$

$$= 6(x - 1)$$

which has sign diagram:



i There is a local maximum at $(-2, 29)$, and a local minimum at $(4, -79)$.

- ii There is a non-horizontal inflection at $(1, -25)$.
- iii $f(x)$ is increasing for $x \leq -2$ or $x \geq 4$, and decreasing for $-2 \leq x \leq 4$.
- iv $f(x)$ is concave down when $x \leq 1$, and concave up when $x \geq 1$.

e $f(x) = 3x^4 + 4x^3 - 2$

$$\therefore f'(x) = 12x^3 + 12x^2 = 12x^2(x+1) \text{ which has sign diagram:}$$

$$\text{and } f''(x) = 36x^2 + 24x = 12x(3x+2) \text{ which has sign diagram:}$$

- i There is a local minimum at $(-1, -3)$, and a horizontal inflection at $(0, -2)$
- ii There is a non-horizontal inflection at $(-\frac{2}{3}, -\frac{70}{27})$ and a horizontal inflection at $(0, -2)$
- iii $f(x)$ is increasing for $x \geq -1$, and decreasing for $x \leq -1$.
- iv $f(x)$ is concave down for $-\frac{2}{3} \leq x \leq 0$, and concave up for $x \leq -\frac{2}{3}$ or $x \geq 0$.

f $f(x) = (x-1)^4$
 $\therefore f'(x) = 4(x-1)^3 \times 1 = 4(x-1)^3$ which has sign diagram:

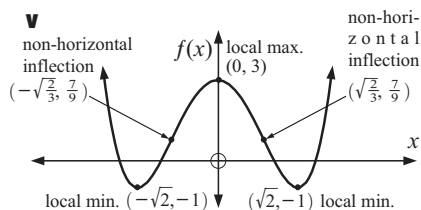
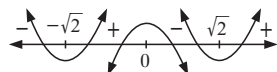
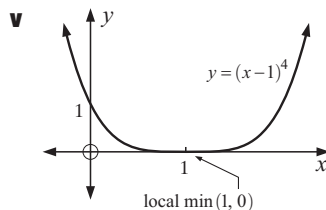
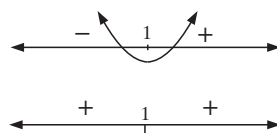
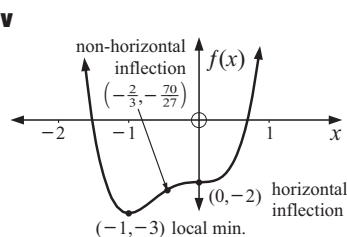
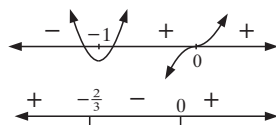
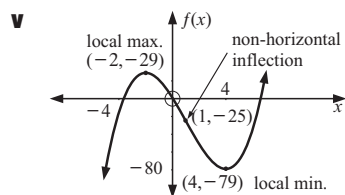
$$\text{and } f''(x) = 12(x-1)^2 \times 1 = 12(x-1)^2 \text{ which has sign diagram:}$$

- i There is a local minimum at $(1, 0)$.
- ii There are no points of inflection.
- iii $f(x)$ is increasing for $x \geq 1$, and decreasing for $x \leq 1$.
- iv $f(x)$ is concave up for all x .

g $f(x) = x^4 - 4x^2 + 3$
 $f'(x) = 4x^3 - 8x = 4x(x^2 - 2) = 4x(x + \sqrt{2})(x - \sqrt{2})$ which has sign diagram:

$$f''(x) = 12x^2 - 8 = 4(3x^2 - 2) = 4(\sqrt{3}x + \sqrt{2})(\sqrt{3}x - \sqrt{2}) \text{ which has sign diagram:}$$

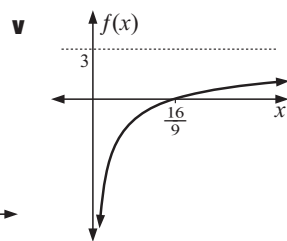
- i There is a local maximum at $(0, 3)$, and local min. at $(\sqrt{2}, -1)$ and $(-\sqrt{2}, -1)$.
- ii There are non-horizontal inflections at $(\sqrt{\frac{2}{3}}, \frac{7}{9})$ and $(-\sqrt{\frac{2}{3}}, \frac{7}{9})$. $\{f''(x) = 0\}$
- iii $f(x)$ is increasing for $-\sqrt{2} \leq x \leq 0$ and $x \geq \sqrt{2}$, and decreasing for $0 \leq x \leq \sqrt{2}$ and $x \leq -\sqrt{2}$.
- iv $f(x)$ is concave down for $-\sqrt{\frac{2}{3}} \leq x \leq \sqrt{\frac{2}{3}}$, and concave up for $x \leq -\sqrt{\frac{2}{3}}$ and $x \geq \sqrt{\frac{2}{3}}$.



$$\begin{aligned} \mathbf{h} \quad f(x) &= 3 - \frac{4}{\sqrt{x}}, \quad x > 0 \\ &= 3 - 4x^{-\frac{1}{2}} \end{aligned}$$

$$\therefore f'(x) = 2x^{-\frac{3}{2}} = \frac{2}{x\sqrt{x}} \quad \text{with sign diag: } \begin{array}{|c|c|} \hline & + \\ \hline 0 & \end{array}$$

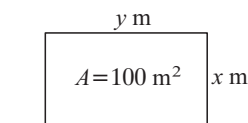
$$\text{and } f''(x) = -3x^{-\frac{5}{2}} = -\frac{3}{x^2\sqrt{x}} \quad \text{with sign diag: } \begin{array}{|c|c|} \hline & - \\ \hline 0 & \end{array}$$



- i** There are no stationary points as $f'(x) \neq 0$.
- ii** There are no points of inflection as $f''(x) \neq 0$.
- iii** $f(x)$ is increasing for all $x > 0$ as $f'(x) > 0$ for all x .
- iv** $f(x)$ is concave down for all $x > 0$ as $f''(x) < 0$ for all x .

EXERCISE 22G

1 a



$$L = 2x + y$$

$$\text{but } xy = 100$$

$$\therefore y = \frac{100}{x}$$

$$\therefore L = 2x + \frac{100}{x}$$

$$\mathbf{c} \quad \frac{dL}{dx} = 2 - 100x^{-2} = 2 - \frac{100}{x^2}$$

$$\text{which is 0 when } \frac{100}{x^2} = 2$$

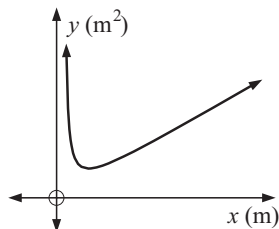
$$\therefore x^2 = 50$$

$$\therefore x = \sqrt{50} \quad \{x > 0\}$$

$$\frac{d^2L}{dx^2} = 200x^{-3} = \frac{200}{x^3} > 0 \text{ for } x > 0$$

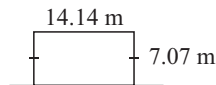
$$\therefore \min L = 28.28 \text{ m when } x = \sqrt{50} \text{ m}$$

b



$$\begin{aligned} \therefore L_{\min} &= 2\sqrt{50} + \frac{100}{\sqrt{50}} \\ &= 2\sqrt{50} + 2\sqrt{50} \\ &= 4\sqrt{50} \\ &= 20\sqrt{2} \text{ m when } x = 5\sqrt{2} \text{ m} \end{aligned}$$

d



2 a Inner length of box = $2x$ cm

b

$$\text{Volume} = 200 \text{ cm}^3$$

$$\therefore x \times 2x \times h = 200$$

$$2x^2h = 200$$

$$\therefore x^2h = 100 \quad \dots\dots (1)$$

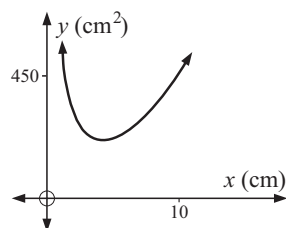
$$\mathbf{c} \quad \text{From (1) } h = \frac{100}{x^2}.$$

Now area of inner surface is

$$\begin{aligned} A(x) &= 2(2x \times x) + 2(2x \times h) + 2(x \times h) \\ &= 4x^2 + 4xh + 2xh \\ &= 4x^2 + 6xh \end{aligned}$$

$$\text{i.e., } A(x) = 4x^2 + \frac{600}{x} \text{ cm}^2$$

d



$$\begin{aligned} \mathbf{e} \quad A(x) &= 4x^2 + 600x^{-1} \\ \therefore A'(x) &= 8x - 600x^{-2} \\ &= 8x - \frac{600}{x^2} \end{aligned}$$

$$\therefore A'(x) = 0 \quad \text{when}$$

$$8x = \frac{600}{x^2}$$

$$8x^3 = 600$$

$$x^3 = 75$$

$$x = \sqrt[3]{75}$$

$$x \doteq 4.127 \text{ cm}$$

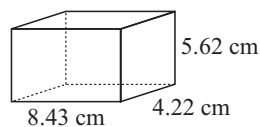
$$A''(x) = 8 + 1200x^{-3}$$

$$= 8 + \frac{1200}{x^3}$$

$$\therefore A''(x) > 0 \quad \{\text{as } x > 0\}$$

$$\therefore \text{minimum when } x \doteq 4.22 \text{ cm}$$

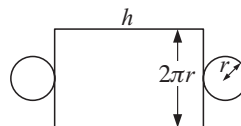
$$\begin{aligned} \therefore A_{\min} &= 4(4.217)^2 + \frac{600}{(4.217)} \\ &\doteq 213.41 \text{ cm}^2 \end{aligned}$$

f

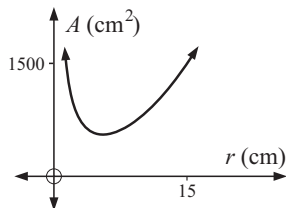
$$\mathbf{3} \quad \mathbf{a} \quad \text{Volume of can} = \pi r^2 h$$

$$\therefore 1000 = \pi r^2 h \quad (\text{in cm})$$

$$\therefore h = \frac{1000}{\pi r^2} \text{ cm}$$

b Opening the can up we get

$$\begin{aligned} \therefore A(r) &= \pi r^2 + \pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + \frac{2000}{r} \text{ cm}^2 \end{aligned}$$

c**d**

$$A(r) = 2\pi r^2 + 2000r^{-1}$$

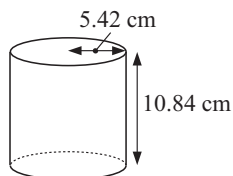
$$A'(r) = 4\pi r - 2000r^{-2} = 4\pi r - \frac{2000}{r^2}$$

$$\text{So, } A'(r) = 0 \quad \text{when} \quad 4\pi r = \frac{2000}{r^2}$$

$$r^3 = \frac{2000}{4\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}}$$

$$\therefore r \doteq 5.419 \text{ cm}$$

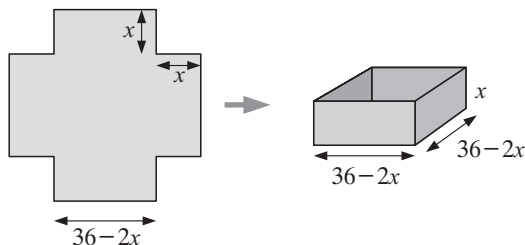
e

$$A''(r) = 4\pi + 4000r^{-3} = 4\pi + \frac{4000}{r^3}$$

$$\text{and as } r > 0, \quad A''(r) > 0$$

$$\therefore \text{area is a minimum when } r \doteq 5.42 \text{ cm}$$

$$\text{and } h = \frac{1000}{\pi r^2} \doteq 10.84 \text{ cm}$$

4

Now volume of container is $V = lbd$
 $= x(36 - 2x)(36 - 2x)$
 $\therefore V = x(36 - 2x)^2 \text{ cm}^3$

let $u = x$, $u' = 1$, $v = (36 - 2x)^2$, $v' = 2(36 - 2x)(-2)$

$$\begin{aligned}\therefore V'(x) &= (36 - 2x)^2 - 4x(36 - 2x) \\ &= (36 - 2x)[(36 - 2x) - 4x] \\ &= (36 - 2x)(36 - 6x)\end{aligned}$$

$\therefore V'(x) = 0$ when $x = 6$ or $x = 18$

Sign diagram of $V'(x)$ is:



$\therefore$ volume is maximised when $x = 6 \text{ cm}$

$\therefore$ cut out $6 \text{ cm} \times 6 \text{ cm}$ squares.

5 a $P = 2\pi r + 2l$
 $\therefore 400 = 2\pi(x) + 2l$
 $\therefore 200 = \pi x + l$
 $\therefore l = 200 - \pi x$

Now $0 \leq l \leq 200$

and $l > 0$ means that

$$\pi x < 200$$

$$\therefore x < \frac{200}{\pi}$$

and $x > 0$

$$\therefore 0 \leq x \leq \frac{200}{\pi}$$

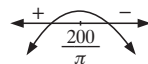
b Area, $A = \pi r^2 + (2x) \times l$
 $= \pi x^2 + 2xl$
 $= \pi x^2 + 2x(200 - \pi x)$
 $= \pi x^2 + 400x - 2\pi x^2$
 $\therefore A = 400x - \pi x^2$

c Now $\frac{dA}{dx} = 400 - 2\pi x$

which is 0 when $2\pi x = 400$

$$x = \frac{200}{\pi} \text{ and } l = 0$$

Sign diagram of $A'(x)$:



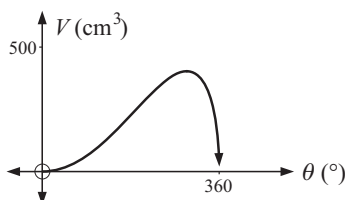
$\therefore$ area will be a maximum when the track is a circle.

6 a Arc AC $= \frac{\theta}{360} \times (2\pi r)$
 $= \frac{\theta}{360} (2 \times \pi \times 10)$
 Arc AC $= \frac{\pi\theta}{18}$

c Height of cone $= \sqrt{10^2 - r^2}$ {Pythagoras}

$$\therefore h = \sqrt{100 - \left(\frac{\theta}{36}\right)^2}$$

e



b Now arc AC forms the base of the cone.

$$\therefore 2\pi r = \frac{\theta}{360} \times 2\pi \times 10$$

$$\therefore r = \frac{\theta}{36}$$

d $V = \frac{1}{3}\pi r^2 h$

$$\begin{aligned}V &= \frac{1}{3}\pi \left(\frac{\theta}{36}\right)^2 \sqrt{100 - \left(\frac{\theta}{36}\right)^2} \\ &= \frac{\pi\theta^2}{3 \times 36^2} \sqrt{\frac{129600 - \theta^2}{36^2}} \\ &= \frac{\pi\theta^2}{139968} \sqrt{129600 - \theta^2}\end{aligned}$$

$$\begin{aligned}
 \text{f Now } V'(\theta) &= \frac{2\pi\theta}{139\,968} (129\,600 - \theta^2)^{\frac{1}{2}} + \frac{\pi\theta^2}{139\,968} \left(\frac{1}{2}\right) (129\,600 - \theta^2)^{-\frac{1}{2}} (-2\theta) \\
 &= \frac{\pi\theta}{139\,968} \left(\frac{2\sqrt{129\,600 - \theta^2}}{1} - \frac{\theta^2}{\sqrt{129\,600 - \theta^2}} \right) \\
 &= \frac{\pi\theta}{139\,968} \left(\frac{2(129\,600 - \theta^2) - \theta^2}{\sqrt{129\,600 - \theta^2}} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{and } V'(\theta) = 0 \quad \text{when } \theta = 0 \quad \text{or} \quad 2(129\,600 - \theta^2) &= \theta^2 \\
 259\,200 - 2\theta^2 &= \theta^2
 \end{aligned}$$

$$\therefore 3\theta^2 = 259\,200$$

$$\begin{aligned}
 \text{Sign diagram of } V'(\theta) \text{ is: } & \begin{array}{c} \text{---} + \text{---} \text{---} - \text{---} \\ \text{0} \quad \quad 293.9 \quad \quad 360 \end{array} \quad \therefore \theta = \sqrt{86\,400} \quad \{\text{as } \theta > 0\} \\
 & \therefore \theta \doteq 293.9
 \end{aligned}$$

$\therefore$ maximum V occurs when $\theta = 293.9^\circ$

7 a X must lie between A and C (or at A or C).

If $x = 0$, then he rows straight to the shore and runs to C.

If $x = 6$, then he rows straight to C.

$$\therefore 0 \leq x \leq 6$$

b Now $XC = 6 - x$

$$\therefore \frac{dT}{dx} = 0$$

$\therefore$ time to row from B to X

$$= \frac{BX}{8} = \frac{\sqrt{5^2 + x^2}}{8}$$

$$\text{when } \frac{x}{8\sqrt{25 + x^2}} = \frac{1}{17}$$

and time to run from X to C

$$= \frac{XC}{17} = \frac{6 - x}{17}$$

$$\therefore 17x = 8\sqrt{25 + x^2}$$

$$\therefore 289x^2 = 64(25 + x^2)$$

$$\therefore 289x^2 = 1600 + 64x^2$$

$$\therefore 225x^2 = 1600$$

$$\therefore x^2 = \frac{1600}{225}$$

$$\therefore \text{total time } T(x) = \frac{\sqrt{25 + x^2}}{8} + \frac{6 - x}{17} \text{ hours}$$

$$\therefore x = \frac{40}{15} = \frac{8}{3} \text{ km}$$

$$= \frac{1}{8}(25 + x^2)^{\frac{1}{2}} + \frac{6}{17} - \frac{x}{17}$$

$$\text{c Now } \frac{dT}{dx} = \frac{1}{16}(25 + x^2)^{-\frac{1}{2}}(2x) - \frac{1}{17}$$

$$\therefore \frac{dT}{dx} = \frac{x}{8\sqrt{25 + x^2}} - \frac{1}{17} \quad \text{Sign diagram of } \frac{dT}{dx}: \begin{array}{c} \text{---} - \text{---} \text{---} + \text{---} \\ \quad \quad \frac{8}{3} \end{array}$$

Thus, the time taken is a minimum if Peter aims for X such that $x = \frac{8}{3}$ km.

8 Let $MX = x$ km, then $XN = 5 - x$ km

$$\therefore AX = \sqrt{4 + x^2} \text{ km} \quad \text{and} \quad XB = \sqrt{1 + (5 - x)^2} \text{ km} \quad \{\text{Pythagoras}\}$$

Now $P = AX + XB$

$$P = (4 + x^2)^{\frac{1}{2}} + (26 - 10x + x^2)^{\frac{1}{2}}$$

$$\therefore \frac{dP}{dx} = \frac{1}{2}(4 + x^2)^{-\frac{1}{2}}(2x) + \frac{1}{2}(26 - 10x + x^2)^{-\frac{1}{2}}(2x - 10)$$

$$= \frac{x}{\sqrt{4 + x^2}} + \frac{x - 5}{\sqrt{x^2 - 10x + 26}}$$

$$\text{Thus, } \frac{dP}{dx} = 0 \quad \text{when} \quad \frac{x}{\sqrt{4 + x^2}} = \frac{5 - x}{\sqrt{x^2 - 10x + 26}}$$

$$\therefore \frac{x^2}{4 + x^2} = \frac{(5 - x)^2}{x^2 - 10x + 26} \quad \{\text{squaring both sides}\}$$

$$\therefore x^2(x^2 - 10x + 26) = (4 + x^2)(25 - 10x + x^2)$$

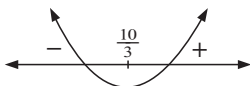
$$\therefore x^4 - 10x^3 + 26x^2 = 100 - 40x + 4x^2 + 25x^2 - 10x^3 + x^4$$

$$\therefore 3x^2 - 40x + 100 = 0$$

$$\therefore (3x - 10)(x - 10) = 0$$

$$\therefore x = \frac{10}{3} \quad \{\text{as } x \text{ cannot be } 10\}$$

Sign diagram of $\frac{dP}{dx}$ is:



$\therefore$ minimum length pipeline occurs when $x = \frac{10}{3}$ km

9

$$V = \pi r^2 h$$

$$\therefore 0.1 = \pi r^2 h \quad \{\text{as } 100 \text{ L} = 0.1 \text{ m}^3\}$$

$$\therefore h = \frac{0.1}{\pi r^2}$$

$$\text{Now } A = \pi r^2 + (2\pi r)h = \pi r^2 + 2\pi r \left(\frac{0.1}{\pi r^2} \right)$$

$$\text{i.e., } A(r) = \pi r^2 + 0.2r^{-1}$$

$$\therefore A'(r) = 2\pi r - 0.2r^{-2} = 2\pi r - \frac{0.2}{r^2}$$

$$\therefore A'(r) = 0 \quad \text{when } 2\pi r = \frac{0.2}{r^2}$$

$$r^3 = \frac{0.2}{2\pi}$$

$$r = \sqrt[3]{\frac{0.2}{2\pi}} \doteq 0.3169 \text{ m}$$

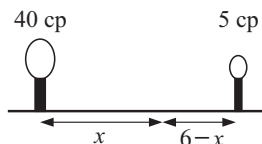
$$r \doteq 31.7 \text{ cm}$$

$$\text{Now } A''(r) = 2\pi + 0.4r^{-3} = 2\pi + \frac{0.4}{r^3} \text{ which is } > 0 \text{ as } r > 0$$

$$\therefore \text{minimum area occurs when } r \doteq 31.7 \text{ cm} \quad \text{and} \quad h \doteq \frac{1}{10\pi(31.69)} \doteq 31.7 \text{ cm}$$

$$\therefore r = h \doteq 31.7 \text{ cm}$$

10



$I \propto \frac{s}{d^2}$ where s is the power of the source and d is the distance from it

$$\therefore I = \frac{kS}{d^2} \quad \{k \text{ is a constant}\}$$

$$\therefore \text{intensity due to } 40 \text{ cp} = \frac{40k}{x^2}$$

$$\text{and intensity due to } 5 \text{ cp} = \frac{5k}{(6-x)^2}$$

$$\therefore \text{total intensity, } I = \frac{40k}{x^2} + \frac{5k}{(6-x)^2}$$

$$= k[40x^{-2} + 5(6-x)^{-2}]$$

$$\text{Now } \frac{dI}{dx} = k[-80x^{-3} - 10(6-x)^{-3}(-1)]$$

$$= k \left[\frac{-80}{x^3} + \frac{10}{(6-x)^3} \right]$$

$$\therefore \frac{dI}{dx} = 0 \quad \text{when} \quad \frac{80}{x^3} = \frac{10}{(6-x)^3}$$

$$\text{i.e., } 8(6-x)^3 = x^3$$

$$\therefore 2(6-x) = x \quad \{\text{finding cube roots}\}$$

$$\therefore 12 - 2x = x$$

$$\therefore 12 = 3x$$

$$\therefore x = 4$$

Sign diagram of $\frac{dI}{dx}$ is: 

$\therefore$ the minimum intensity, i.e., the darkest point occurs when $x = 4$ m i.e., at 4 m from the 40 cp lamp.

11 a $AB = x$ m

$$\therefore BC = (24 - x) \text{ m}$$

$$\therefore AC^2 = AB^2 + BC^2 \quad \{\text{Pythagoras}\}$$

$$= x^2 + (24 - x)^2$$

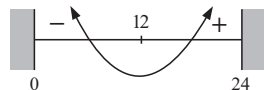
$$= x^2 + 576 - 48x + x^2$$

$$= 2x^2 - 48x + 576$$

$$\text{i.e., } [D(x)]^2 = 2x^2 - 48x + 576 \quad \text{and so } D(x) = \sqrt{2x^2 - 48x + 576}$$

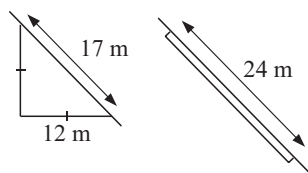
b $\therefore \frac{d[D(x)]^2}{dx} = 4x - 48$

$$\therefore \frac{d[D(x)]^2}{dx} = 0 \quad \text{when } x = 12 \quad \text{and the sign diagram is:}$$



c i.e., when $AB = BC = 12$ m, $D(x)$ is a minimum, and minimum $D(x) = 12\sqrt{2}$ m $\doteq 16.97$ m.

$D(x)$ is a maximum when either $x = 0$ or $x = 24$,
i.e., when the pen ceases to exist and $D(x) = 24$ m.



12 a Consider each boat's position t hours after 1.00 pm.

$$AP = 12t \quad BQ = 8t$$

$$\therefore PB = 100 - 8t$$

Using the Cosine rule in $\triangle PAB$

$$D(t)^2 = AP^2 + BP^2 - 2AP \times BP \cos 60^\circ$$

$$= (12t)^2 + (100 - 8t)^2 - 2(12t)(100 - 8t)\frac{1}{2}$$

$$= 144t^2 + (100 - 8t)^2 - 12t(100 - 8t)$$

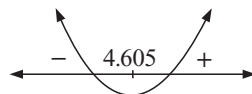
$$= 144t^2 + 10\,000 - 1600t + 64t^2 - 1200t + 96t^2$$

$$= 304t^2 - 2800t + 10\,000 \quad \text{and so } D(t) = \sqrt{304t^2 - 2800t + 10\,000}$$

b Now $\frac{d[D(t)]^2}{dt} = 608t - 2800$

$$\therefore \frac{d[D(t)]^2}{dt} = 0 \quad \text{when } t = \frac{2800}{608} \doteq 4.605\,26$$

and has sign diagram:



$\therefore D(t)$ is a minimum when $t \doteq 4.60526$ hours after 1.00 pm

$$\text{and } [D(t)]_{\min}^2 \doteq 304(4.6053)^2 - 2800(4.6053) + 10\,000$$

$$\therefore [D(t)]_{\min}^2 = 3552.63 \text{ km}^2$$

c The ships are closest when $t = 4.60526$ hours

i.e., when the time is 4 hours 36 minutes

$\therefore$ time is approximately 5.36 pm

- 13 a** $\triangle$ s PAB and PRQ are similar.

$$\therefore \frac{PA}{PR} = \frac{PB}{PQ} = \frac{AB}{RQ}$$

$$\therefore \frac{x}{x+2} = \frac{1}{QR} \quad \text{and} \quad \therefore QR = \frac{x+2}{x}$$

b Now $[L(x)]^2 = RP^2 + QR^2$ {Pythagoras}

$$= (x+2)^2 + \left(\frac{x+2}{x}\right)^2$$

$$= (x+2)^2 \times 1 + (x+2)^2 \times \frac{1}{x^2}$$

$$\therefore [L(x)]^2 = (x+2)^2 \left[1 + \frac{1}{x^2}\right] \quad \text{as required}$$

$$\therefore [L(x)]^2 = (x+2)^2 [1 + x^{-2}]$$

c

$$\frac{d[L(x)]^2}{dx} = 2(x+2)[1 + x^{-2}] + (x+2)^2[-2x^{-3}] \quad \{\text{product rule}\}$$

$$= 2(x+2)[1 + x^{-2} - (x+2)x^{-3}]$$

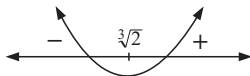
$$= 2(x+2)[1 + x^{-2} - x^{-2} - 2x^{-3}]$$

$$= 2(x+2)\left(1 - \frac{2}{x^3}\right)$$

$$= 2(x+2)\left(\frac{x^3 - 2}{x^3}\right)$$

$$\therefore \frac{d[L(x)]^2}{dx} = 0 \quad \text{when} \quad x = \sqrt[3]{2} \div 1.2599 \quad \{\text{as } x > 0 \text{ and } L(x) > 0\}$$

- d** Sign diagram of $L'(x)$ is:



$$\therefore \text{the ladder is shortest when } x = \sqrt[3]{2} \text{ m and minimum } L \div \sqrt{(x+2)^2 \left(1 + \frac{1}{x^2}\right)} \div 4.16 \text{ m}$$

- 14** Suppose $PN = x$ m

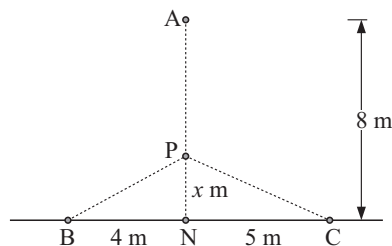
$$\therefore \text{length of cable} = PA + PB + PC$$

$$\therefore L = 8 - x + \sqrt{x^2 + 16} + \sqrt{x^2 + 25}$$

Using technology we graph this and find the minimum,

i.e., the minimum length occurs when $x = 2.57798$

i.e., $x \div 2.578$ m from N



- 15** Suppose P has coordinates $(x, 8)$

$$\therefore PC = \sqrt{(x-3)^2 + (8-11)^2} \quad AP = \sqrt{(x-1)^2 + (8-2)^2}$$

$$= \sqrt{x^2 - 6x + 9 + 9} \quad = \sqrt{x^2 - 2x + 1 + 36}$$

$$= \sqrt{x^2 - 6x + 18} \quad = \sqrt{x^2 - 2x + 37}$$

$$PB = \sqrt{(x-7)^2 + (8-3)^2}$$

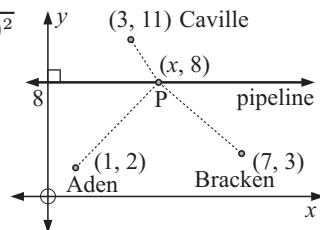
$$= \sqrt{x^2 - 14x + 49 + 25}$$

$$= \sqrt{x^2 - 14x + 74}$$

$$\therefore \text{length of pipeline } L = \sqrt{x^2 - 6x + 18} + \sqrt{x^2 - 2x + 37} + \sqrt{x^2 - 14x + 74}$$

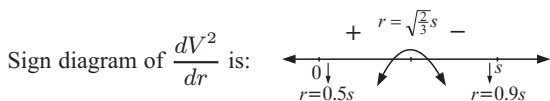
Use technology to graph L and find a minimum. This occurs when $x \div 3.54366$,

i.e., P is at $(3.544, 8)$.



$$\begin{aligned}
 16 \quad r^2 + h^2 &= s^2 \\
 \therefore h^2 &= s^2 - r^2 \\
 h &= \sqrt{s^2 - r^2} \\
 \text{But } V &= \frac{1}{3}\pi r^2 h \\
 \therefore V &= \frac{1}{3}\pi r^2 \sqrt{s^2 - r^2} \\
 \therefore V^2 &= \frac{\pi^2}{9} r^4 (s^2 - r^2) \\
 &= \frac{\pi^2}{9} (r^4 s^2 - r^6) \\
 \therefore \frac{dV^2}{dr} &= \frac{\pi^2}{9} (4r^3 s^2 - 6r^5) \\
 &= \frac{\pi^2}{9} 2r^3 (2s^2 - 3r^2)
 \end{aligned}$$

$$\begin{aligned}
 \frac{dV^2}{dr} \text{ is 0 when} \\
 2s^2 - 3r^2 &= 0 \\
 \therefore 2s^2 &= 3r^2 \quad \{\text{as } r > 0\} \\
 \therefore \frac{s^2}{r^2} &= \frac{3}{2} \\
 \frac{s}{r} &= \sqrt{\frac{3}{2}} \\
 \text{i.e., } s : r &= \sqrt{\frac{3}{2}} : 1
 \end{aligned}$$



$$\therefore V \text{ is a maximum when } s : r = \sqrt{\frac{3}{2}} : 1 = \sqrt{3} : \sqrt{2}$$

$$\begin{aligned}
 17 \quad a \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\
 \therefore x^2 b^2 + y^2 a^2 &= a^2 b^2 \quad \{\times \text{ by } a^2 b^2\} \\
 \therefore a^2 y^2 &= a^2 b^2 - x^2 b^2 \\
 \therefore y^2 &= \frac{a^2 b^2 - x^2 b^2}{a^2} \\
 \therefore y &= \pm \sqrt{b^2 - \frac{b^2}{a^2} x^2} \\
 \text{Since A lies in } Q_1, y > 0 \\
 \therefore y &= \sqrt{b^2 - \frac{b^2}{a^2} x^2} \\
 \therefore y &= \frac{b}{a} \sqrt{a^2 - x^2}
 \end{aligned}$$

$$\begin{aligned}
 b \quad \text{Seating area is} \\
 A &= 2x \times 2y \\
 &= 4xy \\
 &= 4x \left[\frac{b}{a} \sqrt{a^2 - x^2} \right] \\
 \therefore A(x) &= \frac{4bx}{a} \sqrt{a^2 - x^2} \quad \text{as required} \\
 \therefore A^2 &= \frac{16b^2 x^2}{a^2} (a^2 - x^2) \\
 &= \frac{16b^2}{a^2} (a^2 x^2 - x^4)
 \end{aligned}$$

$$c \quad \frac{dA^2}{dx} = \frac{16b^2}{a^2} [2a^2 x - 4x^3]$$

which is 0 when

$$\begin{aligned}
 2a^2 x - 4x^3 &= 0 \\
 \therefore 2x(a^2 - 2x^2) &= 0 \\
 \therefore 2x^2 &= a^2 \quad \{\text{as } x > 0\} \\
 \therefore x &= \pm \frac{a}{\sqrt{2}} \\
 \therefore x &= \frac{a}{\sqrt{2}} \quad \{\text{as } x \text{ is in } Q_1\}
 \end{aligned}$$

[Note that A is a maximum when A^2 is a maximum.]

$$e \quad \% \text{ occupied} = \frac{2ab}{\pi ab} \times 100\% = 63.7\%$$

$$d \quad \text{Sign diagram of } \frac{dA^2}{dx} \text{ is:}$$

$$\therefore \text{maximum area occurs when } x = \frac{a}{\sqrt{2}}$$

$$\begin{aligned}
 \text{Max. area} &= \frac{4b}{a} \frac{a}{\sqrt{2}} \sqrt{a^2 - \left(\frac{a}{\sqrt{2}}\right)^2} \\
 &= \frac{4b}{\sqrt{2}} \times \sqrt{\frac{a^2}{2}} \\
 &= \frac{4b}{\sqrt{2}} \times \frac{a}{\sqrt{2}} \\
 &= 2ab
 \end{aligned}$$

EXERCISE 22H

1 $C(x) = 38\,000 + 250x + x^2$

a Cost = $C(800)$
 $= 38\,000 + 250(800) + (800)^2$
 $= \$878\,000$

Average cost = $\frac{\$878\,000}{800}$
 $= \$1097.50$

Marginal cost = $C'(x)$
 $= 250 + 2x$
 $\therefore C'(800) = 250 + 2(800)$
 $= \$1850$

b $A(x)$ is minimised when $A(x) = C'(x)$

i.e., $\frac{C(x)}{x} = C'(x)$

$C(x) = xC'(x)$

$38\,000 + 250x + x^2 = x(250 + 2x)$

$38\,000 + 250x + x^2 = 250x + 2x^2$

$\therefore 38\,000 = x^2$

$\therefore x = \sqrt{38\,000} \quad \{\text{as } x > 0\}$

$\therefore x \div 195$

Average cost = $\frac{C(x)}{x}$
 $= \frac{C(195)}{195}$
 $= \frac{\$124\,775}{195}$
 $\div \$639.87$

i.e., A is minimised when approximately
 195 items are produced.

2 a $C(x) = 295 + 24x - 0.08x^2 + 0.0008x^3$

Average cost, $A(x) = \frac{C(x)}{x} = \frac{295}{x} + 24 - 0.08x + 0.0008x^2$

Marginal cost = $C'(x) = 24 - 0.16x + 0.0024x^2$

b Average cost is minimised when $C'(x) = A(x)$

$\therefore 24 - 0.16x + 0.0024x^2 = \frac{295}{x} + 24 - 0.08x + 0.0008x^2$

$\therefore 24x - 0.16x^2 + 0.0024x^3 = 295 + 24x - 0.08x^2 + 0.0008x^3$

$\therefore 0.0016x^3 - 0.08x^2 - 295 = 0$

Using technology, $x \div 79.311$ and so, $x > 79$ items

$\therefore$ minimum average cost = $A(79)$

$= \frac{295}{79} + 24 - 0.08(79) + 0.0008(79)$

$= \$26.41$

c $C'(x) = 24 - 0.16x + 0.0024x^2$

$C''(x) = -0.16 + 0.0048x$

which is 0 when $0.0048x = 0.16$

$\therefore x = \frac{0.16}{0.0048} \div 33.33$

i.e., $x = 33$ items

Sign diagram for $C''(x)$ is: $\leftarrow \begin{array}{c} - \quad \quad + \\ \hline 33 \end{array} \rightarrow$

$\therefore C'(x)$ is a minimum when $x = 33$ items

$\therefore$ minimum marginal cost = $C'(33) = \$21.33$

3 Suppose x fittings are produced daily.

$$\therefore C(x) = 1000 + 2x + \frac{5000}{x} = 1000 + 2x + 5000x^{-1} \text{ dollars}$$

$$\therefore C'(x) = 2 - \frac{5000}{x^2}$$

$$C'(x) = 0 \quad \text{when} \quad x^2 = 2500$$

$$\text{i.e., } x = 50 \quad \{\text{as } x > 0\}$$

$$C''(x) = 10000x^{-3}$$

$$= \frac{10000}{x^3}$$

which is > 0 when $x > 0$

$\therefore$ there is minimum cost when 50 fittings are produced.

4 $C(x) = 720 + 4x + 0.02x^2$ dollars and $p(x) = 15 - 0.002x$ dollars

$$\therefore R(x) = xp(x) = 15x - 0.002x^2 \quad \text{and} \quad R'(x) = 15 - 0.004x$$

If $C'(x) = R'(x)$ then

$$4 + 0.04x = 15 - 0.004x$$

$$\therefore 0.044x = 11$$

$$\therefore x = 250$$

And as $R''(x) = -0.004$ and $C''(x) = 0.04$ i.e., $R''(250) < C''(250)$

then maximum profit is made when 250 items are produced.

5 $C(x) = \frac{1}{4}x^2 + 8x + 200$ and $P(x) = 23 - \frac{1}{2}x$

$$\therefore R(x) = xP(x) = x(23 - \frac{1}{2}x) = 23x - \frac{1}{2}x^2 \quad \therefore R'(x) = 23 - x$$

If $C'(x) = R'(x)$ then

$$\frac{1}{2}x + 8 = 23 - x$$

$$\therefore \frac{3}{2}x = 15$$

$$\therefore x = 10$$

And as $R''(x) = -1$ and $C''(x) = \frac{1}{2}$ i.e., $R''(10) < C''(10)$

then maximum profit is made when 10 blankets/day are produced.

6 Cost/hour = running costs + other costs = $\frac{v^2}{10} + 62.5$

$$\text{Now } C(v) = \frac{\frac{v^2}{10} + 62.5}{v} \quad \left\{ \text{cost/km} = \frac{\text{cost/hour}}{\text{km/hour}} \right\}$$

$$\therefore C(v) = \frac{v}{10} + \frac{62.5}{v} = 0.1v + 62.5v^{-1}$$

$$\therefore C'(v) = 0.1 - 62.5v^{-2}$$

$$\text{which is } 0 \quad \text{when} \quad 0.1 = \frac{62.5}{v^2}$$

$$\therefore v^2 = 625$$

$$\therefore v = 25 \quad \{\text{as } v > 0\}$$

and $C''(x) = 62.5 \times 2v^{-3} = \frac{125}{v^3}$ which is > 0 when $v > 0$

$\therefore$ minimum cost/km occurs when $v = 25$ km/h

EXERCISE 22I

$$\begin{array}{lll}
 \mathbf{1} \quad \mathbf{a} \quad \frac{d}{dx}(2y) = 2 \frac{dy}{dx} & \mathbf{b} \quad \frac{d}{dx}(-3y) = -3 \frac{dy}{dx} & \mathbf{c} \quad \frac{d}{dx}(y^3) = 3y^2 \frac{dy}{dx} \\
 \mathbf{d} \quad \frac{d}{dx}\left(\frac{1}{y}\right) = \frac{d}{dx}(y^{-1}) & \mathbf{e} \quad \frac{d}{dx}(y^4) = 4y^3 \frac{dy}{dx} & \mathbf{f} \quad \frac{d}{dx}(\sqrt{y}) = \frac{d}{dx}(y^{\frac{1}{2}}) \\
 & & = \frac{1}{2}y^{-\frac{1}{2}} \frac{dy}{dx} \\
 \mathbf{g} \quad \frac{d}{dx}\left(\frac{1}{y^2}\right) = \frac{d}{dx}(y^{-2}) & \mathbf{h} \quad \frac{d}{dx}(xy) = y + x \frac{dy}{dx} & \mathbf{i} \quad \frac{d}{dx}(x^2y) = 2xy + x^2 \frac{dy}{dx} \\
 & \{\text{product rule}\} & \{\text{product rule}\} \\
 \mathbf{j} \quad \frac{d}{dx}(xy^2) = y^2 + x(2y) \frac{dy}{dx} = y^2 + 2xy \frac{dy}{dx} & \{\text{product rule}\} &
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{2} \quad \mathbf{a} \quad x^2 + y^2 = 25 \quad \text{differentiating both sides with respect to } x, \\
 2x + 2y \frac{dy}{dx} = 0 \\
 \therefore 2y \frac{dy}{dx} = -2x \\
 \text{i.e., } \frac{dy}{dx} = -\frac{2y}{2x} = -\frac{y}{x} \\
 \mathbf{b} \quad x^2 + 3y^2 = 9 \quad \text{differentiating both sides with respect to } x, \\
 2x + 6y \frac{dy}{dx} = 0 \\
 \therefore 6y \frac{dy}{dx} = -2x \\
 \text{i.e., } \frac{dy}{dx} = -\frac{2x}{6y} = -\frac{x}{3y} \\
 \mathbf{c} \quad y^2 - x^2 = 8 \quad \text{differentiating both sides with respect to } x, \\
 2y \frac{dy}{dx} - 2x = 0 \\
 \therefore 2y \frac{dy}{dx} = 2x \\
 \text{i.e., } \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y} \\
 \mathbf{d} \quad x^2 - y^3 = 10 \quad \text{differentiating both sides with respect to } x, \\
 2x - 3y^2 \frac{dy}{dx} = 0 \\
 \therefore 3y^2 \frac{dy}{dx} = 2x \\
 \text{i.e., } \frac{dy}{dx} = \frac{2x}{3y^2} \\
 \mathbf{e} \quad x^2 + xy = 4 \quad \text{differentiating both sides with respect to } x, \\
 2x + \left(y + x \frac{dy}{dx}\right) = 0 \quad \{\text{using the product rule}\} \\
 x \frac{dy}{dx} = -2x - y \quad \text{and so} \quad \frac{dy}{dx} = \frac{-2x - y}{x} \\
 \mathbf{f} \quad x^3 - 2xy = 5 \quad \text{differentiating both sides with respect to } x, \\
 3x^2 - \left[2y + 2x \frac{dy}{dx}\right] = 0 \quad \{\text{using the product rule}\} \\
 3x^2 - 2y = 2x \frac{dy}{dx} \quad \text{and so, } \frac{dy}{dx} = \frac{3x^2 - 2y}{2x}
 \end{array}$$

$$\begin{array}{l}
 \mathbf{3} \quad \mathbf{a} \quad x + y^3 = 4y. \quad \text{Differentiating both sides of the equation with respect to } x, \\
 1 + 3y^2 \frac{dy}{dx} = 4 \frac{dy}{dx} \\
 \therefore \text{ at } y = 1, \quad 1 + 3 \frac{dy}{dx} = 4 \frac{dy}{dx} \\
 \therefore \frac{dy}{dx} = 1 \quad \text{i.e., the slope of the tangent is 1.}
 \end{array}$$

b $x + y = 8xy$. Now when $x = \frac{1}{2}$, $\frac{1}{2} + y = 4y \therefore y = \frac{1}{6}$

Thus the point of contact is $(\frac{1}{2}, \frac{1}{6})$

Differentiating both sides of $x + y = 8xy$ with respect to x ,

$$1 + \frac{dy}{dx} = 8y + 8x \frac{dy}{dx}$$

$$\therefore 1 + \frac{dy}{dx} = 8(\frac{1}{6}) + 8(\frac{1}{2}) \frac{dy}{dx} \quad \text{at } (\frac{1}{2}, \frac{1}{6})$$

$$\therefore 1 + \frac{dy}{dx} = \frac{4}{3} + 4 \frac{dy}{dx}$$

$$\therefore -\frac{1}{3} = 3 \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{9} \quad \text{i.e., the slope of the tangent is } -\frac{1}{9}.$$

REVIEW SET 22A

1 a $s(t) = 2t^3 - 9t^2 + 12t - 5$ cm, $t \geq 0$

$$v(t) = 6t^2 - 18t + 12$$

$$= 6(t^2 - 3t + 2)$$

$$= 6(t-2)(t-1) \text{ cm s}^{-1}$$

b When $t = 0$, $s(0) = -5$ cm
 $v(0) = 12 \text{ cm s}^{-1}$
 $a(0) = -18 \text{ cm s}^{-2}$

c When $t = 2$, $s(2) = -1$ cm
 $v(2) = 0 \text{ cm s}^{-1}$
 $a(2) = 6 \text{ cm s}^{-2}$

i.e., when $t = 2$, the particle is 1 cm to the left of O, instantaneously at rest and increasing in speed towards O.

d Particle changes direction when $t = 1$ and $t = 2$ $s(1) = 0$ cm $s(2) = -1$ cm



f Speed is increasing when $1 \leq t \leq \frac{3}{2}$ and $t \geq 2$
 $\{v(t) \text{ and } a(t) \text{ have the same sign}\}$

2 $C(v) = 10v + \frac{90}{v}$ dollars/hour

a i $t = 2$ hours at $v = 15$ kmph
 $\therefore \text{cost} = \$ \left(10 \times 15 + \frac{90}{15} \right) \times 2$
 $= \$312.00$

ii $t = 5$ hours at $v = 24$ kmph
 $\therefore \text{cost} = \$ \left(10 \times 24 + \frac{90}{24} \right) \times 5$
 $= \$1218.75$

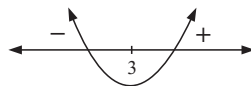
b $C'(v) = 10 - 90v^{-2} = 10 - \frac{90}{v^2}$

i If $v = 10$ kmph
 $\therefore C'(10) = 10 - \frac{90}{10^2}$
 $= 10 - 0.9$
 $= \$9.10 \text{ per kmph}$

ii If $v = 6$ kmph
 $\therefore C'(6) = 10 - \frac{90}{6^2}$
 $= \$7.50 \text{ per kmph}$

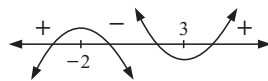
c Now $C'(v) = 10 - \frac{90}{v^2} = \frac{10v^2 - 90}{v^2}$
 $\therefore C'(v) = 0$ when $v^2 = 9$
i.e., $v = 3$ {as $v > 0$ }

$\therefore$ minimum cost occurs when $v = 3$ kmph



$$\begin{aligned}
 \mathbf{3 \quad a} \quad f(x) &= 2x^3 - 3x^2 - 36x + 7 \\
 \therefore f'(x) &= 6x^2 - 6x - 36 \\
 &= 6(x^2 - x - 6) \\
 &= 6(x - 3)(x + 2)
 \end{aligned}$$

and $f'(x)$ has sign diagram:

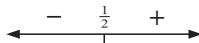


and $f(-2) = 51$, $f(3) = -74$

So there is a local maximum at $(-2, 51)$, and a local minimum at $(3, -74)$

$$\begin{aligned}
 f''(x) &= 12x - 6 \\
 &= 6(2x - 1)
 \end{aligned}$$

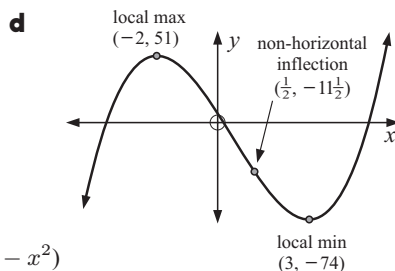
and $f''(x)$ has sign diagram:



and $f(\frac{1}{2}) = -\frac{23}{2}$ so there is a non-horizontal inflection at $(\frac{1}{2}, -\frac{23}{2})$

b $f(x)$ is increasing when $x \leq -2$ or $x \geq 3$,
and decreasing when $-2 \leq x \leq 3$.

c $f(x)$ is concave up when $x \geq \frac{1}{2}$,
and concave down when $x \leq \frac{1}{2}$.



4 a Now if $OD = x$, the coordinates of C are $(x, k - x^2)$

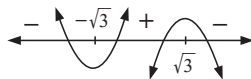
$$\begin{aligned}
 \therefore \text{area of ABCD} &= 2x \times (k - x^2) \\
 \text{i.e., } A &= 2kx - 2x^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{dA}{dx} &= 2k - 6x^2 \quad \text{and} \quad \frac{dA}{dx} = 0 \quad \text{when} \quad AD = 2\sqrt{3} \quad \text{i.e.,} \quad x = \sqrt{3} \\
 \therefore 2k - 6(\sqrt{3})^2 &= 0 \\
 \therefore 2k - 18 &= 0 \\
 \therefore 2k &= 18 \\
 \therefore k &= 9
 \end{aligned}$$

$$\begin{aligned}
 \text{Checking} \quad \frac{dA}{dx} &= 18 - 6x^2 \\
 &= 6(3 - x^2) \\
 &= 6(\sqrt{3} + x)(\sqrt{3} - x)
 \end{aligned}$$

i.e., maximum occurs at $x = \sqrt{3}$ i.e., when $AD = 2\sqrt{3}$

which has sign diagram:



5 a Volume = lbd

$$\begin{aligned}
 \therefore x^2 y &= 1 \\
 \therefore y &= \frac{1}{x^2}, \quad x > 0
 \end{aligned}$$

b area = $x^2 + 4xy$

$$\begin{aligned}
 \therefore \text{cost} &= (x^2 + 4xy) \times 2 \\
 &= 2x^2 + 8xy
 \end{aligned}$$

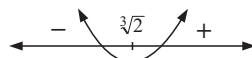
$$\text{But } y = \frac{1}{x^2}$$

$$\therefore C = 2x^2 + \frac{8}{x} \text{ dollars}$$

$$\begin{aligned}
 \mathbf{c} \quad \text{Thus } \frac{dC}{dx} &= 4x - 8x^{-2} \\
 &= 4x - \frac{8}{x^2} \\
 &= \frac{4(x^3 - 2)}{x^2}
 \end{aligned}$$

which is 0 when $x = \sqrt[3]{2}$ m

and $\frac{dC}{dx}$ has sign diagram:



$\therefore$ minimum cost when $x = \sqrt[3]{2}$ m i.e., $x = 1.26$

$$\therefore y = \frac{1}{x^2} \div 0.630$$

and the box is 1.26 m by 1.26 m by 0.630 m

- 6 a** $x^2y + 2xy^3 = -18$ Differentiating with respect to x ,

$$\frac{d}{dx}(x^2y) + \frac{d}{dx}(2xy^3) = \frac{d}{dx}(-18)$$

$$\therefore \frac{d}{dx}(x^2)y + x^2\frac{dy}{dx} + \frac{d}{dx}(2x)y^3 + 2x\frac{d}{dx}(y^3) = 0 \quad \{\text{product rule}\}$$

$$\therefore 2xy + x^2\frac{dy}{dx} + 2y^3 + 6xy^2\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx}(x^2 + 6xy^2) = -2xy - 2y^3$$

$$\therefore \frac{dy}{dx} = \frac{-2y(x + y^2)}{x(x + 6y^2)}$$

b At $(1, -2)$ $\frac{dy}{dx} = \frac{-2(-2)(1 + (-2))^2}{1(1 + 6(-2)^2)} = \frac{4 \times 5}{25} = \frac{4}{5}$

$\therefore$ the tangent has equation $\frac{y - (-2)}{x - 1} = \frac{4}{5}$ i.e., $y + 2 = \frac{4}{5}x - \frac{4}{5}$ i.e., $y = \frac{4}{5}x - \frac{14}{5}$

REVIEW SET 22B

- 1 a** $AC = 2x$ m

Now, ABC is an isosceles triangle.

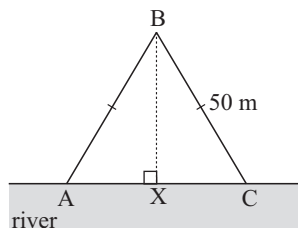
$$\therefore XC = x$$

But, $BC^2 = BX^2 + XC^2$ {Pythagoras}

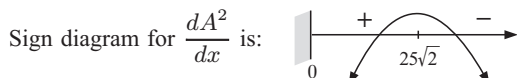
$$\therefore 2500 = BX^2 + x^2$$

$$\therefore BX = \sqrt{2500 - x^2}$$

$$\therefore A(x) = \frac{1}{2}(2x)\sqrt{2500 - x^2} = x\sqrt{2500 - x^2}$$



b Now $[A(x)]^2 = x^2(2500 - x^2)$ $\therefore \frac{dA^2}{dx} = 5000x - 4x^3$
i.e., $A^2 = 2500x^2 - x^4$ $= 4x(1250 - x^2)$
 $= 2x(\sqrt{1250} + x)(\sqrt{1250} - x)$



$\therefore$ maximum area occurs when $x = 25\sqrt{2}$ m $\div 35.4$ m

The corresponding maximum area $\div 1250$ m².

2 $s(t) = 15t - \frac{60}{(t-1)^2}$ cm, $t \geq 0$

a $\therefore s(t) = 15t - 60(t-1)^{-2}$ cm

$$\therefore v(t) = 15 + 120(t-1)^{-3} \text{ cm s}^{-1}$$

$$\therefore a(t) = -360(t-1)^{-4} \text{ cm s}^{-2}$$

b When $t = 3$, $s(t) = 30$ cm

$$v(t) = 30 \text{ cm s}^{-1}$$

$$a(t) = -22.5 \text{ cm s}^{-2}$$

The particle is 30 cm right of O, travelling right at 30 cm s^{-1} and is slowing down at 22.5 cm s^{-2}

c $v(t) = 15 + \frac{120}{(t-1)^3} \text{ cm s}^{-1}$

$$\therefore v(t) = 0$$

when $15 + \frac{120}{(t-1)^3} = 0$

$$\therefore 15(t-1)^3 + 120 = 0$$

$$\therefore (t-1)^3 = -8$$

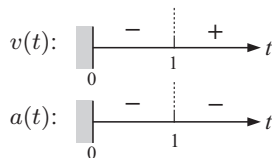
$$\therefore t = -1$$

$$a(t) = -360(t-1)^{-4} = \frac{-360}{(t-1)^4} \text{ cm s}^{-2} \quad \text{Sign diagrams:}$$

where $(t-1)^4$ is always positive. $\therefore a(t) < 0$ for all $t > 0$

So, speed increases for $0 \leq t < 1$ as

$v(t)$, $a(t)$ have the same sign.



- 3** Suppose the sheet is bent x cm from each end. To maximise the water carried we need to maximise the area of cross-section.

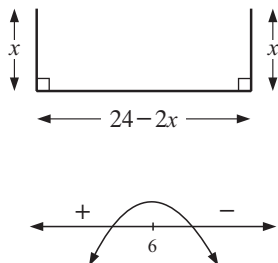
$$A = x(24 - 2x)$$

$$\therefore A = 24x - 2x^2$$

$$\text{and } \frac{dA}{dx} = 24 - 4x$$

which is 0 when $x = 6$

Sign diagram of $\frac{dA}{dx}$ is:



$\therefore$ maximum water when $x = 6$ cm

i.e., bends must be made 6 cm from each end.

- 4 a** $(2, -1)$ lies on the curve, so

$$k = 2^2 - 2(2)(-1)^2 + (-1)^3$$

$$= 4 - 4 - 1$$

$$= -1$$

- b** $x^2 - 2xy^2 + y^3 = -1$

Differentiating with respect to x ,

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(2xy^2) + \frac{d}{dx}(y^3) = \frac{d}{dx}(-1)$$

$$\therefore 2x - \frac{d}{dx}(2x)y^2 - 2x\frac{d}{dx}(y^2) + 3y^2\frac{dy}{dx} = 0$$

$$\therefore 2x - 2y^2 - 4xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$\text{c At } (2, -1), \frac{dy}{dx} = \frac{2(1-2)}{(-1)(-3-4 \times 2)} \quad \therefore \frac{dy}{dx}(3y^2 - 4xy) = 2y^2 - 2x$$

$$= -\frac{2}{11}$$

$$\therefore \frac{dy}{dx} = \frac{2(y^2 - x)}{y(3y - 4x)}$$

$\therefore$ the slope of the normal is $\frac{11}{2}$

$$\text{and its equation is } \frac{y - (-1)}{x - 2} = \frac{11}{2}$$

$$\therefore y + 1 = \frac{11}{2}x - 11$$

$$\therefore y = \frac{11}{2}x - 12$$

$$\text{or } 11x - 2y = 24$$

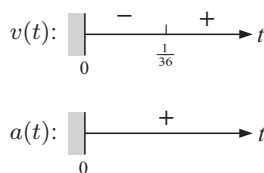
- 5** $x(t) = 3t - \sqrt{t}$ cm, $t \geq 0$

a $x(t) = 3t - t^{\frac{1}{2}}$

$$\therefore v(t) = 3 - \frac{1}{2}t^{-\frac{1}{2}} = 3 - \frac{1}{2\sqrt{t}} = \frac{6\sqrt{t} - 1}{2\sqrt{t}}$$

$$\text{which is 0 when } \sqrt{t} = \frac{1}{6}, \text{ i.e., } t = \frac{1}{36}$$

$$\text{and } a(t) = \frac{1}{4}t^{-\frac{3}{2}} = \frac{1}{4t\sqrt{t}} \text{ which is always positive}$$



- b** When $t = 0$, $x(0) = 0$ cm

$v(0)$ is infinitely large and negative, so is not defined

$a(0)$ is infinitely large and positive, so is not defined

i.e., the particle is at O, moving left and slowing down.

- c** When $t = 9$, $x(9) = 24$ cm $v(9) = \frac{17}{6}$ cm s⁻¹ $a(9) = \frac{1}{108}$ cm s⁻²
the particle is 24 cm right of O, moving right at $\frac{17}{6}$ cm s⁻¹ and increasing its speed.
- d** The particle reverses direction when $t = \frac{1}{36}$ seconds.
 $x\left(\frac{1}{36}\right) = \frac{3}{36} - \frac{1}{6} = -\frac{3}{36} = -\frac{1}{12}$, i.e., it is $\frac{1}{12} \div 0.083$ cm to the left of O.
- e** The particle's speed decreases when $v(t)$ and $a(t)$ have different signs, i.e., for $\leq t \leq \frac{1}{36}$.

6 $f(x) = \frac{x^2 - 1}{x^2 + 1}$

- a** Cuts x -axis when $y = 0$
i.e., $x^2 - 1 = 0$, $\therefore x = \pm 1$
i.e., at $(1, 0)$ and $(-1, 0)$
Cuts y -axis when $x = 0$, i.e., $(0, -1)$

- b** As $x^2 \geq 0$, $x^2 + 1$ can never be 0.
 $\therefore$ there are no vertical asymptotes.

c $f'(x) = \frac{2x(x^2 + 1) - (x^2 - 1)2x}{(x^2 + 1)^2}$
 $= \frac{4x}{(x^2 + 1)^2}$

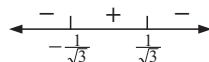
Sign diagram of $f'(x)$ is:

$\therefore$ there is a local minimum at $(0, -1)$

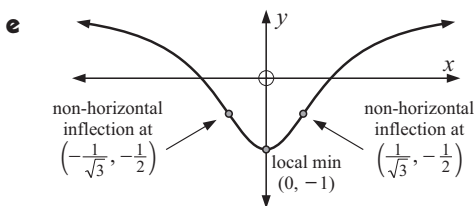
d $f''(x) = \frac{4(x^2 + 1)^2 - 4x \times 2(x^2 + 1)(2x)}{(x^2 + 1)^4}$
 $= \frac{4(x^2 + 1) - 16x^2}{(x^2 + 1)^3}$
 $= \frac{4 - 12x^2}{(x^2 + 1)^3}$
 $= \frac{4(1 + \sqrt{3}x)(1 - \sqrt{3}x)}{(x^2 + 1)^3}$

$\therefore f''(x) = 0$ when $x = \pm \frac{1}{\sqrt{3}}$

and sign diagram is:



$\therefore f(x)$ has nonstationary inflections at $x = \pm \frac{1}{\sqrt{3}}$.



REVIEW SET 22C

1 $f(x) = x^3 - 4x^2 + 4x$
 $= x(x^2 - 4x + 4)$
 $= x(x - 2)^2$

- a** Cuts the y -axis when $x = 0$ i.e., at $(0, 0)$
Cuts the x -axis when $y = 0$ i.e., $x(x - 2)^2 = 0$,
i.e., when $x = 0$ or 2
i.e., at $(0, 0)$ and $(2, 0)$

b $f'(x) = 3x^2 - 8x + 4$
 $= (3x - 2)(x - 2)$
which is 0 when $x = \frac{2}{3}$ or 2

Sign diagram of $f'(x)$ is:

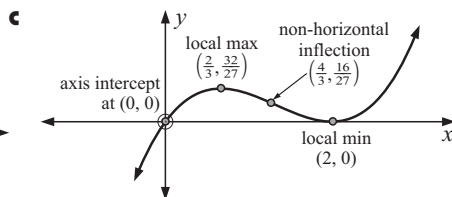
$\therefore$ there is a local maximum at $(\frac{2}{3}, \frac{32}{27})$,

and a local minimum at $(2, 0)$

$f''(x) = 6x - 8 = 2(3x - 4)$

Sign diagram of $f''(x)$ is:

$\therefore$ there is a non-horizontal inflection at $(\frac{4}{3}, \frac{16}{27})$



2 a $P = 200$ m

$$\therefore P = 2x + 2y + \pi x$$

$$\therefore 200 = 2x + 2y + \pi x$$

$$\therefore 2y = 200 - 2x - \pi x$$

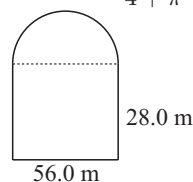
$$\therefore y = 100 - x - \frac{\pi}{2}x$$

c $\frac{dA}{dx} = 200 - 2\left(2 + \frac{\pi}{2}\right)x$

$$= 200 - (4 + \pi)x \quad \text{which is 0 when } (4 + \pi)x = 200 \quad \text{i.e., } x = \frac{200}{4 + \pi}$$

and the sign diagram for $\frac{dA}{dx}$ is:

$$\therefore \text{maximum area occurs when } x = \frac{200}{4 + \pi} \div 28.00 \text{ m}$$



3 a $f(x) = \frac{x^2 + 2x}{x - 2} = \frac{(x - 2)(x + 4) + 8}{x - 2} = x + 4 + \frac{8}{x - 2}$

$\therefore$ there is a vertical asymptote of $x = 2$ {as $x \rightarrow 2$, $|f(x)| \rightarrow \infty$ }

and an oblique asymptote of $y = x + 4$ {as $|x| \rightarrow \infty$, $f(x) \rightarrow x + 4$ }

b $f'(x) = 1 - \frac{8}{(x - 2)^2}$

$$= \frac{(x - 2)^2 - 8}{(x - 2)^2}$$

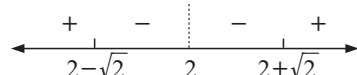
$$= \frac{x^2 - 4x - 4}{(x - 2)^2}$$

$$\therefore f'(x) = 0 \quad \text{if } x^2 - 4x - 4 = 0$$

$$\therefore x = \frac{4 \pm \sqrt{(-4)^2 - 4(-4)}}{2}$$

$$= 2 \pm 2\sqrt{2}$$

$\therefore f'(x)$ has sign diagram



$$\begin{aligned} f(2 + 2\sqrt{2}) &= 2 + 2\sqrt{2} + 4 + \frac{8}{2\sqrt{2}} \\ &= 6 + 2\sqrt{2} + 2\sqrt{2} \\ &= 6 + 4\sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{and } f(2 - 2\sqrt{2}) &= 2 - 2\sqrt{2} + 4 + \frac{8}{-2\sqrt{2}} \\ &= 6 - 2\sqrt{2} - 2\sqrt{2} \\ &= 6 - 4\sqrt{2} \end{aligned}$$

c When $f(x) = 0$, $x^2 + 2x = 0$

$$\therefore x(x + 2) = 0$$

$$\therefore x = -2 \text{ or } 0$$

$\therefore$ the x -intercepts are -2 and 0

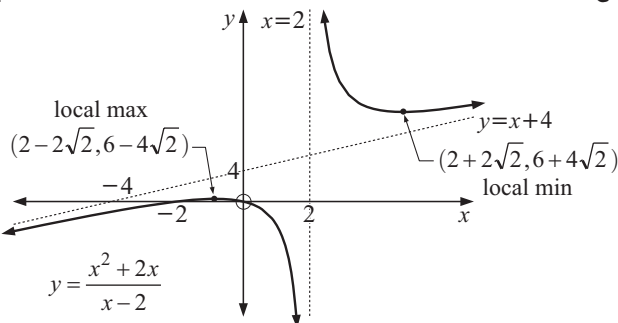
Also, $f(0) = 0$, so the y -intercept is 0

$\therefore f(x)$ has a local maximum at

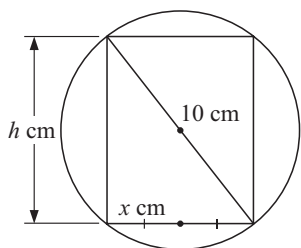
$$(2 - 2\sqrt{2}, 6 - 4\sqrt{2}) \quad \text{and}$$

a local minimum at $(2 + 2\sqrt{2}, 6 + 4\sqrt{2})$.

d



e $f(x) = p$ has two real distinct roots if $p < 6 - 4\sqrt{2}$ or $p > 6 + 4\sqrt{2}$

4 aLet the height of the cylinder be h cm.

$$\therefore (2x)^2 + h^2 = 10^2 \quad \{\text{Pythagoras}\}$$

$$\therefore h = \sqrt{100 - 4x^2}$$

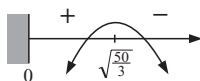
$$\therefore V(x) = \text{area of base} \times \text{height} \\ = \pi x^2 \times \sqrt{100 - 4x^2}$$

$$\text{So, } V(x) = \pi x^2 \sqrt{100 - 4x^2} \text{ cm}^2$$

$$\begin{aligned} \text{b Now } V^2 &= \pi^2 x^4 (100 - 4x^2) \\ &= \pi^2 (100x^4 - 4x^6) \end{aligned}$$

$$\begin{aligned} \therefore \frac{dV^2}{dx} &= \pi^2 (400x^3 - 24x^5) \\ &= 8\pi^2 x^3 (50 - 3x^2) \\ &= 8\pi^2 x^3 (\sqrt{50} + \sqrt{3}x)(\sqrt{50} - \sqrt{3}x) \end{aligned}$$

$$\text{which is 0 when } x = \sqrt{\frac{50}{3}} \quad \{\text{as } x > 0\}$$

and $\frac{dV^2}{dx}$ has sign diagram:

$$\therefore \text{maximum } V \text{ occurs when } x = \sqrt{\frac{50}{3}} \div 4.08$$

$$\therefore \text{radius} = 4.08 \text{ cm, height} = \sqrt{100 - 4\left(\frac{50}{3}\right)} \div 5.77 \text{ cm}$$

5 a Δs LQX and XPM are similar.

$$\therefore \frac{LQ}{XP} = \frac{LX}{XM} = \frac{QX}{PM}$$

$$\therefore \frac{LQ}{1} = \frac{8}{PM}$$

$$\therefore LQ = \frac{8}{PM}$$

$$\therefore LQ = \frac{8}{x} \text{ km}$$

b $L(x) = LX + XM$

$$= \sqrt{\left(\frac{8}{x}\right)^2 + 8^2} + \sqrt{1^2 + x^2}$$

$$= 8\sqrt{\frac{1}{x^2} + 1} + \sqrt{x^2 + 1}$$

$$= \frac{8}{x}\sqrt{1+x^2} + 1\sqrt{1+x^2}$$

$$= \sqrt{x^2 + 1} \left(\frac{8}{x} + 1 \right)$$

$$\therefore L(x) = \sqrt{x^2 + 1} \left(1 + \frac{8}{x} \right)$$

$$\text{and } L^2 = (x^2 + 1) \left(1 + \frac{8}{x} \right)^2$$

$$\text{c } \frac{d[L(x)]^2}{dx} = 2x \left(1 + \frac{8}{x} \right)^2 + (x^2 + 1) 2 \left(1 + \frac{8}{x} \right) \left(-\frac{8}{x^2} \right) \quad \{\text{product rule}\}$$

$$= 2 \left(1 + \frac{8}{x} \right) \left[x \left(1 + \frac{8}{x} \right) - (x^2 + 1) \left(\frac{8}{x^2} \right) \right]$$

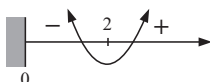
$$= 2 \left(1 + \frac{8}{x} \right) \left[x + 8 - 8 - \frac{8}{x^2} \right]$$

$$= 2 \left(\frac{x+8}{x} \right) \left(\frac{x^3 - 8}{x^2} \right)$$

$$\frac{d[L(x)]^2}{dx} = 0 \quad \text{when } x = -8 \quad \text{or } x^3 = 8$$

but $x > 0$, $\therefore$ when $x = 2$

Sign diagram for $\frac{d[L(x)]^2}{dx}$ is:



$\therefore$ minimum $L(x)$ occurs when $x = 2$ and the shortest length is $\sqrt{2^2 + 1} \left(1 + \frac{8}{2}\right)$
 $= 5\sqrt{5}$
 $\doteq 11.2 \text{ km}$

- 6** When the box is manufactured its base is $(k - 2x)$ by $(k - 2x)$ and height is x cm.

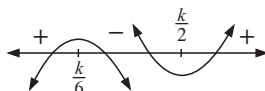
$$\therefore V = x(k - 2x)(k - 2x)$$

$$\text{i.e., } V = x(k - 2x)^2$$

$$\begin{aligned} \therefore \frac{dV}{dx} &= 1(k - 2x)^2 + x \cdot 2(k - 2x)^1(-2) \\ &= (k - 2x)^2 - 4x(k - 2x) \\ &= (k - 2x)[k - 2x - 4x] \\ &= (k - 2x)(k - 6x) \end{aligned}$$

which is 0 when $x = \frac{k}{6}$ or $\frac{k}{2}$, but $k - 2x > 0$ and so $x < \frac{k}{2}$.

and the sign diagram of $\frac{dV}{dx}$ is:



$\therefore$ maximum capacity occurs when $x = \frac{k}{6}$.

Chapter 23

DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

EXERCISE 23A

1 a $f(x) = e^{4x}$
 $\therefore f'(x) = 4e^{4x}$

b $f(x) = e^x + 3$
 $\therefore f'(x) = e^x + 0$
 $= e^x$

c $f(x) = \exp(-2x)$
 $\therefore f'(x) = e^{-2x}$
 $\therefore f'(x) = -2e^{-2x}$

d $f(x) = e^{\frac{x}{2}}$
 $\therefore f'(x) = \frac{1}{2}e^{\frac{x}{2}}$

e $f(x) = 2e^{-\frac{x}{2}}$
 $\therefore f'(x) = 2e^{-\frac{x}{2}} \left(-\frac{1}{2}\right)$
 $= -e^{-\frac{x}{2}}$

f $f(x) = 1 - 2e^{-x}$
 $\therefore f'(x) = 0 - 2e^{-x}(-1)$
 $= 2e^{-x}$

g $f(x) = 4e^{\frac{x}{2}} - 3e^{-x}$
 $\therefore f'(x) = 4e^{\frac{x}{2}} \left(\frac{1}{2}\right)$
 $- 3e^{-x}(-1)$
 $= 2e^{\frac{x}{2}} + 3e^{-x}$

h $f(x) = \frac{e^x + e^{-x}}{2}$
 $\therefore f'(x) = \frac{1}{2}(e^x + e^{-x})$
 $\therefore f'(x) = \frac{1}{2}(e^x + e^{-x}(-1))$
 $= \frac{1}{2}(e^x - e^{-x})$

i $f(x) = e^{-x^2}$
 $\therefore f'(x) = e^{-x^2}(-2x)$
 $= -2xe^{-x^2}$

j $f(x) = e^{\frac{1}{x}}$
 $\therefore f'(x) = e^{\frac{1}{x}} \left(-\frac{1}{x^2}\right)$
 $= -\frac{e^{\frac{1}{x}}}{x^2}$

k $f(x) = 10(1 + e^{2x})$
 $= 10 + 10e^{2x}$
 $\therefore f'(x) = 0 + 10e^{2x}(2)$
 $= 20e^{2x}$

l $f(x) = 20(1 - e^{-2x})$
 $= 20 - 20e^{-2x}$
 $\therefore f'(x) = 0 - 20e^{-2x}(-2)$
 $= 40e^{-2x}$

m $f(x) = e^{2x+1}$
 $\therefore f'(x) = e^{2x+1} \times 2$
 $= 2e^{2x+1}$

n $y = e^{\frac{x}{4}}$
 $\therefore \frac{dy}{dx} = e^{\frac{x}{4}} \times \frac{1}{4}$
 $= \frac{1}{4}e^{\frac{x}{4}}$

o $y = e^{1-2x^2}$
 $\therefore \frac{dy}{dx} = e^{1-2x^2}(-4x)$
 $= -4xe^{1-2x^2}$

p $y = e^{-0.02x} \quad \therefore \frac{dy}{dx} = e^{-0.02x} \times (-0.02) = -0.02e^{-0.02x}$

2 a $f(x) = xe^x$
 $u = x, v = e^x, u' = 1, v' = e^x$
 $\therefore f'(x) = 1e^x + e^xx$
 $= e^x + xe^x$

b $f(x) = x^3e^{-x}$
 $u = x^3, v = e^{-x}, u' = 3x^2, v' = -e^{-x}$
 $\therefore f'(x) = 3x^2e^{-x} + x^3(-e^{-x})$
 $= 3x^2e^{-x} - x^3e^{-x}$

c $f(x) = \frac{e^x}{x}$
 $u = e^x, v = x, u' = e^x, v' = 1$
 $\therefore f'(x) = \frac{e^xx - e^x(1)}{x^2}$
 $= \frac{xe^x - e^x}{x^2}$

d $f(x) = \frac{x}{e^x}$
 $u = x, v = e^x, u' = 1, v' = e^x$
 $\therefore f'(x) = \frac{1e^x - xe^x}{(e^x)^2}$
 $= \frac{e^x(1 - x)}{(e^x)^2}$
 $= \frac{1 - x}{e^x}$

$$\begin{aligned} \mathbf{e} \quad f(x) &= x^2 e^{3x} \\ u &= x^2, \quad v = e^{3x}, \\ u' &= 2x, \quad v' = 3e^{3x} \\ \therefore f'(x) &= 2xe^{3x} + 3x^2 e^{3x} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad f(x) &= \frac{e^x}{\sqrt{x}} \\ u &= e^x, \quad v = x^{\frac{1}{2}}, \quad u' = e^x, \quad v' = \frac{1}{2}x^{-\frac{1}{2}} \\ \therefore f'(x) &= \frac{e^x \sqrt{x} - \frac{e^x}{2\sqrt{x}}}{(\sqrt{x})^2} \\ &= \frac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}} \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad f(x) &= \sqrt{x}e^{-x} \\ u &= x^{\frac{1}{2}}, \quad v = e^{-x}, \\ u' &= \frac{1}{2}x^{-\frac{1}{2}}, \quad v' = -e^{-x} \\ \therefore f'(x) &= \frac{1}{2\sqrt{x}}e^{-x} - \sqrt{x}e^{-x} \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad f(x) &= \frac{e^x + 2}{e^{-x} + 1} \\ u &= e^x + 2, \quad v' = e^{-x} + 1, \quad u' = e^x, \quad v' = -e^{-x} \\ \therefore f'(x) &= \frac{e^x(e^{-x} + 1) - (e^x + 2)(-e^{-x})}{(e^{-x} + 1)^2} \\ &= \frac{1 + e^x + 1 + 2e^{-x}}{(e^{-x} + 1)^2} \\ &= \frac{2 + e^x + 2e^{-x}}{(e^{-x} + 1)^2} \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad f(x) &= (e^x + 2)^4 \\ &= u^4 \quad \text{where } u = e^x + 2 \\ \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= 4u^3(e^x) \\ \therefore f'(x) &= 4(e^x + 2)^3(e^x) \\ &= 4e^x(e^x + 2)^3 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad f(x) &= \frac{1}{1 - e^{-x}} \\ &= u^{-1} \quad \text{where } u = 1 - e^{-x} \\ \therefore f'(x) &= \frac{dy}{du} \frac{du}{dx} \\ &= -u^{-2}(e^{-x}) \\ \therefore f'(x) &= -\frac{e^{-x}}{(1 - e^{-x})^2} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad f(x) &= \sqrt{e^{2x} + 10} \\ &= u^{\frac{1}{2}} \quad \text{where } u = e^{2x} + 10 \\ \therefore \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= \frac{1}{2}u^{-\frac{1}{2}}(2e^{2x}) \\ \therefore f'(x) &= \frac{e^{2x}}{\sqrt{e^{2x} + 10}} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad f(x) &= \frac{1}{(1 - e^{3x})^2} \\ &= u^{-2} \quad \text{where } u = 1 - e^{3x} \\ \therefore \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= -2u^{-3}(-3e^{3x}) = \frac{6e^{3x}}{u^3} \\ \therefore f'(x) &= \frac{6e^{3x}}{(1 - e^{3x})^3} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad f(x) &= \frac{1}{\sqrt{1 - e^{-x}}} \\ &= u^{-\frac{1}{2}} \quad \text{where } u = 1 - e^{-x} \\ \therefore \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= -\frac{1}{2}u^{-\frac{3}{2}}(e^{-x}) = \frac{-e^{-x}}{2u^{\frac{3}{2}}} \\ \therefore f'(x) &= \frac{-e^{-x}}{2(1 - e^{-x})^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad f(x) &= x\sqrt{1 - 2e^{-x}} \\ &= xu^{\frac{1}{2}} \quad \text{where } u = 1 - 2e^{-x} \\ \therefore f'(x) &= 1u^{\frac{1}{2}} + x\frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx} \\ &= 1\sqrt{u} + x\frac{1}{2}u^{-\frac{1}{2}}2e^{-x} \\ &= \frac{\sqrt{1 - 2e^{-x}}}{1} + \frac{xe^{-x}}{\sqrt{1 - 2e^{-x}}} \\ \therefore f'(x) &= \frac{1 - 2e^{-x} + xe^{-x}}{\sqrt{1 - 2e^{-x}}} \end{aligned}$$

4 a $y = Ae^{kx}$ **i** $\frac{dy}{dx} = Ae^{kx}(k)$ **ii** $\frac{d^2y}{dx^2} = k\frac{dy}{dx}$ **b** Prediction:

$$\begin{aligned} &= k(Ae^{kx}) &= k(ky) &\frac{d^ny}{dx^n} = k^ny \\ &= ky &= k^2y \end{aligned}$$

5 $y = 2e^{3x} + 5e^{4x}$ $\therefore \frac{dy}{dx} = 6e^{3x} + 20e^{4x}$ and $\frac{d^2y}{dx^2} = 18e^{3x} + 80e^{4x}$

Now $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = (18e^{3x} + 80e^{4x}) - 7(6e^{3x} + 20e^{4x}) + 12(2e^{3x} + 5e^{4x})$

$$\begin{aligned} &= 18e^{3x} + 80e^{4x} - 42e^{3x} - 140e^{4x} + 24e^{3x} + 60e^{4x} \\ &= e^{3x}[18 - 42 + 24] + e^{4x}[80 - 140 + 60] \\ &= e^{3x}(0) + e^{4x}(0) \\ &= 0 \end{aligned}$$

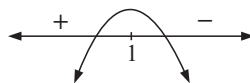
i.e., $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

6 a $y = xe^{-x}$ $u = x, \quad v = e^{-x}, \quad u' = 1, \quad v' = -e^{-x}$

$\therefore \frac{dy}{dx} = 1e^{-x} - xe^{-x}$ {product rule}

$$\begin{aligned} &= e^{-x}(1 - x) \\ &= \frac{1 - x}{e^x} \end{aligned}$$

which has sign diagram:



$\therefore$ at $x = 1, y = 1e^{-1} = \frac{1}{e}$ we have a maximum turning point.

$\therefore$ has a local maximum at $(1, \frac{1}{e})$

b $y = x^2e^x$ $u = x^2, \quad v = e^x, \quad u' = 2x, \quad v' = e^x$

$\therefore \frac{dy}{dx} = 2xe^x + x^2e^x$ {product rule}

$$= xe^x(2 + x)$$

which has sign diagram:



$\therefore$ at $x = -2, y = 4e^{-2}$, we have a maximum turning point

and at $x = 0, y = 0$, we have a minimum turning point.

$\therefore$ we have a local maximum at $(-2, \frac{4}{e^2})$, and a local minimum at $(0, 0)$.

c $y = \frac{e^x}{x}$ $u = e^x, \quad v = x, \quad u' = e^x, \quad v' = 1$

$\therefore \frac{dy}{dx} = \frac{e^x x - e^x(1)}{x^2}$ {quotient rule}

$$= \frac{e^x(x - 1)}{x^2}$$

which has sign diagram:



$\therefore$ at $x = 1, y = \frac{e}{1} = e$ we have a minimum turning point.

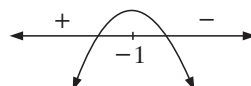
$\therefore$ we have a local minimum at $(1, e)$

d $y = e^{-x}(x + 2)$ $u = e^{-x}, \quad v = x + 2, \quad u' = -e^{-x}, \quad v' = 1$

$\therefore \frac{dy}{dx} = -e^{-x}(x + 2) + e^{-x}$ {product rule}

$$\begin{aligned} &= e^{-x}(-x - 2 + 1) \\ &= e^{-x}(-x - 1) \end{aligned}$$

which has sign diagram:



$\therefore$ at $x = -1, y = e(-1 + 2) = e$ we have a maximum turning point.

$\therefore$ we have a local maximum at $(-1, e)$

EXERCISE 23B

1 a $N = 50e^{2t}$

$$\begin{aligned}\therefore \ln N &= \ln(50e^{2t}) && \{\ln ab = \ln a + \ln b\} \\ &= \ln 50 + \ln e^{2t} && \{\text{as } \ln e^n = n\}\end{aligned}$$

$$\therefore \ln N = \ln 50 + 2t$$

b $P = 8.69e^{-0.0541t}$

$$\begin{aligned}\therefore \ln P &= \ln(8.69e^{-0.0541t}) \\ &= \ln 8.69 + \ln e^{-0.0541t} \\ \therefore \ln P &= \ln 8.69 - 0.0541t\end{aligned}$$

c $S = a^2e^{-kt}$

$$\begin{aligned}\therefore \ln S &= \ln(a^2e^{-kt}) \\ &= \ln a^2 + \ln e^{-kt} \\ \therefore \ln S &= 2 \ln a - kt && \{\ln a^n = n \ln a\}\end{aligned}$$

2 a $\ln e^2 = 2 \ln e$
 $= 2(1)$
 $= 2$

b $\ln \sqrt{e} = \ln e^{\frac{1}{2}}$
 $= \frac{1}{2} \ln e$
 $= \frac{1}{2}$

c $\ln\left(\frac{1}{e}\right) = \ln e^{-1}$
 $= -1 \ln e$
 $= -1$

d $\ln\left(\frac{1}{\sqrt{e}}\right) = \ln e^{-\frac{1}{2}}$
 $= -\frac{1}{2} \ln e$
 $= -\frac{1}{2}(1)$
 $= -\frac{1}{2}$

e $e^{\ln 3} = 3$

f $e^{2 \ln 3} = e^{\ln 3^2}$
 $= e^{\ln 9}$
 $= 9$

g $e^{-\ln 5} = e^{\ln 5^{-1}}$
 $= e^{\ln \frac{1}{5}}$
 $= \frac{1}{5}$

h $e^{-2 \ln 2} = e^{\ln 2^{-2}}$
 $= e^{\ln \frac{1}{4}}$
 $= \frac{1}{4}$

3 a Let $2 = e^x$
 $\therefore \ln 2 = x$
 $\therefore 2 = e^{\ln 2}$

b Let $10 = e^x$
 $\therefore \ln 10 = x$
 $\therefore 10 = e^{\ln 10}$

c Let $a = e^x$
 $\therefore \ln a = x$
 $\therefore a = e^{\ln a}$

d Let $a^x = e^x$
 $\therefore \ln a^x = x$
 $\therefore x \ln a = x$
 $\therefore a^x = e^{x \ln a}$

4 a $e^x = 2$
 $\therefore \ln e^x = \ln 2$
 $x = \ln 2$

b $e^x = -2$ has no solutions
as $e^x > 0$ for all x

c $e^x = 0$ has no solutions
as $e^x > 0$ for all x

d $e^{2x} = 2e^x$
 $\therefore e^x(e^x - 2) = 0$
 $\therefore e^x = 2$ {as $e^x > 0$ }
 $\therefore x = \ln 2$

e $e^x = e^{-x}$
 $\therefore x = -x$
 $\therefore 2x = 0$
 $\therefore x = 0$

f $e^{2x} - 5e^x + 6 = 0$
 $\therefore (e^x - 3)(e^x - 2) = 0$
 $\therefore e^x = 3$ or 2
 $\therefore x = \ln 3$ or $\ln 2$

g $e^x + 2 = 3e^{-x}$
 $\therefore e^{2x} + 2e^x = 3$ { $\times e^x$ }
 $\therefore e^{2x} + 2e^x - 3 = 0$
 $\therefore (e^x + 3)(e^x - 1) = 0$
 $\therefore e^x = -3$ or 1
 $\therefore e^x = 1$ {as $e^x > 0$ }
 $\therefore x = \ln 1$
 $\therefore x = 0$

h $1 + 12e^{-x} = e^x$
 $\therefore e^x + 12 = e^{2x}$ { $\times e^x$ }
 $\therefore e^{2x} - e^x - 12 = 0$
 $\therefore (e^x - 4)(e^x + 3) = 0$
 $\therefore e^x = 4$ or -3
 $\therefore e^x = 4$ {as $e^x > 0$ }
 $\therefore x = \ln 4$

$$\begin{aligned}
 \text{i} \quad & e^x + e^{-x} = 3 \\
 & \therefore e^{2x} + 1 = 3e^x \quad \{\times e^x\} \\
 \therefore & e^{2x} - 3e^x + 1 = 0 \\
 & \therefore e^x = \frac{3 \pm \sqrt{9-4}}{2} \\
 & \therefore e^x = \frac{3 \pm \sqrt{5}}{2} \\
 & \therefore x = \ln \left(\frac{3 + \sqrt{5}}{2} \right) \text{ or } \ln \left(\frac{3 - \sqrt{5}}{2} \right) \\
 & \quad \div 0.962 \quad \text{or} \quad -0.962
 \end{aligned}$$

$$\begin{aligned}
 \text{5 a} \quad & y = 2^x \\
 \therefore & \frac{dy}{dx} = 2^x \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & y = x2^x \\
 \therefore & \frac{dy}{dx} = \frac{d}{dx}(x)2^x + x\frac{d}{dx}(2^x) \\
 & \quad \{\text{product rule}\} \\
 & = 2^x + x2^x \ln 2
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad & y = \frac{2^x}{x} \\
 \therefore & \frac{dy}{dx} = \frac{\frac{d}{dx}(2^x)x - 2^x \frac{d}{dx}(x)}{x^2} \\
 & = \frac{2^x \ln 2 \times x - 2^x}{x^2} \\
 & = \frac{2^x(x \ln 2 - 1)}{x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & y = 5^x \\
 \therefore & \frac{dy}{dx} = 5^x \ln 5
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & y = x^3 6^{-x} = \frac{x^3}{6^x} \\
 \therefore & \frac{dy}{dx} = \frac{\frac{d}{dx}(x^3)6^x - x^3 \frac{d}{dx}(6^x)}{6^{2x}} \\
 & = \frac{3x^2 6^x - x^3 \times 6^x \ln 6}{6^{2x}} \\
 & = \frac{x^2(3 - x \ln 6)}{6^x}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad & y = \frac{x}{3^x} \\
 \therefore & \frac{dy}{dx} = \frac{\frac{d}{dx}(x)3^x - x \frac{d}{dx}(3^x)}{3^{2x}} \\
 & = \frac{3^x - x \times 3^x \ln 3}{3^{2x}} \\
 & = \frac{1 - x \ln 3}{3^x}
 \end{aligned}$$

$$\begin{aligned}
 \text{6 a} \quad & y_1 = e^x, \quad y_2 = e^{2x} - 6 \quad \text{meet when } y_1 = y_2 \\
 & \therefore e^x = e^{2x} - 6 \\
 & \therefore e^{2x} - e^x - 6 = 0 \\
 \therefore & (e^x - 3)(e^x + 2) = 0 \\
 & \therefore e^x = 3 \text{ or } -2 \\
 & \therefore e^x = 3 \quad \{\text{as } e^x > 0\} \\
 & \therefore x = \ln 3 \quad \text{and } y = e^x = 3 \quad \therefore \text{meet at } (\ln 3, 3)
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & y_1 = 2e^x + 1, \quad y_2 = 7 - e^x \quad \text{meet when } y_1 = y_2 \\
 & \therefore 2e^x + 1 = 7 - e^x \\
 & \therefore 3e^x = 6 \\
 & \therefore e^x = 2 \\
 \therefore & \text{meet at } (\ln 2, 5) \quad \therefore x = \ln 2 \quad \text{and } y = 7 - e^x = 7 - 2 = 5
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & y_1 = 3 - e^x, \quad y_2 = 5e^{-x} - 3 \quad \text{meet when } y_1 = y_2 \\
 & \therefore 3 - e^x = 5e^{-x} - 3 \\
 & \therefore 3e^x - e^{2x} = 5 - 3e^x \quad \{\times \text{ by } e^x\} \\
 \therefore & e^{2x} - 6e^x + 5 = 0 \\
 \therefore & (e^x - 5)(e^x - 1) = 0 \\
 & \therefore e^x = 1 \text{ or } 5 \\
 & \therefore x = 0 \text{ or } \ln 5 \\
 \text{and } & y = 3 - e^0 \quad \text{or} \quad 3 - e^{\ln 5} \\
 & = 3 - 1 \quad \quad \quad = 3 - 5 \\
 & = 2 \quad \quad \quad = -2 \\
 \text{i.e., } & \text{meet at } (0, 2) \text{ and } (\ln 5, -2)
 \end{aligned}$$

7 a $y = e^x - 3e^{-x}$

cuts the x -axis at P when $y = 0 \quad \therefore e^x - 3e^{-x} = 0$

$$\therefore e^{2x} - 3 = 0 \quad \{\times \text{ each term by } e^x\}$$

$$\therefore e^{2x} = 3$$

$$\therefore 2x = \ln 3$$

$$\therefore x = \frac{1}{2} \ln 3 \quad \therefore \text{P is } \left(\frac{1}{2} \ln 3, 0\right)$$

cuts the y -axis at Q when $x = 0 \quad \therefore y = e^0 - 3e^0$

$$= 1 - 3$$

$$= -2 \quad \therefore \text{Q is } (0, -2)$$

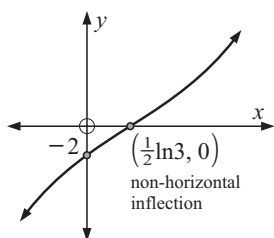
b $\frac{dy}{dx} = e^x + 3e^{-x}$
 $= e^x + \frac{3}{e^x}$

Since $e^x > 0$ for all x

$$\frac{dy}{dx} > 0 \text{ for all } x$$

$\therefore$ the function is increasing for all x

d



c $\frac{dy}{dx} = e^x + 3e^{-x}$
 $\therefore \frac{d^2y}{dx^2} = e^x - 3e^{-x}$
 $= y$

Above x -axis $y > 0 \quad \therefore \frac{d^2y}{dx^2} > 0$

$\therefore$ the function is concave up

Below x -axis $y < 0 \quad \therefore \frac{d^2y}{dx^2} < 0$

$\therefore$ the function is concave down

i.e., a non-horizontal inflection occurs when $y = 0$

8 a For $f(x) = e^x - 3$ and $g(x) = 3 - \frac{5}{e^x}$

the x -intercept occurs when $y = 0 \quad \therefore e^x - 3 = 0$

$$\therefore e^x = 3$$

$$\therefore x = \ln 3$$

i.e., at $(\ln 3, 0)$

and $3 - \frac{5}{e^x} = 0$

$$\therefore \frac{3e^x - 5}{e^x} = 0$$

$$\therefore 3e^x - 5 = 0$$

$$\therefore e^x = \frac{5}{3}$$

$$\therefore x = \ln \frac{5}{3}$$

i.e., at $(\ln \frac{5}{3}, 0)$

the y -intercept occurs when $x = 0 \quad \therefore y = e^0 - 3$

$$y = -2$$

i.e., at $(0, -2)$

and $y = 3 - \frac{5}{e^0}$

$$= 3 - 5$$

$$= -2$$

i.e., $(0, -2)$

b as $x \rightarrow +\infty \quad f(x) \rightarrow +\infty$ as $x \rightarrow +\infty \quad g(x) \rightarrow 3$ (below)
 $x \rightarrow -\infty \quad f(x) \rightarrow -3$ (above) $x \rightarrow -\infty \quad g(x) \rightarrow -\infty$

c $f(x)$ and $g(x)$ meet when $e^x - 3 = 3 - 5e^{-x}$

$$\therefore e^{2x} - 3e^x = 3e^x - 5 \quad \{\times e^x\}$$

$$\therefore e^{2x} - 6e^x + 5 = 0$$

$$\therefore (e^x - 5)(e^x - 1) = 0$$

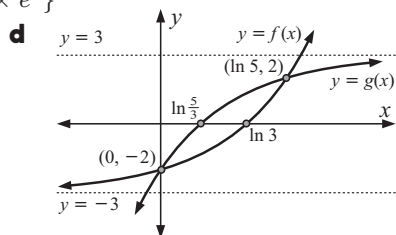
$$\text{i.e., when } e^x = 5 \text{ or } 1$$

$$\therefore x = \ln 5 \text{ or } 0$$

$$\text{when } x = \ln 5, \quad f(x) = e^{\ln 5} - 3 = 5 - 3 = 2$$

$$\text{when } x = 0, \quad f(x) = e^0 - 3 = 1 - 3 = -2$$

$\therefore f(x)$ and $g(x)$ meet at $(0, -2)$ and $(\ln 5, 2)$



EXERCISE 23C

1 a $y = \ln(7x)$ or $y = \ln(7x)$

$$\therefore y = \ln 7 + \ln x$$

$$\therefore \frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{7 \leftarrow f'(x)}{7x \leftarrow f(x)} \\ &= \frac{1}{x} \end{aligned}$$

b $y = \ln(2x + 1)$

$$\therefore \frac{dy}{dx} = \frac{2}{2x + 1} \leftarrow f'(x)$$

c $y = \ln(x - x^2)$

$$\therefore \frac{dy}{dx} = \frac{1 - 2x}{x - x^2} \leftarrow f'(x)$$

d $y = 3 - 2 \ln x$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 0 - 2 \left(\frac{1}{x} \right) \\ &= -\frac{2}{x} \end{aligned}$$

e $y = x^2 \ln x$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 2x \ln x + x^2 \left(\frac{1}{x} \right) \\ &= 2x \ln x + x \end{aligned}$$

f $y = \frac{\ln x}{2x}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\left(\frac{1}{x} \right) 2x - \ln x (2)}{(2x)^2} \\ &= \frac{2 - 2 \ln x}{4x^2} \\ &= \frac{1 - \ln x}{2x^2} \end{aligned}$$

g $y = e^x \ln x$

$$\therefore \frac{dy}{dx} = e^x \ln x + \frac{e^x}{x}$$

h $y = (\ln x)^2$

$$\begin{aligned} \therefore \frac{dy}{dx} &= 2 (\ln x)^1 \left(\frac{1}{x} \right) \\ &= \frac{2 \ln x}{x} \end{aligned}$$

i $y = \sqrt{\ln x} = (\ln x)^{\frac{1}{2}}$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{1}{2} (\ln x)^{-\frac{1}{2}} \left(\frac{1}{x} \right) \\ &= \frac{1}{2x\sqrt{\ln x}} \end{aligned}$$

j $y = e^{-x} \ln x$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -e^{-x} \ln x + e^{-x} \left(\frac{1}{x} \right) \\ &= \frac{e^{-x}}{x} - e^{-x} \ln x \end{aligned}$$

k $y = \sqrt{x} \ln 2x$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{1}{2\sqrt{x}} \ln 2x + \sqrt{x} \left(\frac{1}{x} \right) \\ &= \frac{\ln 2x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \end{aligned}$$

$$\mathbf{1} \quad y = \frac{2\sqrt{x}}{\ln x} \quad \therefore \quad \frac{dy}{dx} = \frac{\frac{1}{\sqrt{x}} \ln x - 2\sqrt{x} \left(\frac{1}{x}\right)}{(\ln x)^2} = \frac{\frac{1}{\sqrt{x}} \ln x - \frac{2}{\sqrt{x}}}{(\ln x)^2} = \frac{\ln x - 2}{\sqrt{x}(\ln x)^2}$$

$$\mathbf{2} \quad \mathbf{a} \quad y = x \ln 5 \\ \therefore \quad \frac{dy}{dx} = \ln 5$$

$$\mathbf{c} \quad y = \ln(x^4 + x) \\ \therefore \quad \frac{dy}{dx} = \frac{4x^3 + 1}{x^4 + x}$$

$$\mathbf{e} \quad y = [\ln(2x + 1)]^3 \\ \therefore \quad \frac{dy}{dx} = 3[\ln(2x + 1)]^2 \times \frac{2}{2x + 1} \\ = \frac{6[\ln(2x + 1)]^2}{2x + 1}$$

$$\mathbf{g} \quad y = \ln\left(\frac{1}{x}\right) = \ln(x^{-1}) \\ \therefore \quad y = -\ln x \quad \therefore \quad \frac{dy}{dx} = -\frac{1}{x}$$

$$\mathbf{i} \quad y = \frac{1}{\ln x} = [\ln x]^{-1} \quad \therefore \quad \frac{dy}{dx} = -1[\ln x]^{-2} \times \frac{1}{x} = \frac{-1}{x[\ln x]^2}$$

$$\mathbf{b} \quad y = \ln(x^3) = 3 \ln x \\ \therefore \quad \frac{dy}{dx} = 3\left(\frac{1}{x}\right) = \frac{3}{x}$$

$$\mathbf{d} \quad y = \ln(10 - 5x) \\ \therefore \quad \frac{dy}{dx} = \frac{-5}{10 - 5x} = \frac{1}{x - 2}$$

$$\mathbf{f} \quad y = \frac{\ln(4x)}{x} \\ \therefore \quad \frac{dy}{dx} = \frac{\left(\frac{4}{4x}\right)x - \ln(4x) \times 1}{x^2} \\ = \frac{1 - \ln(4x)}{x^2}$$

$$\mathbf{h} \quad y = \ln(\ln x) \\ \therefore \quad \frac{dy}{dx} = \frac{\frac{1}{x}}{\ln(\ln x)} = \frac{1}{x \ln(\ln x)}$$

$$\mathbf{3} \quad \mathbf{a} \quad y = \ln \sqrt{1 - 2x} \\ = \ln(1 - 2x)^{\frac{1}{2}} \\ = \frac{1}{2} \ln(1 - 2x) \\ \therefore \quad \frac{dy}{dx} = \frac{1}{2} \times \frac{-2}{1 - 2x} \\ = \frac{1}{2x - 1}$$

$$\mathbf{b} \quad y = \ln \frac{1}{(2x + 3)} \\ = -\ln(2x + 3) \\ \therefore \quad \frac{dy}{dx} = -\frac{2}{2x + 3}$$

$$\mathbf{c} \quad y = \ln(e^x \sqrt{x}) \\ = \ln e^x + \ln x^{\frac{1}{2}} \\ = \ln e^x + \frac{1}{2} \ln x \\ = x + \frac{1}{2} \ln x \\ \therefore \quad \frac{dy}{dx} = 1 + \frac{1}{2} \left(\frac{1}{x}\right) \\ \therefore \quad \frac{dy}{dx} = 1 + \frac{1}{2x}$$

$$\mathbf{d} \quad y = \ln(x\sqrt{2-x}) \\ = \ln x + \ln(2-x)^{\frac{1}{2}} \\ = \ln x + \frac{1}{2} \ln(2-x) \\ \therefore \quad \frac{dy}{dx} = \frac{1}{x} + \frac{1}{2} \left(\frac{-1}{2-x}\right) \\ = \frac{1}{x} - \frac{1}{2(2-x)}$$

$$\mathbf{e} \quad y = \ln\left(\frac{x+3}{x-1}\right) \\ = \ln(x+3) - \ln(x-1) \\ \therefore \quad \frac{dy}{dx} = \frac{1}{x+3} - \frac{1}{x-1}$$

$$\mathbf{f} \quad y = \ln\left(\frac{x^2}{3-x}\right) \\ = \ln x^2 - \ln(3-x) \\ = 2 \ln x - \ln(3-x) \\ \therefore \quad \frac{dy}{dx} = \frac{2}{x} - \frac{-1}{3-x} \\ = \frac{2}{x} + \frac{1}{3-x}$$

$$\mathbf{g} \quad f(x) = \ln((3x-4)^3) \\ = 3 \ln(3x-4) \\ \therefore \quad f'(x) = 3 \times \frac{3}{3x-4} \\ = \frac{9}{3x-4}$$

$$\mathbf{h} \quad f(x) = \ln(x(x^2+1)) \\ = \ln x + \ln(x^2+1) \\ \therefore \quad f'(x) = \frac{1}{x} + \frac{2x}{x^2+1}$$

$$\mathbf{i} \quad f(x) = \ln\left(\frac{x^2+2x}{x-5}\right) \\ = \ln(x^2+2x) - \ln(x-5) \\ f'(x) = \frac{2x+2}{x^2+2x} - \frac{1}{x-5}$$

4 a For this question, we remember that $\log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a}$

i $y = \log_2 x = \frac{\ln x}{\ln 2}$

ii $y = \log_{10} x = \frac{\ln x}{\ln 10}$

iii $y = x \log_3 x = \frac{x \ln x}{\ln 3}$

$\therefore \frac{dy}{dx} = \frac{1}{x \ln 2}$

$\therefore \frac{dy}{dx} = \frac{1}{x \ln 10}$

$\therefore \frac{dy}{dx} = \frac{\frac{d}{dx}(x) \ln x + x \frac{d}{dx}(\ln x)}{\ln 3}$
 $= \frac{\ln x + x \left(\frac{1}{x}\right)}{\ln 3}$
 $= \frac{1 + \ln x}{\ln 3}$

b $y = 2^x \quad \therefore \frac{dy}{dx} = e^{x \ln 2} \times \ln 2$
 $= (e^{\ln 2})^x = 2^x \ln 2$
 $= e^{x \ln 2}$

c $y = a^x \quad \therefore \frac{dy}{dx} = e^{x \ln a} \times \ln a$
 $= (e^{\ln a})^x = a^x \ln a$
 $= e^{x \ln a}$

5 $f(x) = \ln(2x - 1) - 3$

a when $y = 0$, $\ln(2x - 1) = 3$

$\therefore 2x - 1 = e^3$

$\therefore 2x = e^3 + 1$

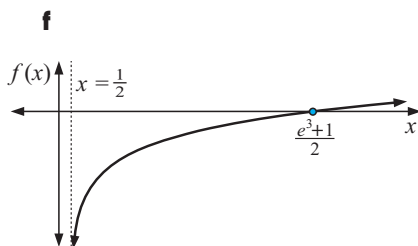
$\therefore x = \frac{e^3 + 1}{2} \div 10.54 \quad \text{i.e., the } x \text{ intercept is } \frac{e^3 + 1}{2}.$

b $f(0) = \ln(-1)$ cannot be found as $\ln(-1)$ is not defined. $\therefore$ there is no y -intercept.

c $f'(x) = \frac{2}{2x - 1}$, $\therefore f'(1) = \frac{2}{2 - 1} = 2 \quad \therefore$ slope of tangent $= 2$

d $\ln(2x - 1)$ has meaning provided $2x - 1 > 0$ i.e., $2x > 1$ and so $x > \frac{1}{2}$
 $\therefore f(x)$ has meaning provided $x > \frac{1}{2}$

e $f'(x) = 2(2x - 1)^{-1}$
 $\therefore f''(x) = -2(2x - 1)^{-2} \cdot 2 = \frac{-4}{(2x - 1)^2}$
 provided $f(x)$ has meaning
 i.e., provided $x > \frac{1}{2}$, $f''(x) < 0$
 $\therefore f(x)$ is always concave down for $x > \frac{1}{2}$



6 Consider $f(x) = \frac{\ln x}{x}$ Let $u = \ln x$, $v = x$, $u' = \frac{1}{x}$, $v' = 1$

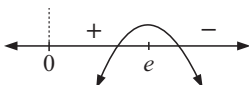
$\therefore f'(x) = \frac{\left(\frac{1}{x}\right)x - \ln x(1)}{x^2} = \frac{1 - \ln x}{x^2}$

Now $f'(x) = 0$ when $1 - \ln x = 0$

$\therefore \ln x = 1$

$\therefore x = e^1$, and $f(e) = \frac{\ln e}{e} = \frac{1}{e}$

Sign diagram of $f'(x)$ is:



$\therefore$ there is a max. turning point at $\left(e, \frac{1}{e}\right)$, i.e., $f(x) \leq \frac{1}{e}$ for all x , i.e., $\frac{\ln x}{x} \leq \frac{1}{e}$ for all x

$$7 \quad f(x) = x - \ln x$$

$$\therefore f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x} \quad \text{and the sign diagram of } f'(x) \text{ is: } \begin{array}{c} \leftarrow - \quad \quad \quad + \rightarrow \\ \quad \quad \quad \cup \end{array}$$

$$\therefore f(x) \text{ has a minimum turning point at } (1, 1 - \ln 1), \text{ i.e., } (1, 1)$$

$$\therefore f(x) \geq 1 \quad \text{for all } x > 0$$

$$x - \ln x \geq 1$$

$$\text{i.e., } \ln x \leq x - 1 \quad \text{for all } x > 0$$

EXERCISE 23D

$$1 \quad f(x) = e^{-x} \quad \therefore f(1) = e^{-1} \quad \text{i.e., the point of contact is } (1, \frac{1}{e}).$$

$$\text{Now } f'(x) = -e^{-x}$$

$$f'(1) = -e^{-1} = -\frac{1}{e} \quad \text{and is the slope of the tangent at } (1, \frac{1}{e}).$$

$$\therefore \text{tangent has equation } \frac{y - \frac{1}{e}}{x - 1} = -\frac{1}{e} \quad \therefore e(y - \frac{1}{e}) = -(x - 1)$$

$$\therefore ey - 1 = -x + 1$$

$$\text{i.e., } x + ey = 2 \quad \text{or } y = -\frac{1}{e}x + \frac{2}{e}$$

$$2 \quad y = \ln(2 - x), \text{ so when } x = -1, y = \ln 3 \quad \text{i.e., the point of contact is } (-1, \ln 3).$$

$$\text{Now } \frac{dy}{dx} = \frac{-1}{2-x}$$

$$\therefore \text{when } x = -1, \frac{dy}{dx} = -\frac{1}{2+1} = -\frac{1}{3} \quad \text{and is the slope of the tangent at } (-1, \ln 3).$$

$$\therefore \text{tangent has equation } \frac{y - \ln 3}{x + 1} = -\frac{1}{3} \quad \text{i.e., } 3(y - \ln 3) = -(x + 1)$$

$$\therefore 3y - 3 \ln 3 = -x - 1$$

$$\text{i.e., } x + 3y = 3 \ln 3 - 1$$

$$3 \quad y = x^2 e^x, \text{ so when } x = 1, y = e, \text{ i.e., the point of contact is } (1, e).$$

$$\text{Now } \frac{dy}{dx} = x^2 e^x + 2x e^x$$

$$\therefore \text{when } x = 1, \frac{dy}{dx} = e + 2e = 3e \quad \text{and is the slope of the tangent at } (1, e).$$

$$\therefore \text{the tangent has equation } \frac{y - e}{x - 1} = 3e \quad \text{i.e., } y - e = 3e(x - 1)$$

$$\therefore y - e = 3ex - 3e$$

$$\therefore y - 3ex = -2e$$

$$\text{i.e., } 3ex - y = 2e$$

$$\text{Cuts the } x\text{-axis when } y = 0 \quad \therefore 3ex = 2e \quad \text{i.e., } x = \frac{2}{3} \quad \text{i.e., at } A(\frac{2}{3}, 0)$$

$$\text{Cuts the } y\text{-axis when } x = 0 \quad \text{i.e., } -y = 2e \quad \text{i.e., at } B(0, -2e)$$

$$4 \quad y = \ln \sqrt{x} \quad \therefore \text{when } y = -1, -1 = \frac{1}{2} \ln x$$

$$= \ln x^{\frac{1}{2}} \quad \therefore \ln x = -2$$

$$= \frac{1}{2} \ln x \quad \therefore x = e^{-2}$$

$$\therefore x = \frac{1}{e^2} \quad \text{i.e., the point of contact is } \left(\frac{1}{e^2}, -1\right)$$

$$\text{Now } \frac{dy}{dx} = \frac{1}{2} \frac{1}{x} = \frac{1}{2x} \quad \therefore \text{at } \left(\frac{1}{e^2}, -1\right), \frac{dy}{dx} = \frac{1}{2e^{-2}} = \frac{e^2}{2}$$

$$\therefore \text{ the tangent has slope} = \frac{e^2}{2} \quad \text{and} \quad \text{the normal has slope} = -\frac{2}{e^2}$$

$$\therefore \text{ the normal has equation } \frac{y+1}{x-\frac{1}{e^2}} = -\frac{2}{e^2} \quad \text{i.e., } e^2(y+1) = -2\left(x - \frac{1}{e^2}\right)$$

$$\therefore e^2 y + e^2 = -2x + \frac{2}{e^2}$$

$$\therefore 2x + e^2 y = -e^2 + \frac{2}{e^2} \quad \text{or} \quad y = -\frac{2}{e^2}x + \frac{2}{e^4} - 1$$

5 $y = e^x$

When $x = a$, $y = e^a$, i.e., the point of contact is (a, e^a) .

Now $\frac{dy}{dx} = e^x$ and so at (a, e^a) , $\frac{dy}{dx} = e^a$ and is the slope of the tangent at (a, e^a) .

$$\therefore \text{ the tangent has equation } \frac{y - e^a}{x - a} = e^a \quad \text{i.e., } y - e^a = e^a(x - a) \dots\dots*$$

if the tangent is through the origin, $(0, 0)$ must satisfy * $\therefore 0 - e^a = e^a(0 - a)$

$$\therefore -e^a = -ae^a$$

$$\therefore e^a(a - 1) = 0$$

$$\therefore a = 1 \quad \{\text{as } e^a > 0\}$$

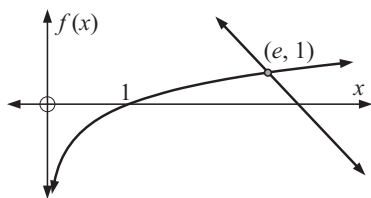
So the equation of the tangent is $y - e = ex - e$ i.e., $y = ex$

6 a $f(x) = \ln x$ is defined for all $x > 0$.

b $f'(x) = \frac{1}{x}$ which is > 0 for all $x > 0$, $\therefore f(x)$ is increasing on $x > 0$.

$f''(x) = -x^{-2} = \frac{-1}{x^2}$ which is < 0 for all $x > 0$, $\therefore f(x)$ is concave down on $x > 0$.

c



$$\text{At } y = 1, \quad 1 = \ln x$$

$$\therefore x = e^1 = e,$$

i.e., the point of contact is $(e, 1)$

$$\text{and } \frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \text{ at } (e, 1), \quad \frac{dy}{dx} = \frac{1}{e}$$

$\therefore$ the slope of the tangent is $\frac{1}{e}$, and the slope of the normal is $-e$

$$\therefore \text{ the equation of the normal is } \frac{y-1}{x-e} = -e \quad \therefore y-1 = -e(x-e)$$

$$\therefore y-1 = -ex + e^2$$

$$\therefore ex + y = 1 + e^2$$

7 $y = 3e^{-x}$ and $y = 2 + e^x$ meet when $3e^{-x} = 2 + e^x$

$$\therefore 3 = 2e^x + e^{2x} \quad \{\times \text{ by } e^x\}$$

$$\therefore e^{2x} + 2e^x - 3 = 0$$

$$\therefore (e^x + 3)(e^x - 1) = 0$$

$$\therefore e^x = -3 \text{ or } 1 \quad \{\text{as } e^x > 0\}$$

$$\therefore x = 0 \quad \{\text{as } e^x > 0\}$$

$$\text{and when } x = 0, \quad y = 3e^0 = 3$$

When $y = 2 + e^x$, $\frac{dy}{dx} = e^x$ $\therefore$ at $(0, 3)$, $\frac{dy}{dx} = e^0 = 1$

i.e., the slope of the tangent is 1

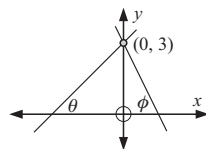
and $1 = \tan \theta$ where θ is the angle between the tangent and the x -axis, $\therefore \theta = 45^\circ$

When $y = 3e^{-x}$, $\frac{dy}{dx} = -3e^{-x} \therefore$ at $(0, 3)$, $\frac{dy}{dx} = -3$

and $\tan \phi = 3$ where ϕ is the angle between the tangent and the x -axis

$$\therefore \phi = \tan^{-1}(3) \doteq 71.6^\circ$$

$\therefore$ angle between the tangents $\doteq (180 - (71.57 + 45))^\circ \doteq 63.4^\circ$



8 a $W = 20e^{-kt}$ and when $t = 50$ hours, $W = 10$ g

$$\therefore 20e^{-50k} = 10$$

$$\therefore e^{-50k} = \frac{1}{2}$$

$$\therefore -50k = \ln \frac{1}{2} = -\ln 2 \text{ and so } k = \frac{1}{50} \ln 2 \doteq 0.0139$$

b i When $t = 0$,

$$W = 20e^{-kt}$$

$$= 20e^0$$

$$\therefore W = 20 \text{ g}$$

ii When $t = 24$,

$$W = 20e^{-24k}$$

$$= 20e^{-24 \frac{\ln 2}{50}}$$

$$\doteq 14.3 \text{ g}$$

iii When $t = 1$ week

$$= 7 \times 24 \text{ hours}$$

$$= 168 \text{ hours}$$

$$W = 20e^{-168 \frac{\ln 2}{50}} \doteq 1.95 \text{ g}$$

c When $W = 1$ g, $20e^{-\frac{\ln 2}{50} \times t} = 1$

$$\therefore e^{-\frac{\ln 2}{50} \times t} = 0.05$$

$$\therefore -\frac{\ln 2}{50} \times t = \ln 0.05$$

$$\therefore t = \frac{-50 \ln 0.05}{\ln 2} \doteq 216.1 \text{ hours}$$

d $\frac{dW}{dt}$

$$= 20e^{-kt}(-k)$$

$$= \left(-20 \frac{\ln 2}{50}\right) \times e^{-\frac{\ln 2}{50} t}$$

i When $t = 100$ hours,

$$\frac{dW}{dt} = \left(\frac{-20 \ln 2}{50}\right) e^{-2 \ln 2}$$

$$\doteq -0.0693 \text{ g/h}$$

ii When $t = 1000$ hours,

$$\frac{dW}{dt} = \left(\frac{-20 \ln 2}{50}\right) e^{-2 \ln 2}$$

$$\doteq -2.64 \times 10^{-7} \text{ g/h}$$

e $\frac{dW}{dt} = -k(20e^{-kt}) = -kW \therefore \frac{dW}{dt} \propto W$

9 $T = 5 + 95e^{-kt} \text{ } ^\circ\text{C}$

a $T = 20^\circ\text{C}$ when $t = 15$

$$\therefore 20 = 5 + 95e^{-15k}$$

$$\therefore 15 = 95e^{-15k}$$

$$\therefore \ln\left(\frac{15}{95}\right) = -15k$$

$$\therefore k = \frac{\ln\left(\frac{15}{95}\right)}{-15} \doteq 0.1231$$

b When $t = 0$,

$$T = 5 + 95e^0$$

$$\therefore T = 5 + 95$$

$$= 100^\circ\text{C}$$

c $\frac{dT}{dt} = 0 + 95e^{-kt}(-k)$

$$= -(95e^{kt})k$$

$$= -k(T - 5)$$

$$\therefore \frac{dT}{dt} \propto T - 5$$

d $\frac{dT}{dt} = -95e^{-kt} \times k$
 $\doteq -11.6902e^{-0.1231t}$

i When $t = 0$,

$$\frac{dT}{dt} \doteq -11.69$$

$\therefore$ temperature is decreasing at 11.7°C/min

ii When $t = 10$,

$$\frac{dT}{dt} \doteq -11.6902e^{-1.231}$$

$$\doteq -3.415$$

$\therefore$ temperature is decreasing at 3.42°C/min

iii When $t = 20$,

$$\frac{dT}{dt} \doteq -11.6902e^{-2.462}$$

$$\doteq -0.998$$

$\therefore$ temperature is decreasing at 0.998°C/min

10 $H(t) = 20 \ln(3t + 2) + 30$ cm, $t \geq 0$

a Planted when $t = 0$

$$\therefore H(0) = 20 \ln(2) + 30 \div 43.9 \text{ cm}$$

b When $H = 1 \text{ m} = 100 \text{ cm}$

$$\therefore 20 \ln(3t + 2) + 30 = 100$$

$$\therefore 20 \ln(3t + 2) = 70$$

$$\therefore \ln(3t + 2) = 3.5$$

$$\therefore 3t + 2 = e^{3.5}$$

$$\therefore 3t = e^{3.5} - 2$$

$$\therefore t = \frac{e^{3.5} - 2}{3} \text{ years}$$

$$\therefore t \div 10.4 \text{ years}$$

c $\frac{dH}{dt} = \frac{20}{(3t + 2)} \times 3$

$$= \frac{60}{3t + 2} \text{ cm/year}$$

i When $t = 3$,

$$\frac{dH}{dt} = \frac{60}{11} \div 5.454$$

$\therefore$ it is growing at 5.45 cm/year

ii When $t = 10$,

$$\frac{dH}{dt} = \frac{60}{32} = 1.875$$

$\therefore$ it is growing at 1.88 cm/year

11 $A = s(1 - e^{-kt})$, $t \geq 0$

a When $t = 0$,

$$A = s(1 - e^0)$$

$$= s(1 - 1)$$

$$= 0$$

b When $t = 3$, $A = 5$ and $s = 10$

$$\therefore 5 = 10(1 - e^{-3k})$$

$$\therefore 0.5 = 1 - e^{-3k}$$

$$\therefore e^{-3k} = 0.5$$

$$\therefore e^{3k} = 2$$

$$\therefore 3k = \ln 2$$

$$\therefore k = \frac{1}{3} \ln 2$$

$$\therefore k \div 0.231$$

c When $t = 5$ and $s = 10$

$$\frac{dA}{dt} = ske^{-kt}$$

$$= 10 \left(\frac{1}{3} \ln 2 \right) \left(e^{-\frac{5}{3} \ln 2} \right)$$

$$\div 0.728 \text{ units/hour}$$

d Now $\frac{dA}{dt} = sk(e^{-kt})$

$$= k(se^{-kt})$$

$$= -k(-se^{-kt})$$

$$= -k(A - s)$$

$$\therefore \frac{dA}{dt} \propto A - s$$

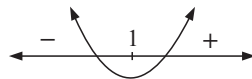
12 $f(x) = \frac{e^x}{x}$

a No x -intercept since $e^x \neq 0$, i.e., $f(x) \neq 0$, and no y -intercept since $\frac{e^0}{0}$ is undefined.

b as $x \rightarrow +\infty$ $f(x) \rightarrow \infty$, and as $x \rightarrow -\infty$ $f(x) \rightarrow 0$ (below)

c By the quotient rule, $f'(x) = \frac{e^x x - e^x(1)}{x^2} = \frac{e^x(x-1)}{x^2}$ with sign diagram:

and when $x = 1$, $f(x) = \frac{e^1}{1} = e$

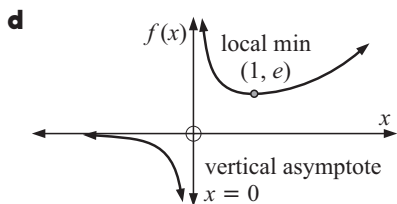


$\therefore$ local minimum when $x = 1$

i.e., local minimum at $(1, e)$.

Note: Vertical asymptote exists at $x = 0$,

i.e., as $x \rightarrow 0$ (from above), $y \rightarrow +\infty$, as $x \rightarrow 0$ (from below), $y \rightarrow -\infty$



e

$$\text{Now } f'(x) = \frac{e^x(x-1)}{x^2}$$

$$\therefore f'(-1) = \frac{e^{-1}(-1-1)}{(-1)^2} = -\frac{2}{e}$$

$$\therefore \text{ slope of the tangent is } = -\frac{2}{e}$$

$$\text{and when } x = -1, y = \frac{e^{-1}}{-1} = -\frac{1}{e}$$

$$\therefore \text{ equation of tangent is } \frac{y - \left(-\frac{1}{e}\right)}{x - (-1)} = -\frac{2}{e} \quad \text{i.e., } \frac{y + \frac{1}{e}}{x + 1} = -\frac{2}{e}$$

$$\therefore e\left(y + \frac{1}{e}\right) = -2(x + 1)$$

$$ey + 1 = -2x - 2$$

$$\therefore 2x + ey = -3$$

13 a $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm, $t \geq 0$

$$v(t) = 100 - 40e^{-\frac{t}{5}} \text{ cm/sec}$$

$$a(t) = 8e^{-\frac{t}{5}} \text{ cm/sec}^2$$

c as $t \rightarrow +\infty$, $e^{-\frac{t}{5}} \rightarrow 0$,
 $\therefore v(t) \rightarrow 100 \text{ cm/sec}$ (below)

e When $v(t) = 80 \text{ cm/sec}$,

$$100 - 40e^{-\frac{t}{5}} = 80$$

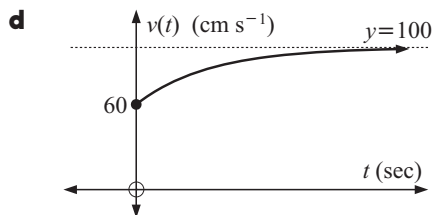
$$\therefore -40e^{-\frac{t}{5}} = -20$$

$$\therefore e^{-\frac{t}{5}} = 0.5$$

$$\therefore -\frac{t}{5} = \ln 0.5$$

$$\therefore t = -5 \ln 0.5 \div 3.47 \text{ sec}$$

b When $t = 0$,
 $s(0) = 200 \text{ cm}$ $v(0) = 60 \text{ cm/sec}$
 $a(0) = 8 \text{ cm/sec}^2$



14 a $A(t) = t \ln t + 1$, $0 < t \leq 5$

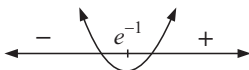
$$\therefore A'(t) = \ln t + t \times \frac{1}{t} + 0 \quad \{\text{product rule}\}$$

$$= \ln t + 1$$

$$\text{which is } 0 \text{ when } \ln t = -1$$

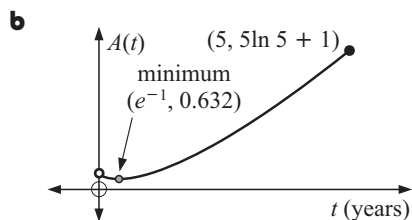
$$\text{i.e., } t = e^{-1}$$

and the sign diagram of $A'(t)$ is:



$$\therefore A(t) \text{ is a minimum when } t = \frac{1}{e} \div 0.3679 \text{ years}$$

i.e., at 4.41 months old.



15 a $f(x) = \frac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}$, $\therefore f'(x) = \frac{1}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}(-x) = \frac{-x}{\pi\sqrt{2}}e^{-\frac{1}{2}x^2}$

$$\text{which is } 0 \text{ only when } x = 0, \text{ and when } x = 0, f(0) = \frac{1}{\pi\sqrt{2}}$$

$$\text{and } f'(x) \text{ has sign diagram: } \begin{array}{c} + \quad \quad - \\ \curvearrowright \quad \quad \curvearrowleft \\ 0 \end{array} \therefore \text{ there is a local maximum at } \left(0, \frac{1}{\pi\sqrt{2}}\right)$$

and the function is increasing for $x \leq 0$, and decreasing for $x \geq 0$

$$\mathbf{b} \quad f'(x) = \frac{-x}{\pi\sqrt{2}} e^{-\frac{1}{2}x^2} = \frac{1}{\pi\sqrt{2}} \left(-xe^{-\frac{1}{2}x^2} \right)$$

$$\therefore f''(x) = \frac{1}{\pi\sqrt{2}} \left((-1)e^{-\frac{1}{2}x^2} + (-x)e^{-\frac{1}{2}x^2}(-x) \right) \quad \{\text{product rule}\}$$

$$= \frac{1}{\pi\sqrt{2}} e^{-\frac{1}{2}x^2} (x^2 - 1)$$

$$= \frac{1}{\pi\sqrt{2}} e^{-\frac{1}{2}x^2} (x+1)(x-1) \quad \text{which has sign diagram: } \begin{array}{c} + \quad - \quad + \\ -1 \quad 1 \end{array}$$

$$\text{When } x = 1, \quad f(x) = \frac{1}{\pi\sqrt{2}} e^{-\frac{1}{2}} = \frac{1}{\pi\sqrt{2}e} \quad \text{and also when } x = -1, \quad f(x) = \frac{1}{\pi\sqrt{2}e}$$

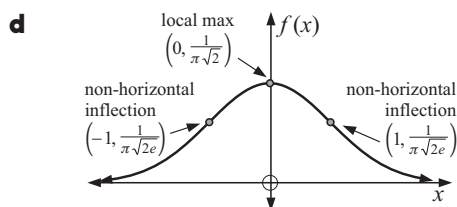
$$\therefore \text{ there are points of inflection at } \left(1, \frac{1}{\pi\sqrt{2}e} \right) \text{ and } \left(-1, \frac{1}{\pi\sqrt{2}e} \right)$$

$$\mathbf{c} \quad \text{as } x \rightarrow \infty, \quad e^{-\frac{1}{2}x^2} \rightarrow 0 \quad (\text{above}),$$

$$\therefore f(x) \rightarrow 0 \quad (\text{above}),$$

$$\text{as } x \rightarrow -\infty, \quad e^{-\frac{1}{2}x^2} \rightarrow 0 \quad (\text{above}),$$

$$\therefore f(x) \rightarrow 0 \quad (\text{above})$$



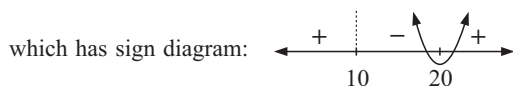
$$\mathbf{16} \quad C(x) = 4 \ln x + \left(\frac{30-x}{10} \right)^2, \quad x \geq 10 \quad \therefore \quad C'(x) = \frac{4}{x} + 2 \left(\frac{30-x}{10} \right) \left(-\frac{1}{10} \right)$$

$$= \frac{4}{x} - \frac{30-x}{50}$$

$$= \frac{200 - x(30-x)}{50x}$$

$$= \frac{200 - 30x + x^2}{50x}$$

$$= \frac{(x-10)(x-20)}{50x}$$



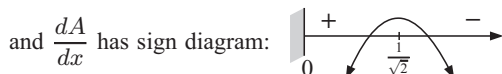
$\therefore$ the minimum cost occurs when $x = 20$, i.e., 20 kettles/day are produced

$$\mathbf{17} \quad \text{Let coordinates of D be } (x, 0), \text{ where } x > 0. \quad \therefore \text{ the coordinates of C are } (x, e^{-x^2})$$

$$\therefore \text{ area ABCD} = 2xe^{-x^2} \quad \therefore \quad \frac{dA}{dx} = 2e^{-x^2} + 2xe^{-x^2}(-2x) \quad \{\text{product rule}\}$$

$$= 2e^{-x^2} (1 - 2x^2)$$

$$= 2e^{-x^2} (1 + \sqrt{2}x)(1 - \sqrt{2}x)$$



$$\therefore \text{ area is a maximum when } x = \frac{1}{\sqrt{2}}, \text{ and so C is } \left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}} \right).$$

$$\mathbf{18} \quad P(x) = R(x) - C(x)$$

$$\therefore P(x) = \left[1000 \ln \left(1 + \frac{x}{400} \right) + 600 \right] - [x(1.5) + 300]$$

$$= 1000 \ln(1 + 0.0025x) - 1.5x + 300$$

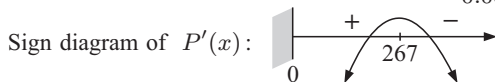
$$\therefore P'(x) = 1000 \left(\frac{0.0025}{1 + 0.0025x} \right) - 1.5 = \frac{2.5}{1 + 0.0025x} - 1.5$$

which is 0 when $\frac{2.5}{1 + 0.0025x} = \frac{3}{2}$

$$\therefore 3 + 0.0075x = 5$$

$$\therefore 0.0075x = 2$$

$$\therefore x = \frac{2}{0.0075} \doteq 266.7 \quad \text{and} \quad P(266) \doteq 410.83$$



$$P(267) \doteq 410.83$$

$\therefore$ to maximise profit, 266 or 267 torches/day should be produced.

- 19 a** $y = ax^2$ ($a > 0$) touches $y = \ln x$ when $ax^2 = \ln x$

If they touch at $x = b$

$$\text{then } ab^2 = \ln b \quad \dots\dots (1)$$

$$\text{Now if } y = ax^2$$

$$\text{and if } y = \ln x$$

$$\text{then } \frac{dy}{dx} = 2ax$$

$$\text{then } \frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \text{ at } x = b, \quad \frac{dy}{dx} = 2ab$$

$$\therefore \text{ at } x = b, \quad \frac{dy}{dx} = \frac{1}{b}$$

and the slope of the tangent at $x = b$ to the curve $y = ax^2$ is also the slope of the tangent to $y = \ln x$ at $x = b$, $\therefore \frac{1}{b} = 2ab \quad \dots\dots (2)$

b Now $ab^2 = \frac{1}{2}$ {from (2)}

$$ab^2 = \ln b \quad \{\text{from (1)}\}$$

$$\therefore \ln b = \frac{1}{2}$$

$$\therefore b = e^{\frac{1}{2}} \quad \text{i.e., } b = \sqrt{e} \quad \text{and when } x = b = \sqrt{e}, \quad y = \ln x = \ln e^{\frac{1}{2}} = \frac{1}{2}$$

$\therefore$ point of contact is $(\sqrt{e}, \frac{1}{2})$

c $a = \frac{1}{2b^2}$ {from (2)} $\therefore a = \frac{1}{2(\sqrt{e})^2} = \frac{1}{2e}$

d The tangent has slope $2ab = 2\left(\frac{1}{2e}\right)\sqrt{e} = \frac{1}{\sqrt{e}}$ and passes through $(\sqrt{e}, \frac{1}{2})$

$$\therefore \text{ tangent is } \frac{y - \frac{1}{2}}{x - \sqrt{e}} = \frac{1}{\sqrt{e}} \quad \therefore y - \frac{1}{2} = \frac{1}{\sqrt{e}}(x - \sqrt{e})$$

$$\therefore y - \frac{1}{2} = \frac{1}{\sqrt{e}}x - 1 \quad \text{i.e., } y = xe^{-\frac{1}{2}} - \frac{1}{2}$$

20 $P(t) = \frac{50\,000}{1 + 1000e^{-0.5t}}, \quad 0 \leq t \leq 25$

$$= 50\,000(1 + 1000e^{-0.5t})^{-1}$$

$$\therefore P'(t) = -50\,000(1 + 1000e^{-0.5t})^{-2}(-500e^{-0.5t})$$

$$= 2.5 \times 10^7 e^{-0.5t}(1 + 1000e^{-0.5t})^{-2}$$

Now the wasp population is growing the fastest when $\frac{dP}{dt}$ is a maximum.

Using technology, the graph of $P'(t)$ can be drawn and the maximum obtained.

Maximum occurs when $x = 13.8155$, i.e., $x \doteq 13.8$ weeks

$$21 \quad f(t) = ate^{bt^2} = (at)e^{bt^2}$$

$$\therefore f'(t) = ae^{bt^2} + ate^{bt^2}(2bt) = ae^{bt^2}(1 + 2bt^2) \quad \{\text{product rule}\}$$

$$\therefore f'(t) \text{ is } 0 \quad \text{when } 1 + 2bt^2 = 0 \quad \text{but } t = 2 \quad \text{at this point,}$$

$$\therefore 1 + 8b = 0$$

$$b = -\frac{1}{8}$$

$$\therefore f(t) = ate^{-\frac{t^2}{8}} \quad \text{and } f(2) = 1, \quad \therefore 2ae^{-\frac{4}{8}} = 1$$

$$\therefore ae^{-\frac{1}{2}} = \frac{1}{2} \quad \text{and so } a = \frac{\sqrt{e}}{2}$$

REVIEW SET 23A

$$1 \quad a \quad y = e^{x^3+2} \\ = e^u \quad \text{where } u = x^3 + 2$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{dy}{du} \frac{du}{dx} \\ &= e^u \times 3x^2 \\ &= 3x^2 e^u \\ &= 3x^2 e^{x^3+2} \end{aligned}$$

$$\begin{aligned} b \quad y &= \frac{e^x}{x^2} = \frac{u}{v} \\ \text{Now } \frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \\ \therefore \frac{dy}{dx} &= \frac{e^x x^2 - e^x 2x}{x^4} \\ &= \frac{xe^x(x-2)}{x^4} \\ &= \frac{e^x(x-2)}{x^3} \end{aligned}$$

$$c \quad x^3 + xy^4 = xe^y$$

Differentiating with respect to x ,

$$\begin{aligned} \frac{d}{dx}(x^3) + \frac{d}{dx}(xy^4) &= \frac{d}{dx}(xe^y) \\ \therefore 3x^2 + \frac{d}{dx}(x)y^4 + x\frac{d}{dx}(y^4) &= \frac{d}{dx}(x)e^y + x\frac{d}{dx}(e^y) \\ \therefore 3x^2 + y^4 + 4xy^3\frac{dy}{dx} &= e^y + xe^y\frac{dy}{dx} \\ \therefore \frac{dy}{dx}(4xy^3 - xe^y) &= e^y - 3x^2 - y^4 \\ \therefore \frac{dy}{dx} &= \frac{e^y - 3x^2 - y^4}{4xy^3 - xe^y} \end{aligned}$$

$$2 \quad y = e^{-x^2} \quad \text{and when } x = 1, \quad y = e^{-x^2} = e^{-1} = \frac{1}{e}, \quad \therefore \text{point of contact is } \left(1, \frac{1}{e}\right)$$

$$\text{Now } \frac{dy}{dx} = -2xe^{-x^2}$$

$$\therefore \text{at } x = 1, \quad \frac{dy}{dx} = -2e^{-1}$$

$$\therefore \text{the slope of the tangent} = -\frac{2}{e} \quad \text{and the slope of the normal} = \frac{e}{2}$$

$$\therefore \text{equation of the normal is } \frac{y - \frac{1}{e}}{x - 1} = \frac{e}{2} \quad \text{i.e., } 2\left(y - \frac{1}{e}\right) = e(x - 1)$$

$$\therefore 2y - \frac{2}{e} = ex - e$$

$$\therefore 2ey - 2 = e^2x - e^2$$

$$\therefore e^2x - 2ey = e^2 - 2 \quad \text{or } y = \frac{e}{2}x - \frac{e}{2} + \frac{1}{e}$$

3 Graphs meet when $e^x + 3 = 9 - e^{-x}$

$$\therefore e^{2x} + 3e^x = 9e^x - 1 \quad \{\times e^x\}$$

$$\therefore e^{2x} - 6e^x + 1 = 0$$

$$\therefore e^x = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm \sqrt{8} = 3 \pm 2\sqrt{2}$$

$$\therefore x = \ln(3+2\sqrt{2}) \text{ or } \ln(3-2\sqrt{2})$$

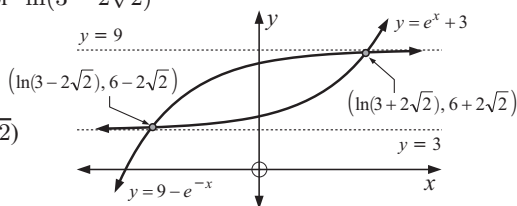
and $y = e^{\ln(3+2\sqrt{2})} + 3 = 6 + 2\sqrt{2}$ or

$$y = e^{\ln(3-2\sqrt{2})} + 3 = 6 - 2\sqrt{2}$$

i.e., graphs meet at $(\ln(3+2\sqrt{2}), 6+2\sqrt{2})$

and $(\ln(3-2\sqrt{2}), 6-2\sqrt{2})$

i.e., (1.76, 8.83) and (-1.76, 3.17)



4 a $f(x) = \frac{e^x}{x-1}$ $\therefore$ when $x = 0$, $f(x) = \frac{e^0}{-1} = -1$ and so the y -intercept is at $(0, -1)$

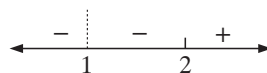
no x -intercept exist as $\frac{e^x}{x-1} \neq 0$

b $f(x)$ is defined for all $x \neq 1$

c Now $f'(x) = \frac{e^x(x-1) - e^x(1)}{(x-1)^2}$ {quotient rule}

$$= \frac{e^x(x-2)}{(x-1)^2}$$

and has sign diagram:



$\therefore f(x)$ is decreasing for $x < 1$ and $1 < x \leq 2$, and increasing for $x \geq 2$.

$$f''(x) = \frac{[e^x(x-2) + e^x(1)](x-1)^2 - e^x(x-2)[2(x-1)^1]}{(x-1)^4}$$
 {product and quotient rules}

$$= \frac{[e^x(x-2+1)](x-1)^2 - 2e^x(x-2)(x-1)}{(x-1)^4}$$

$$= \frac{e^x(x-1)(x-1)^2 - 2e^x(x-2)(x-1)}{(x-1)^4}$$

$$= \frac{e^x(x-1)[(x-1)^2 - 2(x-2)]}{(x-1)^4}$$

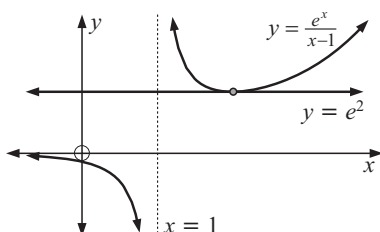
$$= \frac{e^x(x-1)[x^2 - 2x + 1 - 2x + 4]}{(x-1)^4}$$

$$= \frac{e^x(x^2 - 4x + 5)}{(x-1)^3} \text{ where the quadratic has } \Delta < 0$$

The sign diagram of $f''(x)$ is:

$\therefore$ concave down for all $x \leq 1$
and concave up for all $x \geq 1$.

d



e Using **c** we have a local minimum at

$$(2, e^2) \quad \{\text{as } f(2) = \frac{e^2}{2-1} = e^2\}$$

$\therefore$ the tangent at $x = 2$ is horizontal and is $y = e^2$.

5 $H(t) = 60 + 40 \ln(2t + 1)$ cm, $t \geq 0$

a when planted $t = 0$ $\therefore H(0) = 60 + 40 \ln(1) = 60 + 40(0) = 60$ cm

b i When $H(t) = 150$ cm,

$$\therefore 60 + 40 \ln(2t + 1) = 150$$

$$\therefore 40 \ln(2t + 1) = 90$$

$$\therefore \ln(2t + 1) = \frac{90}{40} = 2.25$$

$$\therefore 2t + 1 = e^{2.25}$$

$$\therefore 2t = e^{2.25} - 1$$

$$\therefore t = \frac{1}{2}(e^{2.25} - 1)$$

$$\therefore t \div 4.244 \text{ years}$$

ii When $H(t) = 300$ cm,

$$\therefore 60 + 40 \ln(2t + 1) = 300$$

$$\therefore 40 \ln(2t + 1) = 240$$

$$\therefore \ln(2t + 1) = 6$$

$$\therefore 2t + 1 = e^6$$

$$\therefore 2t = e^6 - 1$$

$$\therefore t = \frac{1}{2}(e^6 - 1)$$

$$\therefore t \div 201.2 \text{ years}$$

c $H'(t) = 40 \left(\frac{2}{2t + 1} \right) = \frac{80}{2t + 1}$ cm/year

i When $t = 2$, $H'(2) = \frac{80}{5} = 16$ cm/year

ii When $t = 20$, $H'(20) = \frac{80}{41} \div 1.95$ cm/year

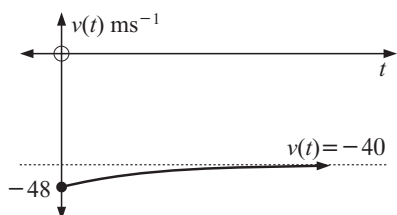
6 $s(t) = 80e^{-\frac{t}{10}} - 40t$ metres, $t \geq 0$

a $\therefore v(t) = -8e^{-\frac{t}{10}} - 40$ m/sec and $a(t) = 0.8e^{-\frac{t}{10}}$ m/sec²

b When $t = 0$, $s(0) = 80$ m $v(0) = -48$ m/sec $a(0) = 0.8$ m/sec²

c as $t \rightarrow \infty$, $e^{-\frac{t}{10}} \rightarrow 0$, $\therefore v(t) \rightarrow -40$ m/sec (below)

d



e

when $v(t) = -44$ m/sec

$$-8e^{-\frac{t}{10}} - 40 = -44$$

$$\therefore -8e^{-\frac{t}{10}} = -4$$

$$\therefore e^{-\frac{t}{10}} = 0.5$$

$$\therefore -\frac{t}{10} = \ln 0.5$$

$$\therefore t = -10 \ln 0.5$$

$$\therefore t \div 6.93 \text{ seconds}$$

7 Let the coordinates of B be $(x, 0)$

$\therefore$ the coordinates of A are (x, e^{-x})

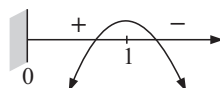
$$\therefore \text{area OABC} = xe^{-x}$$

$$\therefore \frac{dA}{dx} = (1)e^{-x} + x(-e^{-x}) \quad \{\text{product rule}\}$$

$$= e^{-x}(1 - x)$$

$$= \frac{1 - x}{e^x}$$

and has sign diagram:



$\therefore$ has a local maximum when $x = 1$

and when $x = 1$, $y = 1e^{-1} = \frac{1}{e}$

$\therefore$ the coordinates of A are $\left(1, \frac{1}{e}\right)$

$$8 \quad P(x) = R(x) - C(x)$$

$$= [200 \ln(1 + \frac{x}{100}) + 1000] - [(x - 100)^2 + 200]$$

$$= 200 \ln(1 + 0.01x) - (x - 100)^2 + 800$$

$$\frac{dP}{dx} = 200 \left(\frac{0.01}{1 + 0.01x} \right) - 2(x - 100)^1$$

$$= \frac{2}{1 + 0.01x} - \frac{2(x - 100)}{1}$$

$$= \frac{2 - 2(x - 100)(1 + 0.01x)}{1 + 0.01x}$$

$$= \frac{2 - 2(x + 0.01x^2 - 100 - x)}{1 + 0.01x}$$

$$= \frac{2 - 0.02x^2 + 200}{1 + 0.01x}$$

$$= \frac{202 - 0.02x^2}{1 + 0.01x} \quad \text{which is 0 when } 0.02x^2 = 202$$

$$\therefore x^2 = 10100$$

$$\therefore x = \sqrt{10100} \quad \{\text{as } x > 0\}$$

$$\therefore x \doteq 100.49$$

and sign diagram of $\frac{dP}{dx}$ is:

$\therefore$ maximum profit when $x \doteq 100.49$

Now $P(100) = \$938.63$ and $P(101) = \$938.63$

$\therefore$ maximum profit of \$938.63 when 100 or 101 shirts are made.

REVIEW SET 23B

1 a

$$y = \ln(x^3 - 3x)$$

$$\therefore y = \ln u \quad \text{where } u = x^3 - 3x$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

$$= \frac{1}{u}(3x^2 - 3)$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 - 3}{x^3 - 3x}$$

c

$$e^{x+y} = xy^2$$

$$\therefore e^x e^y = xy^2$$

Differentiating with respect to x ,

$$\frac{d}{dx}(e^x e^y) = \frac{d}{dx}(xy^2)$$

$$\therefore \frac{d}{dx}(e^x) e^y + e^x \frac{d}{dx}(e^y) = \frac{d}{dx}(x)y^2 + x \frac{d}{dx}(y^2) \quad \{\text{product rule}\}$$

$$\therefore e^x e^y + e^x e^y \frac{dy}{dx} = y^2 + 2xy \frac{dy}{dx}$$

$$\therefore e^{x+y} \left(1 + \frac{dy}{dx} \right) = y^2 + 2xy \frac{dy}{dx}$$

b

$$y = \ln \left(\frac{x+3}{x^2} \right)$$

$$= \ln(x+3) - \ln x^2$$

$$= \ln(x+3) - 2 \ln x$$

$$\therefore \frac{dy}{dx} = \frac{1}{x+3} - \frac{2}{x}$$

2 $y = \ln(x^2 + 3)$

$\therefore y = \ln u$ where $u = x^2 + 3$

Now $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{u} \times 2x$

$\therefore \frac{dy}{dx} = \frac{2x}{x^2 + 3}$ and when $x = 0$, $\frac{dy}{dx} = 0$, i.e., the slope of the tangent is 0.

But at $x = 0$, $y = \ln(0 + 3) = \ln 3$

$\therefore$ tangent is $y = \ln 3$ which does not cut the x -axis.

3 a $e^{2x} = 3e^x$

$\therefore e^{2x} - 3e^x = 0$

$\therefore e^x(e^x - 3) = 0$

$\therefore e^x = 0$ or 3

$\therefore e^x = 3$ {as $e^x > 0$ }

$\therefore x = \ln 3$

b $e^{2x} - 7e^x + 12 = 0$

$\therefore (e^x - 3)(e^x - 4) = 0$

$\therefore e^x = 3$ or 4

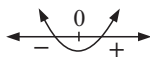
$\therefore x = \ln 3$ or $\ln 4$

4 $f(x) = e^x - x$

a $f'(x) = e^x - 1$

which is 0 when $e^x = 1$
i.e., $x = 0$

Sign diagram of $f'(x)$ is:



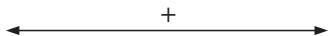
and at $x = 0$, $f(0) = e^0 - 0 = 1$

$\therefore$ there is a local minimum at $(0, 1)$

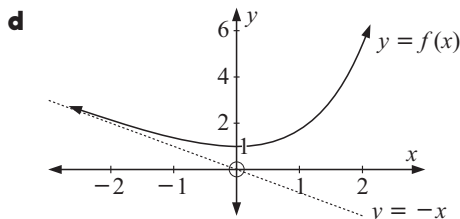
c $f''(x) = e^x$

$\therefore f''(x) > 0$ for all x

$\therefore f(x)$ is concave up for all x



b as $x \rightarrow +\infty$, $e^x \rightarrow \infty$ faster than x $\therefore f(x) \rightarrow +\infty$,
as $x \rightarrow -\infty$, $e^x \rightarrow 0$
 $\therefore f(x) \rightarrow -x$ (from above)



e Since a local minimum exists at $(0, 1)$, $f(x) \geq 1$ for all x .

$\therefore e^x - x \geq 1$

$\therefore e^x \geq x + 1$ for all x

5 a $f(x) = \ln(e^x + 3)$

$\therefore y = \ln u$ where $u = e^x + 3$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$

$= \frac{1}{u} \times e^x$

$= \frac{e^x}{u}$

$\therefore f'(x) = \frac{e^x}{e^x + 3}$

b $f(x) = \ln\left(\frac{(x+2)^3}{x}\right)$

$= \ln(x+2)^3 - \ln x$

$= 3 \ln(x+2) - \ln x$

$\therefore f'(x) = \frac{3}{x+2} - \frac{1}{x}$

$= \frac{3x - (x+2)}{x(x+2)}$

$\therefore f'(x) = \frac{2x-2}{x(x+2)}$

$$6 \quad a \quad 3e^x - 5 = -2e^{-x}$$

$$\therefore 3e^{2x} - 5e^x = -2 \quad \{\times \text{ by } e^x\}$$

$$\therefore 3e^{2x} - 5e^x + 2 = 0$$

$$\therefore (3e^x - 2)(e^x - 1) = 0$$

$$\therefore e^x = \frac{2}{3} \text{ or } 1$$

$$\therefore x = \ln \frac{2}{3} \text{ or } 0$$

$$b \quad 2 \ln x - 3 \ln \left(\frac{1}{x} \right) = 10$$

$$\therefore 2 \ln x - 3 \ln(x^{-1}) = 10$$

$$\therefore 2 \ln x + 3 \ln x = 10$$

$$\therefore 5 \ln x = 10$$

$$\therefore \ln x = 2$$

$$\therefore x = e^2$$

$$7 \quad s(t) = 25t - 10 \ln t \text{ cm} \quad t \geq 1$$

$$a \quad v(t) = 25 - \frac{10}{t} \text{ cm/min}$$

$$\therefore a(t) = 10t^{-2}$$

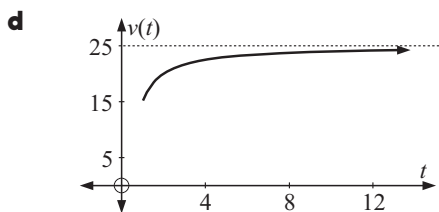
$$= \frac{10}{t^2} \text{ cm/min}^2$$

$$b \quad \text{when } t = e, \quad s(e) = 25e - 10 \ln e = 25e - 10 \text{ cm} \div 57.96 \text{ cm}$$

$$v(e) = 25 - \frac{10}{e} \text{ cm/min} \div 21.32 \text{ cm/min}$$

$$a(e) = \frac{10}{e^2} \text{ cm/min}^2 \div 1.35 \text{ cm/min}^2$$

$$c \quad \text{as } t \rightarrow +\infty, \quad \frac{10}{t} \rightarrow 0, \quad \therefore v(t) \rightarrow 25 \text{ cm/min (below)}$$



$$e \quad \text{when } v(t) = 12 \text{ cm/min,}$$

$$25 - \frac{10}{t} = 12$$

$$\therefore \frac{10}{t} = 13$$

$$\therefore t = \frac{10}{13} \text{ minutes}$$

$$8 \quad C(x) = 10 \ln x + \left(20 - \frac{x}{10} \right)^2 = 10 \ln x + 400 - 4x + \frac{x^2}{100}$$

$$\therefore C'(x) = \frac{10}{x} - 4 + \frac{x}{50}$$

$$= \frac{500 - 200x + x^2}{50x}$$

$$\text{which is } 0 \quad \text{when } x^2 - 200x + 500 = 0$$

$$\text{i.e., when } x = \frac{200 \pm \sqrt{38\,000}}{2} \div 2.53 \text{ or } 197.47$$

$$\text{But } x \geq 50, \quad x \div 197.47 \quad \text{and } C(197) \div 52.92$$

$$C(198) \div 52.92$$

$$\text{Now } C''(x) = -10x^{-2} + \frac{1}{50}, \quad \therefore C''(197.47) \div -0.1(197.47)^{-2} + 0.02 \div 0.02 \text{ which is } > 0$$

$$\therefore \text{minimum cost when } x \div 197.47$$



i.e., need to produce 197 or 198 clocks/day

Chapter 24

DERIVATIVES OF CIRCULAR FUNCTIONS AND RELATED RATES

EXERCISE 24A

1 a $y = \sin(2x)$
 $\therefore \frac{dy}{dx} = \cos(2x) \frac{d}{dx}(2x)$
 $= 2 \cos(2x)$

c $y = \cos(3x) - \sin x$
 $\therefore \frac{dy}{dx} = -\sin(3x) \times 3 - \cos x$
 $= -3 \sin(3x) - \cos x$

e $y = \cos(3 - 2x)$
 $\therefore \frac{dy}{dx} = -\sin(3 - 2x) \times -2$
 $= 2 \sin(3 - 2x)$

g $y = \sin\left(\frac{x}{2}\right) - 3 \cos x$
 $\therefore \frac{dy}{dx} = \frac{1}{2} \cos\left(\frac{x}{2}\right) + 3 \sin x$

i $y = 4 \sin x - \cos(2x)$
 $\therefore \frac{dy}{dx} = 4 \cos x + \sin(2x) \times 2$
 $= 4 \cos x + 2 \sin(2x)$

2 a $y = x^2 + \cos x$
 $\therefore \frac{dy}{dx} = 2x - \sin x$

c $y = e^x \cos x$
 $\therefore \frac{dy}{dx} = e^x(-\sin x) + e^x \cos x$
 $= e^x \cos x - e^x \sin x$

e $y = \ln(\sin x)$
 $\therefore \frac{dy}{dx} = \frac{\cos x}{\sin x}$
 $= \cot x$

g $y = \sin(3x)$
 $\therefore \frac{dy}{dx} = 3 \cos(3x)$

b $y = \sin x + \cos x$
 $\therefore \frac{dy}{dx} = \cos x - \sin x$

d $y = \sin(x + 1)$
 $\therefore \frac{dy}{dx} = \cos(x + 1) \frac{d}{dx}(x + 1)$
 $= 1 \cos(x + 1)$
 $= \cos(x + 1)$

f $y = \tan(5x)$
 $\therefore \frac{dy}{dx} = \frac{1}{\cos^2(5x)} \times 5$
 $= \frac{5}{\cos^2(5x)}$

h $y = 3 \tan(\pi x)$
 $\therefore \frac{dy}{dx} = 3 \times \frac{1}{\cos^2(\pi x)} \times \pi$
 $= \frac{3\pi}{\cos^2(\pi x)}$

b $y = \tan x - 3 \sin x$
 $\therefore \frac{dy}{dx} = \frac{1}{\cos^2 x} - 3 \cos x$

d $y = e^{-x} \sin x$
 $\therefore \frac{dy}{dx} = -e^{-x} \sin x + e^{-x} \cos x$

f $y = e^{2x} \tan x$
 $\therefore \frac{dy}{dx} = 2e^{2x} \tan x + e^{2x} \times \frac{1}{\cos^2 x}$
 $\therefore \frac{dy}{dx} = 2e^{2x} \tan x + \frac{e^{2x}}{\cos^2 x}$

h $y = \cos\left(\frac{x}{2}\right)$
 $\therefore \frac{dy}{dx} = -\frac{1}{2} \sin\left(\frac{x}{2}\right)$

$$\begin{aligned}\mathbf{i} \quad y &= 3 \tan(2x) \\ \therefore \frac{dy}{dx} &= 3 \times \frac{1}{\cos^2(2x)} \times 2 \\ &= \frac{6}{\cos^2(2x)}\end{aligned}$$

$$\begin{aligned}\mathbf{k} \quad y &= \frac{\sin x}{x} \\ \therefore \frac{dy}{dx} &= \frac{x(\cos x) - \sin x \times 1}{x^2} \\ &= \frac{x \cos x - \sin x}{x^2}\end{aligned}$$

$$\begin{aligned}\mathbf{3} \quad \mathbf{a} \quad y &= \sin(x^2) \\ \therefore \frac{dy}{dx} &= 2x \cos(x^2)\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad y &= \sqrt{\cos x} \\ &= (\cos x)^{\frac{1}{2}} \\ \therefore \frac{dy}{dx} &= \frac{1}{2}(\cos x)^{-\frac{1}{2}} \times (-\sin x) \\ &= -\frac{\sin x}{2\sqrt{\cos x}}\end{aligned}$$

$$\begin{aligned}\mathbf{e} \quad y &= \cos^3 x \\ &= (\cos x)^3 \\ \therefore \frac{dy}{dx} &= 3 \cos^2 x \times (-\sin x) \\ &= -3 \sin x \cos^2 x\end{aligned}$$

$$\begin{aligned}\mathbf{g} \quad y &= \cos(\cos x) \\ \therefore \frac{dy}{dx} &= -\sin(\cos x) \times (-\sin x) \\ &= \sin x \sin(\cos x)\end{aligned}$$

$$\begin{aligned}\mathbf{i} \quad y &= \frac{1}{\sin x} \\ &= (\sin x)^{-1} \\ \therefore \frac{dy}{dx} &= -1(\sin x)^{-2} \times \cos x \\ &= -\frac{\cos x}{\sin^2 x}\end{aligned}$$

$$\begin{aligned}\mathbf{j} \quad y &= x \cos x \\ \therefore \frac{dy}{dx} &= 1 \times \cos x + x(-\sin x) \\ &= \cos x - x \sin x\end{aligned}$$

$$\begin{aligned}\mathbf{l} \quad y &= x \tan x \\ \therefore \frac{dy}{dx} &= 1 \times \tan x + x \times \frac{1}{\cos^2 x} \\ \therefore \frac{dy}{dx} &= \tan x + \frac{x}{\cos^2 x}\end{aligned}$$

$$\begin{aligned}\mathbf{b} \quad y &= \cos(\sqrt{x}) \\ &= \cos(x^{\frac{1}{2}}) \\ \therefore \frac{dy}{dx} &= -\sin(x^{\frac{1}{2}}) \times \frac{1}{2}x^{-\frac{1}{2}} \\ &= -\frac{1}{2\sqrt{x}} \sin(\sqrt{x})\end{aligned}$$

$$\begin{aligned}\mathbf{d} \quad y &= \sin^2 x \\ &= (\sin x)^2 \\ \therefore \frac{dy}{dx} &= 2 \sin x \cos x\end{aligned}$$

$$\begin{aligned}\mathbf{f} \quad y &= \cos x \sin(2x) \\ \therefore \frac{dy}{dx} &= (-\sin x) \sin(2x) + \cos x(2 \cos(2x)) \\ &= -\sin x \sin(2x) + 2 \cos x \cos(2x)\end{aligned}$$

$$\begin{aligned}\mathbf{h} \quad y &= \cos^3 4x \\ &= (\cos(4x))^3 \\ \therefore \frac{dy}{dx} &= 3(\cos(4x))^2 \times (-4 \sin(4x)) \\ &= -12 \sin(4x) \cos^2(4x)\end{aligned}$$

$$\begin{aligned}\mathbf{j} \quad y &= \frac{1}{\cos(2x)} \\ &= (\cos(2x))^{-1} \\ \therefore \frac{dy}{dx} &= -1(\cos(2x))^{-2} \times (-2 \sin(2x)) \\ &= \frac{2 \sin(2x)}{\cos^2(2x)}\end{aligned}$$

$$\begin{aligned} \mathbf{k} \quad y &= \frac{2}{\sin^2(2x)} \\ &= 2(\sin(2x))^{-2} \end{aligned}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -4(\sin(2x))^{-3} \times 2 \cos(2x) \\ &= -\frac{8 \cos(2x)}{\sin^3(2x)} \end{aligned}$$

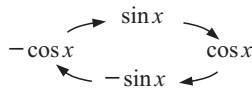
$$\mathbf{l} \quad y = \frac{8}{\tan^3\left(\frac{x}{2}\right)} = 8 \left[\tan\left(\frac{x}{2}\right) \right]^{-3}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -24 \left[\tan\left(\frac{x}{2}\right) \right]^{-4} \times \frac{1}{2} \times \frac{1}{\cos^2\left(\frac{x}{2}\right)} \\ &= \frac{-12}{\cos^2\left(\frac{x}{2}\right) \tan^4\left(\frac{x}{2}\right)} \end{aligned}$$

$$\mathbf{4} \quad \mathbf{a} \quad \text{If } y = \sin x, \text{ then } \frac{dy}{dx} = \cos x, \quad \frac{d^2y}{dx^2} = -\sin x, \quad \frac{d^3y}{dx^3} = -\cos x \quad \text{and} \quad \frac{d^4y}{dx^4} = \sin x$$

b Successive derivatives will cycle through the pattern,

so $\frac{d^n y}{dx^n}$ may only take the values found in **a**.



$$\mathbf{5} \quad \mathbf{a} \quad \text{If } y = \sin(2x + 3), \text{ then } \frac{dy}{dx} = 2 \cos(2x + 3) \quad \text{and} \quad \frac{d^2y}{dx^2} = -4 \sin(2x + 3)$$

$$\therefore \frac{d^2y}{dx^2} + 4y = -4 \sin(2x + 3) + 4 \sin(2x + 3) = 0$$

$$\mathbf{b} \quad \text{If } y = 2 \sin x + 3 \cos x$$

$$\text{then } y' = 2 \cos x - 3 \sin x$$

$$\text{and } y'' = -2 \sin x - 3 \cos x$$

$$\therefore y'' + y = -2 \sin x - 3 \cos x + 2 \sin x + 3 \cos x = 0$$

$$\mathbf{c} \quad y = \frac{\cos x}{1 + \sin x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{(-\sin x)(1 + \sin x) - \cos x(\cos x)}{(1 + \sin x)^2} \\ &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \\ &= \frac{-1 - \sin x}{(1 + \sin x)^2} \quad \{\text{as } \sin^2 x + \cos^2 x = 1\} \\ &= -\frac{(1 + \sin x)}{(1 + \sin x)^2} \\ &= \frac{-1}{1 + \sin x} \end{aligned}$$

Since $\frac{-1}{1 + \sin x}$ never equals 0, there are no horizontal tangents.

$$\mathbf{6} \quad \mathbf{a} \quad y = \sin x$$

$$\therefore \frac{dy}{dx} = \cos x$$

$\therefore$ At $x = 0$, the tangent has slope $\cos 0 = 1$

$$\begin{aligned} \therefore \text{the equation is } \frac{y - 0}{x - 0} &= 1 \\ \text{i.e., } y &= x \end{aligned}$$

$$\mathbf{b} \quad y = \tan x$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos^2 x}$$

$\therefore$ at $x = 0$, the tangent has slope

$$\frac{1}{\cos^2 0} = 1$$

$$\begin{aligned} \text{so the equation is } \frac{y - 0}{x - 0} &= 1 \\ \text{i.e., } y &= x \end{aligned}$$

$$\text{c} \quad y = \cos x \text{ and at } x = \frac{\pi}{6}, y = \frac{\sqrt{3}}{2}$$

$$\therefore \frac{dy}{dx} = -\sin x$$

$$\text{At } x = \frac{\pi}{6}, y = \frac{\sqrt{3}}{2} \text{ and the tangent}$$

$$\text{has slope } -\sin \frac{\pi}{6} = -\frac{1}{2}$$

$$\therefore \text{ the normal has slope } 2,$$

$$\text{so its equation is } \frac{y - \frac{\sqrt{3}}{2}}{x - \frac{\pi}{6}} = 2$$

$$\text{i.e. } y - \frac{\sqrt{3}}{2} = 2x - \frac{\pi}{3}$$

$$\text{i.e. } 2x - y = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$$\text{d} \quad y = \frac{1}{\sin(2x)} = (\sin(2x))^{-1}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= -1(\sin(2x))^{-2} \times 2 \cos(2x) \\ &= -\frac{2 \cos(2x)}{(\sin(2x))^2} \end{aligned}$$

$$\text{At } x = \frac{\pi}{4}, y = 1 \text{ and the tangent}$$

$$\text{has slope } -\frac{2 \cos \frac{\pi}{2}}{(\sin \frac{\pi}{2})^2} = 0$$

$$\therefore \text{ the slope of the normal is undefined, so the normal is } x = \frac{\pi}{4}.$$

$$\text{7} \quad d = 9.3 + 6.8 \cos(0.507t) \text{ m} \quad \therefore \frac{dd}{dt} = -6.8 \sin(0.507t) \times 0.507 = -3.4476 \sin(0.507t)$$

$$\text{a} \quad \text{When } t = 8, \frac{dd}{dt} \div 2.731 > 0$$

$\therefore$ the tide is rising.

$$\text{b} \quad \text{When } t = 8, \text{ the tide is rising at the rate of } 2.731 \text{ m per hour.}$$

$$\text{8} \quad \text{a} \quad V(t) = 340 \sin(100\pi t)$$

$$\begin{aligned} \therefore \frac{dV}{dt} &= 340 \cos(100\pi t) \times 100\pi \\ &= 34\,000\pi \cos(100\pi t) \end{aligned}$$

$$\text{When } t = 0.01,$$

$$\begin{aligned} \frac{dV}{dt} &= 34\,000\pi \times \cos \pi \\ &= -34\,000\pi \text{ units/second} \end{aligned}$$

$$\text{b} \quad V(t) \text{ is a maximum when } \sin(100\pi t) = 1.$$

$$\text{This occurs when } 100\pi t = \frac{\pi}{2}, \text{ and at this time,}$$

$$\begin{aligned} \frac{dV}{dt} &= 34\,000\pi \cos(100\pi t) \\ &= 34\,000\pi \cos\left(\frac{\pi}{2}\right) \\ &= 0 \text{ units/second} \end{aligned}$$

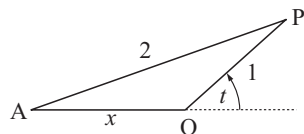
$$\text{9} \quad \text{a} \quad \text{The distance from } A(-x, 0) \text{ to } P(\cos t, \sin t) \text{ is fixed at } 2 \text{ m.}$$

$$\therefore (\cos t + x)^2 + \sin^2 t = 2^2$$

$$\therefore (\cos t + x)^2 = 4 - \sin^2 t$$

$$\therefore x + \cos t = \pm \sqrt{4 - \sin^2 t}$$

$$\therefore \text{ since } x > 0, x = \sqrt{4 - \sin^2 t} - \cos t$$



$$\begin{aligned} \text{b} \quad \text{Now } \frac{dx}{dt} &= \frac{1}{2}(4 - \sin^2 t)^{-\frac{1}{2}}(-2 \sin t \cos t) + \sin t \\ &= \frac{-\sin t \cos t}{\sqrt{4 - \sin^2 t}} + \sin t \end{aligned}$$

$$\begin{aligned} \text{i} \quad \text{When } t &= 0, \\ \sin t &= 0 \text{ and } \cos t = 1 \\ \therefore \frac{dx}{dt} &= 0 + 0 = 0 \end{aligned}$$

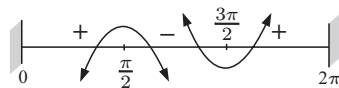
$$\begin{aligned} \text{ii} \quad \text{When } t &= \frac{\pi}{2}, \\ \sin t &= 1 \text{ and } \cos t = 0 \\ \therefore \frac{dx}{dt} &= 0 + \sin \frac{\pi}{2} = 1 \end{aligned}$$

$$\begin{aligned} \text{iii} \quad \text{When } t &= \frac{2\pi}{3}, \\ \sin t &= \frac{\sqrt{3}}{2} \text{ and } \cos t = -\frac{1}{2} \\ \therefore \frac{dx}{dt} &= \frac{-\frac{\sqrt{3}}{2}(-\frac{1}{2})}{\sqrt{4 - \frac{3}{4}}} + \frac{\sqrt{3}}{2} \div 1.106 \end{aligned}$$

10 a If $y = \sin x$ then $\frac{dy}{dx} = \cos x$

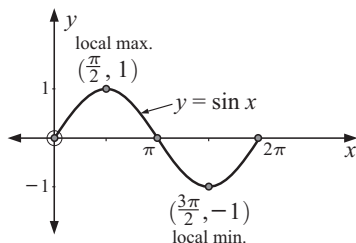
Stationary points occur when $\frac{dy}{dx} = 0$, i.e., when $x = \frac{\pi}{2}, \frac{3\pi}{2}$

Sign diagram for $\frac{dy}{dx}$ is:



Local maximum at $(\frac{\pi}{2}, 1)$.

Local minimum at $(\frac{3\pi}{2}, -1)$.



b If $y = \cos(2x)$ then $\frac{dy}{dx} = -2 \sin(2x)$

$$\frac{dy}{dx} = 0 \quad \text{when} \quad -2 \sin(2x) = 0$$

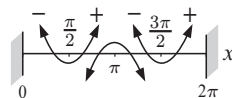
$$\therefore \sin(2x) = 0$$

$$\therefore 2x = k\pi \quad \text{for any integer } k$$

$$\therefore x = \frac{k\pi}{2}$$

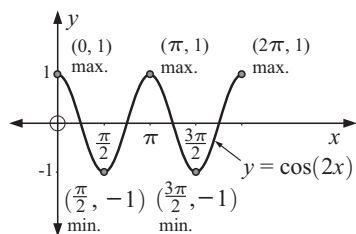
$$\therefore x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \text{ or } 2\pi \quad \text{for } 0 \leq x \leq 2\pi$$

Sign diagram for $\frac{dy}{dx}$ is:



Local maxima at $(0, 1)$, $(\pi, 1)$, $(2\pi, 1)$.

Local minima at $(\frac{\pi}{2}, -1)$, $(\frac{3\pi}{2}, -1)$.



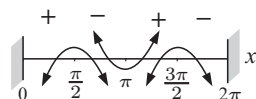
c If $y = \sin^2 x$ then $\frac{dy}{dx} = 2 \sin x \cos x = \sin(2x)$

$$\therefore \frac{dy}{dx} = 0 \quad \text{when} \quad \sin(2x) = 0$$

$$\text{i.e., when } x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \text{ or } 2\pi \quad \text{for } 0 \leq x \leq 2\pi$$

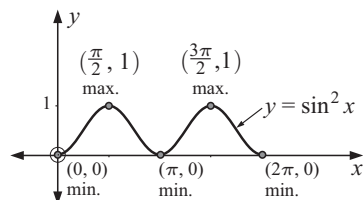
{using **b**}

Sign diagram for $\frac{dy}{dx}$ is:



Local minima at $(0, 0)$, $(\pi, 0)$, $(2\pi, 0)$.

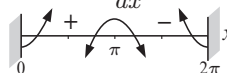
Local maxima at $(\frac{\pi}{2}, 1)$, $(\frac{3\pi}{2}, 1)$.



11 a $f(x) = \frac{1}{\cos x}$ for $0 \leq x \leq 2\pi$
 $\therefore f(x)$ is undefined whenever
 $\cos x = 0$
 i.e., $x = \frac{\pi}{2}, \frac{3\pi}{2}$

b $f(x) = (\cos x)^{-1}$
 $\therefore f'(x) = -1(\cos x)^{-2}(-\sin x)$
 $= \frac{\sin x}{\cos^2 x}$
 $\therefore f'(x) = 0$ when $\sin x = 0$
 i.e., $x = 0, \pi, 2\pi$

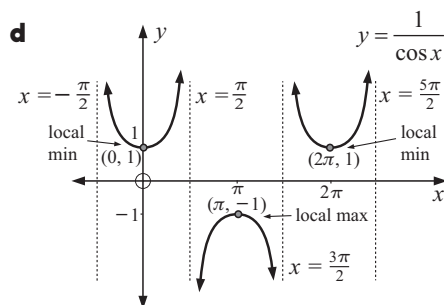
Sign diagram for $\frac{dy}{dx}$ is:



Local minima at $(0, 1)$, $(2\pi, 1)$

Local maximum at $(\pi, -1)$

c $f(x) = \frac{1}{\cos x}$
 $\therefore f(x + 2\pi) = \frac{1}{\cos(x + 2\pi)}$
 $= \frac{1}{\cos x}$
 $= f(x)$
 i.e., $f(x)$ has a period of 2π .

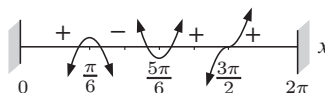


12 If $y = \sin(2x) + 2\cos x$
 then $\frac{dy}{dx} = 2\cos(2x) + (-2\sin x)$
 $= 2[1 - 2\sin^2 x] - 2\sin x$
 $= 2 - 4\sin^2 x - 2\sin x$

At the stationary points, $\frac{dy}{dx} = 0$.

$$\begin{aligned} \therefore -4\sin^2 x - 2\sin x + 2 &= 0 \\ \therefore 2\sin^2 x + \sin x - 1 &= 0 \\ \therefore (2\sin x - 1)(\sin x + 1) &= 0 \\ \therefore \sin x &= \frac{1}{2} \text{ or } \sin x = -1 \\ \therefore x &= \frac{\pi}{6}, \frac{5\pi}{6} \text{ or } x = \frac{3\pi}{2} \end{aligned}$$

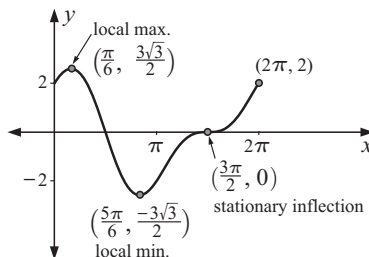
Sign diagram for $\frac{dy}{dx}$ is:



Local maximum at $(\frac{\pi}{6}, \frac{3\sqrt{3}}{2})$.

Local minimum at $(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2})$.

Stationary inflection at $(\frac{3\pi}{2}, 0)$.



13 $x(t) = 1 - 2 \cos t$ cm

$\therefore v(t) = x'(t) = 2 \sin t$ and $a(t) = v'(t) = 2 \cos t$

a When $t = 0$

$$x(0) = 1 - 2 \cos(0)$$

$$= -1 \text{ cm}$$

$$v(0) = 2 \sin(0)$$

$$= 0 \text{ cm/s}$$

$$a(0) = 2 \cos(0)$$

$$= 2 \text{ cm/s}^2$$

b When $t = \frac{\pi}{4}$

$$x\left(\frac{\pi}{4}\right) = 1 - \frac{2}{\sqrt{2}} = 1 - \sqrt{2} \text{ cm}$$

$$v\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} \text{ cm/s} = \sqrt{2} \text{ cm/s}$$

$$a\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} \text{ cm/s} = \sqrt{2} \text{ cm/s}^2$$

The particle is $(\sqrt{2} - 1)$ cm left of the origin, moving right at $\sqrt{2}$ cm/s with increasing speed.

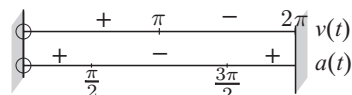
c The particle stops momentarily before it reverses direction. Therefore we need to look for the points where the velocity equals zero, i.e., when

$$v(t) = 2 \sin t = 0$$

$$\therefore \sin t = 0$$

$$\therefore t = 0, \pi, 2\pi \quad (0 \leq t \leq 2\pi)$$

Sign diagrams for $v(t)$ and $a(t)$ are:



The particle reverses direction at $t = 0, \pi, 2\pi$ ($0 \leq t \leq 2\pi$)

At $t = 0$, $x(0) = -1$ cm At $t = \pi$, $x(\pi) = 3$ cm At $t = 2\pi$, $x(2\pi) = -1$ cm

d The particle's speed is increasing when $v(t) = 2 \sin t$ and $a(t) = 2 \cos t$ have the same sign.

Considering $a(t) = 0$

i.e., $2 \cos t = 0$

$\cos t = 0$

$t = \frac{\pi}{2}, \frac{3\pi}{2}$

i.e., $0 \leq t \leq \frac{\pi}{2}$ and $\pi \leq t \leq \frac{3\pi}{2}$

EXERCISE 24B

1 a $y = (\cos x)^{-1} = \frac{1}{\cos x} = \sec x$

Now $\frac{dy}{dx} = -\frac{(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos x \times \cos x} = \frac{\tan x}{\cos x} = \sec x \tan x$

$\therefore \frac{d}{dx}(\sec x) = \sec x \tan x$ as required.

b $y = \cot x = \frac{\cos x}{\sin x} \quad \therefore \frac{dy}{dx} = \frac{\frac{d}{dx}(\cos x) \sin x - \cos x \frac{d}{dx}(\sin x)}{\sin^2 x}$ {quotient rate}

$$= \frac{-\sin x \times \sin x - \cos x \times \cos x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= -\csc^2 x \text{ as required}$$

2 a $y = x \sec x \quad \therefore \frac{dy}{dx} = \sec x + x \sec x \tan x$ {product rule}

$$= \sec x(x \tan x + 1)$$

b $y = e^x \cot x \quad \therefore \frac{dy}{dx} = e^x \cot x + e^x(-\csc^2 x)$ {product rule}

$$= e^x(\cot x - \csc^2 x)$$

c $y = 4 \sec(2x) \quad \therefore \frac{dy}{dx} = 4 \sec(2x) \tan(2x)(2)$

$$= 8 \sec(2x) \tan(2x)$$

$$\begin{aligned} \mathbf{d} \quad y &= e^{-x} \cot\left(\frac{x}{2}\right) & \therefore \frac{dy}{dx} &= -e^{-x} \cot\left(\frac{x}{2}\right) + e^{-x} \left(-\csc^2\left(\frac{x}{2}\right)\right) \left(\frac{1}{2}\right) \quad \{\text{product rule}\} \\ & & &= -e^{-x} \left(\cot\left(\frac{x}{2}\right) + \frac{1}{2} \csc^2\left(\frac{x}{2}\right)\right) \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad y &= x^2 \csc x & \therefore \frac{dy}{dx} &= 2x \csc x + x^2(-\csc x \cot x) \quad \{\text{product rule}\} \\ & & &= x \csc x (2 - x \cot x) \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad y &= x\sqrt{\csc x} = x(\csc x)^{\frac{1}{2}} & \therefore \frac{dy}{dx} &= (\csc x)^{\frac{1}{2}} + \frac{1}{2}x(\csc x)^{-\frac{1}{2}}(-\csc x \cot x) \\ & & &= (\csc x)^{\frac{1}{2}} - \frac{1}{2}x(\csc x)^{\frac{1}{2}} \cot x \\ & & &= \sqrt{\csc x} \left(1 - \frac{1}{2}x \cot x\right) \end{aligned}$$

$$\mathbf{g} \quad y = \ln(\sec x) \quad \therefore \frac{dy}{dx} = \frac{\frac{d}{dx}(\sec x)}{\sec x} = \frac{\cancel{\sec x} \tan x}{\cancel{\sec x}} = \tan x$$

$$\begin{aligned} \mathbf{h} \quad y &= x \csc(x^2) & \therefore \frac{dy}{dx} &= \csc(x^2) + x[-\csc(x^2) \cot(x^2)] (2x) \\ & & &= \csc(x^2)(1 - 2x^2 \cot(x^2)) \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad y &= \frac{\cot x}{\sqrt{x}} = x^{-\frac{1}{2}} \cot x & \therefore \frac{dy}{dx} &= -\frac{1}{2}x^{-\frac{3}{2}} \cot x + x^{-\frac{1}{2}}(-\csc^2 x) \quad \{\text{product rule}\} \\ & & &= -\frac{\cot x + 2x \csc^2 x}{2x\sqrt{x}} \\ & & &= -\frac{\cos x \sin x + 2x}{2x\sqrt{x} \sin^2 x} \end{aligned}$$

$$\mathbf{3} \quad \mathbf{a} \quad y = \sec x \quad \therefore \frac{dy}{dx} = \sec x \tan x$$

$$\text{When } x = \frac{\pi}{4}, \quad y = \sec \frac{\pi}{4} = \frac{1}{\cos \frac{\pi}{4}} = \sqrt{2}$$

$$\text{and } \frac{dy}{dx} = \sec \frac{\pi}{4} \tan \frac{\pi}{4} = \sqrt{2} \times 1 = \sqrt{2}$$

$$\begin{aligned} \therefore \text{ the tangent has equation } \frac{y - \sqrt{2}}{x - \frac{\pi}{4}} &= \sqrt{2} & \therefore y - \sqrt{2} &= x\sqrt{2} - \frac{\pi\sqrt{2}}{4} \\ & & \therefore y &= x\sqrt{2} - \frac{\pi\sqrt{2}}{4} + \sqrt{2} \end{aligned}$$

$$\mathbf{b} \quad y = \cot\left(\frac{x}{2}\right) \quad \therefore \frac{dy}{dx} = -\csc^2\left(\frac{x}{2}\right) \left(\frac{1}{2}\right)$$

$$\text{When } x = \frac{\pi}{3}, \quad \frac{x}{2} = \frac{\pi}{6} \quad \therefore y = \cot\left(\frac{\pi}{6}\right) = -\sqrt{3}$$

$$\text{and } \frac{dy}{dx} = -\frac{1}{2 \sin^2\left(\frac{\pi}{6}\right)} = -\frac{1}{2\left(\frac{1}{2}\right)^2} = -2$$

$$\begin{aligned} \therefore \text{ the tangent has equation } \frac{y - \sqrt{3}}{x - \frac{\pi}{3}} &= -2 & \therefore y - \sqrt{3} &= -2x + \frac{2\pi}{3} \\ & & \therefore y &= -2x + \frac{2\pi}{3} + \sqrt{3} \end{aligned}$$

$$\mathbf{4} \quad \mathbf{a} \quad y = \csc x \quad \therefore \frac{dy}{dx} = -\csc x \cot x$$

$$\text{When } x = \frac{\pi}{6}, \quad y = \frac{1}{\sin \frac{\pi}{6}} = 2 \quad \text{and} \quad \frac{dy}{dx} = -\csc \frac{\pi}{6} \cot \frac{\pi}{6} = -2\sqrt{3}$$

$$\therefore \text{ the normal has slope } \frac{1}{2\sqrt{3}}$$

$$\begin{aligned} \text{and its equation is } \frac{y - 2}{x - \frac{\pi}{6}} &= \frac{1}{2\sqrt{3}} & \therefore 2\sqrt{3}y - 4\sqrt{3} &= x - \frac{\pi}{6} \\ & & \therefore x - 2\sqrt{3}y &= \frac{\pi}{6} - 4\sqrt{3} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad y &= \sqrt{\sec\left(\frac{x}{3}\right)} = \left(\sec\left(\frac{x}{3}\right)\right)^{\frac{1}{2}} \quad \therefore \frac{dy}{dx} = \frac{1}{2} \left(\sec\left(\frac{x}{3}\right)^{-\frac{1}{2}}\right) \sec\left(\frac{x}{3}\right) \tan\left(\frac{x}{3}\right) \left(\frac{1}{3}\right) \\ &= \frac{\sqrt{\sec\left(\frac{x}{3}\right) \tan\left(\frac{x}{3}\right)}}{6} \end{aligned}$$

$$\text{When } x = \pi, \quad \frac{x}{3} = \frac{\pi}{3}$$

$$\therefore y = \frac{1}{\sqrt{\cos\left(\frac{\pi}{3}\right)}} = \frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2}$$

$$\text{and } \frac{dy}{dx} = \frac{\sqrt{\sec\left(\frac{\pi}{3}\right) \tan\left(\frac{\pi}{3}\right)}}{6} = \frac{\sqrt{2}\sqrt{3}}{6} = \frac{1}{\sqrt{6}}$$

$\therefore$ the normal has slope $-\sqrt{6}$

$$\begin{aligned} \text{and its equation is } \frac{y - \sqrt{2}}{x - \pi} &= -\sqrt{6} \quad \therefore y - \sqrt{2} = -x\sqrt{6} + \pi\sqrt{6} \\ \therefore \sqrt{6}x + y &= \pi\sqrt{6} + \sqrt{2} \end{aligned}$$

EXERCISE 24C.1

1 Refer to page 641 of the text.

2 a $\arccos(1) = 0$

b $\arcsin(-1) = -\frac{\pi}{2}$

c $\arctan(1) = \frac{\pi}{4}$

d $\arctan(-1) = -\frac{\pi}{4}$

e $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

f $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

g $\arctan(\sqrt{3}) = \frac{\pi}{3}$

h $\arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

i $\arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

j $\sin^{-1}(-0.767) \div -0.874$

k $\cos^{-1}(0.327) \div 1.24$

l $\tan^{-1}(-50) \div -1.55$

3 a If $\arcsin x = \frac{\pi}{3}$ then $x = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

b If $\arctan(3x) = -\frac{\pi}{4}$ then $3x = \tan\left(-\frac{\pi}{4}\right) = -1 \quad \therefore x = -\frac{1}{3}$

4 Now $\tan\left(\arctan(5) - \arctan\left(\frac{2}{3}\right)\right) = \frac{\tan(\arctan(5)) - \tan\left(\arctan\left(\frac{2}{3}\right)\right)}{1 + \tan(\arctan(5)) \tan\left(\arctan\left(\frac{2}{3}\right)\right)} \quad \{\text{using formula}\}$

$$= \frac{5 - \frac{2}{3}}{1 + 5 \times \frac{2}{3}} = \frac{\frac{13}{3}}{\frac{13}{3}} = 1$$

But $\tan\left(\frac{\pi}{4}\right) = 1$, so $\arctan(5) - \arctan\left(\frac{2}{3}\right) = \frac{\pi}{4}$

* Note there is an error in the question in early print runs of the text.

EXERCISE 24C.2

1 If $y = \arccos x$ then $x = \cos y$

$$\therefore \frac{dy}{dx} = -\sin y = -\sqrt{1 - \cos^2 y} = -\sqrt{1 - x^2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}, \quad x \in [-1, 1]$$

2 a $y = \arctan(2x)$

$$\therefore \frac{dy}{dx} = 2 \times \frac{1}{1 + (2x)^2} = \frac{2}{1 + 4x^2}$$

b $y = \arccos(3x)$

$$\therefore \frac{dy}{dx} = 3 \times \left(\frac{-1}{\sqrt{1 - (3x)^2}} \right) = -\frac{3}{\sqrt{1 - 9x^2}}$$

$$\begin{aligned}
 \text{c} \quad y &= \arcsin\left(\frac{x}{4}\right) \\
 \therefore \frac{dy}{dx} &= \frac{1}{4} \times \frac{1}{\sqrt{1 - \left(\frac{x}{4}\right)^2}} \\
 &= \frac{1}{4\sqrt{1 - \frac{x^2}{16}}} \\
 &= \frac{1}{\sqrt{16 - x^2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad y &= \arccos\left(\frac{x}{5}\right) \\
 \therefore \frac{dy}{dx} &= \frac{1}{5} \times \left(\frac{-1}{\sqrt{1 - \left(\frac{x}{5}\right)^2}}\right) \\
 &= -\frac{1}{5\sqrt{1 - \frac{x^2}{25}}} \\
 &= -\frac{1}{\sqrt{25 - x^2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{e} \quad y &= \arctan(x^2) \\
 \therefore \frac{dy}{dx} &= 2x \times \frac{1}{1 + (x^2)^2} \\
 &= \frac{2x}{1 + x^4}
 \end{aligned}$$

$$\begin{aligned}
 \text{f} \quad y &= \arccos(\sin x) \\
 \therefore \frac{dy}{dx} &= \cos x \times \left(-\frac{1}{\sqrt{1 - \sin^2 x}}\right) \\
 &= -\frac{\cos x}{\cos x} \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{3 a} \quad y &= x \arcsin x \quad \therefore \frac{dy}{dx} = \arcsin x + x \left(\frac{1}{\sqrt{1 - x^2}}\right) \quad \{\text{product rule}\} \\
 &= \arcsin x + \frac{x}{\sqrt{1 - x^2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad y &= e^x \arccos x \quad \therefore \frac{dy}{dx} = e^x \arccos x + e^x \left(-\frac{1}{\sqrt{1 - x^2}}\right) \quad \{\text{product rule}\} \\
 &= e^x \arccos x - \frac{e^x}{\sqrt{1 - x^2}}
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad y &= e^{-x} \arctan x \quad \therefore \frac{dy}{dx} = -e^{-x} \arctan x + e^{-x} \left(\frac{1}{1 + x^2}\right) \quad \{\text{product rule}\} \\
 &= -e^{-x} \arctan x + \frac{e^{-x}}{1 + x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{4 a} \quad \text{If } y &= \arcsin\left(\frac{x}{a}\right), \\
 \text{then } \frac{dy}{dx} &= \left(\frac{1}{a}\right) \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \\
 &= \frac{1}{a\sqrt{1 - \frac{x^2}{a^2}}} \\
 &= \frac{1}{\sqrt{a^2 - x^2}} \quad \text{as required,}
 \end{aligned}$$

and this is defined for $x \in [-a, a]$.

$$\begin{aligned}
 \text{b} \quad \text{If } y &= \arctan\left(\frac{x}{a}\right), \\
 \text{then } \frac{dy}{dx} &= \frac{1}{a} \left(\frac{1}{1 + \left(\frac{x}{a}\right)^2}\right) \\
 &= \frac{a}{a^2} \left(\frac{1}{1 + \frac{x^2}{a^2}}\right) \\
 &= \frac{a}{a^2 + x^2} \quad \text{as required,}
 \end{aligned}$$

and this is defined for $x \in \mathcal{R}$.

$$\begin{aligned}
 \text{c} \quad \text{If } y &= \arccos\left(\frac{x}{a}\right), \\
 \text{then } \frac{dy}{dx} &= \frac{1}{a} \left(-\frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}}\right) = -\frac{1}{a\sqrt{1 - \frac{x^2}{a^2}}} \\
 &= -\frac{1}{\sqrt{a^2 - x^2}} \quad \text{and this is defined for } x \in [-a, a].
 \end{aligned}$$

5 a $\tan \alpha = \frac{2}{x}$ and $\tan(\alpha + \theta) = \frac{3}{x}$

b Now $\theta = (\alpha + \theta) - \alpha = \arctan\left(\frac{3}{x}\right) - \arctan\left(\frac{2}{x}\right)$

c
$$\begin{aligned}\frac{d\theta}{dx} &= \left(-\frac{3}{x^2}\right) \left(\frac{1}{1 + \left(\frac{3}{x}\right)^2}\right) - \left(-\frac{2}{x^2}\right) \left(\frac{1}{1 + \left(\frac{2}{x}\right)^2}\right) \\ &= -\frac{3}{x^2 + 9} + \frac{2}{x^2 + 4} \\ &= \frac{2}{x^2 + 4} - \frac{3}{x^2 + 9} \text{ as required.}\end{aligned}$$

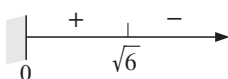
$\frac{d\theta}{dx} = 0$ when $2(x^2 + 9) - 3(x^2 + 4) = 0$

$\therefore 2x^2 + 18 - 3x^2 - 12 = 0$

$\therefore x^2 = 6$

$\therefore x = \sqrt{6} \quad \{x > 0\}$

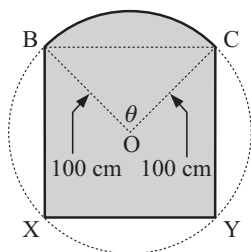
and $\frac{d\theta}{dx}$ has sign diagram:



- d** The maximum viewing angle occurs when $x = \sqrt{6}$, i.e., when Sonia is $\sqrt{6}$ m from the wall.

EXERCISE 24D

1



Using the cosine rule in $\triangle BCO$,

$$BC^2 = 10^2 + 10^2 - 2 \times 10 \times 10 \cos \theta$$

$$\therefore BC = \sqrt{200 - 200 \cos \theta}$$

and $XY = BC$

Now $BY^2 = BX^2 + XY^2$ {Pythagoras}

$$\therefore 400 = BX^2 + (200 - 200 \cos \theta)$$

$$\therefore BX^2 = 200 + 200 \cos \theta$$

$$\therefore BX = \sqrt{200 + 200 \cos \theta}$$

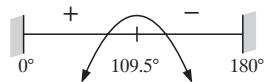
The shaded area is equal to the area of the sector plus $\frac{3}{4}$ of the area of BCYX

$$\begin{aligned}\therefore A &= \frac{1}{2}(10)^2 \theta + \frac{3}{4}[BX \times BC] \\ &= 50\theta + \frac{3}{4}\sqrt{200 + 200 \cos \theta}\sqrt{200 - 200 \cos \theta} \\ &= 50\theta + \frac{3}{4} \times 200\sqrt{1 + \cos \theta}\sqrt{1 - \cos \theta} \\ &= 50\theta + 150\sqrt{1 - \cos^2 \theta} \\ &= 50\theta + 150 \sin \theta \\ &= 50(\theta + 3 \sin \theta) \text{ as required}\end{aligned}$$

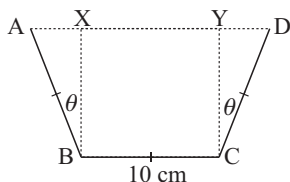
$$\therefore \frac{dA}{d\theta} = 50 + 150 \cos \theta = 50(1 + 3 \cos \theta),$$

which is zero when $\cos \theta = -\frac{1}{3}$

The sign diagram of $\frac{dA}{d\theta}$ is:



$\therefore$ since $0 < \theta < 180$, max A is when $\theta \doteq 109.5^\circ$.

2 a

 The triangles have height $10 \cos \theta$ and width $10 \sin \theta$.

$$\begin{aligned}
 \therefore \text{area } A &= \text{area of } \Delta s + \text{area of rectangle} \\
 &= 2 \times \frac{1}{2} \times 10 \cos \theta \times 10 \sin \theta + 10 \times 10 \cos \theta \\
 &= 100 \sin \theta \cos \theta + 100 \cos \theta \\
 &= 100 \cos \theta (1 + \sin \theta)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \frac{dA}{d\theta} &= -100(2 \sin \theta - 1)(\sin \theta + 1) \\
 &= 100(-\sin \theta(1 + \sin \theta) + \cos \theta \times \cos \theta) \\
 &= 100(-\sin \theta - \sin^2 \theta + \cos^2 \theta) \\
 &= 100(-\sin \theta - \sin^2 \theta + 1 - \sin^2 \theta) \\
 &= -100(2 \sin^2 \theta + \sin \theta - 1) \\
 &= -100(2 \sin \theta - 1)(\sin \theta + 1)
 \end{aligned}$$

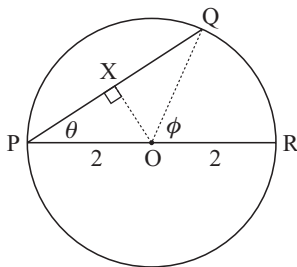
$$\therefore \frac{dA}{d\theta} = 0 \quad \text{when} \quad 2 \sin \theta - 1 = 0 \quad \text{or} \quad \sin \theta + 1 = 0$$

$$\text{i.e., } \sin \theta = \frac{1}{2} \quad \text{or} \quad \sin \theta = -1$$

$$\mathbf{c} \quad \text{Using } \mathbf{b}, \quad \frac{dA}{d\theta} = 0 \quad \text{when} \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6} \text{ or } \frac{3\pi}{2}.$$

Sign diagram for $\frac{dA}{d\theta}$ is:

so the maximum area is when $\theta = \frac{\pi}{6} = 30^\circ$

3


$$\frac{PX}{2} = \cos \theta, \quad \therefore PQ = 2PX = 4 \cos \theta$$

 $\therefore$ the time taken to row from P to Q is

$$\frac{4 \cos \theta}{3} \text{ hours}$$

 Now $\phi = 2\theta$ {angle at the centre}

 But, arc length $QR_{\text{arc}} = 2\phi$,

$$\therefore QR_{\text{arc}} = 4\theta,$$

 and the time taken to walk from Q to R is $\frac{4\theta}{5}$

$$\therefore \text{total time from P to R, } T = \frac{4}{3} \cos \theta + \frac{4\theta}{5}$$

$$\therefore \frac{dT}{d\theta} = -\frac{4}{3} \sin \theta + \frac{4}{5}$$

$$\therefore \frac{dT}{d\theta} = 0 \quad \text{when} \quad -\frac{4}{3} \sin \theta = -\frac{4}{5}$$

$$\therefore \sin \theta = \frac{3}{5}$$

$$\therefore \theta \doteq 0.6435 \text{ radians}$$

$$\text{i.e., } \theta \doteq 36.87^\circ$$

sign diagram of $\frac{dT}{d\theta}$ is:

i.e., the maximum time is when $\theta \doteq 36.87^\circ$

$$\text{and the maximum time is } \frac{4}{3} \cos 0.6435 + \frac{4}{5} \times 0.6435$$

$$\doteq 1.581 \text{ hours}$$

$$\doteq 1 \text{ hour } 34 \text{ min } 53 \text{ sec}$$

$$\begin{aligned} \mathbf{4} \quad \mathbf{a} \quad \tan \theta &= \frac{2}{AX} & \text{and} \quad \sin \theta &= \frac{2}{BX} \\ \therefore AX &= \frac{2}{\tan \theta} = \frac{2 \cos \theta}{\sin \theta} & \text{and} \quad BX &= \frac{2}{\sin \theta} \end{aligned}$$

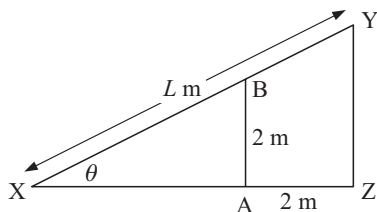
From the similar Δ 's, $\frac{L}{BX} = \frac{AX+2}{AX} = 1 + \frac{2}{AX}$

$$\therefore L = BX + \frac{2BX}{AX}$$

$$\therefore L = \frac{2}{\sin \theta} + \frac{2 \left(\frac{2}{\sin \theta} \right)}{\left(\frac{2 \cos \theta}{\sin \theta} \right)}$$

$$\therefore L = \frac{2}{\sin \theta} + 2 \left(\frac{2}{\sin \theta} \right) \left(\frac{\sin \theta}{2 \cos \theta} \right)$$

$$\therefore L = \frac{2}{\sin \theta} + \frac{2}{\cos \theta}, \text{ as required.}$$

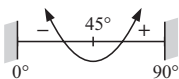


$$\mathbf{b} \quad \text{Now } L = 2(\cos \theta)^{-1} + 2(\sin \theta)^{-1}$$

$$\begin{aligned} \therefore \frac{dL}{d\theta} &= -2(\cos \theta)^{-2}(-\sin \theta) + -2(\sin \theta)^{-2}(\cos \theta) \\ &= \frac{2 \sin \theta}{\cos^2 \theta} - \frac{2 \cos \theta}{\sin^2 \theta} \\ &= \frac{2 \sin^3 \theta - 2 \cos^3 \theta}{\sin^2 \theta \cos^2 \theta} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \text{Now } \frac{dL}{d\theta} &= 0 \quad \text{when } 2 \sin^3 \theta - 2 \cos^3 \theta = 0 \\ &\therefore 2 \sin^3 \theta = 2 \cos^3 \theta \\ &\therefore \tan^3 \theta = 1 \\ &\therefore \tan \theta = 1 \\ &\therefore \theta = 45^\circ \quad \text{since } 0 < \theta < 90^\circ \end{aligned}$$

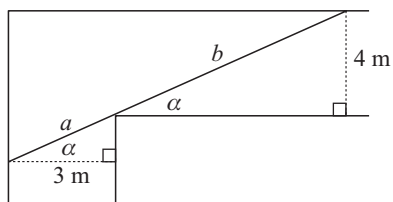
Sign diagram of $\frac{dL}{d\theta}$ is:



$\therefore$ the shortest ladder is required when $\theta = 45^\circ$

$$\text{and } L_{\min} = \frac{2}{\frac{1}{\sqrt{2}}} + \frac{2}{\frac{1}{\sqrt{2}}} = 2\sqrt{2} + 2\sqrt{2} = 4\sqrt{2} \text{ m}$$

5



$$\begin{aligned} \cos \alpha &= \frac{3}{a} & \text{and} \quad \sin \alpha &= \frac{4}{b} \\ \therefore a &= \frac{3}{\cos \alpha} & \text{and} \quad b &= \frac{4}{\sin \alpha} \end{aligned}$$

Now $L = a + b$

$$\therefore L = \frac{3}{\cos \alpha} + \frac{4}{\sin \alpha}$$

$$\therefore L = 3(\cos \alpha)^{-1} + 4(\sin \alpha)^{-1}$$

$$\begin{aligned} \therefore \frac{dL}{d\alpha} &= -3(\cos \alpha)^{-2}(-\sin \alpha) + 4(-(\sin \alpha)^{-2}) \cos \alpha \\ &= \frac{3 \sin \alpha}{\cos^2 \alpha} - \frac{4 \cos \alpha}{\sin^2 \alpha} \\ &= \frac{3 \sin^3 \alpha - 4 \cos^3 \alpha}{\cos^2 \alpha \sin^2 \alpha} \end{aligned}$$

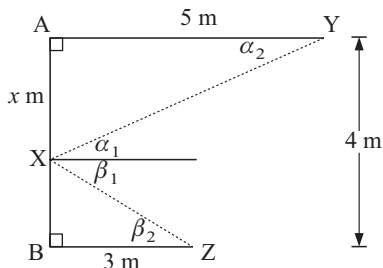
$$\begin{aligned}\therefore \frac{dL}{d\alpha} &= 0 \quad \text{when} \quad 3\sin^3\alpha - 4\cos^3\alpha = 0 \\ &\quad \text{i.e.,} \quad 3\sin^3\alpha = 4\cos^3\alpha \\ &\quad \tan^3\alpha = \frac{4}{3}\end{aligned}$$

$$\therefore \tan\alpha = \sqrt[3]{\frac{4}{3}} \quad \text{and so} \quad \alpha \doteq 47.74^\circ$$

Sign diagram of $\frac{dL}{d\alpha}$ is:

$$\therefore \text{AB is minimised when } \alpha = 47.74^\circ \quad \text{and} \quad L = \frac{3}{\cos\alpha} + \frac{4}{\sin\alpha} \doteq 9.866 \text{ m}$$

6


 Now $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ {alternate angles}

$$\therefore \theta = \alpha + \beta$$

$$\text{Let } AX = x \text{ m}$$

$$\therefore XB = (4 - x) \text{ m}$$

$$\therefore \tan\alpha = \frac{x}{5} \quad \therefore x = 5 \tan\alpha$$

$$\text{and } \tan\beta = \frac{4-x}{3}$$

$$\text{Now } \tan\theta = \tan(\alpha + \beta)$$

$$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

$$\text{i.e., } \tan\theta = \frac{\frac{x}{5} + \frac{4-x}{3}}{1 - \frac{x}{5} \left(\frac{4-x}{3} \right)}$$

$$\text{i.e., } \tan\theta = \frac{3x + 20 - 5x}{15 - x(4-x)} \quad \{\text{multiplying top and bottom by 15}\}$$

$$\therefore \tan\theta = \frac{20 - 2x}{x^2 - 4x + 15}$$

 Differentiating both sides with respect to x

$$\frac{1}{\cos^2\theta} \frac{d\theta}{dx} = \frac{-2(x^2 - 4x + 15) - (20 - 2x)(2x - 4)}{(x^2 - 4x + 15)^2} \quad \{\text{Chain and Quotient rules}\}$$

$$\therefore \frac{1}{\cos^2\theta} \frac{d\theta}{dx} = \frac{[-2x^2 + 8x - 30 - 40x + 80 + 4x^2 - 8x]}{(x^2 - 4x + 15)^2}$$

$$\therefore \frac{1}{\cos^2\theta} \frac{d\theta}{dx} = \frac{[2x^2 - 40x + 50]}{(x^2 - 4x + 15)^2}$$

$$\therefore \frac{d\theta}{dx} = 2\cos^2\theta \frac{[x^2 - 20x + 25]}{(x^2 - 4x + 15)^2}$$

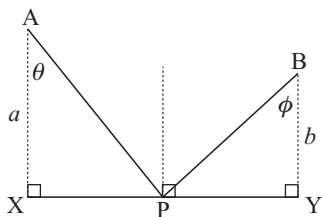
$$\text{Now by inspection, } \theta < 90^\circ, \quad \text{so } \frac{d\theta}{dx} = 0 \quad \text{when } x^2 - 20x + 25 = 0$$

$$\therefore x \doteq 1.3397 \quad \text{or } x \doteq 18.660 \quad (\text{where } 18.660 \text{ is not physically possible})$$

$$\therefore x \doteq 1.340 \text{ m from A}$$

Sign diagram for $\frac{d\theta}{dx}$ is:

$$\therefore \theta \text{ is a maximum when } x \doteq 1.340 \text{ m from A}$$

7 a

$$\frac{a}{AP} = \cos \theta \quad \text{and} \quad \frac{b}{BP} = \cos \phi$$

$$AP = \frac{a}{\cos \theta} \quad \text{and} \quad BP = \frac{b}{\cos \phi}$$

$$\text{Now } L = AP + BP$$

$$\therefore L = \frac{a}{\cos \theta} + \frac{b}{\cos \phi}$$

$$\mathbf{b} \quad L = a(\cos \theta)^{-1} + b(\cos \phi)^{-1}$$

$$\frac{dL}{d\theta} = -a(\cos \theta)^{-2}(-\sin \theta) - b(\cos \phi)^{-2}(-\sin \phi) \frac{d\phi}{d\theta}$$

$$= \frac{a \sin \theta}{\cos^2 \theta} + \frac{b \sin \phi}{\cos^2 \phi} \frac{d\phi}{d\theta}, \quad \text{as required.}$$

$$\mathbf{c} \quad \text{Now } \frac{XP}{a} = \tan \theta \quad \text{and} \quad \frac{YP}{b} = \tan \phi$$

$$\therefore XP = a \tan \theta \quad \text{and} \quad YP = b \tan \phi$$

$$\therefore XY = a \tan \theta + b \tan \phi$$

But XY is a fixed distance, so $a \tan \theta + b \tan \phi$ is a constant,

$$\text{i.e., } a \tan \theta + b \tan \phi = c$$

$$\therefore \frac{a}{\cos^2 \theta} + \frac{b}{\cos^2 \phi} \frac{d\phi}{d\theta} = 0 \quad \{\text{differentiating with respect to } \theta\}$$

$$\therefore \frac{b}{\cos^2 \phi} \frac{d\phi}{d\theta} = -\frac{a}{\cos^2 \theta}$$

$$\therefore \frac{d\phi}{d\theta} = -\frac{a \cos^2 \phi}{b \cos^2 \theta}, \quad \text{as required.}$$

$$\mathbf{d} \quad \frac{dL}{d\theta} = \frac{a \sin \theta}{\cos^2 \theta} + \frac{b \sin \phi}{\cos^2 \phi} \left(\frac{-a \cos^2 \phi}{b \cos^2 \theta} \right) \quad \{\text{using } \mathbf{b} \text{ and } \mathbf{c}\}$$

$$= \frac{a \sin \theta}{\cos^2 \theta} - \frac{a \sin \phi}{\cos^2 \theta}$$

$$= \frac{a(\sin \theta - \sin \phi)}{\cos^2 \theta}$$

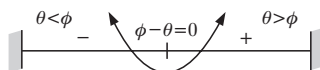
$$\therefore \frac{dL}{d\theta} = 0 \quad \text{when} \quad \sin \theta - \sin \phi = 0$$

$$\text{i.e., when } \sin \phi = \sin \theta$$

$$\mathbf{e} \quad \text{Since } \frac{dL}{d\theta} = 0 \quad \text{when} \quad \sin \phi = \sin \theta,$$

$L = AP + PB$ is either a maximum or minimum when $\phi = \theta$

Sign diagram of $\frac{dL}{d\theta}$ is:



$$\therefore AP + PB \quad \text{is a minimum} \quad \text{when } \theta - \phi = 0$$

$$\text{i.e., when } \theta = \phi$$

$\therefore$ it will be cheapest for the pump house to be located at the point such that $\theta = \phi$.

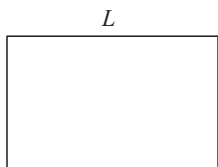
EXERCISE 24E

$$1 \quad ab^3 = 40, \quad \therefore \frac{da}{dt}b^3 + a(3b^2)\frac{db}{dt} = 0$$

Particular case: when $a = 5$, $b = 2$ and $\frac{db}{dt} = +1$, $8\frac{da}{dt} + 5(12)(1) = 0$

$$\therefore \frac{da}{dt} = -\frac{60}{8} = -7.5$$

$\therefore a$ decreases at 7.5 units per second

2

$$A = LB = 100 \text{ cm}^2$$

$$\therefore \frac{dL}{dt}B + L\frac{dB}{dt} = 0$$

Particular case: When a square, $L = B = 10 \text{ cm}$ and $\frac{dL}{dt} = -1$

$$\therefore 10\frac{dB}{dt} + 10(-1) = 0$$

$$\therefore \frac{dB}{dt} = 1 \text{ cm/min}$$

$\therefore$ the breadth is increasing at 1 cm/min

$$3 \quad A = \pi r^2 \quad \therefore \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\text{and } r = 1 \times t$$

Particular cases:

a When $t = 2$, $r = 2$ and $\frac{dr}{dt} = 1$ then $\frac{dA}{dt} = 2\pi(2)(1) = 4\pi \text{ m}^2/\text{sec}$

b When $t = 4$, $r = 4$ and $\frac{dr}{dt} = 1$ then $\frac{dA}{dt} = 2\pi(4)(1) = 8\pi \text{ m}^2/\text{sec}$

4

$$V = \frac{4}{3}\pi r^3 \quad \therefore \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 6\pi \text{ m}^3/\text{min}$$

$$\therefore \frac{dr}{dt} = \frac{6\pi}{4\pi r^2} = \frac{3}{2r^2} \text{ m/min}$$

$$\text{Now } A = 4\pi r^2$$

$$\therefore \frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi r \times \frac{3}{2r^2}$$

Particular case: At $r = 2$, $\frac{dA}{dt} = \frac{8\pi \times 2 \times 3}{2 \times 4} \text{ m}^2/\text{min} = 6\pi \text{ m}^2/\text{min}$

i.e., the surface area is increasing at $6\pi \text{ m}^2/\text{min}$.

$$5 \quad pv^{\frac{3}{2}} = 400 \quad \therefore \frac{dp}{dt}v^{\frac{3}{2}} + \frac{3}{2}pv^{\frac{1}{2}}\frac{dv}{dt} = 0$$

Particular case: When $p = 50 \text{ Nm}^{-2}$, $v^{\frac{3}{2}} = 8$ and so $v = 4$

$$\therefore 3(8) + \frac{3}{2}(50)^2 \frac{dv}{dt} = 0 \quad \text{as } \frac{dp}{dt} = +3 \text{ Nm}^{-2}$$

$$\therefore \frac{dv}{dt} = -\frac{24}{150} \text{ m}^3/\text{min}$$

i.e., the volume is decreasing at $0.16 \text{ m}^3/\text{min}$.

6 $V = \frac{1}{3}\pi r^2 h$ and $r = 3h$

$$\therefore V = \frac{1}{3}\pi(3h)^2 h = 3\pi h^3 \quad \dots *$$

Particular case: After 1 min, the volume $V = 3\pi(20)^3 \text{ cm}^3 = 24\,000\pi \text{ cm}^3$

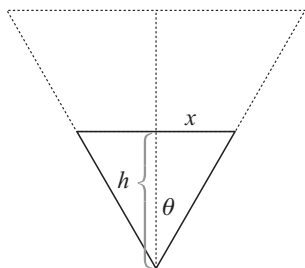
$$\text{i.e., } \frac{dV}{dt} = 24\,000\pi \text{ cm}^3/\text{min}$$

$$\text{But } \frac{dV}{dt} = 9\pi h^2 \frac{dh}{dt} \quad \{\text{from } *\}$$

$$\therefore \text{ when } h = 20, \quad 24\,000\pi = 9\pi \times (20^2) \times \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{24\,000\pi}{400 \times 9\pi} = \frac{20}{3} \text{ cm/min}$$

7



$$\theta = 30^\circ$$

$$\therefore \frac{x}{h} = \tan 30^\circ$$

$$x = h \tan 30^\circ = \frac{h}{\sqrt{3}}$$

$$\therefore V = \frac{h}{\sqrt{3}} \times h \times 600 = 200\sqrt{3}h^2 \text{ cm}^3$$

$$\therefore \frac{dV}{dt} = 400\sqrt{3}h \frac{dh}{dt}$$

Particular case: When $h = 20$, $-100\,000 = 400\sqrt{3}(20) \frac{dh}{dt}$ as $\frac{dV}{dt} = -0.1 \text{ m}^3 = -100\,000 \text{ cm}^3$

$$\therefore \frac{dh}{dt} = \frac{-100\,000}{400\sqrt{3} \times 20} = -\frac{25}{6}\sqrt{3} \text{ cm/min}$$

$\therefore$ the water level is falling at $\frac{25\sqrt{3}}{6} \text{ cm/min}$

- 8** Let P_1 in the diagram be the faster jet and P_2 be the slower jet. Let y m be the distance that P_2 is ahead of P_1 , and x m be the distance between them.

$$\text{Now } x^2 = y^2 + (12\,000)^2 \quad \{\text{Pythagoras}\}$$

$$\therefore 2x \frac{dx}{dt} = 2y \frac{dy}{dt}$$

Particular case:

As P_1 is behind P_2 , then it is catching up at a rate of 50 ms^{-1} .

$\therefore$ distance y is decreasing at a rate of 50 ms^{-1}

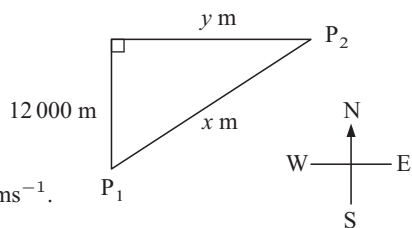
$$\therefore \frac{dy}{dt} = -50 \text{ ms}^{-1}$$

$$\text{When } y = 5000, \quad x = 13\,000$$

$$\therefore 26\,000 \times \frac{dx}{dt} = 10\,000 \times (-50)$$

$$\frac{dx}{dt} = \frac{10}{26} \times (-50) = -\frac{250}{13} \text{ ms}^{-1}$$

$\therefore$ their separation is decreasing at $\frac{250}{13} \text{ ms}^{-1}$.



- 9 Let S m be the height of the person's shadow and x m be his distance from the building.

$\triangle ABC$ and $\triangle XY$ are similar

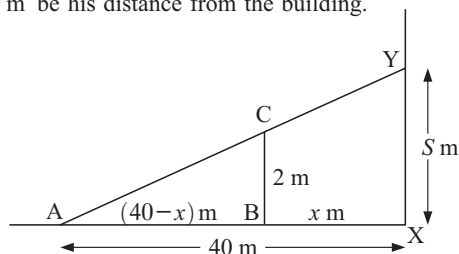
$$\therefore \frac{AB}{AX} = \frac{BC}{XY}$$

$$\therefore \frac{40-x}{40} = \frac{2}{S}$$

$$\therefore S = \frac{80}{40-x}$$

$$= 80(40-x)^{-1}$$

$$\therefore \frac{dS}{dt} = -80(40-x)^{-2}(-1)\frac{dx}{dt} = \frac{80}{(40-x)^2} \frac{dx}{dt}$$



Particular cases: **a** When $x = 20$ m, $\frac{dS}{dt} = \frac{80}{(40-20)^2}(-1)$ as $\frac{dx}{dt} = -1 \text{ ms}^{-1}$

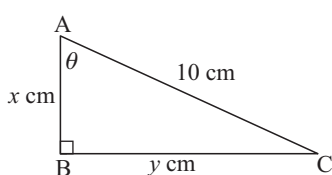
$$\therefore \frac{dS}{dt} = -\frac{80}{400} = -0.2$$

i.e., the person's shadow is shortening at 0.2 ms^{-2}

b When $x = 10$ m, $\frac{dS}{dt} = \frac{80}{(40-10)^2}(-1) = -\frac{80}{900} = -\frac{8}{90}$

i.e., the person's shadow is shortening at $\frac{8}{90} \text{ ms}^{-1}$

10



$$\cos \theta = \frac{x}{10}$$

$$\therefore -\sin \theta \frac{d\theta}{dt} = \frac{1}{10} \frac{dx}{dt} \quad \{\text{differentiating with respect to } t\}$$

If side AB increases at 0.1 cm/sec , i.e., $\frac{dx}{dt} = 0.1 \text{ cm/sec}$,

Particular case: When $\triangle ABC$ is isosceles, $\theta = 45^\circ$

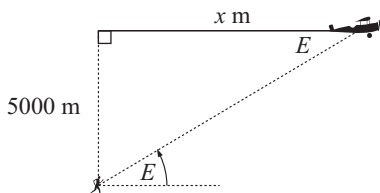
$$\therefore \sin \theta = \frac{1}{\sqrt{2}}$$

$$\therefore -\frac{1}{\sqrt{2}} \frac{d\theta}{dt} = \frac{1}{10} \times 0.1$$

$$\therefore \frac{d\theta}{dt} = -\frac{\sqrt{2}}{100} \text{ radians/sec}$$

i.e., $\angle CAB$ is decreasing at $\frac{\sqrt{2}}{100}$ radians per second

11



$$\text{Now } \tan E = \frac{5000}{x}$$

$$\therefore \tan E = 5000x^{-1}$$

Differentiating with respect to t ,

$$\sec^2 E \frac{dE}{dt} = -5000x^{-2} \frac{dx}{dt}$$

a Particular case: When $E = 60^\circ$,

$$\cos E = \frac{1}{2} \text{ and } \tan E = \sqrt{3} = \frac{5000}{x} \text{ and } \therefore x = \frac{5000}{\sqrt{3}}$$

$$\therefore 2^2 \frac{dE}{dt} = -5000 \times \left(\frac{\sqrt{3}}{5000}\right)^2 \times 200$$

$$\therefore \frac{dE}{dt} = -0.03$$

i.e., the angle of elevation is decreasing at 0.03 radians per second

b Particular case: When $E = 30^\circ$

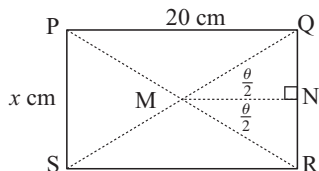
$$\cos E = \frac{\sqrt{3}}{2} \quad \text{and} \quad \tan E = \frac{1}{\sqrt{3}} = \frac{5000}{x} \quad \text{and} \quad \therefore x = 5000\sqrt{3} \text{ m}$$

$$\therefore \left(\frac{2}{\sqrt{3}}\right)^2 \frac{dE}{dt} = -5000 \times (5000\sqrt{3})^{-2} \times 200$$

$$\therefore \frac{dE}{dt} = -5000 \times (5000\sqrt{3})^{-2} \times 200 \times \frac{3}{4} = -0.01$$

i.e., the angle of elevation is decreasing at 0.01 radians per second

12



Let N be the midpoint of QR in isosceles triangle QMR

$$\therefore MN = 10 \text{ cm}$$

Let QR = x cm,

and let $\angle QMR = \theta$.

$$\text{Now in triangle MNQ, } \tan\left(\frac{\theta}{2}\right) = \frac{QN}{MN} = \frac{\frac{x}{2}}{10} = \frac{x}{20}$$

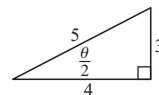
$$\therefore \frac{1}{2} \sec^2\left(\frac{\theta}{2}\right) \frac{d\theta}{dt} = \frac{1}{20} \frac{dx}{dt}$$

$$\therefore \frac{d\theta}{dt} = \frac{1}{10} \cos^2\left(\frac{\theta}{2}\right) \frac{dx}{dt} \quad \text{where} \quad \frac{dx}{dt} = 2 \text{ cms}^{-1}$$

$$\text{Particular case:} \quad \text{When } x = 15 \text{ cm, } \tan\left(\frac{\theta}{2}\right) = \frac{15}{20} = \frac{3}{4}$$

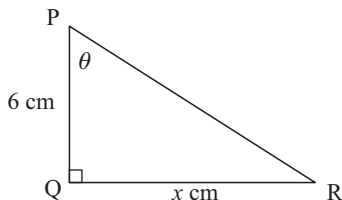
$$\therefore \cos\left(\frac{\theta}{2}\right) = \frac{4}{5}$$

$$\therefore \frac{d\theta}{dt} = \frac{1}{10} \left(\frac{4}{5}\right)^2 2 = 0.128$$



i.e., θ is increasing at 0.128 radians per second

13



Let QR = x cm and the angle at P be θ .

$$\text{Then } \tan \theta = \frac{x}{6}$$

$$\therefore \sec^2 \theta \frac{d\theta}{dt} = \frac{1}{6} \frac{dx}{dt}$$

$$\therefore \frac{d\theta}{dt} = \frac{\cos^2 \theta}{6} \frac{dx}{dt} \quad \text{and} \quad \frac{dx}{dt} = 2 \text{ cm/min}$$

Particular case: When $x = 8$ cm, PR = 10 cm

$$\text{Now } \cos \theta = \frac{6}{10}$$

$$\therefore \frac{d\theta}{dt} = \left(\frac{6}{10}\right)^2 \times \frac{1}{6} \times 2 = 0.12$$

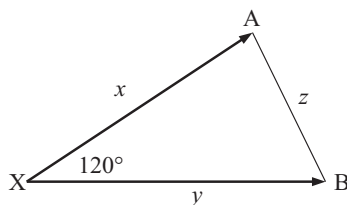
$\therefore \theta$ is increasing at a rate of 0.12 radians per minute.

14 Let x and y be the distances the trains have travelled (respectively) at time t , and let z be the distance between them.

$$\text{Then } z^2 = x^2 + y^2 - 2xy \cos 120^\circ$$

$$z^2 = x^2 + y^2 + xy$$

$$\therefore 2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + \frac{dx}{dt} y + x \frac{dy}{dt}$$



Particular case: After 2 minutes, i.e., $t = 120$ secs,

$$x = 1440 \text{ m}, \quad y = 1920 \text{ m}, \quad \frac{dx}{dt} = 12 \text{ ms}^{-1}, \quad \frac{dy}{dt} = 16 \text{ ms}^{-1}$$

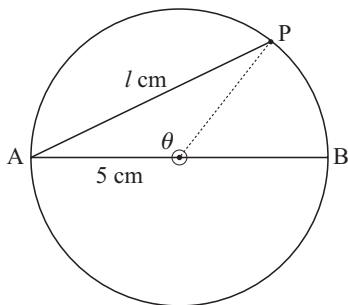
$$\therefore z^2 = 1440^2 + 1920^2 + 1440 \times 1920 = 8\,524\,800$$

$$\therefore 2\sqrt{8\,524\,800} \frac{dz}{dt} = 2880(12) + 3840(16) + (12)1920 + 1440(16) = 1.4208 \times 10^5$$

$$\therefore 2\sqrt{8\,524\,800} \frac{dz}{dt} = 142080 \quad \therefore \frac{dz}{dt} \doteq 24.33$$

$\therefore z$ is increasing at 24.33 ms^{-1}

15



Let $AP = l \text{ cm}$ and let $\angle AOP = \theta$

$$\therefore l^2 = 5^2 + 5^2 - 2 \times 5 \times 5 \cos \theta \quad \{\text{cosine rule}\}$$

$$\therefore l^2 = 50 - 50 \cos \theta$$

$$\therefore 2l \frac{dl}{dt} = 50 \sin \theta \frac{d\theta}{dt}$$

$$\therefore \frac{dl}{dt} = \frac{25 \sin \theta}{l} \frac{d\theta}{dt}$$

Now the point moves at one revolution every 10 seconds.

$$\therefore \frac{d\theta}{dt} = \frac{2\pi}{10} = \frac{\pi}{5} \text{ radians/sec}$$

Particular cases:

a If $AP = l = 5 \text{ cm}$, $\frac{dl}{dt} > 0$,

then $\theta = \frac{\pi}{3}$

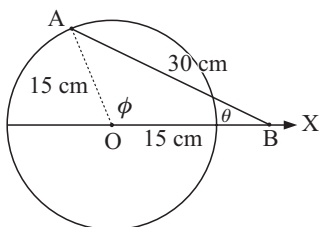
$$\begin{aligned} \therefore \frac{dl}{dt} &= \frac{25 \sin(\frac{\pi}{3})}{5} \times \frac{\pi}{5} \\ &= \frac{\sqrt{3}}{2} \pi \text{ cms}^{-1} \end{aligned}$$

b If P is at B, then $l = 10 \text{ cm}$

and $\theta = \pi$

$$\begin{aligned} \therefore \frac{dl}{dt} &= \frac{25 \sin \pi}{10} \times \frac{\pi}{5} \\ &= 0 \text{ cms}^{-1} \end{aligned}$$

16



Let $\angle AOB = \phi$ and $\angle ABO = \theta$

$$\begin{aligned} \text{Then } \frac{d\phi}{dt} &= -100 \text{ revolutions per second} \quad \{\text{clockwise rotation}\} \\ &= -200\pi \text{ radians per second} \end{aligned}$$

$$\text{Also, } \frac{30}{\sin \phi} = \frac{15}{\sin \theta} \quad \{\text{sine rule}\}$$

$$\therefore \sin \phi = 2 \sin \theta$$

$$\therefore \cos \phi \frac{d\phi}{dt} = 2 \cos \theta \frac{d\theta}{dt} \quad \text{and so} \quad \frac{d\theta}{dt} = \frac{1}{2} \frac{\cos \phi}{\cos \theta} \frac{d\phi}{dt}$$

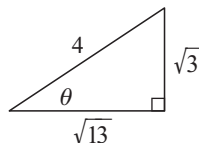
Particular cases:

a When $\angle AOX = 120^\circ$, i.e., $\phi = \frac{2\pi}{3}$

$$\text{then } \cos \phi = -\frac{1}{2} \quad \text{and} \quad \sin \phi = \frac{\sqrt{3}}{2}$$

$$\therefore \sin \theta = \frac{1}{2} \sin \phi = \frac{1}{2} \left(\frac{\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{4}$$

$$\text{and so } \cos \theta = \frac{\sqrt{13}}{4}$$



$$\begin{aligned}
 \therefore \frac{d\theta}{dt} &= \frac{1}{2} \frac{\cos \phi}{\cos \theta} \frac{d\phi}{dt} \\
 &= \frac{1}{2} \times \frac{-\frac{1}{2}}{\frac{\sqrt{13}}{4}} \times (-200\pi) \quad \left\{ \text{as } \frac{d\phi}{dt} = -200\pi \text{ c/sec} \right\} \\
 &= \frac{200\pi}{\sqrt{13}}, \quad \text{i.e., } \theta \text{ is increasing at } \frac{200\pi}{\sqrt{13}} \text{ radians per second}
 \end{aligned}$$

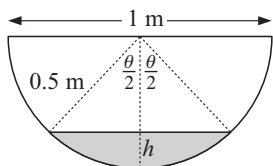
b When $\angle AOX = 180^\circ$, i.e., $\phi = \pi$,

then $\cos \phi = -1$ and $\sin \phi = 0$

$$\therefore \sin \theta = 0 \quad \text{and} \quad \cos \theta = 1$$

$$\begin{aligned}
 \therefore \frac{d\theta}{dt} &= \frac{1}{2} \frac{\cos \phi}{\cos \theta} \frac{d\phi}{dt} \\
 &= \frac{1}{2} \times \frac{-1}{1} \times (-200\pi) \\
 &= 100\pi, \quad \text{i.e., } \theta \text{ is increasing at } 100\pi \text{ radians per second}
 \end{aligned}$$

17



Denote the radius of the semicircle $r = \frac{1}{2}$ m.

Let h be the depth and V be the volume of water in the trough at time t .

a The cross-sectional area of water in the trough

= area of sector – area of triangle

$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$$

$$= \frac{1}{2}r^2(\theta - \sin\theta)$$

$$= \frac{1}{8}(\theta - \sin\theta)$$

$\therefore$ the volume of water, $V = \text{area of water} \times \text{length of trough}$

$$= \frac{1}{8}(\theta - \sin\theta) \times 8$$

$$= \theta - \sin\theta \quad \text{as required}$$

b Now $\frac{dV}{dt} = \frac{d\theta}{dt} - \cos\theta \frac{d\theta}{dt} \quad \therefore \frac{dV}{dt} = \frac{d\theta}{dt}(1 - \cos\theta) \quad \text{where } \frac{dV}{dt} = 0.1 \text{ m}^3/\text{min}$

Particular case: When $h = 0.25$ m,

$$\cos\left(\frac{\theta}{2}\right) = \frac{r-h}{r} = \frac{\frac{1}{2} - \frac{1}{4}}{\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} \quad \text{Note: } \frac{\theta}{2} = 60^\circ$$

$$\text{and } \therefore \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1 = 2\left(\frac{1}{2}\right)^2 - 1 = -\frac{1}{2}$$

$$\therefore 0.1 = \frac{d\theta}{dt} \left(1 - \left(-\frac{1}{2}\right)\right) = \frac{1}{15} \text{ radians per minute}$$

$$\therefore \frac{1}{10} = \frac{d\theta}{dt} \left(\frac{3}{2}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{15} \quad \text{i.e., } \theta \text{ is increasing at } \frac{1}{15} \text{ radians per minute}$$

$$\text{Now } \cos\left(\frac{\theta}{2}\right) = \frac{\frac{1}{2} - h}{\frac{1}{2}} = 1 - 2h,$$

$$\text{so } -\sin\left(\frac{\theta}{2}\right) \frac{1}{2} \frac{d\theta}{dt} = -2 \frac{dh}{dt} \quad \text{and so } -\left(\frac{\sqrt{3}}{2}\right) \left(\frac{1}{30}\right) = -2 \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{\sqrt{3}}{120} \quad \text{i.e., } h \text{ is increasing at } \frac{\sqrt{3}}{120} \text{ metres per minute.}$$

REVIEW SET 24A

$$\begin{aligned} 1 \quad a \quad \frac{d}{dx}(\sin(5x) \ln x) &= \frac{d}{dx}(\sin(5x)) \ln x + \sin(5x) \frac{d}{dx}(\ln x) \quad \{\text{product rule}\} \\ &= 5 \cos(5x) \ln x + \frac{\sin(5x)}{x} \end{aligned}$$

$$\begin{aligned} b \quad \frac{d}{dx}(\sin x \cos(2x)) &= \frac{d}{dx}(\sin x) \cos(2x) + \sin x \frac{d}{dx}(\cos(2x)) \quad \{\text{product rule}\} \\ &= \cos x \cos(2x) + \sin x(-2 \sin(2x)) \\ &= \cos x \cos(2x) - 2 \sin x \sin(2x) \end{aligned}$$

$$\begin{aligned} c \quad \frac{d}{dx}(e^{-2x} \tan x) &= \frac{d}{dx}(e^{-2x}) \tan x + e^{-2x} \frac{d}{dx}(\tan x) \quad \{\text{product rule}\} \\ &= -2e^{-2x} \tan x + e^{-2x} \sec^2 x \end{aligned}$$

$$2 \quad y = x \tan x \quad \therefore \quad \frac{dy}{dx} = 1 \times \tan x + x \times \frac{1}{\cos^2 x}$$

$$\text{Now } \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad \text{and} \quad \tan \frac{\pi}{4} = 1$$

$$\therefore \text{ at } x = \frac{\pi}{4}, \quad y = \frac{\pi}{4} \quad \text{and} \quad \frac{dy}{dx} = 1 + \frac{\pi}{2}$$

$$\begin{aligned} \therefore \text{ the equation of the tangent is } \frac{y - \frac{\pi}{4}}{x - \frac{\pi}{4}} &= 1 + \frac{\pi}{2} \quad \therefore \quad y - \frac{\pi}{4} = \left(1 + \frac{\pi}{2}\right)\left(x - \frac{\pi}{4}\right) \\ &= x - \frac{\pi}{4} + \frac{\pi}{2}x - \frac{\pi^2}{8} \\ \therefore \quad y &= \left(1 + \frac{\pi}{2}\right)x - \frac{\pi^2}{8} \\ \therefore \quad 2y &= (2 + \pi)x - \frac{\pi^2}{4} \\ \therefore \quad (2 + \pi)x - 2y &= \frac{\pi^2}{4} \quad \text{as required} \end{aligned}$$

$$\begin{aligned} 3 \quad a \quad f(x) &= 3 \sin x - 4 \cos(2x) \\ \therefore \quad f'(x) &= 3 \cos x + 8 \sin(2x) \quad \text{and} \quad f''(x) = -3 \sin x + 16 \cos(2x) \end{aligned}$$

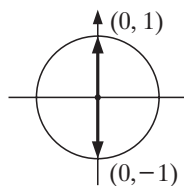
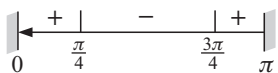
$$\begin{aligned} b \quad f(x) &= x^{\frac{1}{2}} \cos(4x) \\ \therefore \quad f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \cos(4x) + x^{\frac{1}{2}}(-4 \sin(4x)) \\ &= \frac{1}{2}x^{-\frac{1}{2}} \cos(4x) - 4x^{\frac{1}{2}} \sin(4x) \\ \text{and } f''(x) &= -\frac{1}{4}x^{-\frac{3}{2}} \cos(4x) + \frac{1}{2}x^{-\frac{1}{2}}(-4 \sin(4x)) - \left[2x^{-\frac{1}{2}} \sin(4x) + 4x^{\frac{1}{2}} \times 4 \cos(4x)\right] \\ &= -\frac{1}{4}x^{-\frac{3}{2}} \cos(4x) - 4x^{-\frac{1}{2}} \sin(4x) - 16x^{\frac{1}{2}} \cos(4x) \end{aligned}$$

$$4 \quad x(t) = 3 + \sin(2t) \text{ cm}, \quad t \geq 0 \text{ sec}$$

$$\begin{aligned} a \quad x'(t) &= 0 + 2 \cos(2t) & \therefore \quad x(0) &= 3 \text{ cm} \\ x''(t) &= -4 \sin(2t) & x'(0) &= 2 \text{ cm/sec} \\ & & x''(0) &= 0 \text{ cm/sec}^2 \end{aligned}$$

$\therefore$ initially the particle is 3 cm right of O, moving right at a speed of 2 cm/sec.

$$\begin{aligned} b \quad x'(t) = 0 \quad \text{when} \quad 2 \cos(2t) &= 0 \\ \therefore \quad \cos(2t) &= 0 \\ \therefore \quad 2t &= \frac{\pi}{2} + k\pi \\ \therefore \quad t &= \frac{\pi}{4} \text{ or } \frac{3\pi}{4} \quad \{\text{as } 0 \leq t \leq \pi\} \\ \therefore \quad \text{reverses direction at } t &= \frac{\pi}{4}, \frac{3\pi}{4} \end{aligned}$$



- c** $x(0) = 3, \quad x\left(\frac{\pi}{4}\right) = 3 + \sin\left(\frac{\pi}{2}\right) = 4$
 $x\left(\frac{3\pi}{4}\right) = 3 + \sin\left(\frac{3\pi}{2}\right) = 3 - 1 = 2 \quad x(\pi) = 3 + \sin(2\pi) = 3$
 $\therefore$ total distance travelled $= 1 + 2 + 1 = 4$ cm.



- 5 a** $f(x) = \sqrt{\cos x}, \quad 0 \leq x \leq 2\pi$

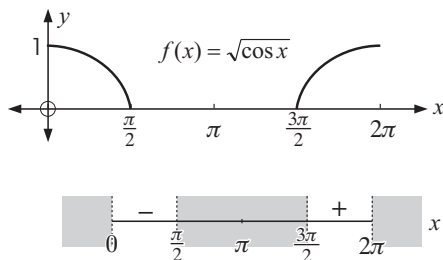
$\therefore f(x)$ is meaningful when $\cos x \geq 0$,
 i.e., when $0 \leq x \leq \frac{\pi}{2}$ and $\frac{3\pi}{2} \leq x \leq 2\pi$

- b** $f(x) = (\cos x)^{\frac{1}{2}}$

$$\begin{aligned} \therefore f'(x) &= \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x) \\ &= \frac{-\sin x}{2\sqrt{\cos x}} \end{aligned}$$

$\therefore f'(x) = 0$ when $-\sin x = 0$
 i.e., $x = 0, \pi, 2\pi$ etc.

Sign diagram for $f'(x)$ is:



$f(x)$ is increasing for $\frac{3\pi}{2} < x < 2\pi$ $f(x)$ is decreasing for $0 < x < \frac{\pi}{2}$

- 6 a** $s(t) = 30 + \cos(\pi t)$ cm, $t \geq 0$

$$\therefore v(t) = -\pi \sin(\pi t)$$

$$\text{So, } v(0) = 0 \text{ cms}^{-1}$$

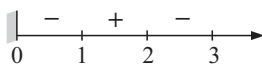
$$v\left(\frac{1}{2}\right) = -\pi \text{ cms}^{-1}$$

$$v(1) = 0 \text{ cms}^{-1}$$

$$v\left(1\frac{1}{2}\right) = \pi \text{ cms}^{-1}$$

$$v(2) = 0 \text{ cms}^{-1}$$

sign diagram of $v(t)$:

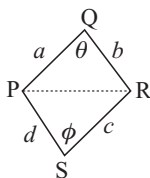


- b** The cork is falling when $v(t) \leq 0$

$$\text{i.e., } 0 \leq t \leq 1, \quad 2 \leq t \leq 3, \text{ etc.}$$

$$\text{i.e., } 2n \leq t \leq 2n + 1 \quad n = 0, 1, 2, 3, \dots$$

- 7 a**



Using the cosine rule,

$$\text{in } \triangle PQR, \quad PR^2 = a^2 + b^2 - 2ab \cos \theta$$

$$\text{in } \triangle PSR, \quad PR^2 = c^2 + d^2 - 2cd \cos \phi$$

$$\therefore a^2 + b^2 - 2ab \cos \theta = c^2 + d^2 - 2cd \cos \phi$$

Now a, b, c and d are constants, so differentiating with respect to ϕ ,

$$\therefore 2ab \sin \theta \frac{d\theta}{d\phi} = 2cd \sin \phi$$

$$\therefore \frac{d\theta}{d\phi} = \frac{2cd \sin \phi}{2ab \sin \theta} = \frac{cd \sin \phi}{ab \sin \theta} \quad \text{as required}$$

- b** Area of quadrilateral, $A = \text{area of } \triangle PQR + \text{area of } \triangle PSR$

$$= \frac{1}{2}ab \sin \theta + \frac{1}{2}cd \sin \phi$$

$$\therefore \frac{dA}{d\phi} = \frac{1}{2}ab \cos \theta \frac{d\theta}{d\phi} + \frac{1}{2}cd \cos \phi$$

$$= \frac{1}{2}ab \cos \theta \left(\frac{cd \sin \phi}{ab \sin \theta} \right) + \frac{1}{2}cd \cos \phi \quad \{\text{using a}\}$$

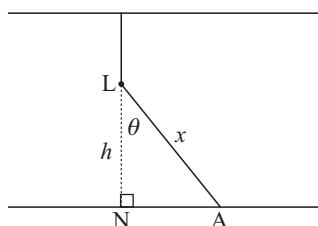
$$= \frac{1}{2}cd \left[\frac{\cos \theta \sin \phi}{\sin \theta} + \cos \phi \right]$$

$$\begin{aligned}
 &= \frac{cd}{2 \sin \theta} (\sin \phi \cos \theta + \cos \phi \sin \theta) \\
 &= \frac{cd}{2 \sin \theta} \sin(\phi + \theta)
 \end{aligned}$$

$$\therefore \frac{dA}{d\phi} = 0 \quad \text{when} \quad \sin(\phi + \theta) = 0 \quad \text{i.e., when} \quad \phi + \theta = 2\pi$$

$\therefore$ the area of PQRS is a maximum when the opposite angles are supplementary, i.e., when PQRS is a cyclic quadrilateral.

8



a

$$\sin \theta = \frac{NA}{x} = \frac{1}{x}$$

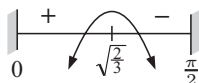
$$\therefore \frac{1}{x^2} = \sin^2 \theta$$

$$\begin{aligned}
 \therefore \text{ at A, } I &= \frac{\sqrt{8} \cos \theta}{x^2} \\
 &= \sqrt{8} \cos \theta \sin^2 \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{b } \frac{dI}{d\theta} &= \sqrt{8}(-\sin \theta) \sin^2 \theta + \sqrt{8} \cos \theta (2 \sin \theta \cos \theta) \\
 &= \sqrt{8} \sin \theta [2 \cos^2 \theta - \sin^2 \theta] \\
 &= \sqrt{8} \sin \theta [2(1 - \sin^2 \theta) - \sin^2 \theta] \\
 &= \sqrt{8} \sin \theta [2 - 3 \sin^2 \theta]
 \end{aligned}$$

$$\frac{dI}{d\theta} = 0 \quad \text{when} \quad \sin \theta = \sqrt{\frac{2}{3}}, \quad 0 < \theta < \frac{\pi}{2}$$

and the sign diagram of $\frac{dI}{d\theta}$ is:

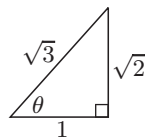


$\therefore$ the maximum illumination at A is obtained when $\sin \theta = \sqrt{\frac{2}{3}}$.

$$\therefore x = \frac{1}{\sin \theta} = \sqrt{\frac{3}{2}}$$

$$\therefore h = \sqrt{x^2 - NA^2} = \sqrt{\frac{3}{2} - 1} = \frac{1}{\sqrt{2}}$$

i.e., the bulb is $\frac{1}{\sqrt{2}}$ m above the floor.



$$\text{c } \cos \theta = \frac{h}{x}$$

$$\therefore I = \frac{\sqrt{8} \cos \theta}{x^2} = \frac{\sqrt{8} \times \frac{h}{x}}{x^2} = \frac{\sqrt{8}h}{x^3}$$

$$\text{Now } x^2 = h^2 + 1$$

$$x^3 = (h^2 + 1)^{\frac{3}{2}}$$

$$\therefore I = \sqrt{8}h (h^2 + 1)^{-\frac{3}{2}}$$

$$\begin{aligned}
 \therefore \frac{dI}{dt} &= \sqrt{8} \frac{dh}{dt} (h^2 + 1)^{-\frac{3}{2}} - \frac{3}{2} \sqrt{8}h (h^2 + 1)^{-\frac{5}{2}} \left(2h \frac{dh}{dt} \right) \\
 &= \sqrt{8} \frac{dh}{dt} \left[(h^2 + 1)^{-\frac{3}{2}} - 3h^2 (h^2 + 1)^{-\frac{5}{2}} \right]
 \end{aligned}$$

$$\text{Particular case: } \frac{dh}{dt} = -0.1 \text{ ms}^{-1}, \quad \text{and when } h = 1 \text{ m}$$

$$\frac{dI}{dt} = \sqrt{8}(-0.1) \left(2^{-\frac{3}{2}} - 3 \times 2^{-\frac{5}{2}} \right) = 0.05$$

$\therefore$ the illumination is increasing at 0.05 units/second.

REVIEW SET 24B

$$\begin{aligned} 1 \quad a \quad y = \frac{x}{\sqrt{\sec x}} &= x(\cos x)^{\frac{1}{2}} \quad \therefore \frac{dy}{dx} = (\cos x)^{\frac{1}{2}} + x \times \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x) \quad \{\text{product rule}\} \\ &= \sqrt{\cos x} - \frac{x \sin x}{2\sqrt{\cos x}} \end{aligned}$$

$$\begin{aligned} b \quad y &= e^x \cot(2x) \quad \therefore \frac{dy}{dx} = e^x \cot(2x) + e^x \times 2(-\csc^2(2x)) \\ &= e^x (\cot(2x) - 2\csc^2(2x)) \end{aligned}$$

$$c \quad y = \arccos\left(\frac{x}{2}\right) \quad \therefore \frac{dy}{dx} = \frac{1}{2} \left(\frac{-1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}} \right) = -\frac{1}{2\sqrt{1 - \frac{x^2}{4}}} = -\frac{1}{\sqrt{4 - x^2}}$$

$$2 \quad a \quad \text{If } y = \sec x \text{ then } \frac{dy}{dx} = \sec x \tan x$$

$$\text{When } x = \frac{\pi}{3}, \quad y = \sec \frac{\pi}{3} = 2 \quad \text{and} \quad \frac{dy}{dx} = \sec \frac{\pi}{3} \tan \frac{\pi}{3} = 2\sqrt{3}$$

$$\therefore \text{the tangent has equation } \frac{y-2}{x-\frac{\pi}{3}} = 2\sqrt{3} \quad \therefore y-2 = 2\sqrt{3}x - \frac{2\pi}{\sqrt{3}}$$

$$\therefore y = 2\sqrt{3}x - \frac{2\pi}{\sqrt{3}} + 2$$

$$b \quad \text{If } y = \arctan x \text{ then } \frac{dy}{dx} = \frac{1}{1+x^2}. \quad \text{When } x = \sqrt{3}, \quad y = \arctan \sqrt{3} = \frac{\pi}{3}$$

$$\text{and } \frac{dy}{dx} = \frac{1}{1+(\sqrt{3})^2} = \frac{1}{1+3} = \frac{1}{4}$$

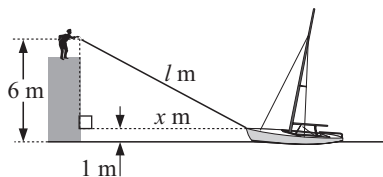
$$\therefore \text{the normal has slope } -4, \text{ so its equation is } \frac{y - \frac{\pi}{3}}{x - \sqrt{3}} = -4 \quad \therefore y - \frac{\pi}{3} = -4x + 4\sqrt{3}$$

$$\therefore y = -4x + 4\sqrt{3} + \frac{\pi}{3}$$

- 3 Let l m be the length of rope and x m be the distance of the boat from the jetty.

$$\text{Then } x^2 + 5^2 = l^2$$

$$\therefore 2x \frac{dx}{dt} = 2l \frac{dl}{dt}$$



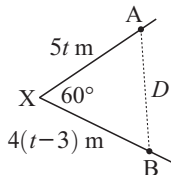
$$\text{Particular case: When } x = 15 \text{ m, } l = \sqrt{15^2 + 5^2} = \sqrt{250} \text{ and } \frac{dl}{dt} = -40 \text{ m/min}$$

$$\therefore 2(15) \frac{dx}{dt} = 2\sqrt{250}(-40) \text{ m/minute}$$

$$\therefore \frac{dx}{dt} = -\frac{80\sqrt{250}}{30} = -42.16$$

$\therefore$ the boat is approaching the jetty at 42.16 m/minute

- 4 **Note:** the question should have stated that B passes through X 3 seconds after A passes through X. Let t be the number of seconds after A passes through X. In this time, A travels $5t$ m. Now B passes through X when $t = 3$.
 $\therefore$ for $t > 3$, B is $4(t-3)$ m from X.

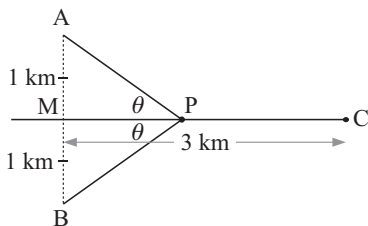


Using the cosine rule,

$$\begin{aligned} D^2 &= 25t^2 + 16(t-3)^2 - 2 \times 5t \times 4(t-3) \times \cos 60^\circ \\ &= 25t^2 + 16(t-3)^2 - 20t(t-3) \end{aligned}$$

$$\therefore 2D \frac{dD}{dt} = 50t + 32(t-3) - 20(t-3) - 20t$$

$$\begin{aligned} \text{when } 5t = 20, \quad t = 4 \quad \text{and} \quad D^2 &= 25 \times 16 + 16 - 20 \times 4 = 336 \\ \therefore 2\sqrt{336} \frac{dD}{dt} &= 200 + 32 - 20 - 20 \times 4 = 132 \\ \therefore \frac{dD}{dt} &= \frac{66}{\sqrt{336}} \div 3.60 \text{ m/s} \end{aligned}$$

5


a The length of cable required
 = (PA + PB + PC) km

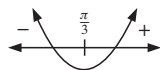
i If P is at M, then PA = PB = 1 km
 and PC = 3 km
 $\therefore$ 5 km of cable is required

ii If P is at C, then PA
 = PB = $\sqrt{1^2 + 3^2} = \sqrt{10}$ km
 and PC = 0 km
 $\therefore$ $2\sqrt{10}$ km of cable is required

b Now $\sin \theta = \frac{1}{AP} = \frac{1}{BP}$ and $\tan \theta = \frac{1}{MP}$
 $\therefore AP = BP = \frac{1}{\sin \theta}$ and $MP = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$
 $\therefore AP + BP + CP = \frac{2}{\sin \theta} + (CM - MP) \therefore L = \frac{2}{\sin \theta} + 3 - \frac{\cos \theta}{\sin \theta}$ as required

c Since $L = 2(\sin \theta)^{-1} + 3 - (\tan \theta)^{-1}$, $\frac{dL}{d\theta} = -2 \cos \theta (\sin \theta)^{-2} + (\tan \theta)^{-2} \times \frac{1}{\cos^2 \theta}$
 $= \frac{-2 \cos \theta}{\sin^2 \theta} + \frac{1}{\cos^2 \theta} \frac{\cos^2 \theta}{\sin^2 \theta}$
 $= \frac{1 - 2 \cos \theta}{\sin^2 \theta}$ as required

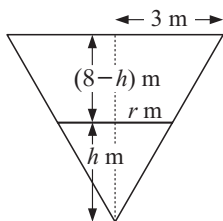
$\therefore \frac{dL}{d\theta} = 0$ if $\cos \theta = \frac{1}{2}$, i.e., $\theta = \frac{\pi}{3}$ and sign diagram of $\frac{dL}{d\theta}$ is:



$\therefore$ the minimum length of cable is required when $\theta = \frac{\pi}{3}$

Then $\sin \theta = \frac{\sqrt{3}}{2}$ and $\tan \theta = \sqrt{3}$,

so $L_{\min} = \frac{4}{\sqrt{3}} + 3 - \frac{1}{\sqrt{3}} = (3 + \sqrt{3})$ km as required

6

a

$$V(r) = \frac{1}{3} \pi r^2 h$$

Using similar triangles $\frac{h}{r} = \frac{8}{3}$

$$\therefore h = \frac{8r}{3}$$

$$\therefore V(r) = \frac{1}{3} \pi r^2 \left(\frac{8r}{3} \right) = \frac{8\pi}{9} r^3 \text{ m}^3$$

b Particular case: When $h = 5$, $r = \frac{3h}{8} = \frac{15}{8}$ and $\frac{dV}{dt} = -0.2 = -\frac{1}{5} \text{ m}^3/\text{min}$

Now $\frac{dV}{dt} = \frac{8\pi}{3} r^2 \frac{dr}{dt} \therefore -\frac{1}{5} = \frac{8\pi}{3} \left(\frac{15}{8} \right)^2 \frac{dr}{dt}$

$$\therefore -\frac{1}{5} = \frac{225}{24} \pi \frac{dr}{dt}$$

$$\therefore \frac{dr}{dt} = -\frac{8}{375\pi}$$

$$\therefore \frac{dr}{dt} = -0.00679 \text{ i.e., radius is decreasing at } 0.00679 \text{ m/min}$$

Chapter 25

INTEGRATION

EXERCISE 25A

1 a i $\frac{d}{dx}(x^2) = 2x$

$$\therefore \frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$$

$$\therefore \text{the antiderivative of } x \text{ is } \frac{1}{2}x^2$$

iii $\frac{d}{dx}(x^6) = 6x^5$

$$\therefore \frac{d}{dx}\left(\frac{1}{6}x^6\right) = x^5$$

$$\therefore \text{the antiderivative of } x^5 \text{ is } \frac{1}{6}x^6$$

v $\frac{d}{dx}(x^{-3}) = -3x^{-4}$

$$\therefore \frac{d}{dx}\left(-\frac{1}{3}x^{-3}\right) = x^{-4}$$

$$\therefore \text{the antiderivative of } x^{-4} \text{ is } -\frac{1}{3}x^{-3}$$

viii $\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} \quad \therefore \frac{d}{dx}(2x^{\frac{1}{2}}) = x^{-\frac{1}{2}}$

$$\therefore \text{the antiderivative of } x^{-\frac{1}{2}} \text{ is } 2x^{\frac{1}{2}} = 2\sqrt{x}$$

b the antiderivative of x^n is $\frac{x^{n+1}}{n+1}$

2 a i $\frac{d}{dx}(e^{2x}) = 2e^{2x}$

$$\therefore \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = e^{2x}$$

$$\therefore \text{the antiderivative of } e^{2x} \text{ is } \frac{1}{2}e^{2x}$$

iii $\frac{d}{dx}(e^{\frac{1}{2}x}) = \frac{1}{2}e^{\frac{1}{2}x}$

$$\therefore \frac{d}{dx}\left(2e^{\frac{1}{2}x}\right) = e^{\frac{1}{2}x}$$

$$\therefore \text{the antiderivative of } e^{\frac{1}{2}x} \text{ is } 2e^{\frac{1}{2}x}$$

v $\frac{d}{dx}(e^{\pi x}) = \pi e^{\pi x}$

$$\therefore \frac{d}{dx}\left(\frac{1}{\pi}e^{\pi x}\right) = e^{\pi x}$$

$$\therefore \text{the antiderivative of } e^{\pi x} \text{ is } \frac{1}{\pi}e^{\pi x}$$

b the antiderivative of e^{kx} is $\frac{1}{k}e^{kx}$

ii $\frac{d}{dx}(x^3) = 3x^2$

$$\therefore \frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$$

$$\therefore \text{the antiderivative of } x^2 \text{ is } \frac{1}{3}x^3$$

iv $\frac{d}{dx}(x^{-1}) = -x^{-2}$

$$\therefore \frac{d}{dx}(-x^{-1}) = x^{-2}$$

$$\therefore \text{the antiderivative of } x^{-2} \text{ is } -x^{-1} \text{ or } -\frac{1}{x}$$

vi $\frac{d}{dx}\left(x^{\frac{4}{3}}\right) = \frac{4}{3}x^{\frac{1}{3}}$

$$\therefore \frac{d}{dx}\left(\frac{3}{4}x^{\frac{4}{3}}\right) = x^{\frac{1}{3}}$$

$$\therefore \text{the antiderivative of } x^{\frac{1}{3}} \text{ is } \frac{3}{4}x^{\frac{4}{3}}$$

ii $\frac{d}{dx}(e^{5x}) = 5e^{5x}$

$$\therefore \frac{d}{dx}\left(\frac{1}{5}e^{5x}\right) = e^{5x}$$

$$\therefore \text{the antiderivative of } e^{5x} \text{ is } \frac{1}{5}e^{5x}$$

iv $\frac{d}{dx}(e^{0.01x}) = 0.01e^{0.01x}$

$$\therefore \frac{d}{dx}(100e^{0.01x}) = e^{0.01x}$$

$$\therefore \text{the antiderivative of } e^{0.01x} \text{ is } 100e^{0.01x}$$

vi $\frac{d}{dx}\left(e^{\frac{x}{3}}\right) = \frac{1}{3}e^{\frac{x}{3}}$

$$\therefore \frac{d}{dx}\left(3e^{\frac{x}{3}}\right) = e^{\frac{x}{3}}$$

$$\therefore \text{the antiderivative of } e^{\frac{x}{3}} \text{ is } 3e^{\frac{x}{3}}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad \frac{d}{dx}(x^3 + x^2) &= 3x^2 + 2x \\ \frac{d}{dx}(2x^3 + 2x^2) &= 6x^2 + 4x \\ \therefore \text{antiderivative of } 6x^2 + 4x & \\ \text{is } 2x^3 + 2x^2 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \frac{d}{dx}(e^{3x+1}) &= 3e^{3x+1} \\ \frac{d}{dx}\left(\frac{1}{3}e^{3x+1}\right) &= e^{3x+1} \\ \therefore \text{antiderivative of } e^{3x+1} &\text{ is } \frac{1}{3}e^{3x+1} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad \frac{d}{dx}(x\sqrt{x}) &= \frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{1}{2}} \\ &= \frac{3}{2}\sqrt{x} \\ \therefore \frac{d}{dx}\left(\frac{2}{3}x\sqrt{x}\right) &= \sqrt{x} \\ \therefore \text{antiderivative of } \sqrt{x} &\text{ is } \frac{2}{3}x\sqrt{x} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad \frac{d}{dx}((2x+1)^4) &= 4(2x+1)^3 \times 2 \\ &= 8(2x+1)^3 \\ \therefore \frac{d}{dx}\left(\frac{1}{8}(2x+1)^4\right) &= (2x+1)^3 \\ \therefore \text{antiderivative of } (2x+1)^3 &\text{ is } \frac{1}{8}(2x+1)^4 \end{aligned}$$

EXERCISE 25B.1

$$\begin{aligned} \mathbf{1} \quad y &= x^7 & \therefore \int 7x^6 dx &= x^7 + c_1 \\ \therefore \frac{dy}{dx} &= 7x^6 & \therefore 7 \int x^6 dx &= x^7 + c_1 \\ & & \therefore \int x^6 dx &= \frac{1}{7}x^7 + c \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad y &= x^3 + x^2 & \therefore \int 3x^2 + 2x dx &= x^3 + x^2 + c \\ \therefore \frac{dy}{dx} &= 3x^2 + 2x \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad y &= e^{2x+1} & \therefore \int 2e^{2x+1} dx &= e^{2x+1} + c_1 \\ \therefore \frac{dy}{dx} &= 2e^{2x+1} & \therefore 2 \int e^{2x+1} dx &= e^{2x+1} + c_1 \\ & & \therefore \int e^{2x+1} dx &= \frac{1}{2}e^{2x+1} + c \end{aligned}$$

$$\begin{aligned} \mathbf{4} \quad y &= (2x+1)^4 & \therefore \int 8(2x+1)^3 dx &= (2x+1)^4 + c_1 \\ \therefore \frac{dy}{dx} &= 4(2x+1)^3 \times 2 & \therefore 8 \int (2x+1)^3 dx &= (2x+1)^4 + c_1 \\ &= 8(2x+1)^3 & \therefore \int (2x+1)^3 dx &= \frac{1}{8}(2x+1)^4 + c \end{aligned}$$

$$\begin{aligned} \mathbf{5} \quad y &= x\sqrt{x} = x^{\frac{3}{2}} & \therefore \int \frac{3}{2}\sqrt{x} dx &= x\sqrt{x} + c_1 \\ \therefore \frac{dy}{dx} &= \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x} & \therefore \frac{3}{2} \int \sqrt{x} dx &= x\sqrt{x} + c_1 \\ & & \therefore \int \sqrt{x} dx &= \frac{2}{3}x\sqrt{x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{6} \quad y &= \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} & \therefore \int -\frac{1}{2}\left(\frac{1}{x\sqrt{x}}\right) dx &= \frac{1}{\sqrt{x}} + c_1 \\ \therefore \frac{dy}{dx} &= -\frac{1}{2}x^{-\frac{3}{2}} = -\frac{1}{2}\left(\frac{1}{x\sqrt{x}}\right) & \therefore -\frac{1}{2} \int \frac{1}{x\sqrt{x}} dx &= \frac{1}{\sqrt{x}} + c_1 \\ & & \therefore \int \frac{1}{x\sqrt{x}} dx &= -\frac{2}{\sqrt{x}} + c \end{aligned}$$

7 Suppose $F(x)$ is the antiderivative of $f(x)$ and $G(x)$ is the antiderivative of $g(x)$.

$$\begin{aligned} \therefore \frac{d}{dx}(F(x) + G(x)) &= f(x) + g(x) \\ \therefore \int (f(x) + g(x)) dx &= F(x) + G(x) + c \\ &= (F(x) + c_1) + (G(x) + c_2) \\ &= \int f(x) dx + \int g(x) dx \end{aligned}$$

$$\begin{aligned}
 8 \quad y &= \sqrt{1-4x} & \therefore \int \frac{-2}{\sqrt{1-4x}} dx &= \sqrt{1-4x} + c_1 \\
 &= (1-4x)^{\frac{1}{2}} & \therefore -2 \int \frac{1}{\sqrt{1-4x}} dx &= \sqrt{1-4x} + c_1 \\
 \therefore \frac{dy}{dx} &= \frac{1}{2}(1-4x)^{-\frac{1}{2}}(-4) & \therefore \int \frac{1}{\sqrt{1-4x}} dx &= -\frac{1}{2}\sqrt{1-4x} + c \\
 &= -2(1-4x)^{-\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 9 \quad \frac{d}{dx}(\ln(5-3x+x^2)) &= \frac{2x-3}{5-3x+x^2} \\
 \therefore \int \frac{2x-3}{5-3x+x^2} dx &= \ln|5-3x+x^2| + c_1 \\
 \therefore \int \frac{4x-6}{5-3x+x^2} dx &= 2\ln(5-3x+x^2) + c \quad \{5-3x+x^2 > 0 \text{ for all } x \\
 &\quad \text{as } a > 0 \text{ and } \Delta = -11 < 0\}
 \end{aligned}$$

$$\begin{aligned}
 10 \quad \frac{d}{dx}(2^x) &= \frac{d}{dx}(e^{\ln 2})^x \quad \{\text{since } 2^x = (e^{\ln 2})^x\} & \therefore \int 2^x \ln 2 \, dx &= 2^x + c_1 \\
 &= e^{(\ln 2)x} \times \ln 2 & \therefore \ln 2 \int 2^x \, dx &= 2^x + c_1 \\
 &= 2^x \ln 2 & \therefore \int 2^x \, dx &= \frac{2^x}{\ln 2} + c
 \end{aligned}$$

$$\begin{aligned}
 11 \quad \frac{d}{dx}(x \ln x) &= 1 \times \ln x + x \times \frac{1}{x} & \therefore \int (\ln x + 1) \, dx &= x \ln x + c_1 \\
 &= \ln x + 1 & \therefore \int \ln x \, dx + x &= x \ln x + c_1 \\
 & & \therefore \int \ln x \, dx &= x \ln x - x + c
 \end{aligned}$$

EXERCISE 25B.2

$$\begin{aligned}
 1 \quad \mathbf{a} \quad \int (x^4 - x^2 - x + 2) \, dx &= \frac{1}{5}x^5 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x + c \\
 \mathbf{b} \quad \int (\sqrt{x} + e^x) \, dx &= \int (x^{\frac{1}{2}} + e^x) \, dx \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + e^x + c \\
 &= \frac{2}{3}x^{\frac{3}{2}} + e^x + c \\
 \mathbf{c} \quad \int (3e^x - \frac{1}{x}) \, dx &= 3e^x - \ln|x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \int (x\sqrt{x} - \frac{2}{x}) \, dx &= \int (x^{\frac{3}{2}} - \frac{2}{x}) \, dx \\
 &= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 2\ln|x| + c \\
 &= \frac{2}{5}x^{\frac{5}{2}} - 2\ln|x| + c \\
 \mathbf{e} \quad \int \left(\frac{1}{x\sqrt{x}} + \frac{4}{x} \right) \, dx &= \int (x^{-\frac{3}{2}} + \frac{4}{x}) \, dx \\
 &= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + 4\ln|x| + c \\
 &= -\frac{2}{\sqrt{x}} + 4\ln|x| + c \\
 \mathbf{f} \quad \int (\frac{1}{2}x^3 - x^4 + x^{\frac{1}{3}}) \, dx &= \frac{1}{2} \frac{x^4}{4} - \frac{x^5}{5} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c \\
 &= \frac{1}{8}x^4 - \frac{1}{5}x^5 + \frac{3}{4}x^{\frac{4}{3}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad \int \left(x^2 + \frac{3}{x} \right) \, dx &= \frac{x^3}{3} + 3\ln|x| + c \\
 \mathbf{h} \quad \int \left(\frac{1}{2x} + x^2 - e^x \right) \, dx &= \frac{1}{2} \ln|x| + \frac{x^3}{3} - e^x + c \\
 \mathbf{i} \quad \int \left(5e^x + \frac{1}{3}x^3 - \frac{4}{x} \right) \, dx &= 5e^x + \frac{1}{3} \frac{x^4}{4} - 4\ln|x| + c \\
 &= 5e^x + \frac{1}{12}x^4 - 4\ln|x| + c
 \end{aligned}$$

$$\begin{aligned} \mathbf{2} \quad \mathbf{a} \quad & \int (x^2 + 3x - 2) dx \\ &= \frac{x^3}{3} + \frac{3x^2}{2} - 2x + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right) dx \\ &= \int x^{\frac{1}{2}} - x^{-\frac{1}{2}} dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \int \left(2e^x - \frac{1}{x^2} \right) dx \\ &= \int (2e^x - x^{-2}) dx \\ &= 2 \int e^x dx - \frac{x^{-1}}{-1} + c \\ &= 2e^x + \frac{1}{x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \int \left(\frac{1-4x}{x\sqrt{x}} \right) dx \\ &= \int \left(\frac{1}{x\sqrt{x}} - \frac{4}{\sqrt{x}} \right) dx \\ &= \int (x^{-\frac{3}{2}} - 4x^{-\frac{1}{2}}) dx \\ &= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= -\frac{2}{\sqrt{x}} - 8\sqrt{x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \int (2x+1)^2 dx \\ &= \int (4x^2 + 4x + 1) dx \\ &= \frac{4x^3}{3} + \frac{4x^2}{2} + x + c \\ &= \frac{4}{3}x^3 + 2x^2 + x + c \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \int \frac{x^2 + x - 3}{x} dx \\ &= \int \left(x + 1 - \frac{3}{x} \right) dx \\ &= \frac{x^2}{2} + x - 3 \ln |x| + c \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \int \frac{2x-1}{\sqrt{x}} dx \\ &= \int \left(2x^{\frac{1}{2}} - x^{-\frac{1}{2}} \right) dx \\ &= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= \frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \int \frac{x^2 - 4x + 10}{x^2\sqrt{x}} dx \\ &= \int \left(\frac{x^2}{x^2\sqrt{x}} - \frac{4x}{x^2\sqrt{x}} + \frac{10}{x^2\sqrt{x}} \right) dx \\ &= \int \left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}} + 10x^{-\frac{5}{2}} \right) dx \\ &= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{4x^{-\frac{1}{2}}}{-\frac{1}{2}} + \frac{10x^{-\frac{3}{2}}}{-\frac{3}{2}} + c \\ &= 2\sqrt{x} + \frac{8}{\sqrt{x}} - \frac{20}{3x\sqrt{x}} + c \end{aligned}$$

$$\begin{aligned} \mathbf{i} \quad & \int (x+1)^3 dx \\ &= \int (x^3 + 3x^2 + 3x + 1) dx \\ &= \frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + c \end{aligned}$$

$$\begin{aligned} \mathbf{3} \quad \mathbf{a} \quad & \frac{dy}{dx} = 6 \\ \therefore y &= \int 6 dx \\ \therefore y &= 6x + c \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad & \frac{dy}{dx} = 4x^2 \\ \therefore y &= \int 4x^2 dx \\ \therefore y &= \frac{4}{3}x^3 + c \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad & \frac{dy}{dx} = 5x - x^2 \\ \therefore y &= \int (5x - x^2) dx \\ \therefore y &= \frac{5}{2}x^2 - \frac{1}{3}x^3 + c \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \frac{dy}{dx} = \frac{1}{x^2} = x^{-2} \\ \therefore y &= \int x^{-2} dx \\ &= \frac{x^{-1}}{-1} + c \\ \therefore y &= -\frac{1}{x} + c \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \frac{dy}{dx} = 2e^x - 5 \\ \therefore y &= \int (2e^x - 5) dx \\ \therefore y &= 2e^x - 5x + c \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \frac{dy}{dx} = 4x^3 + 3x^2 \\ \therefore y &= \int (4x^3 + 3x^2) dx \\ &= \frac{4x^4}{4} + \frac{3x^3}{3} + c \\ \therefore y &= x^4 + x^3 + c \end{aligned}$$

$$4 \quad a \quad \frac{dy}{dx} = (1 - 2x)^2$$

$$\begin{aligned} \therefore y &= \int (1 - 2x)^2 dx \\ &= \int (1 - 4x + 4x^2) dx \\ &= x - \frac{4x^2}{2} + \frac{4x^3}{3} + c \\ &= x - 2x^2 + \frac{4}{3}x^3 + c \end{aligned}$$

$$c \quad \frac{dy}{dx} = \frac{x^2 + 2x - 5}{x^2}$$

$$= 1 + 2x^{-1} - 5x^{-2}$$

$$\therefore y = \int (1 + 2x^{-1} - 5x^{-2}) dx = x + 2 \ln |x| - \frac{5x^{-1}}{-1} + c = x + 2 \ln |x| + \frac{5}{x} + c$$

$$5 \quad a \quad f'(x) = x^3 - 5x + 3$$

$$\begin{aligned} \therefore f(x) &= \int (x^3 - 5x + 3) dx \\ &= \frac{x^4}{4} - \frac{5x^2}{2} + 3x + c \end{aligned}$$

$$c \quad f'(x) = 3e^x - \frac{4}{x}$$

$$\therefore f(x) = \int \left(3e^x - \frac{4}{x} \right) dx$$

$$= 3e^x - 4 \ln |x| + c$$

$$6 \quad a \quad f'(x) = 2x - 1 \quad \text{and} \quad f(0) = 3$$

$$\begin{aligned} f(x) &= \int (2x - 1) dx \\ &= \frac{2x^2}{2} - x + c \\ &= x^2 - x + c \end{aligned}$$

$$\text{But } f(0) = 3$$

$$\therefore 0 - 0 + c = 3$$

$$\text{i.e., } c = 3$$

$$\therefore f(x) = x^2 - x + 3$$

$$c \quad f'(x) = e^x + \frac{1}{\sqrt{x}} \quad \text{and} \quad f(1) = 1$$

$$\begin{aligned} f'(x) &= e^x + x^{-\frac{1}{2}} \\ f(x) &= \int \left(e^x + x^{-\frac{1}{2}} \right) dx \\ &= e^x + 2x^{\frac{1}{2}} + c \end{aligned}$$

$$\text{But } f(1) = 1$$

$$\therefore e^1 + 2 + c = 1$$

$$\text{i.e., } c = -1 - e$$

$$\therefore f(x) = e^x + 2\sqrt{x} - 1 - e$$

$$b \quad \frac{dy}{dx} = \sqrt{x} - \frac{2}{\sqrt{x}}$$

$$= x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$$

$$\begin{aligned} \therefore y &= \int \left(x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} \right) dx \\ &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + c \end{aligned}$$

$$b \quad f'(x) = 2\sqrt{x}(1 - 3x)$$

$$= 2x^{\frac{1}{2}} - 6x^{\frac{3}{2}}$$

$$\begin{aligned} \therefore f(x) &= \int \left(2x^{\frac{1}{2}} - 6x^{\frac{3}{2}} \right) dx \\ &= \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{6x^{\frac{5}{2}}}{\frac{5}{2}} + c \\ &= \frac{4}{3}x^{\frac{3}{2}} - \frac{12}{5}x^{\frac{5}{2}} + c \end{aligned}$$

$$b \quad f'(x) = 3x^2 + 2x \quad \text{and} \quad f(2) = 5$$

$$\begin{aligned} f(x) &= \int (3x^2 + 2x) dx \\ &= \frac{3x^3}{3} + \frac{2x^2}{2} + c \\ &= x^3 + x^2 + c \end{aligned}$$

$$\text{But } f(2) = 5$$

$$\therefore 8 + 4 + c = 5$$

$$\text{i.e., } c = -7$$

$$\therefore f(x) = x^3 + x^2 - 7$$

$$d \quad f'(x) = x - \frac{2}{\sqrt{x}} \quad \text{and} \quad f(1) = 2$$

$$\begin{aligned} f'(x) &= x - 2x^{-\frac{1}{2}} \\ f(x) &= \int \left(x - 2x^{-\frac{1}{2}} \right) dx \\ &= \frac{x^2}{2} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ &= \frac{1}{2}x^2 - 4\sqrt{x} + c \end{aligned}$$

$$\text{But } f(1) = 2$$

$$\therefore \frac{1}{2} - 4 + c = 2 \quad \text{i.e., } c = \frac{11}{2}$$

$$\therefore f(x) = \frac{1}{2}x^2 - 4\sqrt{x} + \frac{11}{2}$$

$$7 \quad a \quad f''(x) = 2x + 1 \quad f'(1) = 3 \quad f(2) = 7$$

$$\text{Now } f'(x) = \frac{2x^2}{2} + x + c = x^2 + x + c$$

$$\text{But } f'(1) = 3$$

$$\text{and } f(2) = 7$$

$$\therefore 1 + 1 + c = 3$$

$$\therefore \frac{8}{3} + 2 + 2 + k = 7$$

$$\therefore c = 1$$

$$k = 7 - 4 - \frac{8}{3}$$

$$\therefore f'(x) = x^2 + x + 1$$

$$\therefore k = \frac{1}{3}$$

$$\therefore f(x) = \frac{x^3}{3} + \frac{x^2}{2} + x + k$$

$$\therefore f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{3}$$

$$b \quad f''(x) = 15\sqrt{x} + \frac{3}{\sqrt{x}} \quad f'(1) = 12 \quad f(0) = 5$$

$$\text{Now } f''(x) = 15x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$$

$$\therefore f'(x) = \frac{15x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + c$$

$$\text{But } f'(1) = 12 \quad \therefore 10 + 6 + c = 12 \quad \text{and so } c = -4$$

$$\therefore f'(x) = 10x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 4$$

$$\therefore f(x) = \frac{10x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} - 4x + k = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + k$$

$$\text{But } f(0) = 5 \quad \therefore k = 5 \quad \therefore f(x) = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + 5$$

$$c \quad f''(x) = 2x \quad (1, 0) \text{ and } (0, 5) \text{ lie on the curve}$$

$$\text{Now } f'(x) = \frac{2x^2}{2} + c = x^2 + c \quad \text{and } f(x) = \frac{x^3}{3} + cx + k$$

$$\text{But } f(0) = 5 \quad \therefore 0 + 0 + k = 5 \quad \text{and so } k = 5$$

$$\text{and } f(1) = 0 \quad \therefore \frac{1}{3} + c + 5 = 0 \quad \text{and so } c = -5\frac{1}{3}$$

$$\therefore f(x) = \frac{1}{3}x^3 - \frac{16}{3}x + 5$$

EXERCISE 25C

$$1 \quad a \quad \int (2x + 5)^3 dx \\ = \frac{1}{2} \times \frac{(2x + 5)^4}{4} + c \\ = \frac{1}{8}(2x + 5)^4 + c$$

$$b \quad \int \frac{1}{(3 - 2x)^2} dx \\ = \int (3 - 2x)^{-2} dx \\ = \frac{1}{-2} \times \frac{(3 - 2x)^{-1}}{-1} + c \\ = \frac{1}{2(3 - 2x)} + c$$

$$c \quad \int \frac{4}{(2x - 1)^4} dx \\ = \int 4(2x - 1)^{-4} dx \\ = 4\left(\frac{1}{2}\right) \times \frac{(2x - 1)^{-3}}{-3} + c \\ = -\frac{2}{3}(2x - 1)^{-3} + c$$

$$d \quad \int (4x - 3)^7 dx \\ = \frac{1}{4} \times \frac{(4x - 3)^8}{8} + c \\ = \frac{1}{32}(4x - 3)^8 + c$$

$$e \quad \int \sqrt{3x - 4} dx \\ = \int (3x - 4)^{\frac{1}{2}} dx \\ = \frac{1}{3} \times \frac{(3x - 4)^{\frac{3}{2}}}{\frac{3}{2}} + c \\ = \frac{2}{9}(3x - 4)^{\frac{3}{2}} + c$$

$$f \quad \int \frac{10}{\sqrt{1 - 5x}} dx \\ = \int 10(1 - 5x)^{-\frac{1}{2}} dx \\ = 10\left(\frac{1}{-5}\right) \times \frac{(1 - 5x)^{\frac{1}{2}}}{\frac{1}{2}} + c \\ = -4(1 - 5x)^{\frac{1}{2}} + c$$

$$\begin{aligned}
 \mathbf{g} \quad & \int 3(1-x)^4 dx \\
 &= 3 \int (1-x)^4 dx \\
 &= 3\left(\frac{1}{-1}\right) \times \frac{(1-x)^5}{5} + c \\
 &= -\frac{3}{5}(1-x)^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int \frac{4}{\sqrt{3-4x}} dx = \int 4(3-4x)^{-\frac{1}{2}} dx \\
 &= 4\left(\frac{1}{-4}\right) \times \frac{(3-4x)^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= -2\sqrt{3-4x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \frac{dy}{dx} = \sqrt{2x-7} \\
 &= (2x-7)^{\frac{1}{2}} \\
 \therefore y &= \frac{1}{2} \times \frac{(2x-7)^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \frac{1}{3}(2x-7)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 & \text{But } y = 11 \text{ when } x = 8 \\
 \therefore \frac{1}{3}(9)^{\frac{3}{2}} + c &= 11 \\
 \frac{1}{3}(27) + c &= 11 \\
 9 + c &= 11 \text{ and so } c = 2 \\
 \therefore y &= \frac{1}{3}(2x-7)^{\frac{3}{2}} + 2
 \end{aligned}$$

$$\mathbf{b} \quad f(x) \text{ has tangent-slope function } \frac{4}{\sqrt{1-x}}$$

$$\begin{aligned}
 \text{i.e., } f'(x) &= \frac{4}{\sqrt{1-x}} = 4(1-x)^{-\frac{1}{2}} \\
 f(x) &= 4\left(\frac{1}{-1}\right) \times \frac{(1-x)^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= -8\sqrt{1-x} + c
 \end{aligned}$$

$$\begin{aligned}
 & \text{But } y = 11 \text{ when } x = -3 \\
 \therefore -8\sqrt{1-(-3)} + c &= 11 \\
 \therefore -8\sqrt{4} + c &= 11 \\
 \therefore -16 + c &= 11 \text{ and } c = 5 \\
 \therefore f(x) &= 5 - 8\sqrt{1-x}
 \end{aligned}$$

$$\text{When } x = -8, y = 5 - 8\sqrt{1-(-8)} = 5 - 8(3) = -19, \therefore \text{point is } (-8, -19).$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & \int 3(2x-1)^2 dx \\
 &= 3 \int (2x-1)^2 dx \\
 &= 3\left(\frac{1}{2}\right) \frac{(2x-1)^3}{3} + c \\
 &= \frac{1}{2}(2x-1)^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int (x^2 - x)^2 dx \\
 &= \int (x^4 - 2x^3 + x^2) dx \\
 &= \frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3} + c \\
 &= \frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int (1-3x)^3 dx \\
 &= \left(\frac{1}{-3}\right) \frac{(1-3x)^4}{4} + c \\
 &= -\frac{1}{12}(1-3x)^4 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int (1-x^2)^2 dx \\
 &= \int (1-2x^2+x^4) dx \\
 &= x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int 4\sqrt{5-x} dx \\
 &= 4 \int (5-x)^{\frac{1}{2}} dx \\
 &= 4\left(\frac{1}{-1}\right) \frac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= -\frac{8}{3}(5-x)^{\frac{3}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int (x^2+1)^3 dx \\
 &= \int (x^6+3x^4+3x^2+1) dx \\
 &= \frac{x^7}{7} + \frac{3x^5}{5} + \frac{3x^3}{3} + x + c \\
 &= \frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & \int (2e^x + 5e^{2x}) dx \\
 &= 2e^x + 5\left(\frac{1}{2}\right)e^{2x} + c \\
 &= 2e^x + \frac{5}{2}e^{2x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int (x^2 - 2e^{-3x}) dx \\
 &= \frac{x^3}{3} - 2\left(\frac{1}{-3}\right)e^{-3x} + c \\
 &= \frac{1}{3}x^3 + \frac{2}{3}e^{-3x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int (\sqrt{x} + 4e^{2x} - e^{-x}) dx \\
 &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 4\left(\frac{1}{2}\right)e^{2x} - (-1)e^{-x} + c \\
 &= \frac{2}{3}x^{\frac{3}{2}} + 2e^{2x} + e^{-x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int \frac{1}{2x-1} dx \\
 &= \frac{1}{2} \ln |2x-1| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int \frac{5}{1-3x} dx \\
 &= 5 \int \frac{1}{1-3x} dx \\
 &= 5\left(\frac{1}{-3}\right) \ln |1-3x| + c \\
 &= -\frac{5}{3} \ln |1-3x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int \left(e^{-x} - \frac{4}{2x+1}\right) dx \\
 &= \frac{1}{-1}e^{-x} - 4\left(\frac{1}{2}\right) \ln |2x+1| + c \\
 &= -e^{-x} - 2 \ln |2x+1| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \int (e^x + e^{-x})^2 dx \\
 &= \int (e^{2x} + 2 + e^{-2x}) dx \\
 &= \frac{1}{2}e^{2x} + 2x + \left(\frac{1}{-2}\right)e^{-2x} + c \\
 &= \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int (e^{-x} + 2)^2 dx \\
 &= \int (e^{-2x} + 4e^{-x} + 4) dx \\
 &= \frac{1}{-2}e^{-2x} + 4\left(\frac{1}{-1}\right)e^{-x} + 4x + c \\
 &= -\frac{1}{2}e^{-2x} - 4e^{-x} + 4x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & \int \left(x - \frac{5}{1-x}\right) dx = \frac{x^2}{2} - 5\left(\frac{1}{-1}\right)\ln|1-x| + c \\
 &= \frac{1}{2}x^2 + 5\ln|1-x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \mathbf{a} \quad & \frac{dy}{dx} = (1 - e^x)^2 \\
 &= 1 - 2e^x + e^{2x} \\
 \therefore y &= x - 2e^x + \frac{1}{2}e^{2x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \frac{dy}{dx} = 1 - 2x + \frac{3}{x+2} \\
 \therefore y &= x - \frac{2x^2}{2} + 3\ln|x+2| + c \\
 &= x - x^2 + 3\ln|x+2| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \frac{dy}{dx} = e^{-2x} + \frac{4}{2x-1} \quad \therefore y = \frac{1}{-2}e^{-2x} + 4\left(\frac{1}{2}\right)\ln|2x-1| + c \\
 &= -\frac{1}{2}e^{-2x} + 2\ln|2x-1| + c
 \end{aligned}$$

6 Differentiating Tracy's answer gives

$$\begin{aligned}
 & \frac{d}{dx} \left(\frac{1}{4}\ln|4x| + c\right) \\
 &= \frac{1}{4} \left(\frac{1}{4x}\right) \times 4 + 0 \\
 &= \frac{1}{4x}
 \end{aligned}$$

The derivative of Nadine's answer is

$$\begin{aligned}
 & \frac{d}{dx} \left(\frac{1}{4}\ln|x| + c\right) \\
 &= \frac{1}{4} \left(\frac{1}{x}\right) + 0 \\
 &= \frac{1}{4x}
 \end{aligned}$$

$\therefore$ both answers give the correct derivative and both are correct.

Note that this result occurs because $\log 4x = \log 4 + \log x$ $\therefore$ their answers differ by a constant ($\log 4$) which is just part of the constant c .

$$\begin{aligned}
 \mathbf{7} \quad \mathbf{a} \quad & f'(x) = 2e^{-2x} \quad f(0) = 3 \\
 \therefore f(x) &= 2\left(\frac{1}{-2}\right)e^{-2x} + c \\
 &= -e^{-2x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{But } f(0) &= 3 \\
 \therefore -e^0 + c &= 3 \\
 \therefore c &= 4 \\
 \therefore f(x) &= -e^{-2x} + 4
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & f'(x) = 2x - \frac{2}{1-x} \quad f(-1) = 3 \\
 \therefore f(x) &= \frac{2x^2}{2} - \frac{2}{-1}\ln|1-x| + c \\
 &= x^2 + 2\ln|1-x| + c
 \end{aligned}$$

$$\begin{aligned}
 \text{But } f(-1) &= 3 \\
 \therefore 1 + 2\ln|2| + c &= 3 \\
 \therefore c &= 2 - 2\ln 2 \\
 \therefore f(x) &= x^2 + 2\ln|1-x| + 2 - 2\ln 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & f'(x) = \sqrt{x} + \frac{1}{2}e^{-4x} \\
 \therefore f(x) &= x^{\frac{1}{2}} + \frac{1}{2}e^{-4x} \\
 \text{and } f(x) &= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2}\left(\frac{1}{-4}\right)e^{-4x} + c \\
 &= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{But } f(1) &= 0 \\
 \therefore \frac{2}{3} - \frac{1}{8}e^{-4} + c &= 0 \\
 \therefore c &= \frac{1}{8}e^{-4} - \frac{2}{3} \\
 \therefore f(x) &= \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + \frac{1}{8}e^{-4} - \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 8 \quad & \frac{3}{x+2} - \frac{1}{x-2} \\
 &= \frac{3(x-2) - 1(x+2)}{(x+2)(x-2)} \\
 &= \frac{3x-6-x-2}{x^2-4} \\
 &= \frac{2x-8}{x^2-4}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \int \frac{2x-8}{x^2-4} dx \\
 &= \int \left(\frac{3}{x+2} - \frac{1}{x-2} \right) dx \\
 &= 3 \ln |x+2| - \ln |x-2| + c
 \end{aligned}$$

$$\begin{aligned}
 9 \quad & \frac{1}{2x-1} - \frac{1}{2x+1} \\
 &= \frac{1(2x+1) - 1(2x-1)}{(2x-1)(2x+1)} \\
 &= \frac{2x+1-2x+1}{(2x-1)(2x+1)} \\
 &= \frac{2}{4x^2-1}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \int \frac{2}{4x^2-1} dx \\
 &= \int \left(\frac{1}{2x-1} - \frac{1}{2x+1} \right) dx \\
 &= \frac{1}{2} \ln |2x-1| - \frac{1}{2} \ln |2x+1| + c
 \end{aligned}$$

EXERCISE 25D

1 a Let $u = x^3 + 1$, $\therefore \frac{du}{dx} = 3x^2$

$$\begin{aligned}
 \therefore \int 3x^2(x^3+1)^4 dx \\
 &= \int u^4 \frac{du}{dx} dx \\
 &= \int u^4 du \\
 &= \frac{1}{5} u^5 + c \\
 &= \frac{1}{5} (x^3+1)^5 + c
 \end{aligned}$$

b Let $u = x^2 + 3$ $\therefore \frac{du}{dx} = 2x$

$$\begin{aligned}
 \therefore \int \frac{2x}{\sqrt{x^2+3}} dx \\
 &= \int \left((x^2+3)^{-\frac{1}{2}} \times 2x \right) dx \\
 &= \int u^{-\frac{1}{2}} \frac{du}{dx} dx \\
 &= \int u^{-\frac{1}{2}} du \\
 &= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \\
 &= 2\sqrt{u} + c \\
 &= 2\sqrt{x^2+3} + c
 \end{aligned}$$

c Let $u = x^3 + x$ $\therefore \frac{du}{dx} = 3x^2 + 1$

$$\begin{aligned}
 \therefore \int \sqrt{x^3+x} (3x^2+1) dx \\
 &= \int \sqrt{u} \frac{du}{dx} dx \\
 &= \int u^{\frac{1}{2}} du \\
 &= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\
 &= \frac{2}{3} u^{\frac{3}{2}} + c \\
 &= \frac{2}{3} (x^3+x)^{\frac{3}{2}} + c
 \end{aligned}$$

d Let $u = 2 + x^4$ $\therefore \frac{du}{dx} = 4x^3$

$$\begin{aligned}
 \therefore \int 4x^3(2+x^4)^3 dx \\
 &= \int u^3 \frac{du}{dx} dx \\
 &= \int u^3 du \\
 &= \frac{u^4}{4} + c \\
 &= \frac{1}{4} (2+x^4)^4 + c
 \end{aligned}$$

e Let $u = x^3 + 2x + 1 \quad \therefore \quad \frac{du}{dx} = 3x^2 + 2$ **f** Let $u = 3x^3 - 1 \quad \therefore \quad \frac{du}{dx} = 9x^2$

$$\begin{aligned} \therefore \int (x^3 + 2x + 1)^4 (3x^2 + 2) dx \\ &= \int u^4 \frac{du}{dx} dx \\ &= \int u^4 du \\ &= \frac{u^5}{5} + c \\ &= \frac{1}{5}(x^3 + 2x + 1)^5 + c \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{x^2}{(3x^3 - 1)^4} dx \\ &= \int (3x^3 - 1)^{-4} \times x^2 dx \\ &= \int u^{-4} \left(\frac{1}{9} \frac{du}{dx} \right) dx \\ &= \frac{1}{9} \int u^{-4} du \\ &= \frac{1}{9} \frac{u^{-3}}{-3} + c \quad \text{i.e.,} \quad -\frac{1}{27(3x^3 - 1)^3} + c \end{aligned}$$

g Let $u = 1 - x^2 \quad \therefore \quad \frac{du}{dx} = -2x$

$$\begin{aligned} \therefore \int \frac{x}{(1 - x^2)^5} dx \\ &= \int ((1 - x^2)^{-5} \times x) dx \\ &= \int u^{-5} \left(-\frac{1}{2} \frac{du}{dx} \right) dx \\ &= -\frac{1}{2} \int u^{-5} du \\ &= -\frac{1}{2} \frac{u^{-4}}{-4} + c \\ &= \frac{1}{8(1 - x^2)^4} + c \end{aligned}$$

h Let $u = x^2 + 4x - 3 \quad \therefore \quad \frac{du}{dx} = 2x + 4$

$$\begin{aligned} \therefore \int \frac{x + 2}{(x^2 + 4x - 3)^2} dx \\ &= \int (x^2 + 4x - 3)^{-2} (x + 2) dx \\ &= \int u^{-2} \left(\frac{1}{2} \frac{du}{dx} \right) dx \\ &= \frac{1}{2} \int u^{-2} du \\ &= \frac{1}{2} \frac{u^{-1}}{-1} + c \\ &= \frac{-1}{2(x^2 + 4x - 3)} + c \end{aligned}$$

i Let $u = x^2 + x \quad \therefore \quad \frac{du}{dx} = 2x + 1 \quad \therefore \quad \int x^4(x + 1)^4(2x + 1) dx$
 $= \int (x^2 + x)^4(2x + 1) dx \quad \{\text{as } a^4b^4 = (ab)^4\}$
 $= \int u^4 \frac{du}{dx} dx = \int u^4 du$
 $= \frac{1}{5}u^5 + c$
 $= \frac{1}{5}(x^2 + x)^5 + c$

2 a Let $u = 1 - 2x \quad \therefore \quad \frac{du}{dx} = -2$

$$\begin{aligned} \therefore \int -2e^{1-2x} dx &= \int e^u \frac{du}{dx} dx \\ &= \int e^u du \\ &= e^u + c \\ &= e^{1-2x} + c \end{aligned}$$

b Let $u = x^2 \quad \therefore \quad \frac{du}{dx} = 2x$

$$\begin{aligned} \therefore \int 2xe^{x^2} dx &= \int e^u \frac{du}{dx} dx \\ &= \int e^u du \\ &= e^u + c \\ &= e^{x^2} + c \end{aligned}$$

c Let $u = x^3 + 1 \quad \therefore \quad \frac{du}{dx} = 3x^2$

$$\begin{aligned} \therefore \int x^2 e^{x^3+1} dx \\ &= \int e^u \left(\frac{1}{3} \frac{du}{dx} \right) dx \\ &= \frac{1}{3} \int e^u du \\ &= \frac{1}{3} e^u + c \\ &= \frac{1}{3} e^{x^3+1} + c \end{aligned}$$

d Let $u = \sqrt{x} \quad \therefore \quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$$\begin{aligned} \therefore \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx \\ &= \int e^u \left(2 \frac{du}{dx} \right) dx \\ &= 2 \int e^u du \\ &= 2e^u + c \\ &= 2e^{\sqrt{x}} + c \end{aligned}$$

$$\text{e} \quad \text{Let } u = x - x^2 \quad \therefore \quad \frac{du}{dx} = 1 - 2x$$

$$\begin{aligned} \therefore \int (2x - 1)e^{x-x^2} dx &= \int e^u \left(-\frac{du}{dx} \right) dx \\ &= \int -e^u du \\ &= -e^u + c \\ &= -e^{x-x^2} + c \end{aligned}$$

$$\text{3 a} \quad \text{Let } u = x^2 + 1 \quad \therefore \quad \frac{du}{dx} = 2x$$

$$\begin{aligned} \therefore \int \frac{2x}{x^2 + 1} dx &= \int \frac{1}{x^2 + 1} (2x) dx \\ &= \int \frac{1}{u} \frac{du}{dx} dx \\ &= \int \frac{1}{u} du \\ &= \ln |u| + c \\ &= \ln(x^2 + 1) + c \quad \text{as } x^2 + 1 > 0 \end{aligned}$$

$$\text{c} \quad \text{Let } u = x^2 - 3x \quad \therefore \quad \frac{du}{dx} = 2x - 3$$

$$\begin{aligned} \therefore \int \frac{2x - 3}{x^2 - 3x} dx &= \int \frac{1}{x^2 - 3x} (2x - 3) dx \\ &= \int \frac{1}{u} \frac{du}{dx} dx \\ &= \int \frac{1}{u} du \\ &= \ln |u| + c \\ &= \ln |x^2 - 3x| + c \end{aligned}$$

$$\text{e} \quad \text{Let } u = 5x - x^2 \quad \therefore \quad \frac{du}{dx} = 5 - 2x$$

$$\begin{aligned} \therefore \int \frac{4x - 10}{5x - x^2} dx &= \int \frac{1}{5x - x^2} (4x - 10) dx \\ &= \int \frac{1}{u} \left(-2 \frac{du}{dx} \right) dx \\ &= -2 \int \frac{1}{u} dx \\ &= -2 \ln |u| + c \\ &= -2 \ln |5x - x^2| + c \end{aligned}$$

$$\text{f} \quad \text{Let } u = \frac{x-1}{x} \quad \therefore \quad \frac{du}{dx} = 0 - (-1)x^{-2} \\ = 1 - x^{-1} = \frac{1}{x^2}$$

$$\begin{aligned} \therefore \int \frac{e^{\frac{x-1}{x}}}{x^2} dx &= \int e^u \frac{du}{dx} dx \\ &= \int e^u du \\ &= e^u + c \\ &= e^{\frac{x-1}{x}} + c \end{aligned}$$

$$\text{b} \quad \text{Let } u = 2 - x^2 \quad \therefore \quad \frac{du}{dx} = -2x$$

$$\begin{aligned} \therefore \int \frac{x}{2 - x^2} dx &= \int \left(\frac{1}{2 - x^2} \times x \right) dx \\ &= \int \frac{1}{u} \left(-\frac{1}{2} \frac{du}{dx} \right) dx \\ &= -\frac{1}{2} \int \frac{1}{u} du \\ &= -\frac{1}{2} \ln |u| + c \\ &= -\frac{1}{2} \ln |2 - x^2| + c \end{aligned}$$

$$\text{d} \quad \text{Let } u = x^3 - x \quad \therefore \quad \frac{du}{dx} = 3x^2 - 1$$

$$\begin{aligned} \therefore \int \frac{6x^2 - 2}{x^3 - x} dx &= \int \frac{1}{x^3 - x} (6x^2 - 2) dx \\ &= \int \frac{1}{u} \left(2 \frac{du}{dx} \right) dx \\ &= 2 \int \frac{1}{u} du \\ &= 2 \ln |u| + c \\ &= 2 \ln |x^3 - x| + c \end{aligned}$$

$$\text{f} \quad \text{Let } u = x^3 - 3x \quad \therefore \quad \frac{du}{dx} = 3x^2 - 3$$

$$\begin{aligned} \therefore \int \frac{1 - x^2}{x^3 - 3x} dx &= \int \frac{1}{x^3 - 3x} (1 - x^2) dx \\ &= \int \frac{1}{u} \left(-\frac{1}{3} \frac{du}{dx} \right) dx \\ &= -\frac{1}{3} \int \frac{1}{u} du \\ &= -\frac{1}{3} \ln |u| + c \\ &= -\frac{1}{3} \ln |x^3 - 3x| + c \end{aligned}$$

$$4 \quad \mathbf{a} \quad \text{Let } u = 3 - x^3 \quad \therefore \quad \frac{du}{dx} = -3x^2$$

$$\begin{aligned} \therefore f(x) &= \int x^2(3 - x^3)^2 dx \\ &= \int u^2 \left(-\frac{1}{3} \frac{du}{dx} \right) dx \quad \{\} \\ &= -\frac{1}{3} \int u^2 du \\ &= -\frac{1}{3} \times \frac{u^3}{3} + c \\ &= -\frac{1}{9}(3 - x^3)^3 + c \end{aligned}$$

$$\mathbf{b} \quad \text{Let } u = \ln x \quad \therefore \quad \frac{du}{dx} = \frac{1}{x}$$

$$\begin{aligned} \therefore f(x) &= \int \frac{4}{x \ln x} dx \\ &= \int u^{-1} \left(4 \frac{du}{dx} \right) dx \\ &= 4 \int \frac{1}{u} du \\ &= 4 \ln |u| + c \\ &= 4 \ln |\ln x| + c \end{aligned}$$

$$\mathbf{c} \quad \text{Let } u = 1 - x^2 \quad \therefore \quad \frac{du}{dx} = -2x$$

$$\begin{aligned} \therefore f(x) &= \int x \sqrt{1 - x^2} dx \\ &= \int \sqrt{u} \left(-\frac{1}{2} \frac{du}{dx} \right) dx \\ &= -\frac{1}{2} \int u^{\frac{1}{2}} du \\ &= -\frac{1}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= -\frac{1}{3} u^{\frac{3}{2}} + c \\ &= -\frac{1}{3} (1 - x^2)^{\frac{3}{2}} + c \end{aligned}$$

$$\mathbf{d} \quad \text{Let } u = 1 - x^2 \quad \therefore \quad \frac{du}{dx} = -2x$$

$$\begin{aligned} \therefore f(x) &= \int x e^{1-x^2} dx \\ &= \int e^u \left(-\frac{1}{2} \frac{du}{dx} \right) dx \\ &= -\frac{1}{2} \int e^u du \\ &= -\frac{1}{2} e^u + c \\ &= -\frac{1}{2} e^{1-x^2} + c \end{aligned}$$

$$\mathbf{e} \quad \text{Let } u = x^3 - x \quad \therefore \quad \frac{du}{dx} = 3x^2 - 1$$

$$\begin{aligned} \therefore f(x) &= \int \frac{1 - 3x^2}{x^3 - x} dx \\ &= \int \frac{1}{u} \left(-\frac{du}{dx} \right) dx \\ &= - \int \frac{1}{u} du \\ &= -\ln |u| + c \\ &= -\ln |x^3 - x| + c \end{aligned}$$

$$\mathbf{f} \quad \text{Let } u = \ln x \quad \therefore \quad \frac{du}{dx} = \frac{1}{x}$$

$$\begin{aligned} \therefore f(x) &= \int \frac{(\ln x)^3}{x} dx \\ &= \int u^3 \frac{du}{dx} dx \\ &= \int u^3 du \\ &= \frac{u^4}{4} + c \\ &= \frac{1}{4} (\ln x)^4 + c \end{aligned}$$

EXERCISE 25E.1

$$\begin{aligned} \mathbf{1} \quad \mathbf{a} \quad &\int_0^1 x^3 dx \\ &= \left[\frac{x^4}{4} \right]_0^1 \\ &= \frac{1}{4} - 0 \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad &\int_0^2 (x^2 - x) dx \\ &= \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_0^2 \\ &= \left(\frac{8}{3} - 2 \right) - (0 - 0) \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \mathbf{c} \quad &\int_0^1 e^x dx \\ &= [e^x]_0^1 \\ &= e^1 - e^0 \\ &= e - 1 \\ &\div 1.718 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int_1^4 \left(x - \frac{3}{\sqrt{x}} \right) dx \\
 &= \int_1^4 \left(x - 3x^{-\frac{1}{2}} \right) dx \\
 &= \left[\frac{x^2}{2} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^4 \\
 &= \left[\frac{x^2}{2} - 6\sqrt{x} \right]_1^4 \\
 &= \left[\frac{16}{2} - 12 \right] - \left(\frac{1}{2} - 6 \right) \\
 &= 1\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int_4^9 \frac{x-3}{\sqrt{x}} dx \\
 &= \int_4^9 \left(x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} \right) dx \\
 &= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} \right]_4^9 \\
 &= \left[\frac{2}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}} \right]_4^9 \\
 &= \left[\frac{2}{3}(27) - 6(3) \right] - \left[\frac{2}{3}(8) - 6(2) \right] \\
 &= (18 - 18) - \left(\frac{16}{3} - 12 \right) \\
 &= 6\frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int_1^3 \frac{1}{x} dx \\
 &= [\ln |x|]_1^3 \\
 &= \ln 3 - \ln 1 \\
 &= \ln 3 - 0 \\
 &= \ln 3 \\
 &\div 1.099
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \int_1^2 (e^{-x} + 1)^2 dx \\
 &= \int_1^2 (e^{-2x} + 2e^{-x} + 1) dx \\
 &= \left[\left(\frac{1}{-2} \right) e^{-2x} + 2 \left(\frac{1}{-1} \right) e^{-x} + x \right]_1^2 \\
 &= \left[-\frac{e^{-2x}}{2} - 2e^{-x} + x \right]_1^2 \\
 &= \left(-\frac{e^{-4}}{2} - 2e^{-2} + 2 \right) - \left(-\frac{e^{-2}}{2} - 2e^{-1} + 1 \right) \\
 &\div 1.524
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int_2^6 \frac{1}{\sqrt{2x-3}} dx \\
 &= \int_2^6 (2x-3)^{-\frac{1}{2}} dx \\
 &= \left[\frac{1}{\frac{1}{2}} \frac{(2x-3)^{\frac{1}{2}}}{\frac{1}{2}} \right]_2^6 \\
 &= [\sqrt{2x-3}]_2^6 \\
 &= \sqrt{9} - \sqrt{1} \\
 &= 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad & \int_0^1 e^{1-x} dx = \left[\left(\frac{1}{-1} \right) e^{1-x} \right]_0^1 \\
 &= \left(\frac{e^0}{-1} \right) - \left(\frac{e^1}{-1} \right) \\
 &= -1 + e \\
 &\div 1.718
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \text{In } \int_1^2 \frac{x}{(x^2+2)^2} dx \quad \therefore \int_1^2 \frac{x}{(x^2+2)^2} dx = \int_1^2 u^{-2} \left(\frac{1}{2} \frac{du}{dx} \right) dx \\
 & \text{we let } u = x^2 + 2 \\
 & \therefore \frac{du}{dx} = 2x \\
 & \text{and when } x = 1, \quad u = 3 \\
 & \text{when } x = 2, \quad u = 6 \\
 &= \frac{1}{2} \int_3^6 u^{-2} du \\
 &= \frac{1}{2} \left[\frac{u^{-1}}{-1} \right]_3^6 \\
 &= \frac{1}{2} \left[-\frac{1}{6} - \left(-\frac{1}{3} \right) \right] \\
 &= \frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \text{In } \int_0^1 x^2 e^{x^3+1} dx \quad \therefore \int_0^1 x^2 e^{x^3+1} dx = \int_0^1 e^u \left(\frac{1}{3} \frac{du}{dx} \right) dx \\
 & \text{we let } u = x^3 + 1 \\
 & \therefore \frac{du}{dx} = 3x^2 \\
 & \text{and when } x = 0, \quad u = 1 \\
 & \text{when } x = 1, \quad u = 2 \\
 &= \frac{1}{3} \int_1^2 e^u du \\
 &= \frac{1}{3} [e^u]_1^2 \\
 &= \frac{1}{3} (e^2 - e) \quad (\div 1.557)
 \end{aligned}$$

- c** In $\int_0^3 x\sqrt{x^2+16} \, dx$ $\therefore \int_0^3 x\sqrt{x^2+16} \, dx = \int_0^3 u^{\frac{1}{2}} \left(\frac{1}{2} \frac{du}{dx}\right) dx$
 we let $u = x^2 + 16$ $= \frac{1}{2} \int_{16}^{25} u^{\frac{1}{2}} du$
 $\therefore \frac{du}{dx} = 2x$ $= \frac{1}{2} \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_{16}^{25}$
 and when $x = 0$, $u = 16$ $= \frac{1}{2} \times \frac{2}{3} \left[u^{\frac{3}{2}} \right]_{16}^{25}$
 when $x = 3$, $u = 25$ $= \frac{1}{3} (125 - 64) = 20\frac{1}{3}$
- d** In $\int_1^2 xe^{-2x^2} \, dx$ $\therefore \int_1^2 xe^{-2x^2} \, dx = \int_1^2 e^u \left(-\frac{1}{4} \frac{du}{dx}\right) dx$
 we let $u = -2x^2$ $= -\frac{1}{4} \int_{-2}^{-8} e^u du$
 $\therefore \frac{du}{dx} = -4x$ $= -\frac{1}{4} [e^u]_{-2}^{-8}$
 and when $x = 1$, $u = -2$ $= -\frac{1}{4} (e^{-8} - e^{-2})$
 when $x = 2$, $u = -8$ $\div 0.0337$
- e** In $\int_2^3 \frac{x}{2-x^2} \, dx$ $\therefore \int_2^3 \frac{x}{2-x^2} \, dx = \int_2^3 \frac{1}{u} \left(-\frac{1}{2} \frac{du}{dx}\right) dx$
 we let $u = 2 - x^2$ $= -\frac{1}{2} \int_{-2}^{-7} \frac{1}{u} du$
 $\therefore \frac{du}{dx} = -2x$ $= -\frac{1}{2} [\ln |u|]_{-2}^{-7}$
 and when $x = 2$, $u = -2$ $= -\frac{1}{2} (\ln 7 - \ln 2)$
 when $x = 3$, $u = -7$ $= -\frac{1}{2} \ln\left(\frac{7}{2}\right) \quad (\div -0.6264)$
- f** In $\int_1^2 \frac{\ln x}{x} \, dx$ $\therefore \int_1^2 \frac{\ln x}{x} \, dx = \int_1^2 u \frac{du}{dx} \, dx$
 we let $u = \ln x$ $= \int_0^{\ln 2} u \, du$
 $\therefore \frac{du}{dx} = \frac{1}{x}$ $= \left[\frac{u^2}{2} \right]_0^{\ln 2}$
 and when $x = 1$, $u = 0$ $= \frac{(\ln 2)^2}{2} - 0$
 when $x = 2$, $u = \ln 2$ $\div 0.2402$
- g** In $\int_0^1 \frac{1-3x^2}{1-x^3+x} \, dx$ we let $u = 1 - x^3 + x$
 $\therefore \frac{du}{dx} = -3x^2 + 1$ and when $x = 0$, $u = 1$
 when $x = 1$, $u = 1$
 $\therefore \int_0^1 \frac{1-3x^2}{1-x^3+x} \, dx = \int_0^1 \frac{1}{u} \frac{du}{dx} \, dx$
 $= \int_1^1 \frac{1}{u} \, du$
 $= 0 \quad \{\text{since limits of integration are the same}\}$

$$\begin{aligned} \text{h In } \int_2^4 \frac{6x^2 - 4x + 4}{x^3 - x^2 + 2x} dx \quad & \text{we let } u = x^3 - x^2 + 2x \\ & \therefore \frac{du}{dx} = 3x^2 - 2x + 2 \quad \text{and} \quad \begin{array}{ll} \text{when } x = 2, & u = 8 \\ \text{when } x = 4, & u = 56 \end{array} \end{aligned}$$

$$\begin{aligned} \therefore \int_2^4 \frac{6x^2 - 4x + 4}{x^3 - x^2 + 2x} dx &= \int_2^4 \frac{1}{u} \left(2 \frac{du}{dx} \right) dx \\ &= 2 \int_8^{56} \frac{1}{u} du \\ &= 2 [\ln |u|]_8^{56} \\ &= 2(\ln 56 - \ln 8) \\ &= 2 \ln 7 \quad (\doteq 3.892) \end{aligned}$$

$$\begin{aligned} \text{i In } \int_0^1 (x^2 + 2x)^n (x + 1) dx \quad & \text{we let } u = x^2 + 2x \\ & \therefore \frac{du}{dx} = 2x + 2 \quad \text{and} \quad \begin{array}{ll} \text{when } x = 0, & u = 0 \\ \text{when } x = 1, & u = 3 \end{array} \end{aligned}$$

$$\therefore \int_0^1 (x^2 + 2x)^n (x + 1) dx = \int_0^1 u^n \left(\frac{1}{2} \frac{du}{dx} \right) dx = \frac{1}{2} \int_0^3 u^n du$$

$$\text{Now if } n \neq -1, \quad \text{the integral} = \frac{1}{2} \left[\frac{u^{n+1}}{n+1} \right]_0^3 = \frac{1}{2} \left(\frac{3^{n+1}}{n+1} \right)$$

$$\text{but if } n = -1, \quad \text{the integral} = \frac{1}{2} \int_0^3 \frac{1}{u} du = \frac{1}{2} [\ln |u|]_0^3 \quad \text{which is undefined as } \ln 0 \text{ is not defined}$$

EXERCISE 25E.2

$$\begin{aligned} \text{1 a } \int_1^4 \sqrt{x} dx &= 4.667 \\ \int_1^4 (-\sqrt{x}) dx &= -4.667 \end{aligned}$$

$$\begin{aligned} \text{b } \int_0^1 x^7 dx &= 0.125 = \frac{1}{8} \\ \int_0^1 (-x^7) dx &= -0.125 = -\frac{1}{8} \end{aligned}$$

$$\text{2 a } \int_0^1 x^2 dx = \frac{1}{3} \quad \text{b } \int_1^2 x^2 dx = \frac{7}{3} \quad \text{c } \int_0^2 x^2 dx = \frac{8}{3} \quad \text{d } \int_0^1 3x^2 dx = 1$$

$$\text{3 a } \int_0^2 (x^3 - 4x) dx = -4 \quad \text{b } \int_2^3 (x^3 - 4x) dx = 6\frac{1}{4} \quad \text{c } \int_0^3 (x^3 - 4x) dx = 2\frac{1}{4}$$

$$\begin{aligned} \text{4 a } \int_a^b -f(x) dx &= - \int_a^b f(x) dx & \text{b } \int_a^b f(x) dx + \int_b^c f(x) dx &= \int_a^c f(x) dx \\ & & \text{and } \int_a^b k f(x) dx &= k \int_a^b f(x) dx \end{aligned}$$

$$\text{5 a } \int_0^1 x^2 dx = \frac{1}{3} \quad \text{b } \int_0^1 \sqrt{x} dx = \frac{2}{3}$$

$$\begin{aligned} \text{c } \int_0^1 (x^2 + \sqrt{x}) dx &= 1 \\ \text{this indicates that } \int_0^1 (x^2 + \sqrt{x}) dx &= \int_0^1 x^2 dx + \int_0^1 \sqrt{x} dx \\ \text{and in general } \int_a^b f(x) dx + \int_a^b g(x) dx &= \int_a^b (f(x) + g(x)) dx \end{aligned}$$

EXERCISE 25E.3

$$\text{1 a } \int_0^3 f(x) dx = 2 + 3 + 1.5 = 6.5 \quad \text{b } \int_3^7 f(x) dx = -\left(\frac{3}{2} + 3 + \frac{5}{2} + 2\right) = -9$$

$$\text{c } \int_2^4 f(x) dx = 1.5 - 1.5 = 0 \quad \text{d } \int_0^7 f(x) dx = 6.5 - 9 = -2.5$$

$$2 \quad \mathbf{a} \quad \int_0^4 f(x) dx = \frac{1}{2}\pi(2)^2 = 2\pi$$

$$\mathbf{c} \quad \int_6^8 f(x) dx = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$$

$$3 \quad \mathbf{a} \quad \int_2^4 f(x) dx + \int_4^7 f(x) dx \\ = \int_2^7 f(x) dx$$

$$4 \quad \mathbf{a} \quad \int_1^3 f(x) dx + \int_3^6 f(x) dx = \int_1^6 f(x) dx$$

$$\therefore \int_3^6 f(x) dx = \int_1^6 f(x) dx - \int_1^3 f(x) dx = (-3) - (2) = -5$$

$$\mathbf{b} \quad \int_0^2 f(x) dx + \int_2^4 f(x) dx + \int_4^6 f(x) dx = \int_0^6 f(x) dx$$

$$\therefore \int_2^4 f(x) dx = \int_0^6 f(x) dx - \int_4^6 f(x) dx - \int_0^2 f(x) dx \\ = (7) - (-2) - (5) \\ = 4$$

$$\mathbf{b} \quad \int_4^6 f(x) dx = -(2 \times 2) = -4$$

$$\mathbf{d} \quad \int_0^8 f(x) dx = 2\pi + (-4) + \frac{\pi}{2} = \frac{5\pi}{2} - 4$$

$$\mathbf{b} \quad \int_1^3 g(x) dx + \int_3^8 g(x) dx + \int_8^9 g(x) dx \\ = \int_1^9 g(x) dx$$

REVIEW SET 25A

$$1 \quad \mathbf{a} \quad \int \frac{4}{\sqrt{x}} dx = 4 \int x^{-\frac{1}{2}} dx \\ = 4 \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c \\ = 8\sqrt{x} + c$$

$$\mathbf{c} \quad \int x e^{1-x^2} dx = \int e^u \left(-\frac{1}{2} \frac{du}{dx} \right) du \\ = -\frac{1}{2} \int e^u du \\ = -\frac{1}{2} e^u + c \\ = -\frac{1}{2} e^{1-x^2} + c$$

$$\mathbf{b} \quad \int \frac{3}{1-2x} dx = 3 \int \frac{1}{1-2x} dx \\ = 3 \left(-\frac{1}{2} \right) \ln |1-2x| + c \\ = -\frac{3}{2} \ln |1-2x| + c$$

$$\{\text{letting } u = 1 - x^2 \quad \therefore \frac{du}{dx} = -2x\}$$

$$2 \quad \mathbf{a} \quad \int_{-5}^{-1} \sqrt{1-3x} dx \\ = \int_{-5}^{-1} (1-3x)^{\frac{1}{2}} dx \\ = \left[\frac{1}{-\frac{3}{2}} \times \frac{(1-3x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{-5}^{-1} \\ = -\frac{2}{9} \left[(1-3x)^{\frac{3}{2}} \right]_{-5}^{-1} \\ = -\frac{2}{9} \left[4^{\frac{3}{2}} - 16^{\frac{3}{2}} \right] \\ = -\frac{2}{9} [8 - 64] \\ = 12\frac{4}{9}$$

$$\mathbf{b} \quad \text{In } \int_0^1 \frac{4x^2}{(x^3+2)^3} dx \\ \text{we let } u = x^3 + 2 \\ \therefore \frac{du}{dx} = 3x^2 \quad \text{and} \quad \begin{array}{l} \text{when } x = 0, u = 2 \\ \text{when } x = 1, u = 3 \end{array} \\ \therefore \int_0^1 \frac{4x^2}{(x^3+2)^3} dx = \int_2^3 \frac{1}{u^3} \left(\frac{4}{3} \frac{du}{dx} \right) dx \\ = \frac{4}{3} \int_2^3 u^{-3} du \\ = \frac{4}{3} \left[\frac{u^{-2}}{-2} \right]_2^3 \\ = \frac{4}{3} \left[-\frac{1}{2u^2} \right]_2^3 \\ = \frac{4}{3} \left[\left(-\frac{1}{18} \right) - \left(-\frac{1}{8} \right) \right] \\ = \frac{5}{54}$$

$$3 \quad y = \sqrt{x^2 - 4} = (x^2 - 4)^{\frac{1}{2}}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{1}{2} (x^2 - 4)^{-\frac{1}{2}} \times 2x \\ &= \frac{x}{(x^2 - 4)^{\frac{1}{2}}} \\ &= \frac{x}{\sqrt{x^2 - 4}} \end{aligned}$$

$$\therefore \int \frac{x}{\sqrt{x^2 - 4}} dx = \sqrt{x^2 - 4} + c$$

$$4 \quad y = f(x) \quad f(0) = -1 \quad f(1) = 13$$

$$f''(x) = 18x + 10$$

$$\therefore f'(x) = 9x^2 + 10x + c$$

$$\text{and } f(x) = 3x^3 + 5x^2 + cx + d$$

$$\text{But } f(0) = -1 \quad \therefore d = -1$$

$$\therefore f(x) = 3x^3 + 5x^2 + ax - 1$$

$$\text{But } f(1) = 13$$

$$\therefore 3 + 5 + a - 1 = 13$$

$$\therefore a + 7 = 13$$

$$\therefore a = 6$$

$$\therefore f(x) = 3x^3 + 5x^2 + 6x - 1$$

$$5 \quad \text{a} \quad \frac{4x - 3}{2x + 1} = \frac{2(2x + 1) - 5}{2x + 1} = 2 + \frac{-5}{2x + 1} \quad \therefore A = 2 \quad B = -5$$

$$\begin{aligned} \text{b} \quad \int_0^2 \frac{4x - 3}{2x + 1} dx &= \int_0^2 \left(2 - 5 \left(\frac{1}{2x + 1} \right) \right) dx \\ &= \left[2x - 5 \left(\frac{1}{2} \right) \ln |2x + 1| \right]_0^2 \\ &= \left[4 - \frac{5}{2} \ln 5 \right] - \left[0 - \frac{5}{2} \ln 1 \right] \\ &= 4 - \frac{5}{2} \ln 5 \\ &\doteq -0.0236 \end{aligned}$$

$$\begin{aligned} 6 \quad \text{a} \quad \int_3^4 \frac{1}{\sqrt{2x + 1}} dx \\ &= \int_3^4 (2x + 1)^{-\frac{1}{2}} dx \\ &= \left[\frac{\frac{1}{2} (2x + 1)^{\frac{1}{2}}}{\frac{1}{2}} \right]_3^4 \\ &= \left[\sqrt{2x + 1} \right]_3^4 \\ &= \sqrt{9} - \sqrt{7} \\ &= 3 - \sqrt{7} \end{aligned}$$

$$\begin{aligned} \text{b} \quad \int_0^1 x^2 e^{x^3 + 1} dx \\ &= \int_1^2 e^u \left(\frac{1}{3} \frac{du}{dx} \right) dx \\ &= \frac{1}{3} \int_1^2 e^u du \\ &= \frac{1}{3} \times [e^u]_1^2 \\ &= \frac{1}{3} (e^2 - e) \end{aligned}$$

$$u = x^3 + 1 \quad \frac{du}{dx} = 3x^2$$

$$u(0) = 1, \quad u(1) = 2$$

$$7 \quad \int_0^a e^{1-2x} dx = \frac{e}{4} \quad \therefore \left[\frac{1}{-2} e^{1-2x} \right]_0^a = \frac{e}{4}$$

$$\therefore -\frac{1}{2} e^{1-2a} - \left(-\frac{1}{2} e^1 \right) = \frac{e}{4}$$

$$\therefore -\frac{1}{2} e^{1-2a} + \frac{e}{2} = \frac{e}{4}$$

$$\therefore \frac{1}{2} e^{1-2a} = \frac{e}{2} - \frac{e}{4}$$

$$\therefore \frac{1}{2} e^{1-2a} = \frac{e}{4}$$

$$\therefore e^{1-2a} = \frac{e}{2}$$

$$\therefore 1 - 2a = \ln \left(\frac{e}{2} \right) = \ln e - \ln 2$$

$$\therefore 1 - 2a = 1 - \ln 2$$

$$\therefore 2a = \ln 2$$

$$\therefore a = \frac{1}{2} \ln 2$$

$$\therefore a = \ln 2^{\frac{1}{2}}$$

$$\text{i.e., } a = \ln \sqrt{2}$$

REVIEW SET 25B

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \int (2e^{-x} - \frac{1}{x} + 3) \, dx \\
 &= (\frac{1}{-1})2e^{-x} - \ln|x| + 3x + c \\
 &= -2e^{-x} - \ln|x| + 3x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \left(\sqrt{x} - \frac{1}{\sqrt{x}} \right)^2 \, dx \\
 &= \int \left(x - 2 + \frac{1}{x} \right) \, dx \\
 &= \frac{x^2}{2} - 2x + \ln|x| + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \int_1^2 (x^2 - 1)^2 \, dx \\
 &= \int_1^2 (x^4 - 2x^2 + 1) \, dx \\
 &= \left[\frac{x^5}{5} - \frac{2x^3}{3} + x \right]_1^2 \\
 &= \left[\frac{32}{5} - \frac{16}{3} + 2 \right] - \left[\frac{1}{5} - \frac{2}{3} + 1 \right] \\
 &= \frac{31}{5} - \frac{14}{3} + 1 \\
 &= 2\frac{8}{15}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \text{In } \int_1^2 x(x^2 - 1)^2 \, dx \\
 & \text{We let } u = x^2 - 1 \quad \therefore \frac{du}{dx} = 2x \\
 & \text{and when } x = 1, \quad u = 0 \\
 & \quad \text{when } x = 2, \quad u = 3 \\
 & \therefore \int_1^2 x(x^2 - 1)^2 \, dx = \int_0^3 u^2 \left(\frac{1}{2} \frac{du}{dx} \, dx \right) \\
 &= \frac{1}{2} \int_0^3 u^2 \, du \\
 &= \frac{1}{2} \left[\frac{u^3}{3} \right]_0^3 \\
 &= \frac{1}{2} \left(\frac{27}{3} - 0 \right) \\
 &= 4\frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad & f(x) = (3x^2 + x)^3 \\
 & \therefore f'(x) = 3(3x^2 + x)^2(6x + 1) \\
 & \therefore \int 3(3x^2 + x)^2(6x + 1) \, dx = (3x^2 + x)^3 + c_1 \\
 & \therefore \int 3(3x^2 + x)^2(6x + 1) \, dx = (3x^2 + x)^3 + c_1 \\
 & \therefore \int (3x^2 + x)^2(6x + 1) \, dx = \frac{1}{3}(3x^2 + x)^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad & f'(x) = x^2 - 3x + 2 \quad \therefore f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + c \\
 \text{But } & f(1) = 3 \quad \therefore \frac{1}{3} - \frac{3}{2} + 2 + c = 3 \\
 & c = 1 - \frac{1}{3} + 1\frac{1}{2} \\
 & \therefore c = 2\frac{1}{6} \\
 \text{i.e., } & f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 2\frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad & \int_2^3 \frac{1}{\sqrt{3x-4}} \, dx \\
 &= \int_2^3 (3x-4)^{-\frac{1}{2}} \, dx \\
 &= \left[\frac{\frac{1}{3}(3x-4)^{\frac{1}{2}}}{\frac{1}{2}} \right]_2^3 \\
 &= \left[\frac{2}{3}\sqrt{3x-4} \right]_2^3 \\
 &= \frac{2}{3}\sqrt{5} - \frac{2}{3}\sqrt{2} \\
 &= \frac{2}{3}(\sqrt{5} - \sqrt{2})
 \end{aligned}$$

6 a $f''(x) = 3x^2 + 2x$

$$\begin{aligned}\therefore f'(x) &= \frac{3x^3}{3} + \frac{2x^2}{2} + c \\ &= x^3 + x^2 + c\end{aligned}$$

and $f(x) = \frac{x^4}{4} + \frac{x^3}{3} + cx + d$

But $f(0) = 3 \quad \therefore d = 3$

and $f(x) = \frac{x^4}{4} + \frac{x^3}{3} + cx + 3$

Now $f(2) = 3 \quad \therefore 4 + \frac{8}{3} + 2c + 3 = 3$

$$\therefore \frac{20}{3} = -2c$$

$$\therefore c = -\frac{10}{3}$$

Thus, $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{10}{3}x + 3$

b Now $f'(2) = 2^3 + 2^2 - \frac{10}{3} = 12 - \frac{10}{3} = \frac{26}{3}$

$\therefore$ normal has slope $-\frac{3}{26}$

$\therefore$ equation is $\frac{y-3}{x-2} = -\frac{3}{26}$

$$\therefore y - 3 = -\frac{3}{26}(x - 2)$$

$$\therefore y = -\frac{3}{26}x + \frac{6}{26} + 3, \text{ i.e., } 3x + 26y = 84$$

7
$$2 \left| \begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 2 & 4 & 2 \\ 1 & 2 & 1 & 4 \end{array} \right| \quad \therefore \frac{x^3 - 3x + 2}{x - 2} = x^2 + 2x + 1 + \frac{4}{x - 2}$$

$$\therefore A = 1, B = 2, C = 1, D = 4$$

and
$$\begin{aligned}\int \frac{x^3 - 3x + 2}{x - 2} dx &= \int x^2 + 2x + 1 + \frac{4}{x - 2} dx \quad \text{or} \quad = \int (x + 1)^2 + \frac{4}{x - 2} dx \\ &= \frac{x^3}{3} + \frac{2x^2}{2} + x + 4 \ln|x - 2| + c = \frac{(x + 1)^3}{3} + 4 \ln|x - 2| + c \\ &= \frac{1}{3}x^3 + x^2 + x + 4 \ln|x - 2| + c\end{aligned}$$

8 a
$$\begin{aligned}\int \frac{1}{x + 2} dx - \int \frac{2}{x - 1} dx &= \ln|x + 2| - 2 \ln|x - 1| + c_1 \\ &= \ln\left(\frac{|x + 2|}{(x - 1)^2}\right) + c_1\end{aligned}$$

b
$$\begin{aligned}\frac{1}{x + 2} - \frac{2}{x - 1} &= \frac{(x - 1) - 2(x + 2)}{(x + 2)(x - 1)} = \frac{-x - 5}{(x + 2)(x - 1)} = -\frac{x + 5}{(x + 2)(x - 1)} \\ \therefore \int \frac{x + 5}{(x + 2)(x - 1)} dx &= -\left[\int \frac{1}{x + 2} dx - \int \frac{2}{x - 1} dx\right] \\ &= -\ln\left(\frac{|x + 2|}{(x - 1)^2}\right) + c \\ &= \ln\left(\frac{(x - 1)^2}{|x + 2|}\right) + c\end{aligned}$$

REVIEW SET 25C

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad \frac{dy}{dx} &= (x^2 - 1)^2 \\
 \therefore y &= \int (x^2 - 1)^2 dx \\
 &= \int (x^4 - 2x^2 + 1) dx \\
 &= \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + c
 \end{aligned}
 \qquad
 \begin{aligned}
 \mathbf{b} \quad \frac{dy}{dx} &= 400 - 20e^{-\frac{x}{2}} \quad \therefore y = \int (400 - 20e^{-\frac{x}{2}}) dx \\
 &= 400x - \frac{20e^{-\frac{x}{2}}}{-\frac{1}{2}} + c \\
 &= 400x + 40e^{-\frac{x}{2}} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad \int_{-2}^0 \frac{4}{2x-1} dx \\
 &= 4 \int_{-2}^0 \frac{1}{2x-1} dx \\
 &= 4 \left[\left(\frac{1}{2} \right) \ln |2x-1| \right]_{-2}^0 \\
 &= 2 [\ln |2x-1|]_{-2}^0 \\
 &= 2 [\ln |-1| - \ln |-5|] \\
 &= 2[0 - \ln 5] \\
 &= -2 \ln 5 \\
 &\doteq -3.219
 \end{aligned}
 \qquad
 \begin{aligned}
 \mathbf{b} \quad \text{In } \int_0^1 \frac{10x}{\sqrt{3x^2+1}} dx \\
 \text{we let } u = 3x^2 + 1, \quad \therefore \frac{du}{dx} = 6x \\
 \text{When } x = 0, u = 1. \quad \text{When } x = 1, u = 4. \\
 \therefore \int_0^1 \frac{10x}{\sqrt{3x^2+1}} dx = \int_1^4 u^{-\frac{1}{2}} \left(\frac{5}{3} \frac{du}{dx} \right) dx \\
 = \frac{5}{3} \int_1^4 u^{-\frac{1}{2}} du \\
 = \frac{5}{3} \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^4 \\
 = \frac{10}{3} (\sqrt{4} - \sqrt{1}) \\
 = \frac{10}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \frac{d}{dx} (\ln x)^2 &= 2 (\ln x)^1 \left(\frac{1}{x} \right) \\
 &= \frac{2 \ln x}{x}
 \end{aligned}$$

$$\therefore \int \frac{2 \ln x}{x} dx = (\ln x)^2 + c_1$$

$$\therefore \int \frac{\ln x}{x} dx = \frac{1}{2} (\ln x)^2 + c$$

$$\mathbf{4} \quad f''(x) = 4x^2 - 3$$

$$\therefore f'(x) = \frac{4x^3}{3} - 3x + c$$

$$\text{But } f'(0) = 6 \quad \therefore c = 6$$

$$\text{i.e., } f'(x) = \frac{4x^3}{3} - 3x + 6$$

$$\therefore f(x) = \frac{4}{3} \frac{x^4}{4} - \frac{3x^2}{2} + 6x + d$$

$$\therefore f(x) = \frac{1}{3}x^4 - \frac{3}{2}x^2 + 6x + d$$

$$\text{But } f(2) = 3 \quad \therefore \frac{16}{3} - 6 + 12 + d = 3$$

$$\therefore d = -3 - \frac{16}{3} = -\frac{25}{3}$$

$$\therefore f(x) = \frac{1}{3}x^4 - \frac{3}{2}x^2 + 6x - \frac{25}{3}$$

$$\begin{aligned}
 \text{and } f(3) &= 27 - \frac{27}{2} + 18 - \frac{25}{3} \\
 &= 23\frac{1}{6}
 \end{aligned}$$

$$\mathbf{5} \quad \int (2x+3)^n dx = \frac{1}{2} \frac{(2x+3)^{n+1}}{n+1} + c \quad \text{if } n \neq -1$$

$$\text{i.e., } \frac{(2x+3)^{n+1}}{2(n+1)} + c \quad \text{if } n \neq -1$$

$$\text{But if } n = -1, \quad \int (2x+3)^{-1} dx = \int \frac{1}{2x+3} dx = \frac{1}{2} \ln |2x+3| + c$$

$$\text{i.e., } \int (2x+3)^n dx = \begin{cases} \frac{(2x+3)^{n+1}}{2(n+1)} + c & \text{if } n \neq -1 \\ \frac{1}{2} \ln |2x+3| + c & \text{if } n = -1 \end{cases}$$

$$\begin{aligned} \mathbf{6} \quad \mathbf{a} \quad & (e^x + 2)^3 \\ &= (e^x)^3 + 3(e^x)^2(2) + 3(e^x)(2)^2 + (2)^3 \\ &= e^{3x} + 6e^{2x} + 12e^x + 8 \end{aligned}$$

$$\begin{aligned} \mathbf{b} \quad \int_0^1 (e^x + 2)^3 dx &= \left[\frac{1}{3} e^{3x} + 3e^{2x} + 12e^x + 8x \right]_0^1 \\ &= \left(\frac{1}{3} e^3 + 3e^2 + 12e + 8 \right) - \left(\frac{1}{3} + 3 + 12 \right) \\ &= \frac{1}{3} e^3 + 3e^2 + 12e - 7\frac{1}{3} \quad (\div 54.14.....) \end{aligned}$$

$$\mathbf{c} \quad \text{By technology} \quad \int_0^1 (e^x + 2)^3 dx = 54.14..... \quad \{\text{using } f_n \text{Int}(\dots)\}$$

$$\mathbf{7} \quad f'(x) = 2\sqrt{x} + \frac{a}{\sqrt{x}} = 2x^{\frac{1}{2}} + ax^{-\frac{1}{2}}$$

$$\therefore f(x) = \frac{4}{3}x^{\frac{3}{2}} + 2ax^{\frac{1}{2}} + c = \frac{4x\sqrt{x}}{3} + 2a\sqrt{x} + c$$

$$\text{Now } f(0) = 2 \quad \therefore c = 2 \quad \therefore f(x) = \frac{4x\sqrt{x}}{3} + 2a\sqrt{x} + 2$$

$$\text{Also } f(1) = 4 \quad \therefore \frac{4}{3} + 2a + 2 = 4 \quad \therefore 2a = \frac{2}{3} \quad \therefore a = \frac{1}{3}$$

$$\text{Now } f'(x) = 2\sqrt{x} + \frac{1}{3\sqrt{x}} = \frac{6x+1}{3\sqrt{x}} > 0 \quad \text{since } f(x) \text{ is only defined for } x > 0$$

$$\text{and for } x > 0, \quad 6x+1 \quad \text{and} \quad 3\sqrt{x} \quad \text{are} > 0$$

$$\therefore \text{the function has no stationary point as } f'(x) \neq 0 \quad \text{for any value of } x.$$

$$\mathbf{8} \quad \int_a^{2a} (x^2 + ax + 2) dx = \frac{73a}{2}$$

$$\therefore \left[\frac{x^3}{3} + \frac{ax^2}{2} + 2x \right]_a^{2a} = \frac{73a}{2}$$

$$\therefore \left(\frac{8a^3}{3} + \frac{a}{2}(4a^2) + 4a \right) - \left(\frac{a^3}{3} + \frac{a^3}{2} + 2a \right) = \frac{73a}{2}$$

$$\frac{8a^3}{3} + 2a^3 + 4a - \frac{a^3}{3} - \frac{a^3}{2} - 2a = \frac{73a}{2}$$

$$\therefore 16a^3 + 12a^3 + 24a - 2a^3 - 3a^3 - 12a = 73a$$

$$\therefore 23a^3 - 207a = 0$$

$$\therefore 23a(a^2 - 9) = 0$$

$$\therefore 23a(a+3)(a-3) = 0$$

$$\therefore a = 0 \quad \text{or} \quad a = \pm 3$$

Chapter 26

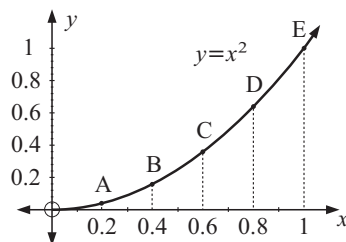
INTEGRATION (AREAS AND OTHER APPLICATIONS)

EXERCISE 26A.1

- 1 a** Divide the region into strips of width $\frac{1}{5}$ unit.

$$O(0, 0) \quad A(\frac{1}{5}, \frac{1}{25}) \quad B(\frac{2}{5}, \frac{4}{25})$$

$$C(\frac{3}{5}, \frac{9}{25}) \quad D(\frac{4}{5}, \frac{16}{25}) \quad E(1, 1)$$



$$\text{Now } A < A_1 + A_2 + A_3 + A_4 + A_5$$

$$\text{i.e., } A < \frac{(0 + \frac{1}{25})}{2} \cdot \frac{1}{5} + \frac{(\frac{1}{25} + \frac{4}{25})}{2} \cdot \frac{1}{5} + \dots + \frac{(\frac{16}{25} + \frac{25}{25})}{2} \cdot \frac{1}{5}$$

$$\therefore A < \frac{1}{10} \left(\frac{85}{25} \right)$$

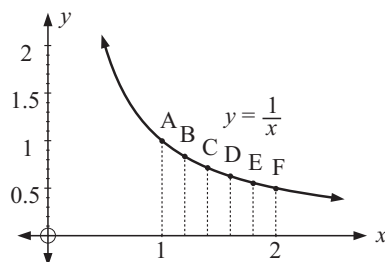
$$\therefore A < 0.34$$

and $A \div 0.34 \text{ units}^2$ is an estimate of the area

- b** Divide the region into strips of width $\frac{1}{5}$ unit.

$$A(1, 1) \quad B(\frac{6}{5}, \frac{5}{6}) \quad C(\frac{7}{5}, \frac{5}{7})$$

$$D(\frac{8}{5}, \frac{5}{8}) \quad E(\frac{9}{5}, \frac{5}{9}) \quad F(2, \frac{1}{2})$$



$$\text{Now } A > A_1 + A_2 + A_3 + A_4 + A_5$$

$$\text{i.e., } A > \frac{(1 + \frac{5}{6})}{2} \cdot \frac{1}{5} + \frac{(\frac{5}{6} + \frac{5}{7})}{2} \cdot \frac{1}{5} + \dots + \frac{(\frac{5}{9} + \frac{1}{2})}{2} \cdot \frac{1}{5}$$

$$\therefore A > \frac{1}{10} \left(\frac{11}{6} + \frac{65}{42} + \frac{75}{56} + \frac{85}{72} + \frac{19}{18} \right)$$

$$\therefore A > 0.6956$$

and $A \div 0.70 \text{ units}^2$ is an estimate of the area

- 2 a** Using 10 vertical strips in question 1 a

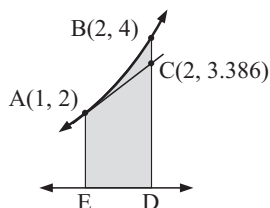
$$\begin{aligned} A &< \frac{(\frac{0}{100} + \frac{1}{100})}{2} \cdot \frac{1}{10} + \frac{(\frac{1}{100} + \frac{4}{100})}{2} \cdot \frac{1}{10} + \dots + \frac{(\frac{81}{100} + \frac{100}{100})}{2} \cdot \frac{1}{10} \\ &< \frac{1}{20} \left(\frac{1}{100} + \frac{5}{100} + \frac{13}{100} + \frac{25}{100} + \frac{41}{100} + \frac{61}{100} + \frac{85}{100} + \frac{113}{100} + \frac{145}{100} + \frac{181}{100} \right) \\ &< \frac{1}{20} \left(\frac{670}{100} \right) \\ &< 0.335 \end{aligned}$$

and $A \div 0.335 \text{ units}^2$ is an estimate of the area

- b** Using 10 vertical strips in question 1 b

$$A > \frac{(\frac{10}{10} + \frac{10}{11})}{2} \cdot \frac{1}{10} + \frac{(\frac{10}{11} + \frac{10}{12})}{2} \cdot \frac{1}{10} + \dots + \frac{(\frac{10}{19} + \frac{10}{20})}{2} \cdot \frac{1}{10}$$

$$\begin{aligned}
 \therefore A &> \frac{10}{20} \left(\left(\frac{1}{10} + \frac{1}{11} \right) + \left(\frac{1}{11} + \frac{1}{12} \right) + \dots + \left(\frac{1}{19} + \frac{1}{20} \right) \right) \\
 &> \frac{1}{2} \left(\frac{21}{110} + \frac{23}{132} + \frac{25}{156} + \frac{27}{182} + \frac{29}{210} + \frac{31}{240} + \frac{33}{272} + \frac{35}{306} + \frac{37}{342} + \frac{39}{380} \right) \\
 &> 0.693\,771 \\
 \text{and } A &\doteq 0.6938 \text{ is an estimate of the area}
 \end{aligned}$$

EXERCISE 26A.2**1 a**

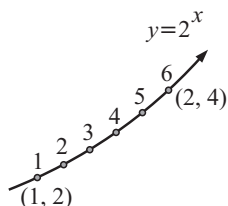
The lower bound is
the area ACDE

$$\begin{aligned}
 &= \frac{2 + 3.386}{2} \\
 &= 2.693
 \end{aligned}$$

The upper bound is
the area ABDE

$$\begin{aligned}
 &= \frac{2 + 4}{2} \\
 &= 3
 \end{aligned}$$

- b** Without the tangent and using 5 subdivisions on $[1, 2]$
 $n = 5$



Point	Coordinates
1	(1, 2)
2	(1.2, 2 ^{1.2})
3	(1.4, 2 ^{1.4})
4	(1.6, 2 ^{1.6})
5	(1.8, 2 ^{1.8})
6	(2, 4)

$$\begin{aligned}
 A_L &= \frac{1}{5}(2^1) + \frac{1}{5}(2^{1.2}) + \frac{1}{5}(2^{1.4}) + \frac{1}{5}(2^{1.6}) + \frac{1}{5}(2^{1.8}) \\
 &= \frac{1}{5}(2 + 2^{1.2} + 2^{1.4} + 2^{1.6} + 2^{1.8}) \\
 &\doteq 2.690
 \end{aligned}$$

$$\begin{aligned}
 A_U &= \frac{1}{5}(2^{1.2} + 2^{1.4} + 2^{1.6} + 2^{1.8} + 2^2) \\
 &\doteq 3.090
 \end{aligned}$$

- c** Using the provided software:

n	A_L	A_U
10	2.786 55	2.986 55
50	2.865 44	2.905 44
100	2.875 40	2.895 40
500	2.883 39	2.887 39
5000	2.885 19	2.885 59

The upper and lower sums converge to 2.885 (approx.)

- 2 a** $y = 1 + e^x$ $[0, 1]$ choosing 5 subdivisions on this $[0, 1]$.

Point	Coordinates
1	(0, 2)
2	(0.2, 1 + e ^{0.2})
3	(0.4, 1 + e ^{0.4})
4	(0.6, 1 + e ^{0.6})
5	(0.8, 1 + e ^{0.8})
6	(1, 1 + e ¹)

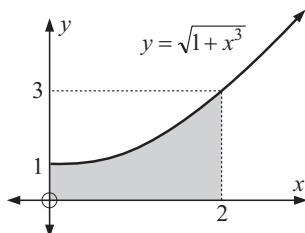
$$\begin{aligned}
 A_L &= 0.2(1 + e^0) + 0.2(1 + e^{0.2}) + \dots + 0.2(1 + e^{0.8}) \\
 &= 0.2(5 + (e^0 + e^{0.2} + e^{0.4} + e^{0.6} + e^{0.8})) \\
 &\doteq 2.552
 \end{aligned}$$

$$\begin{aligned}
 \therefore A_U &= 0.2(1 + e^{0.2}) + 0.2(1 + e^{0.4}) + \dots + 0.2(1 + e^1) \\
 &= 0.2(5 + (e^{0.2} + e^{0.4} + e^{0.6} + e^{0.8} + e^1)) \\
 &\doteq 2.896
 \end{aligned}$$

b

n	A_L	A_U
100	2.709 70	2.726 89
1000	2.717 42	2.719 14
10 000	2.718 20	2.718 37
100 000	2.718 27	2.718 29

c It appears that as $t \rightarrow \infty$ both upper and lower sums converge to e , where $e \doteq 2.718\,28$

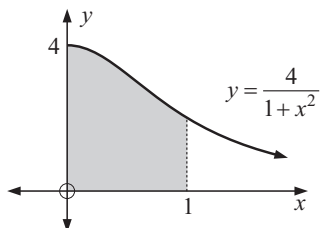
3 a

b

n	A_L	A_U
100	3.221 38	3.261 38
10 000	3.241 11	3.241 51

c A good estimate would be the average

$$\text{i.e., } \frac{3.241\,11 + 3.241\,51}{2}$$

$$\therefore \int_0^2 \sqrt{1+x^3} dx \doteq 3.2413$$

4 a

b

n	A_L	A_U
100	3.131 58	3.151 58
1000	3.140 59	3.142 59
10 000	3.141 49	3.141 69

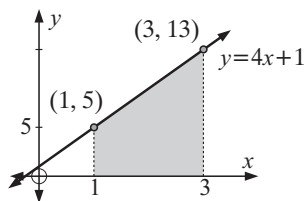
c Area $= \int_0^1 \frac{4}{1+x^2} dx = \frac{3.141\,69 + 3.141\,49}{2}$
 $\doteq 3.141\,59$
 $\doteq 3.1416$
 (We suspect this is π)

5 For the curve $y = \sqrt[3]{x^2+2}$, $y > 0$ for $0 \leq x \leq 5$.

Using the software provided

n	A_L	A_U
100	10.159 79	10.246 80
1000	10.198 87	10.207 57
10 000	10.202 78	10.203 65

A good estimate is $\int_0^5 \sqrt[3]{x^2+2} dx = \frac{10.202\,78 + 10.203\,65}{2} \doteq 10.203\,22$

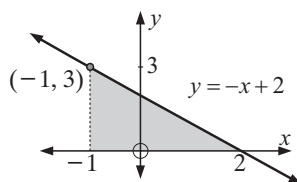
6 a


$$\int_1^3 (1+4x) dx$$

= area of the shaded trapezium

$$= \left(\frac{5+13}{2} \right) \times 2$$

$$= 18$$

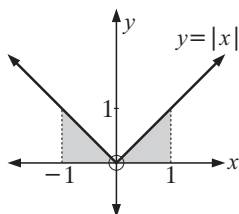
b


$$\int_{-1}^2 (2-x) dx$$

= area of shaded triangle

$$= \frac{1}{2}(3 \times 3)$$

$$= 4.5$$

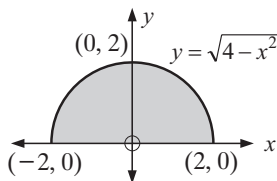
c

$$\int_{-1}^1 |x| dx = \text{shaded area shown}$$

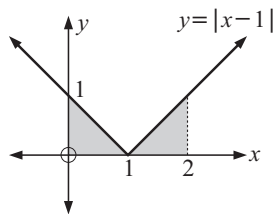
Due to symmetry we need to only consider the RHS.

$$\begin{aligned}\therefore \int_0^1 |x| dx &= \text{area of a shaded } \triangle \\ &= \frac{1}{2}(1 \times 1) = 0.5\end{aligned}$$

$$\therefore \int_{-1}^1 |x| dx = 2 \times 0.5 = 1$$

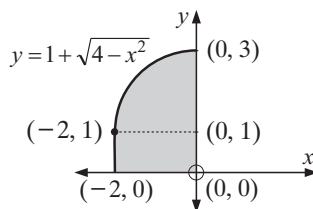
e

$$\begin{aligned}\int_{-2}^2 \sqrt{4 - x^2} dx \\ &= \text{area of semi-circle radius 2} \\ &= \frac{1}{2}(\pi \times 2^2) \\ &= 2\pi\end{aligned}$$

d

$$y = \sqrt{4 - x^2}$$

$$\begin{aligned}\therefore \int_0^2 |x - 1| dx &= 2 \int_1^2 |x - 1| dx \\ &\quad \{\text{by symmetry}\} \\ &= 2 \times \left[\frac{1}{2}(1 \times 1) \right] \\ &= 1\end{aligned}$$

f

$$\begin{aligned}\int_{-2}^0 (1 + \sqrt{4 - x^2}) dx \\ &= \text{area of rectangle} \\ &\quad + \text{area of quarter of circle with radius 2} \\ &= 2 \times 1 + \frac{1}{4}(\pi \times 2^2) \\ &= 2 + \pi\end{aligned}$$

EXERCISE 26B

- 1 a** $\int_a^a f(x) dx = F(a) - F(a) = 0$
graphically: $\int_a^a f(x) dx$ = area of
the strip between $x = a$ and $x = a$,
i.e., width = 0 $\therefore$ area = 0

$$\begin{aligned}\text{c } \int_b^a f(x) dx &= F(a) - F(b) \\ &= -[F(b) - F(a)] \\ &= -\int_a^b f(x) dx\end{aligned}$$

$$\begin{aligned}\text{e } \int_a^b (f(x) + g(x)) dx &= [F(b) + G(b)] - [F(a) + G(a)] \\ &= [F(b) - F(a)] + [G(b) - G(a)] \\ &= \int_a^b f(x) dx + \int_a^b g(x) dx\end{aligned}$$

- 2 a** $f(x) = x^3$ has antiderivative
 $F(x) = \frac{x^4}{4}$

$$\begin{aligned}\text{So, area} &= \int_0^1 x^3 dx \\ &= F(1) - F(0) \\ &= \frac{1}{4} - 0 \\ &= \frac{1}{4} \text{ units}^2\end{aligned}$$

$$\begin{aligned}\text{b } \int_a^b c dx &= F(b) - F(a) \\ &= cb - ca \quad \{\text{antideriv. of } c \text{ is } cx\} \\ &= c(b - a)\end{aligned}$$

$$\begin{aligned}\text{d } \text{If } \frac{d}{dx} F(x) &= f(x) \text{ then} \\ \frac{d}{dx} cF(x) &= cf(x) \\ \therefore \int_a^b cf(x) dx &= c[F(b) - F(a)] \\ &= c \int_a^b f(x) dx\end{aligned}$$

- b** $f(x) = x^3$ has antiderivative
 $F(x) = \frac{x^4}{4}$

$$\begin{aligned}\text{So, area} &= \int_1^2 x^3 dx \\ &= F(2) - F(1) \\ &= \frac{16}{4} - \frac{1}{4} \\ &= 3\frac{3}{4} \text{ units}^2\end{aligned}$$

c $f(x) = x^2 + 3x + 2$ has antiderivative

$$F(x) = \frac{x^3}{3} + \frac{3x^2}{2} + 2x$$

$$\begin{aligned}\text{So, area} &= \int_1^3 (x^2 + 3x + 2) \, dx \\ &= F(3) - F(1) \\ &= \left(\frac{27}{3} + \frac{27}{2} + 6\right) - \left(\frac{1}{3} + \frac{3}{2} + 2\right) \\ &= 24\frac{2}{3} \text{ units}^2\end{aligned}$$

e $f(x) = e^x$ has antiderivative

$$F(x) = e^x$$

$$\begin{aligned}\text{So, area} &= \int_0^{1.5} e^x \, dx \\ &= F(1.5) - F(0) \\ &= e^{1.5} - e^0 \\ &= e^{1.5} - 1 \\ &\div 3.482 \text{ units}^2\end{aligned}$$

g $f(x) = x^3 + 2x^2 + 7x + 4$ has antiderivative

$$F(x) = \frac{x^4}{4} + \frac{2x^3}{3} + \frac{7x^2}{2} + 4x$$

d $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$$F(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$$

$$\begin{aligned}\text{So the area} &= \int_0^2 \sqrt{x} \, dx \\ &= F(2) - F(0) \\ &= \frac{2}{3} \times 2\sqrt{2} - 0 \\ &= \frac{4\sqrt{2}}{3} \text{ units}^2\end{aligned}$$

f $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ has antiderivative

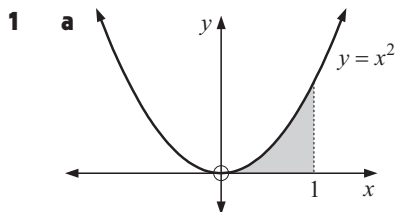
$$F(x) = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{x}$$

$$\begin{aligned}\text{So, area} &= \int_1^4 \frac{1}{\sqrt{x}} \, dx = F(4) - F(1) \\ &= 2\sqrt{4} - 2\sqrt{1} \\ &= 2 \text{ units}^2\end{aligned}$$

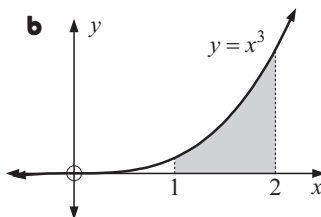
So the area

$$\begin{aligned}&= \int_1^{1.25} x^3 + 2x^2 + 7x + 4 \, dx \\ &= F(1.25) - F(1) \\ &= [12.381 \, 18 - 8.416 \, 67] \\ &\div 3.965 \text{ units}^2\end{aligned}$$

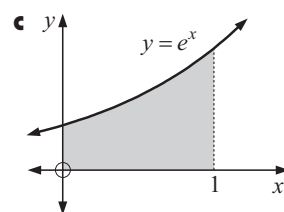
EXERCISE 26C



$$\begin{aligned}\text{Area} &= \int_0^1 x^2 \, dx \\ &= \left[\frac{x^3}{3}\right]_0^1 \\ &= \frac{1}{3} - 0 = \frac{1}{3} \text{ units}^2\end{aligned}$$



$$\begin{aligned}\text{Area} &= \int_1^2 x^3 \, dx \\ &= \left[\frac{x^4}{4}\right]_1^2 \\ &= \frac{16}{4} - \frac{1}{4} = 3\frac{3}{4} \text{ units}^2\end{aligned}$$



$$\begin{aligned}\text{Area} &= \int_0^1 e^x \, dx \\ &= [e^x]_0^1 \\ &= e - 1 \\ &\div 1.718 \text{ units}^2\end{aligned}$$

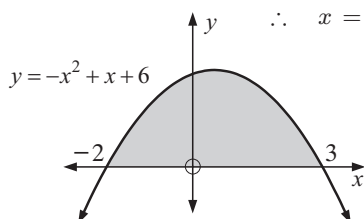
d

Cuts the x -axis at $y = 0$

$$\therefore 6 + x - x^2 = 0$$

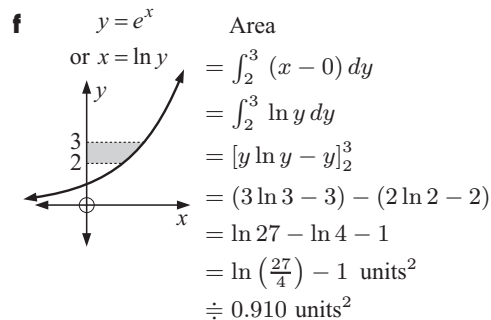
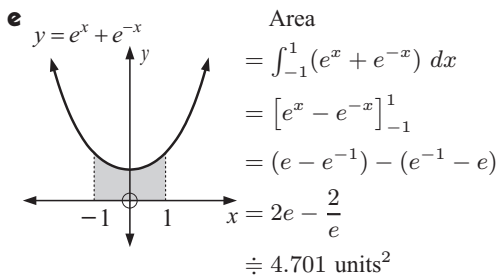
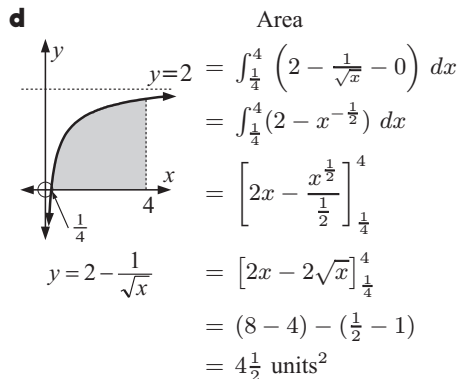
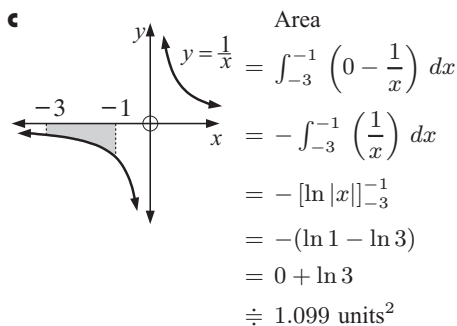
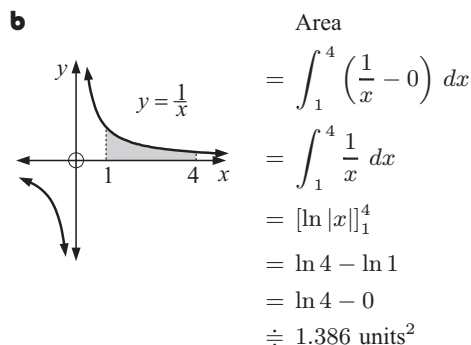
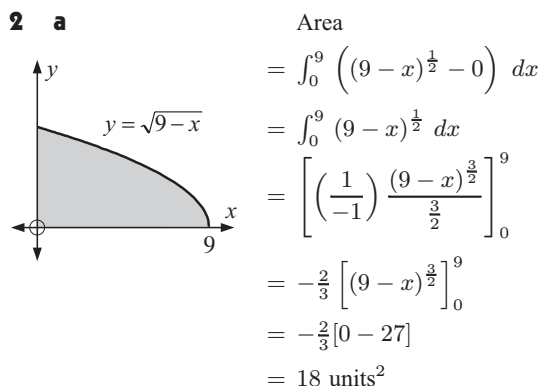
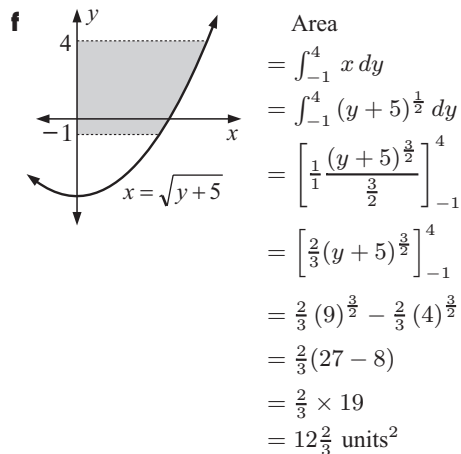
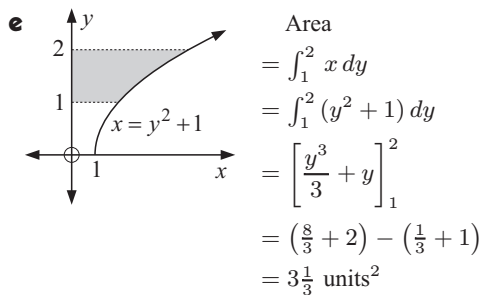
$$\therefore (3 - x)(2 + x) = 0$$

$$\therefore x = 3 \text{ or } -2$$

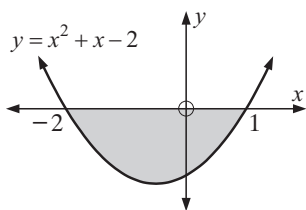


i.e., x -intercepts are 3 and -2

$$\begin{aligned}\text{Area} &= \int_{-2}^3 (6 + x - x^2) \, dx \\ &= \left[6x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^3 \\ &= \left(18 + \frac{9}{2} - 9\right) - \left(-12 + 2 + \frac{8}{3}\right) \\ &= 20\frac{5}{6} \text{ units}^2\end{aligned}$$



- 3 a** The curve cuts the x -axis when $y = 0$ $\therefore x^2 + x - 2 = 0$



$$\therefore (x+2)(x-1) = 0$$

$$\therefore x = -2 \text{ or } 1$$

i.e., x -intercepts are -2 and 1

$$\text{Area} = \int_{-2}^1 [0 - (x^2 + x - 2)] dx$$

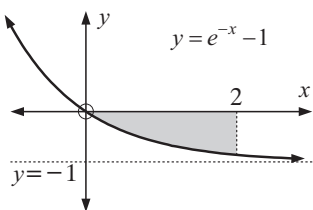
$$= \int_{-2}^1 (-x^2 - x + 2) dx$$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^1$$

$$= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right)$$

$$= 4\frac{1}{2} \text{ units}^2$$

- b** The curve cuts the x -axis at $(0, 0)$.



$$\text{Area} = \int_0^2 [0 - (e^{-x} - 1)] dx$$

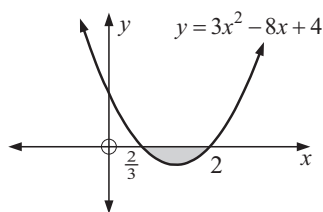
$$= \int_0^2 (1 - e^{-x}) dx$$

$$= [x + e^{-x}]_0^2$$

$$= \left(2 + \frac{1}{e^2} \right) - (0 + e^0)$$

$$= 1 + \frac{1}{e^2} \quad (\div 1.135 \text{ units}^2)$$

- c** The curve cuts the x -axis when $y = 0$ $\therefore 3x^2 - 8x + 4 = 0$



$$\therefore (3x-2)(x-2) = 0$$

$$\therefore x = 2 \text{ or } \frac{2}{3}$$

i.e., x -intercepts are 2 and $\frac{2}{3}$

$$\text{Area} = \int_{\frac{2}{3}}^2 [0 - (3x^2 - 8x + 4)] dx$$

$$= \int_{\frac{2}{3}}^2 (-3x^2 + 8x - 4) dx$$

$$= \left[-x^3 + 4x^2 - 4x \right]_{\frac{2}{3}}^2$$

$$= (-8 + 16 - 8) - \left(-\frac{8}{27} + \frac{16}{3} - \frac{8}{3} \right)$$

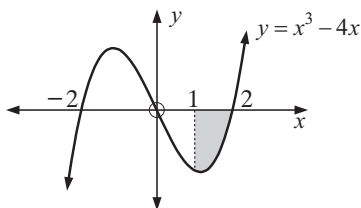
$$= 1\frac{5}{27} \text{ units}^2$$

- d** The curve cuts the x -axis when $y = 0$ $\therefore x^3 - 4x = 0$

$$\therefore x(x^2 - 4) = 0$$

$$\therefore x(x+2)(x-2) = 0$$

i.e., x -intercepts are 0 and ± 2



$$\text{Area} = \int_1^2 [0 - f(x)] dx$$

$$= \int_1^2 (-x^3 + 4x) dx$$

$$= \left[-\frac{x^4}{4} + 2x^2 \right]_1^2$$

$$= (-4 + 8) - \left(-\frac{1}{4} + 2 \right)$$

$$= 2\frac{1}{4} \text{ units}^2$$

4 a $y = x^2 - 2x$ meets $y = 3$ where

$$x^2 - 2x = 3$$

$$\therefore x^2 - 2x - 3 = 0$$

$$\therefore (x-3)(x+1) = 0$$

$$\therefore x = 3 \text{ and } x = -1$$

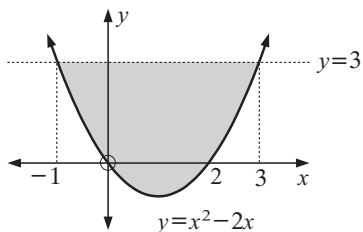
$$A = \int_{-1}^3 [3 - (x^2 - 2x)] dx$$

$$= \int_{-1}^3 (3 + 2x - x^2) dx$$

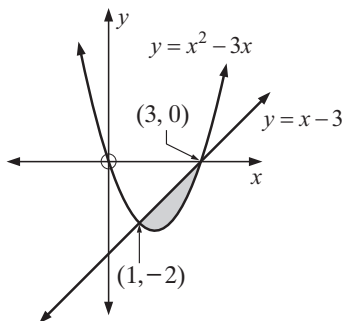
$$= \left[3x + x^2 - \frac{x^3}{3} \right]_{-1}^3$$

$$= (9 + 9 - 9) - (-3 + 1 + \frac{1}{3})$$

$$= 10\frac{2}{3} \text{ units}^2$$



b i



ii $y = x - 3$ meets $y = x^2 - 3x$

where $x = 3$ i.e., $(3, 0)$

and where $x = 1$ i.e., $(1, -2)$

Checking algebraically:

the graphs meet where $x - 3 = x^2 - 3x$

$$\therefore x^2 - 3x - x + 3 = 0$$

$$\therefore x^2 - 4x + 3 = 0$$

$$\therefore (x-1)(x-3) = 0$$

$$\therefore x = 1 \text{ or } 3 \quad \checkmark$$

iii Area $= \int_1^3 [(x-3) - (x^2-3x)] dx$

$$= \int_1^3 (-3 + 4x - x^2) dx$$

$$= \left[-3x + 2x^2 - \frac{x^3}{3} \right]_1^3$$

$$= (-9 + 18 - 9) - (-3 + 2 - \frac{1}{3})$$

$$= 1\frac{1}{3} \text{ units}^2$$

c $y = \sqrt{x}$ meets $y = x^2$ where $\sqrt{x} = x^2$

$$\therefore x = x^4$$

$$\therefore x^4 - x = 0$$

$$\therefore x(x^3 - 1) = 0$$

$$\therefore x(x-1)(x^2+x+1) = 0$$

$$\therefore x = 0 \text{ or } 1$$

The factor $(x^2 + x + 1)$ has no real root since $\Delta = -3$ which is < 0 .

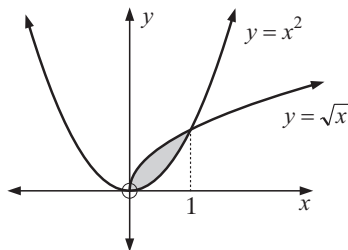
$$\text{Area} = \int_0^1 (\sqrt{x} - x^2) dx$$

$$= \int_0^1 (x^{\frac{1}{2}} - x^2) dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3}$$

$$= \frac{1}{3} \text{ unit}^2$$



d $y = e^x - 1$ meets $y = 2 - 2e^{-x}$ where

$$e^x - 1 = 2 - 2e^{-x}$$

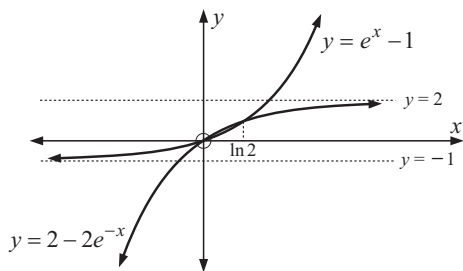
$$\therefore e^{2x} - e^x = 2e^x - 2 \quad \{\times e^x\}$$

$$\therefore e^{2x} - 3e^x + 2 = 0$$

$$\therefore (e^x - 1)(e^x - 2) = 0$$

$$\therefore e^x = 1 \text{ or } 2$$

$$\therefore x = 0 \text{ or } \ln 2$$



$$A = \int_0^{\ln 2} [(2 - 2e^{-x}) - (e^x - 1)] dx$$

$$= \int_0^{\ln 2} (3 - e^x - 2e^{-x}) dx$$

$$= [3x - e^x + 2e^{-x}]_0^{\ln 2}$$

$$= (3 \ln 2 - 2 + 1) - (0 - 1 + 2)$$

$$= 3 \ln 2 - 2$$

$$\doteq 0.0794 \text{ units}^2$$

e $y = 2e^x$ meets $y = e^{2x}$ where

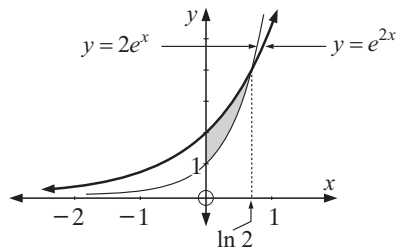
$$2e^x = e^{2x}$$

$$\therefore e^{2x} - 2e^x = 0$$

$$\therefore e^x(e^x - 2) = 0$$

$$\therefore e^x = 2 \quad \{\text{as } e^x > 0 \text{ for all } x\}$$

$$\therefore x = \ln 2$$



$$A = \int_0^{\ln 2} (2e^x - e^{2x}) dx$$

$$= [2e^x - \frac{1}{2}e^{2x}]_0^{\ln 2}$$

$$= (4 - 2) - (2 - \frac{1}{2})$$

$$= \frac{1}{2} \text{ unit}^2$$

5 $y = 2x$ meets $y^2 = 4x$ where

$$(2x)^2 = 4x$$

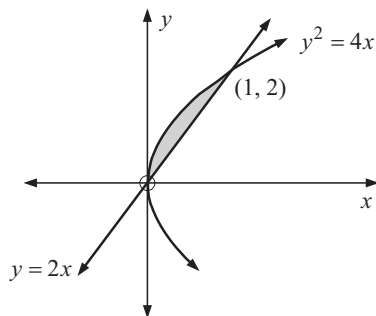
$$\therefore 4x^2 = 4x$$

$$\therefore 4x^2 - 4x = 0$$

$$\therefore 4x(x - 1) = 0$$

$$\therefore x = 0 \text{ or } 1$$

The upper part of $y^2 = 4x$
is $y = \sqrt{4x}$
i.e., $y = 2\sqrt{x}$



$$\text{Area} = \int_0^1 (2\sqrt{x} - 2x) dx$$

$$= \int_0^1 (2x^{\frac{1}{2}} - 2x) dx$$

$$= \left[\frac{4}{3}x^{\frac{3}{2}} - x^2 \right]_0^1$$

$$= \frac{4}{3} - 1$$

$$= \frac{1}{3} \text{ unit}^2$$

6 a Now $x^2 + y^2 = 9 \quad \therefore y^2 = 9 - x^2$
 $y = \pm\sqrt{9 - x^2}$

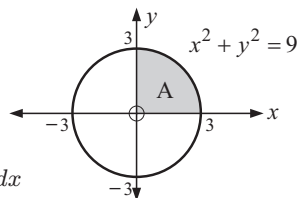
In the upper half of the circle all y -values are ≥ 0

$\therefore y = +\sqrt{9 - x^2}$ is the required equation.

b Now the shaded area above is A where $A = \int_0^3 \sqrt{9 - x^2} dx$

This is a quarter of the area of a circle radius 3 units $\therefore A = \frac{1}{4}(\pi \times 3^2) = \frac{9}{4}\pi \div 7.069 \text{ units}^2$

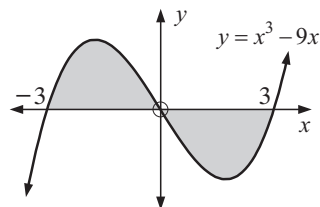
c The answer checks using technology.



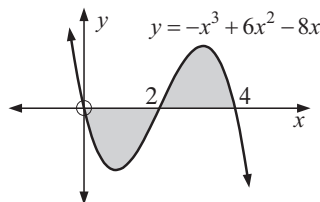
7 a $f(x) = x^3 - 9x$
 $= x(x^2 - 9)$
 $= x(x + 3)(x - 3)$

$\therefore y = f(x)$ cuts the x -axis at $0, \pm 3$

$$\begin{aligned} \text{Area} &= \int_{-3}^0 (x^3 - 9x) dx + \int_0^3 [0 - (x^3 - 9x)] dx \\ &= \left[\frac{x^4}{4} - \frac{9x^2}{2} \right]_{-3}^0 + \left[-\frac{x^4}{4} + \frac{9x^2}{2} \right]_0^3 \\ &= (0 - [\frac{81}{4} - \frac{81}{2}]) + ([-\frac{81}{4} + \frac{81}{2}] - 0) \\ &= 40\frac{1}{2} \text{ units}^2 \end{aligned}$$

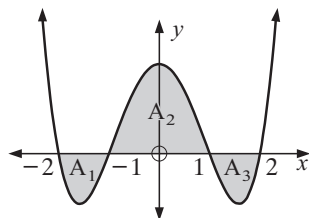


b $f(x) = -x(x - 2)(x - 4)$
 $= -x^3 + 6x^2 - 8x$
 $\therefore y = f(x)$ cuts the x -axis at $0, 2$ and 4



$$\begin{aligned} \text{Area} &= \int_0^2 [0 - (-x^3 + 6x^2 - 8x)] dx + \int_2^4 [(-x^3 + 6x^2 - 8x) - 0] dx \\ &= \int_0^2 (x^3 - 6x^2 + 8x) dx + \int_2^4 (-x^3 + 6x^2 - 8x) dx \\ &= \left[\frac{x^4}{4} - 2x^3 + 4x^2 \right]_0^2 + \left[-\frac{x^4}{4} + 2x^3 - 4x^2 \right]_2^4 \\ &= ([4 - 16 + 16] - 0) + ([-64 + 128 - 64] - [-4 + 16 - 16]) \\ &= 8 \text{ units}^2 \end{aligned}$$

c $f(x) = x^4 - 5x^2 + 4$
 $= (x^2 - 1)(x^2 - 4)$
 $= (x + 1)(x - 1)(x + 2)(x - 2)$
 $\therefore y = f(x)$ cuts the x -axis at $\pm 1, \pm 2$



$$\begin{aligned} A_1 &= \int_{-2}^{-1} [0 - (x^4 - 5x^2 + 4)] dx \\ &= \int_{-2}^{-1} (-x^4 + 5x^2 - 4) dx \\ &= \left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x \right]_{-2}^{-1} \\ &= (\frac{1}{5} - \frac{5}{3} + 4) - (\frac{32}{5} - \frac{40}{3} + 8) \\ &= \frac{22}{15} \text{ units}^2 \end{aligned}$$

$$\begin{aligned} A_2 &= \int_{-1}^1 (x^4 - 5x^2 + 4) dx \\ &= \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x \right]_{-1}^1 \\ &= (\frac{1}{5} - \frac{5}{3} + 4) - (-\frac{1}{5} + \frac{5}{3} - 4) \\ &= \frac{76}{15} \text{ units}^2 \end{aligned}$$

Now by symmetry, $A_3 = A_1 \quad \therefore A = \frac{22}{15} + \frac{76}{15} + \frac{22}{15} = \frac{120}{15} = 8 \text{ units}^2$

- 8 a i** The graphs meet where $x^3 - 4x = 3x + 6$

$$\therefore x^3 - 7x - 6 = 0$$

$$\therefore (x+1)(x^2 - x - 6) = 0$$

$$\therefore (x+1)(x-3)(x+2) = 0$$

$$\therefore x = -2, -1 \text{ or } 3$$

$$\begin{aligned} \therefore \text{Area} &= \int_{-2}^{-1} ([x^3 - 4x] - [3x + 6]) \, dx + \int_{-1}^3 ([3x + 6] - [x^3 - 4x]) \, dx \\ &= \int_{-2}^{-1} (x^3 - 7x - 6) \, dx + \int_{-1}^3 (-x^3 + 7x + 6) \, dx \end{aligned}$$

ii $\text{Area} = \int_{-2}^3 |x^3 - 7x - 6| \, dx$

- b** Using technology, $\text{area} = 32\frac{3}{4} \text{ units}^2$

$$-1 \left| \begin{array}{rrrr} 1 & 0 & -7 & -6 \\ 0 & -1 & 1 & 6 \\ 1 & -1 & -6 & 0 \end{array} \right|$$

- 9 a** The graphs meet where

$$x^3 - 5x = 2x^2 - 6$$

$$\therefore x^3 - 2x^2 - 5x + 6 = 0$$

$$\therefore (x-1)(x^2 - x - 6) = 0$$

$$\therefore (x-1)(x-3)(x+2) = 0$$

$$\therefore x = -2, 1 \text{ or } 3$$

$$\text{So, Area} = \int_{-2}^3 |x^3 - 2x^2 - 5x + 6| \, dx = 21\frac{1}{12} \text{ units}^2 \quad \{\text{technology}\}$$

- b** The graphs meet where

$$-x^3 + 3x^2 + 6x - 8 = 5x - 5$$

$$\therefore x^3 - 3x^2 - x + 3 = 0$$

$$\therefore (x-1)(x^2 - 2x - 3) = 0$$

$$\therefore (x-1)(x-3)(x+1) = 0$$

$$\therefore x = -1, 1 \text{ or } 3$$

$$\text{So, Area} = \int_{-1}^3 |x^3 - 3x^2 - x + 3| \, dx = 8 \text{ units}^2 \quad \{\text{technology}\}$$

- c** $y = 2x^3 - 3x^2 + 18$ meets $y = x^3 + 10x - 6$

$$\text{where } 2x^3 - 3x^2 + 18 = x^3 + 10x - 6$$

$$\therefore x^3 - 3x^2 - 10x + 24 = 0$$

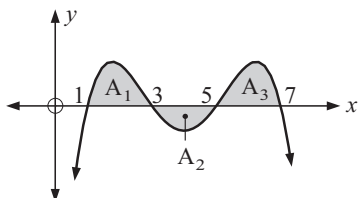
$$\therefore (x-2)(x^2 - x - 12) = 0$$

$$\therefore (x-2)(x-4)(x+3) = 0$$

$$\therefore x = -3, 2 \text{ or } 4$$

$$\text{So, Area} = \int_{-3}^4 |x^3 - 3x^2 - 10x + 24| \, dx = 101\frac{3}{4} \text{ units}^2 \quad \{\text{technology}\}$$

10



- a** $\int_1^7 f(x) \, dx$ only gives us the correct area provided that $f(x)$ is positive on the interval $1 \leq x \leq 7$. But $f(x)$ is not positive for $3 \leq x \leq 5$, so $\int_1^7 f(x) \, dx$ will give us
- $$\begin{aligned} &(\text{Area enclosed in } 1 \leq x \leq 3) \\ &\quad - (\text{Area enclosed in } 3 \leq x \leq 5) \\ &\quad + (\text{Area enclosed in } 5 \leq x \leq 7) \end{aligned}$$
- which is NOT the shaded area.

- b**
$$\begin{aligned} \text{shaded area} &= \int_1^3 f(x) \, dx + \int_3^5 [0 - f(x)] \, dx + \int_5^7 f(x) \, dx \\ &= \int_1^3 f(x) \, dx - \int_3^5 f(x) \, dx + \int_5^7 f(x) \, dx \end{aligned}$$

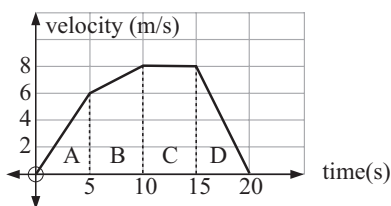
$$\begin{aligned}
 \mathbf{11} \quad \text{Area} &= \int_1^k \frac{1}{1+2x} dx = 0.2 \text{ units}^2 \\
 \therefore \left[\frac{1}{2} \ln |1+2x| \right]_1^k &= 0.2 \\
 \therefore [\ln |1+2x|]_1^k &= 0.4 \\
 \therefore \ln |1+2k| - \ln 3 &= 0.4 \\
 \therefore \ln(1+2k) - \ln 3 &= 0.4 \quad \{\text{as } k > 0, \text{ then } 1+2k > 0 \text{ also}\} \\
 \therefore \ln \left(\frac{1+2k}{3} \right) &= 0.4 \\
 \therefore \frac{1+2k}{3} &= e^{0.4} \\
 \therefore 1+2k &= 3e^{0.4} \\
 \therefore k &= \frac{3e^{0.4} - 1}{2} \doteq 1.7377
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{12} \quad \text{Area} &= \int_0^b \sqrt{x} dx \quad \therefore \int_0^b x^{\frac{1}{2}} dx = 1 \\
 \therefore \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^b &= 1 \\
 \therefore \frac{2}{3} b\sqrt{b} - 0 &= 1 \\
 \therefore b\sqrt{b} &= \frac{3}{2} \\
 \therefore b^{\frac{3}{2}} &= 1.5 \\
 \therefore b &= (1.5)^{\frac{2}{3}} \doteq 1.3104
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{13} \quad y = x^2 \text{ meets } y = k \text{ where } x^2 = k \quad \therefore x &= \pm\sqrt{k} \\
 \text{Now area} &= \int_0^{\sqrt{k}} (k - x^2) dx \quad \therefore \int_0^{\sqrt{k}} (k - x^2) dx = 2.4 \\
 \therefore \left[kx - \frac{x^3}{3} \right]_0^{\sqrt{k}} &= 2.4 \\
 \therefore k\sqrt{k} - \frac{k\sqrt{k}}{3} - 0 &= 2.4 \\
 \therefore \frac{2k\sqrt{k}}{3} &= 2.4 \\
 \therefore k^{\frac{3}{2}} &= 3.6 \\
 \therefore k &= (3.6)^{\frac{2}{3}} \doteq 2.3489
 \end{aligned}$$

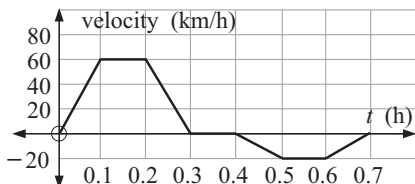
14 By symmetry, the area bounded by $x = 0$ and $x = a$ is $\frac{1}{2}(6a)$ units².

$$\begin{aligned}
 \therefore \int_0^a (x^2 + 2) dx &= 3a \\
 \therefore \left[\frac{x^3}{3} + 2x \right]_0^a &= 3a \\
 \therefore \frac{a^3}{3} + 2a - 0 &= 3a \\
 \therefore a^3 + 6a &= 9a \\
 \therefore a^3 - 3a &= 0 \\
 \therefore a(a^2 - 3) &= 0 \\
 \therefore a = 0 \text{ or } \pm\sqrt{3} \quad \therefore a &= \sqrt{3} \quad \{\text{as } a > 0\}
 \end{aligned}$$

EXERCISE 26D.1
1


Total distance travelled

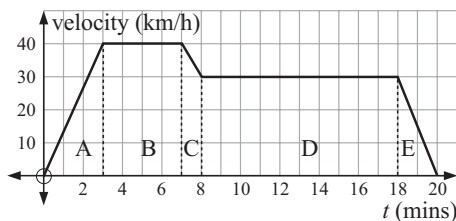
$$\begin{aligned}
 &= \text{area A} + \text{area B} + \text{area C} + \text{area D} \\
 &= \frac{1}{2}(5 \times 6) + \left(\frac{6+8}{2}\right) 5 + 5 \times 8 + \frac{1}{2}(5 \times 8) \\
 &= 15 + 35 + 40 + 20 \\
 &= 110 \text{ m}
 \end{aligned}$$

2


- a i** The graph above the t -axis indicates that the velocity is positive and the car is travelling forwards.
- ii** The graph below the t -axis indicates that the velocity is negative and the car is travelling backwards.

b Final displacement = area above the t -axis – area below the t -axis

$$\begin{aligned}
 &= \left(\frac{0.1}{2} + 0.1 + \frac{0.1}{2}\right) 60 - \left(\frac{0.1}{2} + 0.1 + \frac{0.1}{2}\right) 20 \\
 &= 12 - 4 \\
 &= 8 \text{ km from the starting point in the positive direction}
 \end{aligned}$$

3


Total distance travelled

$$\begin{aligned}
 &= \text{area A} + \text{area B} + \text{area C} + \text{area D} + \text{area E} \\
 &= \frac{1}{60} \left[\frac{1}{2}(3 \times 40) + (40 \times 4) + \left(\frac{40+30}{2}\right) 2 + (10 \times 30) + \frac{1}{2}(2 \times 30) \right] \\
 &= \frac{1}{60} [60 + 160 + 35 + 300 + 30] \\
 &= 9.75 \text{ km}
 \end{aligned}$$

{the factor $\frac{1}{60}$ accounts for the fact that the times are in minutes while the speeds are in km/h}

EXERCISE 26D.2
1

a $v(t) = 1 - 2t \text{ cms}^{-1}, t \geq 0$

$v(t) = s'(t) = 1 - 2t$ which has sign diagram:

$\therefore$ a direction reversal occurs at $t = \frac{1}{2}$.



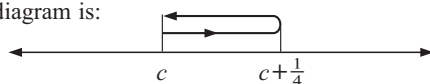
Now $s(t) = \int (1 - 2t) dt = t - \frac{2t^2}{2} + c = t - t^2 + c$

$\therefore s(0) = c$

and $s(\frac{1}{2}) = \frac{1}{4} + c$

and $s(1) = c$

$\therefore$ motion diagram is:



$$\begin{aligned}
 \therefore \text{total distance travelled} &= (c + \frac{1}{4} - c) + (c + \frac{1}{4} - c) \\
 &= \frac{1}{2} \text{ cm}
 \end{aligned}$$

b Displacement $= s(1) - s(0)$ $= c - c$ $= 0 \text{ cm}$	or a	total distance travelled $= \int_0^1 1 - 2t dt$ $= 0.5 \text{ cm}$	b displacement $= \int_0^1 (1 - 2t) dt$ $= [t - t^2]_0^1$ $= 0 - 0$ $= 0 \text{ cm}$
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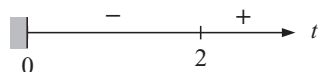
2 a $v(t) = t^2 - t - 2 \text{ cms}^{-1}, t \geq 0$

$$v(t) = s'(t) = t^2 - t - 2$$

$$= (t - 2)(t + 1)$$

$\therefore$ a direction reversal occurs at $t = 2$.

which has sign diagram:



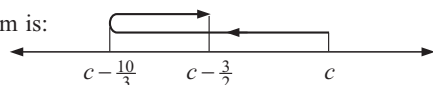
Now $s(t) = \int (t^2 - t - 2) dt = \frac{t^3}{3} - \frac{t^2}{2} - 2t + c$

$$s(0) = c$$

$$s(2) = c - \frac{10}{3}$$

$$s(3) = c - \frac{3}{2}$$

$\therefore$ motion diagram is:



$$\begin{aligned} \therefore \text{total distance travelled} &= \left(c - \left[c - \frac{10}{3}\right]\right) + \left(c - \frac{3}{2} - \left[c - \frac{10}{3}\right]\right) \\ &= \frac{10}{3} - \frac{3}{2} + \frac{10}{3} \\ &= \frac{31}{6} \\ &= 5\frac{1}{6} \text{ cm} \end{aligned}$$

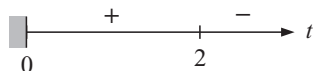
b Displacement $= s(3) - s(0)$ $= c - \frac{3}{2} - c$ $= -\frac{3}{2} \text{ cm}$	or a	total distance travelled $= \int_0^3 t^2 - t - 2 dt$ $\div 5.17 \text{ cm}$	b displacement $= \int_0^3 (t^2 - t - 2) dt$ $= -1.5 \text{ cm}$
--	-------------	---	---

3 $x'(t) = 16t - 4t^3 \text{ units s}^{-1}, t \geq 0$

$$= 4t(4 - t^2)$$

$$= 4t(2 + t)(2 - t)$$

which has sign diagram:



$\therefore$ a direction reversal occurs at $t = 2$.

Now $x(t) = \int (16t - 4t^3) dt = 8t^2 - t^4 + c$

a $0 \leq t \leq 3$

$$x(0) = c$$

$$x(2) = 32 - 16 + c = c + 16$$

$$x(3) = 72 - 81 + c = c - 9$$

$\therefore$ motion diagram is:

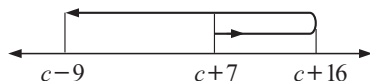


$$\begin{aligned} \therefore \text{total distance travelled} &= (c + 16 - c) + (c + 16 - [c - 9]) \\ &= 41 \text{ units} \end{aligned}$$

b $1 \leq t \leq 3$

$$x(1) = 7 + c = c + 7$$

$\therefore$ motion diagram is:



$$\begin{aligned} \therefore \text{total distance travelled} &= (c + 16 - [c + 7]) + (c + 16 - [c - 9]) \\ &= 34 \text{ units} \end{aligned}$$

or a total distance travelled $= \int_0^3 16t - 4t^3 dt$ $= 41 \text{ units}$	b total distance travelled $= \int_1^3 16t - 4t^3 dt$ $= 34 \text{ units}$
--	---

4 $v(t) = 50 - 10e^{-0.5t} \text{ ms}^{-1} \quad t \geq 0$

a $v(0) = 50 - \frac{10}{e^0} = 50 - 10 = 40 \text{ ms}^{-1}$

b $v(3) = 50 - \frac{10}{e^{1.5}} \doteq 47.8 \text{ ms}^{-1}$

c The velocity reaches 45 ms^{-1} when

$$45 = 50 - 10e^{-0.5t}$$

$$\therefore 10e^{-\frac{t}{2}} = 5$$

$$\therefore e^{-\frac{t}{2}} = \frac{1}{2}$$

$$\therefore e^{\frac{t}{2}} = 2$$

$$\therefore \frac{t}{2} = \ln 2$$

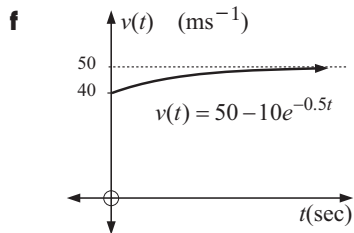
$$\therefore t = 2 \ln 2 \doteq 1.39 \text{ sec}$$

d $v(t) = 50 - \frac{10}{e^{\frac{t}{2}}}$, $\therefore$ as $t \rightarrow \infty$, $\frac{10}{e^{\frac{t}{2}}} \rightarrow +0$ and so $v(t) \rightarrow 50 \text{ ms}^{-1}$ (below)

e $a(t) = v'(t)$
 $= -10e^{-0.5t}(-0.5)$
 $= 5e^{-0.5t} \text{ ms}^{-2}$
 $= \frac{5}{e^{0.5t}} \text{ ms}^{-2}$

$$\therefore a(t) > 0 \text{ for all } t \quad \{e^x > 0 \text{ for all } x\}$$

i.e., the acceleration is always positive



g total distance travelled $= \int_0^3 |50 - 10e^{-0.5t}| dt$
 $\doteq 134.5 \text{ m}$

5 $a(t) = \frac{t}{10} - 3 \text{ ms}^{-2} \quad \therefore v(t) = \int \left(\frac{t}{10} - 3 \right) dt = \frac{t^2}{20} - 3t + c$

But $v(0) = 45 \quad \therefore c = 45$

Now $v(t) = \frac{t^2}{20} - 3t + 45$
 $= \frac{t^2 - 60t + 900}{20}$
 $= \frac{(t - 30)^2}{20}$

Total distance travelled in the first minute $= \int_0^{60} \left| \frac{(t - 30)^2}{20} \right| dt$
 $= \frac{1}{20} \int_0^{60} |(t - 30)^2| dt$
 $= \frac{1}{20} \int_0^{60} (t - 30)^2 dt \quad \{\text{as } (t - 30)^2 \geq 0\}$
 $= \frac{1}{20} \left[\frac{(t - 30)^3}{3} \right]_0^{60}$
 $= \frac{1}{60} ((30)^3 - (-30)^3)$
 $= \frac{1}{60} (30^3 + 30^3)$
 $= 900 \text{ m}$

$$6 \quad a(t) = 4e^{-\frac{t}{20}} \text{ ms}^{-2} \quad \therefore \quad v(t) = \int 4e^{-\frac{t}{20}} dt = 4 \frac{1}{-\frac{1}{20}} e^{-\frac{t}{20}} + c = -80e^{-\frac{t}{20}} + c$$

$$\text{Now } v(0) = 20 \text{ ms}^{-1} \quad \therefore \quad c = 100 \quad \therefore \quad v(t) = 100 - 80e^{-\frac{t}{20}}$$

a as $t \rightarrow \infty$, $e^{-\frac{t}{20}} \rightarrow 0$ (above) $\therefore v(t) \rightarrow 100$ (below)
i.e., the body approaches a limiting velocity of 100 ms^{-1}

$$\mathbf{b} \quad \text{total distance travelled} = \int_0^{10} \left| 100 - 80e^{-\frac{t}{20}} \right| dt \doteq 370.4 \text{ m}$$

EXERCISE 26E

1 The marginal cost is $C'(x)$ and $C'(x) = 3.15 + 0.004x$ \$/gadget

$$\therefore C(x) = \int (3.15 + 0.004x) dx \\ = 3.15x + 0.002x^2 + c$$

$$\text{But } C(0) = 450 \quad \therefore \quad c = 450$$

$$\therefore C(x) = 3.15x + 0.002x^2 + 450 \text{ dollars}$$

$$C(800) = 3.15(800) + 0.002(800)^2 + 450 \\ = \$4250$$

$$\therefore \text{ total cost is } \$4250$$

2 a The marginal profit is $P'(x)$ and $P'(x) = 15 - 0.03x$ \$/plate

$$\therefore P(x) = \int (15 - 0.03x) dx \\ = 15x - 0.015x^2 + c$$

$$\text{But } P(0) = -650 \quad \therefore \quad c = -650$$

$$\therefore P(x) = 15x - 0.015x^2 - 650 \text{ dollars}$$

b The maximum profit occurs when $P'(x) = 0$ i.e., when $15 - 0.03x = 0$

$$\therefore 0.03x = 15 \\ \therefore x = \frac{15}{0.03} \\ \text{i.e., } x = 500$$

$$\text{and } P''(x) = -0.03 < 0 \quad \therefore \text{ profit is at a maximum when } x = 500$$

$$\therefore \text{ maximum profit} = P(500) \\ = 15(500) - 0.015(500)^2 - 650 \\ = \$3100$$

c In order for a profit to be made $P(x)$ must be greater than 0

$$\text{i.e., } 15x - 0.015x^2 - 650 > 0$$

Using technology to graph $P(x)$ and find x intercepts we get $x_1 = 45.39$ and $x_2 = 954.6$

$$\therefore 46 \leq x \leq 954 \quad \{\text{We cannot produce part plates.}\}$$

3 $E(t) = 350(80 + 0.15t)^{0.8} - 120(80 + 0.15t)$ calories/day

$$\begin{aligned} \text{Total energy needs over the first week} &= \int_0^7 E(t) dt \\ &= \int_0^7 [350(80 + 0.15t)^{0.8} - 120(80 + 0.15t)] dt \\ &= \left[\frac{1}{0.15} \times \frac{350(80 + 0.15t)^{1.8}}{1.8} - 9600t - 9t^2 \right]_0^7 \\ &\doteq 14\,400 \text{ calories} \end{aligned}$$

$$4 \quad \frac{dT}{dx} = \frac{-20}{x^{0.63}} = -20x^{-0.63} \quad \therefore \quad T = \int -20x^{-0.63} dx = \frac{-20x^{0.37}}{0.37} + c$$

$$\text{But when } x = 3, \quad T = 100, \quad \therefore \quad \frac{-20(3^{0.37})}{0.37} + c = 100$$

$$\therefore \quad c = 100 + \frac{20(3^{0.37})}{0.37} \doteq 181.1639$$

$$\text{i.e., } T = \frac{-20x^{0.37}}{0.37} + 181.1639$$

$$\text{and when } x = 6, \quad T = -104.8925 + 181.1639 \\ \doteq 76.27$$

$\therefore$ the outer surface temperature is about 76.3°C

$$5 \quad \mathbf{a} \quad \frac{d^2y}{dx^2} = -\frac{1}{10}(1-x)^2 \quad \text{and} \quad \frac{dy}{dx} = \int \frac{d^2y}{dx^2} dx \\ = \int -\frac{1}{10}(1-x)^2 dx \\ = -\frac{1}{10}\left(\frac{1}{-1}\right) \times \frac{(1-x)^3}{3} + c \\ = \frac{1}{30}(1-x)^3 + c$$

$$\text{But when } x = 0, \quad \frac{dy}{dx} = 0 \quad \{\text{horizontal tangent}\}$$

$$\therefore \quad \frac{1}{30}(1-0)^3 + c = 0 \quad \text{and so} \quad c = -\frac{1}{30}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{30}(1-x)^3 - \frac{1}{30}$$

$$\therefore \quad y = \int \left[\frac{1}{30}(1-x)^3 - \frac{1}{30} \right] dx \\ = \frac{1}{30} \left(\frac{-1}{-1} \right) \frac{(1-x)^4}{4} - \frac{1}{30}x + d \\ = -\frac{(1-x)^4}{120} - \frac{x}{30} + d$$

$$\text{Also, when } x = 0, \quad y = 0 \quad \therefore \quad -\frac{1}{120} - 0 + d = 0 \quad \therefore \quad d = \frac{1}{120}$$

$$\text{and so } y = \frac{1}{120} - \frac{(1-x)^4}{120} - \frac{x}{30}$$

b Maximum deflection occurs at the right hand end where $x \doteq 1$

$$\text{and at } x \doteq 1, \quad y \doteq \frac{1}{120} - 0 - \frac{1}{30} \doteq -0.025 \text{ m}$$

i.e., maximum deflection is about 2.5 cm.

$$6 \quad \mathbf{a} \quad \frac{d^2y}{dx^2} = \frac{1}{100} \left(2x - \frac{x^2}{2} \right) = \frac{1}{50}x - \frac{1}{200}x^2$$

$$\therefore \quad \frac{dy}{dx} = \int \left(\frac{1}{50}x - \frac{1}{200}x^2 \right) dx = \frac{1}{100}x^2 - \frac{1}{600}x^3 + c$$

$$\text{The sag, } y = \int \left(\frac{1}{100}x^2 - \frac{1}{600}x^3 + c \right) dx$$

$$\text{i.e., } y = \frac{1}{300}x^3 - \frac{1}{2400}x^4 + cx + d$$

$$\text{Now when } x = 0, \quad y = 0 \quad \therefore \quad 0 - 0 + 0 + d = 0$$

$$\therefore \quad d = 0$$

$$\therefore \quad y = \frac{1}{300}x^3 - \frac{1}{2400}x^4 + cx$$

Also when $x = 4$, $y = 0 \quad \therefore \quad \frac{1}{300}(4^3) - \frac{1}{2400}(4^4) + 4c = 0$

$$\therefore 4c = \frac{1}{2400}(4^4) - \frac{1}{300}(4^3)$$

$$\therefore c = \frac{1}{2400}(4^3) - \frac{1}{300}(4^2)$$

$$\therefore c = -\frac{2}{75}$$

$$\therefore y = \left(\frac{1}{300}x^3 - \frac{1}{2400}x^4 - \frac{2}{75}x\right) \text{ m}$$

b The maximum sag occurs when $\frac{dy}{dx} = 0$ i.e., $\frac{1}{100}x^2 - \frac{1}{600}x^3 - \frac{2}{75} = 0$
 $6x^2 - x^3 - 16 = 0$

Using technology the three solutions are: $x = -1.464$, 2 or 5.4641

But the maximum lies between 0 and 4 , $\therefore$ maximum sag occurs when $x = 2$.

When $x = 2$, $y = \frac{1}{300}(2^3) - \frac{1}{2400}(2^4) - \frac{2}{75}(2)$

$$\div -0.03333 \text{ m}$$

$$\div -3.333 \text{ cm} \quad \therefore \text{ the maximum sag is } \div 3.33 \text{ cm}$$

c When 1 m from P , i.e., $x = 3 \text{ m}$, $y = \frac{1}{300}(3^3) - \frac{1}{2400}(3^4) - \frac{2}{75}(3)$
 $= -0.02375 \text{ m}$
 $= -2.375 \text{ cm} \quad \therefore \text{ the sag is } 2.375 \text{ cm}$

d When 1 m from P , i.e., $x = 3 \text{ m}$, $\frac{dy}{dx} = \frac{1}{100}(3^2) - \frac{1}{600}(3^3) - \frac{2}{75} \div 0.0183$

i.e., the angle θ , that the plank makes with the horizontal is such that $\tan \theta \div 0.0183$

i.e., $\theta = \tan^{-1}(0.0183) \div 1.05^\circ$

7 The cost per unit volume V is $\frac{1}{2}x^2 + 4 \text{ } \$/\text{m}^3$

$$\therefore \frac{dC}{dV} = \left(\frac{1}{2}x^2 + 4\right) \$/\text{m}^3 \quad \text{and as } V = \pi r^2 x, \quad \frac{dV}{dx} = \pi r^2$$

Now $\frac{dC}{dx} = \frac{dC}{dV} \frac{dV}{dx} \quad \{\text{Chain rule}\}$

$$\therefore \frac{dC}{dx} = \left(\frac{1}{2}x^2 + 4\right) \times \pi r^2$$

$$\begin{aligned} \therefore \frac{dC}{dx} &= \pi r^2 \left(\frac{1}{2}x^2 + 4\right) \quad \text{and the cost } C, \text{ of digging a well} = \int \frac{dC}{dx} dx \\ &= \int [\pi r^2 (\frac{1}{2}x^2 + 4)] dx \\ &= \pi r^2 \left(\frac{x^3}{6} + 4x\right) + c \end{aligned}$$

$$\therefore \text{ cost of digging a well } h \text{ metres deep} = \pi r^2 \left(\frac{h^3}{6} + 4h\right) + c$$

Now if the initial cost $= C_0$, $\pi r^2(\frac{0}{6} + 0) + c = C_0 \quad \therefore \quad c = C_0$

$$\therefore C(h) = \pi r^2 \left(\frac{h^3 + 24h}{6}\right) + C_0$$

8 a The yield Y per unit area A is proportional to $\frac{1}{\sqrt{x+4}}$.

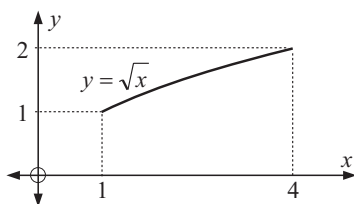
i.e., $\frac{dY}{dA} \propto \frac{1}{\sqrt{x+4}}$

$$\therefore \frac{dY}{dA} = k \left(\frac{1}{\sqrt{x+4}}\right) = \frac{k}{\sqrt{x+4}} \quad \text{where } k \text{ is a constant}$$

- b** Now the shaded area $A = \text{length} \times \text{width}$ $\therefore \frac{dA}{dx} = 4 - 2p$
 $\therefore A = (4 - 2p)x$ Now $\frac{dY}{dx} = \frac{dY}{dA} \frac{dA}{dx}$ {Chain rule}
 $\therefore \frac{dY}{dx} = \frac{k}{\sqrt{x+4}} \times (4 - 2p)$
 i.e., $\frac{dY}{dx} = \frac{k(4 - 2p)}{\sqrt{x+4}}$
- c** $\frac{dY}{dx}$ is the instantaneous rate of change of the yield with respect to the distance x from the canal.
 $\therefore \text{total yield} = Y = \int_0^p \frac{dY}{dx} dx = \int_0^p \frac{k(4 - 2p)}{\sqrt{x+4}} dx$
- d** $Y = k(4 - 2p) \int_0^p (x + 4)^{-\frac{1}{2}} dx$ {using **c**}
 $= k(4 - 2p) \times \left[\frac{(x + 4)^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^p$
 $= 2k(4 - 2p) [\sqrt{x+4}]_0^p$
 $= 4k(2 - p) [\sqrt{p+4} - \sqrt{4}]$
 $\therefore Y = 4k(2 - p) (\sqrt{p+4} - 2)$
- e** For yield to be a maximum we need to maximise Y . Using technology to graph Y and find the maximum, we find that the maximum occurs when $p \doteq 0.9735$ km
 i.e., orchard is $0.974 \text{ km} \times 2.05 \text{ km}$

REVIEW SET 26A

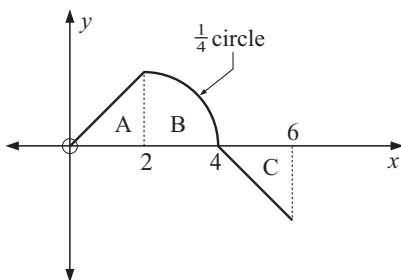
1 a



x	$\sqrt{x}$
1	1
1.5	1.2247
2	1.4142
2.5	1.5811
3	1.7321
3.5	1.8708
4	2

- b** $A_L = 0.5(1 + 1.2247 + 1.4142 + 1.5811 + 1.7321 + 1.8708)$
 $= 0.5(8.8229)$
 $\doteq 4.41 \text{ units}^2$

2



- a** $\int_0^4 f(x) dx = \text{area of triangle} + \text{area of } \frac{1}{4} \text{ circle}$
 $= \frac{1}{2}(2 \times 2) + \frac{1}{4}\pi(2)^2$
 $= 2 + \pi$
- b** $\int_4^6 f(x) dx = - \text{area of triangle below } x\text{-axis}$
 $= -\frac{1}{2}(2 \times 2)$
 $= -2$
- c** $\int_0^6 f(x) dx = \int_0^4 f(x) dx + \int_4^6 f(x) dx$
 $= (2 + \pi) + (-2)$
 $= \pi$

3 a shaded area $= \int_a^b [f(x) - g(x)] dx + \int_b^c [g(x) - f(x)] dx + \int_c^d [f(x) - g(x)] dx$

b shaded area $= \int_a^d |f(x) - g(x)| dx$

4 $a(t) = 6t - 30 \text{ cms}^{-2} \quad \therefore v(t) = \int (6t - 30) dt = 3t^2 - 30t + c$

But $v(0) = 27 \quad \therefore c = 27 \quad \therefore v(t) = 3t^2 - 30t + 27 \text{ cms}^{-1}$

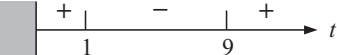
$\therefore s(t) = \int (3t^2 - 30t + 27) dt = t^3 - 15t^2 + 27t + d$

But $s(0) = 0 \quad \therefore d = 0 \quad \therefore s(t) = t^3 - 15t^2 + 27t$

Now $v(t) = 3t^2 - 30t + 27$

$= 3(t^2 - 10t + 9)$

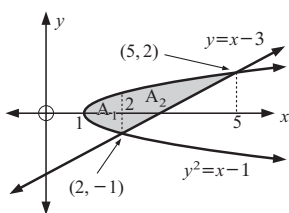
$= 3(t-1)(t-9)$

which has sign diagram: 

The particle comes to rest for the second time at $t = 9$ seconds.

$\therefore$ total distance travelled $= \int_0^9 |3t^2 - 30t + 27| dt = 269 \text{ cm}$

5 a



$y^2 = x - 1$ meets $y = x - 3$ where

$x - 1 = (x - 3)^2$

$\therefore x - 1 = x^2 - 6x + 9$

$\therefore x^2 - 7x + 10 = 0$

$\therefore (x - 5)(x - 2) = 0$

$\therefore x = 2 \text{ or } x = 5$

$\therefore$ at $(5, 2)$ and $(2, -1)$

b Area $= A_1 + A_2$

$= 2 \int_1^2 (x - 1)^{\frac{1}{2}} dx + \int_2^5 [(x - 1)^{\frac{1}{2}} - (x - 3)] dx$

$= 2 \left[\frac{2}{3} (x - 1)^{\frac{3}{2}} \right]_1^2 + \left[\frac{2}{3} (x - 1)^{\frac{3}{2}} - \frac{x^2}{2} + 3x \right]_2^5$

$= 2 \left[\frac{2}{3} - 0 \right] + \left[\left(\frac{2}{3} (8) - \frac{25}{2} + 15 \right) - \left(\frac{2}{3} - 2 + 6 \right) \right] = 4\frac{1}{2} \text{ units}^2$

6 $y = k$ meets $y = x^2$ where $x^2 = k \quad \therefore x = \pm\sqrt{k}$

By symmetry, $\int_0^{\sqrt{k}} (k - x^2) dx = \frac{1}{2} \times 5\frac{1}{3} = \frac{1}{2} \times \frac{16}{3} \quad \therefore \frac{2}{3} k\sqrt{k} = \frac{8}{3}$

$\therefore \left[kx - \frac{x^3}{3} \right]_0^{\sqrt{k}} = \frac{8}{3}$

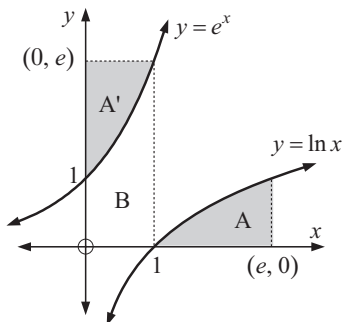
$\therefore k\sqrt{k} = 4$

$\therefore k^{\frac{3}{2}} = 4$

$\therefore k = 4^{\frac{2}{3}} = \sqrt[3]{16}$

$\therefore k\sqrt{k} - \frac{k\sqrt{k}}{3} = \frac{8}{3}$

7



$y = e^x$ and $y = \ln x$ are inverse functions, i.e., they are symmetrical about $y = x$

$\therefore$ area A $=$ area A'

but area A' + area B = area of rectangle

$\therefore$ area A + area B $= e \times 1 = e$

and as area A $= \int_1^e \ln x dx$

and area B $= \int_0^1 e^x dx$

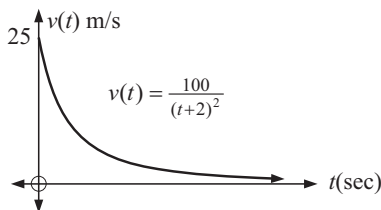
then $\int_1^e \ln x dx + \int_0^1 e^x dx = e$

$$8 \quad v(t) = \frac{100}{(t+2)^2} = 100(t+2)^{-2} \text{ ms}^{-1}$$

$$a \quad \text{At } t = 0, \quad v(0) = \frac{100}{2^2} = 25 \text{ ms}^{-1} \quad \text{At } t = 3, \quad v(3) = \frac{100}{5^2} = 4 \text{ ms}^{-1}$$

$$b \quad \text{as } t \rightarrow +\infty, \quad v(t) \rightarrow 0 \text{ ms}^{-1} \quad (\text{above})$$

c



$$e \quad a(t) = v'(t) \\ = -200(t+2)^{-3} \text{ ms}^{-2} \\ = \frac{-200}{(t+2)^3}$$

$$f \quad \frac{dv}{dt} = \frac{-200}{(t+2)^3} \\ = -\frac{1}{5} \frac{1000}{(t+2)^3} \\ = -\frac{1}{5} \left(\frac{100}{(t+2)^2} \right)^{\frac{3}{2}} \\ = -\frac{1}{5} v^{\frac{3}{2}} \\ \therefore \frac{dv}{dt} = -kv^{\frac{3}{2}} \quad \text{where } k = \frac{1}{5}$$

$$9 \quad y = x^3 \text{ meets } y = 7x^2 - 10x$$

$$\text{when } x^3 = 7x^2 - 10x$$

$$\text{i.e., } x^3 - 7x^2 + 10x = 0$$

$$\therefore x(x^2 - 7x + 10) = 0$$

$$\therefore x(x-2)(x-5) = 0$$

$$\therefore x = 0, 2 \text{ or } 5$$

10 The area between $x = 0$ and $x = a$ is 2 units².

$$\therefore \int_0^a e^x dx = 2$$

$$\therefore [e^x]_0^a = 2$$

$$\therefore e^a - e^0 = 2$$

$$\therefore e^a = 3$$

$$\therefore a = \ln 3$$

d As $v(t)$ is always positive, the boat is always travelling forwards.

$$s(t) = \int v(t) dt \\ = \int 100(t+2)^{-2} dt \\ = -100(t+2)^{-1} + c \\ = \frac{-100}{t+2} + c$$

$$\therefore s(0) = c - 50 \text{ m}$$

$\therefore$ when the boat has travelled 30 m,

$$s(t) = c - 20$$

$$\therefore c - 20 = \frac{-100}{t+2} + c$$

$$\therefore \frac{-100}{t+2} = -20$$

$$\therefore t+2 = 5$$

$$\therefore t = 3 \text{ seconds}$$

$$\therefore \text{total area} = \int_0^5 |x^3 - 7x^2 + 10x| dx \\ = 21\frac{1}{12} \text{ units}^2$$

The area between $x = a = \ln 3$ and $x = b$ is 2 units².

$$\therefore \int_{\ln 3}^b e^x dx = 2$$

$$\therefore [e^x]_{\ln 3}^b = 2$$

$$\therefore e^b - e^{\ln 3} = 2$$

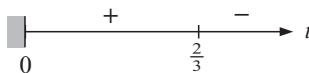
$$\therefore e^b - 3 = 2$$

$$\therefore e^b = 5$$

$$\therefore b = \ln 5$$

REVIEW SET 26B

- 1 $v(t) = 2t - 3t^2 = t(2 - 3t)$ which has sign diagram:



Now $s(t) = \int (2t - 3t^2) dt$

$$s(t) = t^2 - t^3 + c \text{ metres}$$

and so $s(0) = c$

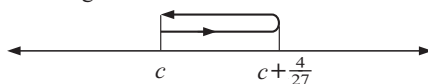
$$s\left(\frac{2}{3}\right) = \frac{4}{9} - \frac{8}{27} + c = c + \frac{4}{27}$$

$$s(1) = 1 - 1 + c = c$$

$\therefore$ total distance travelled $= (c + \frac{4}{27} - c) + (c + \frac{4}{27} - c) = \frac{8}{27} \text{ m} \doteq 29.6 \text{ cm}$

or total distance travelled $= \int_0^1 |2t - 3t^2| dt$
 $\doteq 0.296 \text{ m}$

with motion diagram:



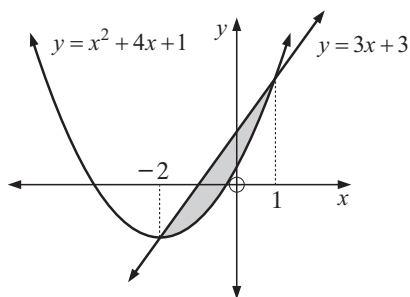
- 2 $y = x^2 + 4x + 1$ meets $y = 3x + 3$ where

$$x^2 + 4x + 1 = 3x + 3$$

$$\therefore x^2 + x - 2 = 0$$

$$\therefore (x+2)(x-1) = 0$$

$$\therefore x = -2 \text{ or } 1$$



$$\begin{aligned} \therefore \text{area} &= \int_{-2}^1 [(3x+3) - (x^2+4x+1)] dx \\ &= \int_{-2}^1 (-x^2 - x + 2) dx \\ &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^1 \\ &= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right) \\ &= -\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + 2 + 4 \\ &= 4\frac{1}{2} \text{ units}^2 \end{aligned}$$

- 3 $\frac{dI}{dt} = -\frac{100}{t^2} \quad t \geq 0.2 \text{ seconds}$

$$\therefore I(t) = \int \frac{dI}{dt} dt = \int -100t^{-2} dt = 100t^{-1} + c$$

Now $I(2) = 150 \text{ amps}$

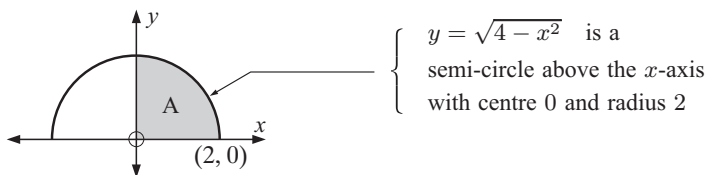
$$\therefore \frac{100}{2} + c = 150 \text{ and so } c = 100 \quad \therefore I(t) = \left(\frac{100}{t} + 100 \right) \text{ amps}$$

a $I(20) = \frac{100}{20} + 100$
 $= 105 \text{ amps}$

b as $t \rightarrow +\infty$

$$I(t) \rightarrow 100 \text{ amps (above)}$$

4



Now $\int_0^2 \sqrt{4-x^2} dx = \text{shaded area} = \frac{1}{4} \text{ of the area of a circle of radius 2 units}$
 $= \frac{1}{4} \pi (2^2)$
 $= \pi \text{ units}^2$

5 $\int_{-1}^3 f(x) dx$ gives us the correct area only if $f(x)$ is positive on the interval $-1 \leq x \leq 3$.

But $f(x)$ is not positive for $1 \leq x \leq 3$ so $\int_{-1}^3 f(x) dx$ does not provide the correct answer.

The shaded area below the x -axis is given by $\int_1^3 [0 - f(x)] dx = -\int_1^3 f(x) dx$

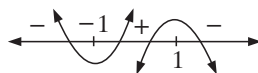
6 a $f(x) = \frac{x}{1+x^2} \quad \therefore \quad f'(x) = \frac{1(1+x^2) - x(2x)}{(1+x^2)^2} \quad \{\text{quotient rule}\}$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2}$$

$$= \frac{1-x^2}{(1+x^2)^2}$$

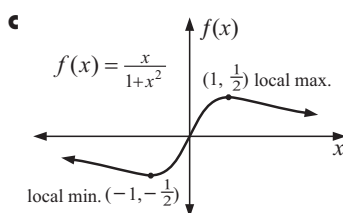
$$= \frac{(1+x)(1-x)}{(1+x^2)^2}$$

which has sign diagram:



$\therefore$ there is a local minimum at $(-1, -\frac{1}{2})$ and a local maximum at $(1, \frac{1}{2})$

b as $x \rightarrow \infty$ $f(x) \rightarrow 0$ (above) and
as $x \rightarrow -\infty$ $f(x) \rightarrow 0$ (below)



d Area $= \int_{-2}^0 \left[0 - \frac{x}{1+x^2} \right] dx \quad \therefore \quad \int_{-2}^0 \frac{-x}{1+x^2} dx = \int_{-2}^0 \frac{1}{u} \left(-\frac{1}{2} \frac{du}{dx} \right) dx$

$$= \int_{-2}^0 \frac{-x}{1+x^2} dx = -\frac{1}{2} \int_5^1 \frac{1}{u} du$$

$$= -\frac{1}{2} [\ln |u|]_5^1$$

$$= -\frac{1}{2} (0 - \ln 5)$$

$$= \frac{1}{2} \ln 5 \text{ units}^2 \quad (\div 0.805 \text{ units}^2)$$

Let $u = 1 + x^2 \quad \therefore \quad \frac{du}{dx} = 2x$
and when $x = 0, \quad u = 1$
when $x = -2, \quad u = 5$

7 The coordinates of B are $(2, 4+k)$

$$\therefore \text{ area rectangle OABC} = 2 \times (4+k)$$

$$= 8 + 2k$$

$\therefore$ since the two shaded regions are equal in area,
each area is $4 + k \text{ units}^2$.

$$\therefore \int_0^2 (x^2 + k) dx = 4 + k$$

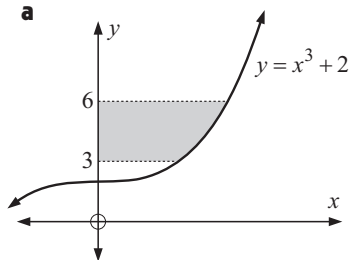
$$\therefore \left[\frac{x^3}{3} + kx \right]_0^2 = 4 + k$$

$$\therefore \frac{8}{3} + 2k = 4 + k$$

$$\therefore k = 4 - \frac{8}{3}$$

$$\therefore k = \frac{4}{3}$$

8 a



b $y = x^3 + 2$
 $\therefore x^3 = y - 2$
 $\therefore x = (y - 2)^{\frac{1}{3}}$

c Area $= \int_3^6 x dy$
 $= \int_3^6 (y - 2)^{\frac{1}{3}} dy$
 $= \left[\frac{1}{\frac{4}{3}} \frac{(y - 2)^{\frac{4}{3}}}{\frac{4}{3}} \right]_3^6$
 $= \frac{3}{4} \left(4^{\frac{4}{3}} - 1^{\frac{4}{3}} \right)$
 $= \frac{3}{4} (4\sqrt[3]{4} - 1) \text{ units}^2$
(or $\div 4.01 \text{ units}^2$)

9 $y = 2x^3 - 9x$ meets $y = 3x^2 - 10$

when $2x^3 - 9x = 3x^2 - 10$

$\therefore 2x^3 - 3x^2 - 9x + 10 = 0$

$$1 \begin{array}{r|rrrr} 2 & 2 & -3 & -9 & 10 \\ & 0 & 2 & -1 & -10 \\ \hline & 2 & -1 & -10 & 0 \end{array}$$

$\therefore (x-1)(2x^2 - x - 10) = 0$

$\therefore (x-1)(2x-5)(x+2) = 0$

$\therefore x = -2, 1 \text{ or } \frac{5}{2}$

$\therefore$ total area

$$= \int_{-2}^{\frac{5}{2}} |2x^3 - 3x^2 - 9x + 10| dx$$

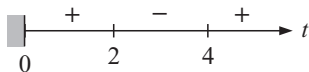
$\doteq 31.2 \text{ units}^2$

REVIEW SET 26C

1 a $v(t) = t^2 - 6t + 8 \text{ ms}^{-1}, t \geq 0$

$= (t-4)(t-2)$

which has sign diagram:



b Now $s(t) = \int (t^2 - 6t + 8) dt = \frac{t^3}{3} - 3t^2 + 8t + c$

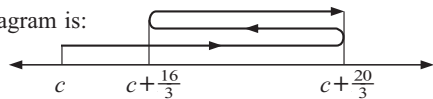
and so $s(0) = c$

$s(2) = c + 6\frac{2}{3}$

$s(4) = c + 5\frac{1}{3}$

$s(5) = c + 6\frac{2}{3}$

the motion diagram is:



The particle moves in the positive direction initially, then at $t = 2$, $6\frac{2}{3}$ m from its starting point, it changes direction. It changes direction again at $t = 4$, $5\frac{1}{3}$ m from its starting point, and at $t = 5$ it is $6\frac{2}{3}$ m from its starting point.

c After 5 seconds, the particle is $6\frac{2}{3}$ m to the right of its starting point.

d total distance travelled $= (c + \frac{20}{3} - c) + [(c + \frac{20}{3}) - (c + \frac{16}{3})] + [(c + \frac{20}{3}) - (c + \frac{16}{3})]$
 $= 9\frac{1}{3} \text{ m}$

2 If $x = \ln\left(\frac{y+3}{2}\right)$

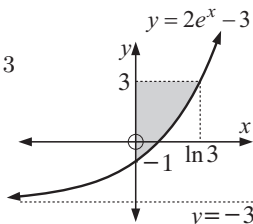
then $\frac{y+3}{2} = e^x \therefore y = 2e^x - 3$

when $y = 3$, $2e^x - 3 = 3$

$\therefore 2e^x = 6$

$\therefore e^x = 3$

$\therefore x = \ln 3$



Area $= \int_0^{\ln 3} (3 - [2e^x - 3]) dx$
 $= \int_0^{\ln 3} (6 - 2e^x) dx$
 $= [6x - 2e^x]_0^{\ln 3}$
 $= (6 \ln 3 - 2e^{\ln 3}) - (0 - 2)$
 $= (6 \ln 3 - 2 \times 3 + 2)$
 $= 6 \ln 3 - 4 \text{ units}^2$
 $(\doteq 2.59 \text{ units}^2)$

3 a From the graph,

area $\triangle OBX <$ area under the curve $<$ area $OXYZ$

$\therefore \frac{1}{2}\pi(1) < \int_0^\pi \sin x dx < \pi(1)$

i.e., $\frac{\pi}{2} < \int_0^\pi \sin x dx < \pi$

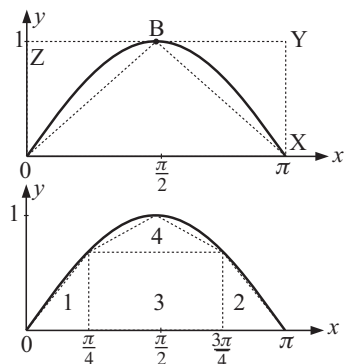
b If we partition the diagram into triangles as shown:

Area 1 $= \frac{1}{2} \left(\frac{\pi}{4}\right) \left(\frac{\sqrt{2}}{2}\right) = \frac{\pi\sqrt{2}}{16}$

Area 2 $= \frac{\pi\sqrt{2}}{16}$

Area 3 $= \frac{\pi}{2} \times \frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{4}$

Area 4 $= \frac{1}{2} \left(\frac{\pi}{2}\right) \left(1 - \frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \left(1 - \frac{\sqrt{2}}{2}\right)$



This total area is just less than the area under the arch

$$\begin{aligned}
 \text{and total area} &= \frac{\pi\sqrt{2}}{16} + \frac{\pi\sqrt{2}}{16} + \frac{\pi\sqrt{2}}{4} + \frac{\pi}{4} \left(1 - \frac{\sqrt{2}}{2}\right) \\
 &= \frac{\pi\sqrt{2} + \pi\sqrt{2} + 4\pi\sqrt{2} + 4\pi - 2\pi\sqrt{2}}{16} \\
 &= \frac{4\pi + 4\pi\sqrt{2}}{16} \\
 &= \frac{\pi + \pi\sqrt{2}}{4} \\
 &= \frac{\pi}{4}(1 + \sqrt{2}) \text{ units}^2
 \end{aligned}$$

- 4 a** A is the *upper half* of a circle centre (2, 0) and radius 2.

$$\begin{aligned}
 \therefore (x-2)^2 + (y-0)^2 &= 2^2 & \text{So, } y &= \sqrt{4x-x^2} \quad \text{---} \\
 (x-2)^2 + y^2 &= 4 & & \\
 y^2 &= 4 - (x-2)^2 & \text{or } y &= -\sqrt{4x-x^2} \quad \text{---} \\
 y^2 &= 4 - x^2 + 4x - 4 & & \\
 y^2 &= 4x - x^2 & \therefore \text{required equation is } y_A &= \sqrt{4x-x^2} \\
 \therefore y &= \pm\sqrt{4x-x^2}
 \end{aligned}$$

- b** Now B is the *lower half* of a circle centre (5, 0) and radius 1.

$$\begin{aligned}
 \therefore (x-5)^2 + (y-0)^2 &= 1^2 \\
 (x-5)^2 + y^2 &= 1 \\
 y^2 &= 1 - (x-5)^2 \\
 y^2 &= 1 - x^2 + 10x - 25 \\
 \therefore y &= \pm\sqrt{10x-x^2-24} \quad \therefore y_B = -\sqrt{10x-x^2-24}
 \end{aligned}$$

$$\begin{array}{lll}
 \text{c} & \int_0^4 y_A dx & \int_4^6 y_B dx \\
 & = \frac{1}{2}\pi r^2 \quad \text{where } r=2 & = -\frac{1}{2}\pi r^2 \quad \text{where } r=1 \\
 & = \frac{1}{2}\pi(2)^2 & = -\frac{1}{2}\pi(1)^2 \\
 & = 2\pi & = -\frac{\pi}{2}
 \end{array}
 \quad
 \begin{array}{l}
 \text{d} \quad \int_0^6 f(x) dx \\
 = \int_0^4 y_A dx + \int_4^6 y_B dx \\
 = 2\pi + \left(-\frac{\pi}{2}\right) \\
 = \frac{3\pi}{2}
 \end{array}$$

- 5** The shaded area $= \int_0^2 ax(x-2) dx = 4 \text{ units}^2$

$$\begin{aligned}
 \therefore \int_0^2 (ax^2 - 2ax) dx &= 4 & \therefore \frac{8a}{3} - \frac{12a}{3} &= 4 \\
 \therefore \left[\frac{ax^3}{3} - ax^2 \right]_0^2 &= 4 & \therefore -\frac{4a}{3} &= 4 \\
 \therefore \left[\frac{8a}{3} - 4a \right] - 0 &= 4 & \therefore a &= -3 \\
 & & \therefore y &= -3x(x-2)
 \end{aligned}$$

Suppose A has coordinates $(k, -3k(k-2))$

$$\therefore \text{slope OA} = \frac{-3k(k-2)-0}{k-0} = -3(k-2) \quad \therefore \text{equation of OA is } y = -3(k-2)x$$

Now if OA divides the shaded area into equal areas,

$$\begin{aligned}
 \int_0^k [-3x(x-2) - (-3(k-2)x)] dx &= 2 & \therefore -k^3 + \frac{3k^3}{2} &= 2 \\
 \therefore \int_0^k (-3x^2 + 6x + 3kx - 6x) dx &= 2 & \therefore \frac{k^3}{2} &= 2 \\
 \therefore \int_0^k (-3x^2 + 3kx) dx &= 2 & \therefore k^3 &= 4 \\
 \therefore \left[-x^3 + \frac{3kx^2}{2} \right]_0^k &= 2 & \therefore k &= \sqrt[3]{4} \\
 & & \therefore \text{the } x\text{-coordinate of A is } &\sqrt[3]{4}
 \end{aligned}$$

- 6** The line $y = mx + c$ passes through $(-1, 0)$. $\therefore 0 = -m + c$ and so $c = m$
 $\therefore$ the line is $y = cx + c$

The curve and the line meet when $cx + c = -x^2 + 2x + 3$

$$\therefore x^2 + (c - 2)x + (c - 3) = 0$$

$$\therefore (x + 1)(x + [c - 3]) = 0 \quad \{\text{as we know } x = -1 \text{ is a solution}\}$$

$$\therefore x = -1 \text{ or } 3 - c$$

$$\begin{aligned} \text{If we let } a = 3 - c, \text{ then the enclosed area} &= \int_{-1}^a [(-x^2 + 2x + 3) - (3 - a)(x + 1)] dx \\ &= \int_{-1}^a [-x^2 + (a - 1)x + a] dx \\ &= \left[-\frac{x^3}{3} + \frac{(a - 1)x^2}{2} + ax \right]_{-1}^a \\ &= \left(-\frac{a^3}{3} + \frac{(a - 1)a^2}{2} + a^2 \right) - \left(\frac{1}{3} + \frac{(a - 1)}{2} - a \right) \\ &= -\frac{1}{3}a^3 + \frac{1}{2}a^3 - \frac{1}{2}a^2 + a^2 - \frac{1}{3} - \frac{1}{2}a + \frac{1}{2} + a \\ &= \frac{1}{6}a^3 + \frac{1}{2}a^2 + \frac{1}{2}a + \frac{1}{6} \end{aligned}$$

$$\text{But this area is } 4.5 \text{ units}^2, \quad \therefore \frac{1}{6}a^3 + \frac{1}{2}a^2 + \frac{1}{2}a + \frac{1}{6} = \frac{9}{2}$$

$$\therefore a^3 + 3a^2 + 3a + 1 = 27$$

$$\therefore a^3 + 3a^2 + 3a - 26 = 0$$

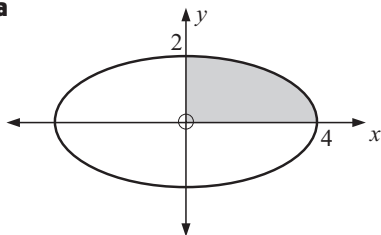
$$\therefore (a - 2)(a^2 + 5a + 13) = 0 \quad \text{where the quadratic has } \Delta < 0, \\ \therefore \text{no real solutions}$$

$$\therefore a = 2 \quad \text{and so } c = 1 \quad (= m)$$

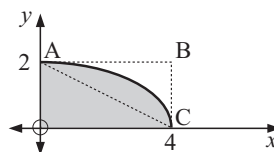
From technology, the only real solution to this equation is $c = 1$.

So $c = 1$, $m = 1$, and the line has equation $y = x + 1$.

7 a



b



Now area $\triangle AOC < \text{shaded area} < \text{area } ABCD$

$$\therefore \frac{1}{2}(2 \times 4) < \int_0^4 \frac{1}{2}\sqrt{16 - x^2} dx < 2 \times 4$$

$$\therefore 4 < \int_0^4 \frac{1}{2}\sqrt{16 - x^2} dx < 8$$

$$\therefore 8 < \int_0^4 \sqrt{16 - x^2} dx < 16 \quad \{(\times 2)\}$$

- 8** The curves meet when

$$x^3 + x^2 + 2x + 6 = 7x^2 - x - 4$$

$$\therefore x^3 - 6x^2 + 3x + 10 = 0$$

$$\begin{array}{r|rrrr} -1 & 1 & -6 & 3 & 10 \\ & 0 & -1 & 7 & -10 \\ \hline & 1 & -7 & 10 & 0 \end{array}$$

$$\therefore (x + 1)(x^2 - 7x + 10) = 0$$

$$\therefore (x + 1)(x - 2)(x - 5) = 0$$

$$\therefore x = -1, 2 \text{ or } 5$$

$\therefore$ area enclosed

$$= \int_{-1}^5 |x^3 - 6x^2 + 3x + 10| dx$$

$$= 40\frac{1}{2} \text{ units}^2$$

Chapter 27

CIRCULAR FUNCTION INTEGRATION

EXERCISE 27A

- 1** **a** $\int (3 \sin x - 2) \, dx$
 $= -3 \cos x - 2x + c$
- b** $\int (4x - 2 \cos x) \, dx$
 $= 2x^2 - 2 \sin x + c$
- c** $\int (2\sqrt{x} + \frac{4}{\cos^2 x}) \, dx$
 $= \int (2x^{\frac{1}{2}} + 4 \frac{1}{\cos^2 x}) \, dx$
 $= \frac{4}{3} x^{\frac{3}{2}} + 4 \tan x + c$
- d** $\int (\frac{1}{\cos^2 x} + 2 \sin x) \, dx$
 $= \tan x - 2 \cos x + c$
- e** $\int (\frac{x}{2} - \frac{1}{\cos^2 x}) \, dx$
 $= \frac{x^2}{4} - \tan x + c$
- f** $\int (\sin x - 2 \cos x + e^x) \, dx$
 $= -\cos x - 2 \sin x + e^x + c$
- 2** **a** $\int (\sqrt{x} + \frac{1}{2} \cos x) \, dx$
 $= \int (x^{\frac{1}{2}} + \frac{1}{2} \cos x) \, dx$
 $= \frac{2}{3} x^{\frac{3}{2}} + \frac{1}{2} \sin x + c$
- b** $\int (\theta - \sin \theta) \, d\theta$
 $= \frac{\theta^2}{2} + \cos \theta + c$
- c** $\int (t\sqrt{t} + \frac{2}{\cos^2 t}) \, dt$
 $= \int (t^{\frac{3}{2}} + 2 \sec^2 t) \, dt$
 $= \frac{2}{5} t^{\frac{5}{2}} + 2 \tan t + c$
- d** $\int (2e^t - 4 \sin t) \, dt$
 $= 2e^t + 4 \cos t + c$
- e** $\int (3 \cos t - \frac{1}{t}) \, dt$
 $= 3 \sin t - \ln |t| + c$
- f** $\int (3 - \frac{2}{\theta} + \frac{1}{\cos^2 \theta}) \, d\theta$
 $= 3\theta - 2 \ln |\theta| + \tan \theta + c$
- 3** **a** $\frac{d}{dx}(e^x \sin x) = e^x \sin x + e^x \cos x \quad \therefore \int e^x (\sin x + \cos x) \, dx = \int (e^x \sin x + e^x \cos x) \, dx$
 $= e^x \sin x + c$
- b** $\frac{d}{dx}(e^{-x} \sin x) = -e^{-x} \sin x + e^{-x} \cos x$
 $= \frac{\cos x - \sin x}{e^x} \quad \therefore \int \frac{\cos x - \sin x}{e^x} \, dx = e^{-x} \sin x + c$
- c** $\frac{d}{dx}(x \cos x) = \cos x + x(-\sin x)$
 $= \cos x - x \sin x \quad \therefore \int (\cos x - x \sin x) \, dx = x \cos x + c_1$
 $\therefore \int \cos x \, dx - \int x \sin x \, dx = x \cos x + c_1$
 $\therefore \sin x - \int x \sin x \, dx = x \cos x + c_1$
 $\therefore \int x \sin x \, dx = -x \cos x + \sin x + c$
- d** $\frac{1}{\cos x} = (\cos x)^{-1} \quad \therefore \frac{d}{dx}(\frac{1}{\cos x}) = -(\cos x)^{-2}(-\sin x)$
 $= \frac{\sin x}{\cos^2 x}$
 $= \frac{\sin x}{\cos x}$
 $= \frac{\tan x}{\cos x} \quad \therefore \int \frac{\tan x}{\sin x} \, dx = \frac{1}{\cos x} + c$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad f'(x) &= x^2 - 4 \cos x \\
 \therefore f(x) &= \int (x^2 - 4 \cos x) \, dx \\
 &= \frac{x^3}{3} - 4 \sin x + c
 \end{aligned}$$

$$\text{But } f(0) = 3$$

$$\therefore 0 - 4 \sin(0) + c = 3$$

$$\therefore c = 3$$

$$\therefore f(x) = \frac{x^3}{3} - 4 \sin x + 3$$

$$\begin{aligned}
 \mathbf{c} \quad f'(x) &= \sqrt{x} - \frac{2}{\cos^2 x} \\
 \therefore f(x) &= \int (x^{\frac{1}{2}} - \frac{2}{\cos^2 x}) \, dx \\
 &= \frac{2}{3} x^{\frac{3}{2}} - 2 \tan x + c
 \end{aligned}$$

$$\text{But } f(\pi) = 0$$

$$\therefore \frac{2}{3} \pi^{\frac{3}{2}} - 2 \tan \pi + c = 0$$

$$\therefore c = -\frac{2}{3} \pi^{\frac{3}{2}}$$

$$\therefore f(x) = \frac{2}{3} x^{\frac{3}{2}} - 2 \tan x - \frac{2}{3} \pi^{\frac{3}{2}}$$

$$\begin{aligned}
 \mathbf{b} \quad f'(x) &= 2 \cos x - 3 \sin x \\
 \therefore f(x) &= \int (2 \cos x - 3 \sin x) \, dx \\
 &= 2 \sin x + 3 \cos x + c
 \end{aligned}$$

$$\text{But } f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\therefore 2 \sin \frac{\pi}{4} + 3 \cos \frac{\pi}{4} + c = \frac{1}{\sqrt{2}}$$

$$\therefore 2\left(\frac{1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) + c = \frac{1}{\sqrt{2}}$$

$$\therefore c = -\frac{4}{\sqrt{2}}$$

$$\therefore c = -2\sqrt{2}$$

$$\therefore f(x) = 2 \sin x + 3 \cos x - 2\sqrt{2}$$

EXERCISE 27B

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad \int \sin(3x) \, dx \\
 = -\frac{1}{3} \cos(3x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \int 2 \cos(4x) \, dx \\
 = 2 \times \frac{1}{4} \sin(4x) + c \\
 = \frac{1}{2} \sin(4x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad \int \frac{1}{\cos^2(2x)} \, dx \\
 = \frac{1}{2} \tan(2x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad \int 3 \cos\left(\frac{x}{2}\right) \, dx \\
 = 6 \sin\left(\frac{x}{2}\right) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad \int (3 \sin(2x) - e^{-x}) \, dx \\
 = -\frac{3}{2} \cos(2x) + e^{-x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad \int \left[e^{2x} - \frac{2}{\sec^2\left(\frac{x}{2}\right)} \right] \, dx \\
 = \frac{1}{2} e^{2x} - 2 \times 2 \tan\left(\frac{x}{2}\right) + c \\
 = \frac{1}{2} e^{2x} - 4 \tan\left(\frac{x}{2}\right) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad \int 2 \sin\left(2x + \frac{\pi}{6}\right) \, dx \\
 = -\frac{2}{2} \cos\left(2x + \frac{\pi}{6}\right) + c \\
 = -\cos\left(2x + \frac{\pi}{6}\right) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad \int -3 \cos\left(\frac{\pi}{4} - x\right) \, dx \\
 = -3(-\sin\left(\frac{\pi}{4} - x\right)) + c \\
 = 3 \sin\left(\frac{\pi}{4} - x\right) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{i} \quad \int \frac{4}{\cos^2\left(\frac{\pi}{3} - 2x\right)} \, dx \\
 = 4 \times \left(-\frac{1}{2}\right) \tan\left(\frac{\pi}{3} - 2x\right) + c \\
 = -2 \tan\left(\frac{\pi}{3} - 2x\right) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{j} \quad \int \cos(2x) + \sin(2x) \, dx \quad \mathbf{k} \quad \int 2 \sin(3x) + 5 \cos(4x) \, dx \\
 = \frac{1}{2} \sin(2x) - \frac{1}{2} \cos(2x) + c \quad = -\frac{2}{3} \cos(3x) + \frac{5}{4} \sin(4x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{l} \quad \int \frac{1}{2} \cos(8x) - 3 \sin x \, dx \\
 = \frac{1}{2} \left(\frac{1}{8}\right) \sin(8x) + 3 \cos x + c \\
 = \frac{1}{16} \sin(8x) + 3 \cos x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \int \cos^2 x \, dx \\
 &= \int \left(\frac{1}{2} + \frac{1}{2} \cos(2x) \right) dx \\
 &= \frac{1}{2}x + \frac{1}{4} \sin(2x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int (1 + \cos^2(2x)) \, dx \\
 &= \int \left(1 + \frac{1}{2} + \frac{1}{2} \cos(4x) \right) dx \\
 &= \int \left(\frac{3}{2} + \frac{1}{2} \cos(4x) \right) dx \\
 &= \frac{3}{2}x + \frac{1}{8} \sin(4x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int \frac{1}{2} \cos^2(4x) \, dx \\
 &= \int \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2} \cos(8x) \right) dx \\
 &= \int \left(\frac{1}{4} + \frac{1}{4} \cos(8x) \right) dx \\
 &= \frac{1}{4}x + \frac{1}{32} \sin(8x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \sin^2 x \, dx \\
 &= \int \left(\frac{1}{2} - \frac{1}{2} \cos(2x) \right) dx \\
 &= \frac{1}{2}x - \frac{1}{4} \sin(2x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int (3 - \sin^2(3x)) \, dx \\
 &= \int \left(3 - \left(\frac{1}{2} - \frac{1}{2} \cos(6x) \right) \right) dx \\
 &= \int \left(\frac{5}{2} + \frac{1}{2} \cos(6x) \right) dx \\
 &= \frac{5}{2}x + \frac{1}{12} \sin(6x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int (1 + \cos x)^2 \, dx \\
 &= \int (1 + 2 \cos x + \cos^2 x) \, dx \\
 &= \int \left(1 + 2 \cos x + \frac{1}{2} + \frac{1}{2} \cos(2x) \right) dx \\
 &= \frac{3}{2}x + 2 \sin x + \frac{1}{4} \sin(2x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \cos^2 \theta &= \frac{1}{2} + \frac{1}{2} \cos(2\theta) \quad \therefore \quad \cos^4 \theta = \left(\frac{1}{2} + \frac{1}{2} \cos(2\theta) \right)^2 \\
 &= \frac{1}{4} + \frac{1}{4} \cos^2(2\theta) + \frac{1}{2} \cos(2\theta) \\
 &= \frac{1}{4} + \frac{1}{4} \left(\frac{1}{2} + \frac{1}{2} \cos(4\theta) \right) + \frac{1}{2} \cos(2\theta) \\
 &= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} \cos(4\theta) + \frac{1}{2} \cos(2\theta) \\
 &= \frac{1}{8} \cos(4\theta) + \frac{1}{2} \cos(2\theta) + \frac{3}{8} \quad \text{as required}
 \end{aligned}$$

$$\therefore \int \cos^4 x \, dx = \int \left(\frac{1}{8} \cos(4x) + \frac{1}{2} \cos(2x) + \frac{3}{8} \right) dx = \frac{1}{32} \sin(4x) + \frac{1}{4} \sin(2x) + \frac{3}{8}x + c$$

$$\mathbf{4} \quad \mathbf{a} \quad \text{Consider } \int \sin^4 x \cos x \, dx$$

$$\text{Let } u = \sin x, \quad \frac{du}{dx} = \cos x$$

$$\therefore \int \sin^4 x \cos x \, dx$$

$$= \int u^4 \frac{du}{dx} \, dx$$

$$= \int u^4 \, du$$

$$= \frac{u^5}{5} + c$$

$$= \frac{1}{5} \sin^5 x + c$$

$$\mathbf{b} \quad \text{Consider } \int \frac{\sin x}{\sqrt{\cos x}} \, dx$$

$$\text{Let } u = \cos x, \quad \frac{du}{dx} = -\sin x$$

$$\therefore \int \frac{\sin x}{\sqrt{\cos x}} \, dx$$

$$= \int u^{-\frac{1}{2}} \left(-\frac{du}{dx} \right) dx$$

$$= \int -u^{-\frac{1}{2}} \, du$$

$$= -\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c \quad \text{or} \quad -2\sqrt{\cos x} + c$$

$$\mathbf{c} \quad \text{Consider } \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

$$\text{Let } u = \cos x, \quad \frac{du}{dx} = -\sin x$$

$$\therefore \int \frac{\sin x}{\cos x} \, dx$$

$$= \int \frac{1}{u} \left(-\frac{du}{dx} \right) dx$$

$$= \int -\frac{1}{u} \, du$$

$$= -\ln|u| + c$$

$$= -\ln|\cos x| + c$$

$$\mathbf{d} \quad \text{Consider } \int \sqrt{\sin x} \cos x \, dx$$

$$\text{Let } u = \sin x, \quad \frac{du}{dx} = \cos x$$

$$\therefore \int \sqrt{\sin x} \cos x \, dx$$

$$= \int u^{\frac{1}{2}} \frac{du}{dx} \, dx$$

$$= \int u^{\frac{1}{2}} \, du$$

$$= \frac{2}{3} u^{\frac{3}{2}} + c$$

$$= \frac{2}{3} (\sin x)^{\frac{3}{2}} + c$$

e Consider $\int \frac{\cos x}{(2 + \sin x)^2} dx$

Let $u = 2 + \sin x$, $\frac{du}{dx} = \cos x$

$$\therefore \int \frac{\cos x}{(2 + \sin x)^2} dx$$

$$= \int u^{-2} \frac{du}{dx} dx$$

$$= \int u^{-2} du$$

$$= -u^{-1} + c$$

$$= \frac{-1}{2 + \sin x} + c$$

f Consider $\int \frac{\sin x}{\cos^3 x} dx$

Let $u = \cos x$, $\frac{du}{dx} = -\sin x$

$$\therefore \int \frac{\sin x}{\cos^3 x} dx = \int u^{-3} \left(-\frac{du}{dx}\right) dx$$

$$= \int -u^{-3} du$$

$$= \frac{-u^{-2}}{-2} + c$$

$$= \frac{1}{2}u^{-2} + c$$

$$= \frac{1}{2}(\cos x)^{-2} + c$$

$$= \frac{1}{2\cos^2 x} + c$$

g Consider $\int \frac{\sin x}{1 - \cos x} dx$

Let $u = 1 - \cos x$, $\frac{du}{dx} = \sin x$

$$\therefore \int \frac{\sin x}{1 - \cos x} dx$$

$$= \int \frac{1}{u} \frac{du}{dx} dx$$

$$= \int \frac{1}{u} du$$

$$= \ln |u| + c$$

$$= \ln |1 - \cos x| + c$$

h Consider $\int \frac{\cos(2x)}{\sin(2x) - 3} dx$

Let $u = \sin(2x) - 3$, $\frac{du}{dx} = 2\cos(2x)$

$$\therefore \cos(2x) = \frac{1}{2} \frac{du}{dx}$$

$$\therefore \int \frac{\cos(2x)}{\sin(2x) - 3} dx$$

$$= \int \frac{1}{u} \left(\frac{1}{2} \frac{du}{dx}\right) dx$$

$$= \frac{1}{2} \int \frac{1}{u} du$$

$$= \frac{1}{2} \ln |u| + c$$

$$= \frac{1}{2} \ln |\sin(2x) - 3| + c$$

5 a $\int \cos^3 x dx = \int \cos^2 x \cos x dx$
 $= \int (1 - \sin^2 x) \cos x dx$

Let $u = \sin x$, $\frac{du}{dx} = \cos x$

$$\therefore \int \cos^3 x dx = \int (1 - u^2) \frac{du}{dx} dx$$

$$= \int (1 - u^2) du$$

$$= u - \frac{u^3}{3} + c$$

$$= \sin x - \frac{\sin^3 x}{3} + c$$

$$= \sin x - \frac{1}{3} \sin^3 x + c$$

b $\int \sin^5 x dx$
 $= \int \sin^4 x \sin x dx$
 $= \int (1 - \cos^2 x)^2 \sin x dx$
 $= \int (1 - 2\cos^2 x + \cos^4 x) \sin x dx$

Let $u = \cos x$, $\frac{du}{dx} = -\sin x$

$$\therefore \int \sin^5 x dx = \int (1 - 2u^2 + u^4) \left(-\frac{du}{dx}\right) dx$$

$$= \int (-1 + 2u^2 - u^4) du$$

$$= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + c$$

$$= -\cos x + \frac{2}{3} \cos^3 x - \frac{1}{5} \cos^5 x + c$$

$$\begin{aligned}
 \text{c} \quad & \int \sin^4 x \cos^3 x \, dx & \therefore \int \sin^4 x \cos^3 x \, dx &= \int (u^4 - u^6) \frac{du}{dx} \, dx \\
 &= \int \sin^4 x \cos^2 x \cos x \, dx & &= \int (u^4 - u^6) \, du \\
 &= \int \sin^4 x (1 - \sin^2 x) \cos x \, dx & &= \frac{1}{5} u^5 - \frac{1}{7} u^7 + c \\
 &= \int (\sin^4 x - \sin^6 x) \cos x \, dx & &= \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + c
 \end{aligned}$$

$$\text{Let } u = \sin x, \quad \frac{du}{dx} = \cos x$$

$$\text{6 a} \quad \text{Let } u = \cos x, \quad \frac{du}{dx} = -\sin x$$

$$\begin{aligned}
 \therefore f(x) &= \int \sin x e^{\cos x} \, dx \\
 &= \int e^{\cos x} \sin x \, dx \\
 &= \int e^u \left(-\frac{du}{dx} \right) dx \\
 &= -\int e^u \, du \\
 &= -e^u + c \\
 &= -e^{\cos x} + c
 \end{aligned}$$

$$\text{c} \quad \text{Let } u = \sin x - \cos x,$$

$$\frac{du}{dx} = \cos x + \sin x$$

$$\begin{aligned}
 \therefore f(x) &= \int \frac{\sin x + \cos x}{\sin x - \cos x} \, dx \\
 &= \int \frac{1}{u} \frac{du}{dx} \, dx \\
 &= \int \frac{1}{u} \, du \\
 &= \ln |u| + c \\
 &= \ln |\sin x - \cos x| + c
 \end{aligned}$$

$$\text{b} \quad \text{Let } u = \sin(2x), \quad \frac{du}{dx} = 2 \cos(2x)$$

$$\begin{aligned}
 \therefore f(x) &= \int \sin^3(2x) \cos(2x) \, dx \\
 &= \int u^3 \left(\frac{1}{2} \frac{du}{dx} \right) dx \\
 &= \frac{1}{2} \int u^3 \, du \\
 &= \frac{1}{2} \times \frac{u^4}{4} + c \\
 &= \frac{1}{8} \sin^4(2x) + c
 \end{aligned}$$

$$\text{d} \quad \text{Let } u = \tan x$$

$$\therefore \frac{du}{dx} = \frac{1}{\cos^2 x}$$

$$\begin{aligned}
 \text{So,} \quad & \int \frac{e^{\tan x}}{\cos^2 x} \, dx \\
 &= \int e^u \left(\frac{du}{dx} \right) dx \\
 &= \int e^u \, du \\
 &= e^u + c \\
 &= e^{\tan x} + c
 \end{aligned}$$

$$\text{7 a} \quad \int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$$

$$\text{We let } u = \sin x, \quad \text{so } \frac{du}{dx} = \cos x$$

$$\begin{aligned}
 \therefore \int \cot x \, dx &= \int \frac{1}{u} \frac{du}{dx} \, dx \\
 &= \int \frac{1}{u} \, du \\
 &= \ln |u| + c \\
 &= \ln |\sin x| + c
 \end{aligned}$$

$$\text{b} \quad \int \cot(3x) \, dx = \int \frac{\cos(3x)}{\sin(3x)} \, dx$$

$$\text{We let } u = \sin(3x),$$

$$\text{so } \frac{du}{dx} = 3 \cos(3x)$$

$$\begin{aligned}
 \therefore \int \cot(3x) \, dx &= \int \frac{\frac{1}{3} \times 3 \cos(3x)}{\sin(3x)} \, dx \\
 &= \frac{1}{3} \int \frac{1}{u} \frac{du}{dx} \, dx \\
 &= \frac{1}{3} \int \frac{1}{u} \, du \\
 &= \frac{1}{3} \ln |u| + c \\
 &= \frac{1}{3} \ln |\sin(3x)| + c
 \end{aligned}$$

c We let $u = \cot x$

$$\text{so } \frac{du}{dx} = -\csc^2 x$$

$$\begin{aligned}\therefore \int \csc^2 x \, dx &= \int \left(-\frac{du}{dx}\right) dx \\ &= -\int du \\ &= -u + c \\ &= -\cot x + c\end{aligned}$$

d We let $u = \sec x$

$$\text{so } \frac{du}{dx} = \sec x \tan x$$

$$\begin{aligned}\therefore \int \sec x \tan x \, dx &= \int \frac{du}{dx} dx \\ &= \int du \\ &= u + c \\ &= \sec x + c\end{aligned}$$

e We let $u = \csc x$,

$$\text{so } \frac{du}{dx} = -\csc x \cot x$$

$$\begin{aligned}\therefore \int \csc x \cot x \, dx &= \int \left(-\frac{du}{dx}\right) dx \\ &= -\int du \\ &= -u + c \\ &= -\csc x + c\end{aligned}$$

f We let $u = \sec(3x)$,

$$\text{so } \frac{du}{dx} = 3 \sec(3x) \tan(3x)$$

$$\begin{aligned}\therefore \int \tan(3x) \sec(3x) \, dx &= \int \frac{1}{3} \times 3 \sec(3x) \tan(3x) \, dx \\ &= \frac{1}{3} \int \frac{du}{dx} dx \\ &= \frac{1}{3} \int du \\ &= \frac{1}{3} u + c \\ &= \frac{1}{3} \sec(3x) + c\end{aligned}$$

g We let $u = \csc\left(\frac{x}{2}\right)$,

$$\text{so } \frac{du}{dx} = -\frac{1}{2} \csc\left(\frac{x}{2}\right) \cot\left(\frac{x}{2}\right)$$

$$\begin{aligned}\therefore \int \csc\left(\frac{x}{2}\right) \cot\left(\frac{x}{2}\right) \, dx &= \int (-2) \times \left(-\frac{1}{2} \csc\left(\frac{x}{2}\right) \cot\left(\frac{x}{2}\right)\right) dx \\ &= -2 \int \frac{du}{dx} dx \\ &= -2 \int du \\ &= -2u + c \\ &= -2 \csc\left(\frac{x}{2}\right) + c\end{aligned}$$

h $\int \sec^3 x \sin x \, dx = \int (\cos x)^{-3} \sin x \, dx$ We let $u = \cos x$

$$\text{so } \frac{du}{dx} = -\sin x$$

$$\begin{aligned}\therefore \int \sec^3 x \sin x \, dx &= \int u^{-3} \left(-\frac{du}{dx}\right) dx \\ &= -\int u^{-3} du \\ &= \frac{1}{2} u^{-2} + c \\ &= \frac{1}{2 \cos^2 x} + c\end{aligned}$$

i We let $u = \cot x$

$$\text{so } \frac{du}{dx} = -\csc^2 x$$

$$\begin{aligned}\therefore \int \frac{\csc^2 x}{\sqrt{\cot x}} \, dx &= \int \frac{1}{\sqrt{u}} \left(-\frac{du}{dx}\right) dx \\ &= -\int u^{-\frac{1}{2}} du \\ &= -2u^{\frac{1}{2}} + c \\ &= -2\sqrt{\cot x} + c\end{aligned}$$

EXERCISE 27C

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \int_0^{\frac{\pi}{6}} \cos x \, dx \\
 &= [\sin x]_0^{\frac{\pi}{6}} \\
 &= \sin \frac{\pi}{6} - \sin 0 \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \, dx \\
 &= [-\cos x]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\
 &= -\cos \frac{\pi}{2} + \cos \frac{\pi}{3} \\
 &= \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x} \, dx \\
 &= [\tan x]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\
 &= \tan \frac{\pi}{3} - \tan \frac{\pi}{4} \\
 &= \sqrt{3} - 1
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int_0^{\frac{\pi}{6}} \sin(3x) \, dx \\
 &= \left[-\frac{1}{3} \cos(3x)\right]_0^{\frac{\pi}{6}} \\
 &= -\frac{1}{3} \left[\cos \frac{\pi}{2} - \cos 0\right] \\
 &= -\frac{1}{3} [0 - 1] \\
 &= \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int_0^{\frac{\pi}{4}} \cos^2 x \, dx \\
 &= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2} \cos(2x)\right) dx \\
 &= \left[\frac{x}{2} + \frac{1}{4} \sin(2x)\right]_0^{\frac{\pi}{4}} \\
 &= \left[\frac{\pi}{8} + \frac{1}{4} \sin \frac{\pi}{2}\right] - 0 \\
 &= \frac{\pi}{8} + \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int_0^{\frac{\pi}{2}} \sin^2 x \, dx \\
 &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2} \cos(2x)\right) dx \\
 &= \left[\frac{x}{2} - \frac{1}{4} \sin(2x)\right]_0^{\frac{\pi}{2}} \\
 &= \left[\frac{\pi}{4} - \frac{1}{4} \sin \pi\right] - 0 \\
 &= \frac{\pi}{4}
 \end{aligned}$$

$$\mathbf{2} \quad \mathbf{a} \quad \text{Let } u = \cos x, \quad \frac{du}{dx} = -\sin x$$

$$\text{When } x = 0, \quad u = \cos 0 = 1$$

$$\text{When } x = \frac{\pi}{3}, \quad u = \cos \frac{\pi}{3} = \frac{1}{2}$$

$$\begin{aligned}
 \therefore \int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}} \, dx &= \int_1^{\frac{1}{2}} u^{-\frac{1}{2}} \left(-\frac{du}{dx}\right) dx \\
 &= \int_{\frac{1}{2}}^1 u^{-\frac{1}{2}} du \\
 &= \left[2u^{\frac{1}{2}}\right]_{\frac{1}{2}}^1 \\
 &= 2\sqrt{1} - 2\sqrt{\frac{1}{2}} \\
 &= 2 - \sqrt{2}
 \end{aligned}$$

$$\mathbf{b} \quad \text{Let } u = \sin x, \quad \frac{du}{dx} = \cos x$$

$$\text{When } x = 0, \quad u = \sin 0 = 0$$

$$\text{When } x = \frac{\pi}{6}, \quad u = \sin \frac{\pi}{6} = \frac{1}{2}$$

$$\begin{aligned}
 \therefore \int_0^{\frac{\pi}{6}} \sin^2 x \cos x \, dx &= \int_0^{\frac{1}{2}} u^2 \frac{du}{dx} dx \\
 &= \int_0^{\frac{1}{2}} u^2 du \\
 &= \left[\frac{u^3}{3}\right]_0^{\frac{1}{2}} \\
 &= \frac{1}{3} \left(\frac{1}{2}\right)^3 \quad \text{or} \quad \frac{1}{24}
 \end{aligned}$$

$$\mathbf{c} \quad \text{Let } u = \cos x \quad \therefore \frac{du}{dx} = -\sin x$$

$$\text{When } x = 0, \quad u = \cos 0 = 1$$

$$\text{When } x = \frac{\pi}{4}, \quad u = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\begin{aligned}
 \int_0^{\frac{\pi}{4}} \tan x \, dx &= \int_1^{\frac{1}{\sqrt{2}}} \frac{\sin x}{\cos x} \, dx \\
 &= \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u} \left(-\frac{du}{dx}\right) dx \\
 &= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u} \, du \\
 &= [\ln |u|]_{\frac{1}{\sqrt{2}}}^1 \\
 &= \ln 1 - \ln \frac{1}{\sqrt{2}} \\
 &= \ln \sqrt{2} \\
 &= \frac{1}{2} \ln 2
 \end{aligned}$$

$$\mathbf{d} \quad \text{Let } u = \sin x, \quad \frac{du}{dx} = \cos x$$

$$\text{When } x = \frac{\pi}{6}, \quad u = \sin \frac{\pi}{6} = \frac{1}{2}$$

$$\text{When } x = \frac{\pi}{2}, \quad u = \sin \frac{\pi}{2} = 1$$

$$\begin{aligned}
 \therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{\tan x} \, dx &= \int_{\frac{1}{2}}^1 \frac{\cos x}{\sin x} \, dx \\
 &= \int_{\frac{1}{2}}^1 \frac{1}{u} \frac{du}{dx} \, dx \\
 &= \int_{\frac{1}{2}}^1 \frac{1}{u} \, du \\
 &= [\ln |u|]_{\frac{1}{2}}^1 \\
 &= \ln 1 - \ln \frac{1}{2} \\
 &= 0 + \ln 2 \quad \{\ln \left(\frac{1}{a}\right) = -\ln a\} \\
 &= \ln 2
 \end{aligned}$$

e Let $u = 1 - \sin x$, $\frac{du}{dx} = -\cos x$

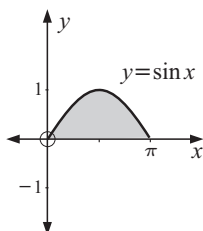
When $x = 0$, $u = 1 - \sin 0 = 1$

When $x = \frac{\pi}{6}$, $u = 1 - \sin \frac{\pi}{6} = \frac{1}{2}$

$$\begin{aligned} \therefore \int_0^{\frac{\pi}{6}} \frac{\cos x}{1 - \sin x} dx &= \int_1^{\frac{1}{2}} \frac{1}{u} \left(-\frac{du}{dx} \right) dx \\ &= \int_{\frac{1}{2}}^1 \frac{1}{u} du \\ &= [\ln |u|]_{\frac{1}{2}}^1 \\ &= \ln 1 - \ln \frac{1}{2} \\ &= \ln 2 \end{aligned}$$

EXERCISE 27D

1 a

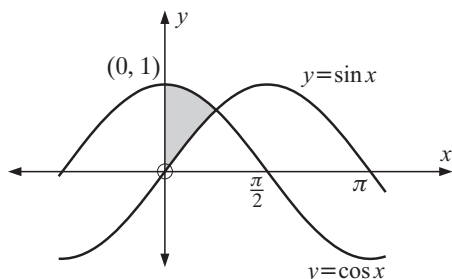


$$\begin{aligned} A &= \int_0^{\pi} \sin x \, dx \\ &= [-\cos x]_0^{\pi} \\ &= [-\cos \pi + \cos 0] \\ &= -(-1) + 1 \\ &= 2 \text{ units}^2 \end{aligned}$$

b We first note that $\sin^2 x \geq 0$ always, so the function never drops below the x -axis.

$$\begin{aligned} \therefore A &= \int_0^{\pi} \sin^2 x \, dx \\ &= \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2} \cos(2x) \right) dx \\ &= \left[\frac{x}{2} - \frac{1}{4} \sin(2x) \right]_0^{\pi} \\ &= \left[\frac{\pi}{2} - \frac{1}{4} \sin(2\pi) \right] - \left[0 - \frac{1}{4} \sin 0 \right] \\ &= \frac{\pi}{2} \text{ units}^2 \end{aligned}$$

2



The curves $y = \cos x$ and $y = \sin x$ meet when $x = \frac{\pi}{4}$.

$$\begin{aligned} \therefore A &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x) \, dx \\ &= [\sin x + \cos x]_0^{\frac{\pi}{4}} \\ &= \left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (\sin 0 + \cos 0) \\ &= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - (0 + 1) \\ &= (\sqrt{2} - 1) \text{ units}^2 \end{aligned}$$

3 a $y = \tan x$

A has y -coordinate 1 and lies on the graph of $y = \tan x$ on the interval $[0, \frac{\pi}{2}]$.

$\therefore \tan x = 1 \quad \therefore x = \frac{\pi}{4}$ i.e., A is at $(\frac{\pi}{4}, 1)$

b $\tan x = \frac{\sin x}{\cos x}$

Let $u = \cos x$, $\frac{du}{dx} = -\sin x$

When $x = 0$, $u = \cos 0 = 1$

When $x = \frac{\pi}{4}$, $u = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\begin{aligned} \therefore \text{area} &= \int_0^{\frac{\pi}{4}} \tan x \, dx \\ &= \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u} \left(-\frac{du}{dx} \right) dx \\ &= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u} du \\ &= [\ln |u|]_{\frac{1}{\sqrt{2}}}^1 \\ &= \ln 1 - \ln \frac{1}{\sqrt{2}} \\ &= \ln \sqrt{2} \\ &= \frac{1}{2} \ln 2 \text{ units}^2 \end{aligned}$$

- 4 a** $y = \sin(2x)$ is the curve C_1
 $y = \sin x$ is the curve C_2

- b** The curves meet when $\sin(2x) = \sin x$
 $\therefore 2 \sin x \cos x - \sin x = 0$
 $\therefore \sin x(2 \cos x - 1) = 0$
 i.e., $\sin x = 0$ or $\cos x = \frac{1}{2}$
 $\therefore x$ coordinate of A = $\frac{\pi}{3}$
 $\therefore A$ is at $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$

$$\begin{aligned} \text{c Area} &= \int_0^{\frac{\pi}{3}} (\sin(2x) - \sin x) dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin(2x)) dx \\ &= \left[-\frac{1}{2} \cos(2x) + \cos x\right]_0^{\frac{\pi}{3}} + \left[-\cos x + \frac{1}{2} \cos(2x)\right]_{\frac{\pi}{3}}^{\pi} \\ &= \left(-\frac{1}{2} \cos \frac{2\pi}{3} + \cos \frac{\pi}{3}\right) - \left(-\frac{1}{2} \cos 0 + \cos 0\right) + \left(-\cos \pi + \frac{1}{2} \cos 2\pi\right) \\ &\quad - \left(-\cos \frac{\pi}{3} + \frac{1}{2} \cos \frac{2\pi}{3}\right) \\ &= \left(\frac{1}{4} + \frac{1}{2}\right) - \left(-\frac{1}{2} + 1\right) + \left(1 + \frac{1}{2}\right) - \left(-\frac{1}{2} - \frac{1}{4}\right) \\ &= 2\frac{1}{2} \text{ units}^2 \end{aligned}$$

- 5 a** $y = \cos(2x)$ is the curve C_2 and $y = \cos^2 x$ is the curve C_1

- b** When $x = 0$, $y = \cos 0 = 1$. $\therefore A$ is at $(0, 1)$
 When $x = \frac{\pi}{4}$, $y = \cos(\frac{\pi}{2}) = 0$. $\therefore B$ is at $(\frac{\pi}{4}, 0)$
 Also, $\cos^2(\frac{\pi}{2}) = 0$. $\therefore C$ is at $(\frac{\pi}{2}, 0)$
 When $x = \frac{3\pi}{4}$, $y = \cos(\frac{3\pi}{2}) = 0$. $\therefore D$ is at $(\frac{3\pi}{4}, 0)$
 $\cos(2\pi) = \cos^2 \pi = 1 \therefore y = \cos(2x)$ and $y = \cos^2 x = 1$ when $x = \pi \therefore E$ is at $(\pi, 1)$

$$\begin{aligned} \text{c } A &= \int_0^{\pi} (\cos^2 x - \cos(2x)) dx \\ &= \int_0^{\pi} \frac{1}{2} + \frac{1}{2} \cos(2x) - \cos(2x) dx \\ &= \int_0^{\pi} \frac{1}{2} - \frac{1}{2} \cos(2x) dx \\ &= \left[\frac{x}{2} - \frac{1}{4} \sin(2x)\right]_0^{\pi} = \left(\frac{\pi}{2} - 0\right) - (0 - 0) = \frac{\pi}{2} \text{ units}^2 \end{aligned}$$

REVIEW SET 27A

1 a $\int \sin^7 x \cos x dx$
 $= \int u^7 \left(\frac{du}{dx}\right) dx$ on letting $u = \sin x$
 $= \int u^7 du$
 $= \frac{u^8}{8} + c$
 $= \frac{1}{8} \sin^8 x + c$

b Consider $\int \tan(2x) dx = \int \frac{\sin(2x)}{\cos(2x)} dx$
 Let $u = \cos(2x)$, $\frac{du}{dx} = -2 \sin(2x)$
 $\therefore \sin(2x) = -\frac{1}{2} \frac{du}{dx}$
 $\therefore \int \tan(2x) dx = \int \frac{1}{u} \left(-\frac{1}{2} \frac{du}{dx}\right) dx$

c Consider $\int e^{\sin x} \cos x dx$
 Let $u = \sin x$, $\frac{du}{dx} = \cos x$
 $\therefore \int e^{\sin x} \cos x dx = \int e^u \frac{du}{dx} dx$
 $= \int e^u du$
 $= e^u + c$
 $= e^{\sin x} + c$

$$\begin{aligned} &= -\frac{1}{2} \int \frac{1}{u} du \\ &= -\frac{1}{2} \ln |u| + c \\ &= -\frac{1}{2} \ln |\cos(2x)| + c \end{aligned}$$

2 If $y = x \tan x$ then $\therefore \int \frac{\sin x}{\cos x} dx + \int \frac{x}{\cos^2 x} dx = x \tan x + c_1$

$$\frac{dy}{dx} = 1 \tan x + x \left(\frac{1}{\cos^2 x} \right) \quad \therefore \int \frac{1}{u} \left(-\frac{du}{dx} \right) dx + \int \frac{x}{\cos^2 x} dx = x \tan x + c_1$$

$$= \frac{\sin x}{\cos x} + \frac{x}{\cos^2 x}$$

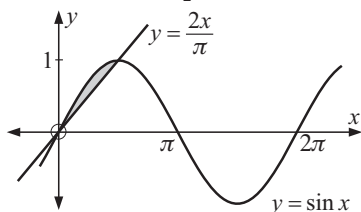
$$\{\text{letting } u = \cos x, \therefore \frac{du}{dx} = -\sin x\}$$

$$\therefore -\int \frac{1}{u} du + \int \frac{x}{\cos^2 x} dx = x \tan x + c_1$$

$$\therefore \int \frac{x}{\cos^2 x} dx = x \tan x + \ln |u| + c$$

$$= x \tan x + \ln |\cos x| + c$$

- 3** Using technology, the graphs meet
when $x = 0$ and $x = \frac{\pi}{2}$.



$\therefore$ enclosed area

$$= \int_0^{\frac{\pi}{2}} \left(\sin x - \frac{2x}{\pi} \right) dx$$

$$= \left[-\cos x - \frac{x^2}{\pi} \right]_0^{\frac{\pi}{2}}$$

$$= \left(0 - \frac{\pi}{4} \right) - (-1 - 0)$$

$$= \left(1 - \frac{\pi}{4} \right) \text{ units}^2$$

4 a $\int_0^{\frac{\pi}{3}} \cos^2 \left(\frac{x}{2} \right) dx = \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} + \frac{1}{2} \cos x \right) dx$

$$= \left[\frac{x}{2} + \frac{1}{2} \sin x \right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{6} + \frac{1}{2} \sin \frac{\pi}{3} - 0 - \frac{1}{2} \sin 0$$

$$= \frac{\pi}{6} + \frac{\sqrt{3}}{4}$$

b $\int_0^{\frac{\pi}{4}} \tan x dx = \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx \quad \therefore \int_0^{\frac{\pi}{4}} \tan x dx = \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u} \left(-\frac{du}{dx} \right) dx$

Let $u = \cos x, \quad \frac{du}{dx} = -\sin x$

When $x = 0, \quad u = \cos 0 = 1$

When $x = \frac{\pi}{4}, \quad u = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{u} du$$

$$= [\ln |u|]_{\frac{1}{\sqrt{2}}}^1$$

$$= \ln 1 - \ln \frac{1}{\sqrt{2}}$$

$$= \ln \sqrt{2}$$

$$= \frac{1}{2} \ln 2$$

5 $y = \ln \left(\frac{1}{\cos x} \right)$

$$= -\ln(\cos x)$$

$$\therefore \int \tan x dx = \ln \left(\frac{1}{\cos x} \right) + c$$

$$= -\ln(\cos x) + c$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\cos x} \times (-\sin x)$$

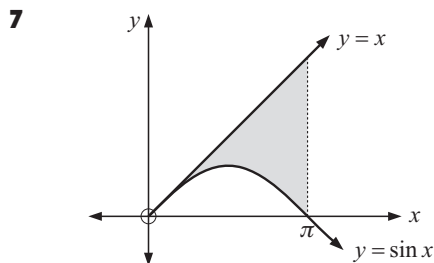
$$\{\text{since } \frac{1}{\cos x} > 0\}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x$$

$$\text{or } \int \tan x dx = \ln(\sec x) + c$$

$$\begin{aligned}
 \mathbf{6} \quad & \text{Let } u = \sin \theta \quad \therefore \frac{du}{d\theta} = \cos \theta & \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos \theta}{\sin \theta} d\theta \\
 & u\left(\frac{\pi}{6}\right) = \sin \frac{\pi}{6} = \frac{1}{2} & = \int_{\frac{1}{2}}^1 \frac{1}{u} \frac{du}{d\theta} d\theta \\
 & u\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1 & = \int_{\frac{1}{2}}^1 \frac{1}{u} du \\
 & & = [\ln |u|]_{\frac{1}{2}}^1 \\
 & & = \ln 1 - \ln \left(\frac{1}{2}\right) \\
 & & = \ln 2
 \end{aligned}$$



$$\begin{aligned}
 & \text{Required area} \\
 & = \text{area of } \triangle - \text{area under sine curve} \\
 & = \frac{1}{2} \pi \times \pi - \int_0^{\pi} \sin x \, dx \\
 & = \frac{\pi^2}{2} - [-\cos x]_0^{\pi} \\
 & = \frac{\pi^2}{2} - [-\cos \pi + \cos 0] \\
 & = \frac{\pi^2}{2} - 2 \text{ units}^2
 \end{aligned}$$

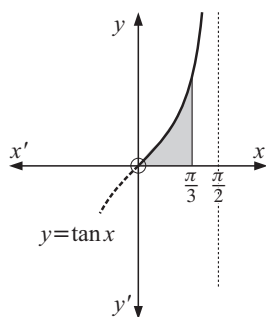
REVIEW SET 27B

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \int 4 \sin^2 \left(\frac{x}{2}\right) dx \\
 & = \int 2 \left(2 \sin^2 \left(\frac{x}{2}\right)\right) dx \\
 \text{Now } \cos(2\theta) & = 1 - 2 \sin^2 \theta \\
 \therefore 2 \sin^2 \theta & = 1 - \cos(2\theta) \\
 \therefore \int 4 \sin^2 \left(\frac{x}{2}\right) dx & = 2 \int (1 - \cos x) dx \\
 & = 2[x - \sin x] + c \\
 & = 2x - 2 \sin x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int (2 - \cos x)^2 dx \\
 & = \int (4 - 4 \cos x + \cos^2 x) dx \\
 & = \int 4 - 4 \cos x + \frac{1}{2} + \frac{1}{2} \cos(2x) dx \\
 & = \frac{9}{2}x - 4 \sin x + \frac{1}{2} \left(\frac{1}{2}\right) \sin(2x) dx \\
 & = \frac{9}{2}x - 4 \sin x + \frac{1}{4} \sin(2x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad & \text{If } u = \cos x, \quad \frac{du}{dx} = -\sin x \quad \therefore \int \frac{\sin x}{\cos^4 x} dx \\
 & = \int u^{-4} \left(-\frac{du}{dx}\right) dx \\
 & = - \int u^{-4} du \\
 & = \frac{u^{-3}}{3} + c \\
 & = \frac{1}{3 \cos^3 x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad & \frac{d}{dx}(\sin(x^2)) \\
 & = 2x \cos(x^2) \\
 \therefore \int x \cos(x^2) dx & = \frac{1}{2} \int 2x \cos(x^2) dx \\
 & = \frac{1}{2} \sin(x^2) + c
 \end{aligned}$$

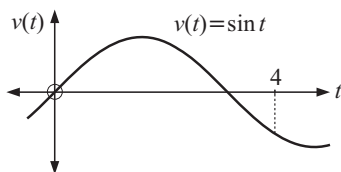
4Now $\tan x \geq 0$ for $0 \leq x \leq \frac{\pi}{3}$

$$\begin{aligned}
 \therefore A &= \int_0^{\frac{\pi}{3}} \tan x \, dx \\
 &= \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} \, dx \\
 &= \int_1^{\frac{1}{2}} \frac{1}{u} \left(-\frac{du}{dx} \right) dx \\
 &= \int_{\frac{1}{2}}^1 \frac{1}{u} \, du = [\ln |u|]_{\frac{1}{2}}^1 \\
 &= \ln 1 - \ln \frac{1}{2} \\
 &= \ln 2 \text{ units}^2
 \end{aligned}$$

Let $u = \cos x$, $\frac{du}{dx} = -\sin x$

When $x = \frac{\pi}{3}$, $u = \cos \frac{\pi}{3} = \frac{1}{2}$

When $x = 0$, $u = \cos 0 = 1$

5

Since $v(t) = \sin t \text{ ms}^{-1}$, $v(t) = 0$ when $t = \pi$ seconds.

We may therefore see from the graph that the particle reverses its direction at time $t = \pi$.

$$\begin{aligned}
 \therefore \text{the distance travelled is } D &= \int_0^{\pi} \sin t \, dt + \int_{\pi}^4 (-\sin t) \, dt \\
 &= [-\cos t]_0^{\pi} + [\cos t]_{\pi}^4 \\
 &= -\cos \pi + \cos 0 + \cos 4 - \cos \pi \\
 &= 3 + \cos 4 \\
 &\doteq 2.35 \text{ m}
 \end{aligned}$$

6

a

$$\begin{aligned}
 &\int_0^{\frac{\pi}{6}} \sin^2 \left(\frac{x}{2} \right) dx \\
 &= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \frac{1}{2} \cos x \right) dx \\
 &= \left[\frac{x}{2} - \frac{1}{2} \sin x \right]_0^{\frac{\pi}{6}} \\
 &= \left(\frac{\pi}{12} - \frac{1}{4} \right) - (0 - 0) \\
 &= \frac{\pi}{12} - \frac{1}{4}
 \end{aligned}$$

b

Let $u = \tan x$, $\frac{du}{dx} = \sec^2 x = \frac{1}{\cos^2 x}$

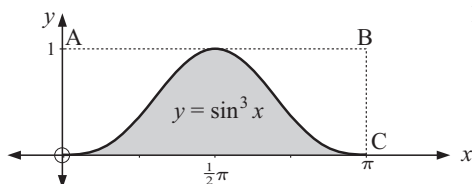
When $x = \frac{\pi}{4}$, $u = \tan \frac{\pi}{4} = 1$

When $x = \frac{\pi}{3}$, $u = \tan \frac{\pi}{3} = \sqrt{3}$

$$\begin{aligned}
 \therefore \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x \tan x} dx \\
 &= \int_1^{\sqrt{3}} \frac{1}{u} \frac{du}{dx} dx \\
 &= \int_1^{\sqrt{3}} \frac{1}{u} du = [\ln |u|]_1^{\sqrt{3}} \\
 &= \ln \sqrt{3} - \ln 1 \\
 &= \frac{1}{2} \ln 3
 \end{aligned}$$

7

Consider the graph of $y = \sin^3 x$, $0 \leq x \leq \pi$



Now $\int_0^{\pi} \sin^3 x \, dx = \text{shaded area}$

But the shaded area < area of rectangle ABCO

i.e., $\int_0^{\pi} \sin^3 x \, dx < \pi$

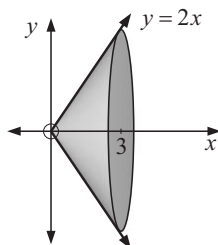
$\therefore \int_0^{\pi} \sin^3 x \, dx < 4$

Chapter 28

VOLUMES OF REVOLUTION

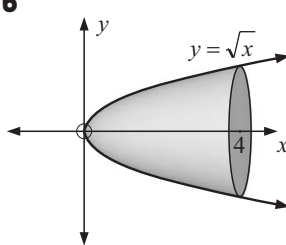
EXERCISE 28A.1

1 a



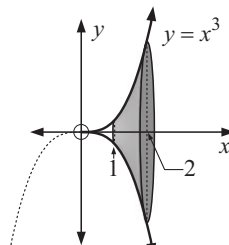
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^3 (2x)^2 dx \\
 &= 4\pi \int_0^3 x^2 dx \\
 &= 4\pi \left[\frac{x^3}{3} \right]_0^3 \\
 &= 4\pi(9 - 0) \\
 &= 36\pi \text{ units}^3
 \end{aligned}$$

b



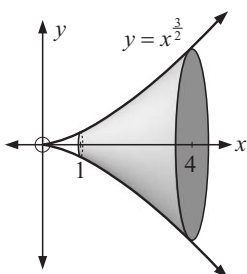
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^4 (\sqrt{x})^2 dx \\
 &= \pi \int_0^4 x dx \\
 &= \pi \left[\frac{x^2}{2} \right]_0^4 \\
 &= \pi(8 - 0) \\
 &= 8\pi \text{ units}^3
 \end{aligned}$$

c



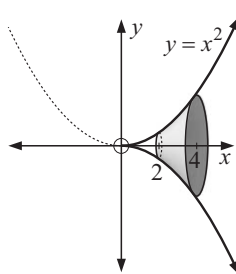
$$\begin{aligned}
 \text{Volume} &= \pi \int_1^2 (x^3)^2 dx \\
 &= \pi \int_1^2 x^6 dx \\
 &= \pi \left[\frac{x^7}{7} \right]_1^2 \\
 &= \pi \left(\frac{128}{7} - \frac{1}{7} \right) \\
 &= \frac{127\pi}{7} \text{ units}^3
 \end{aligned}$$

d



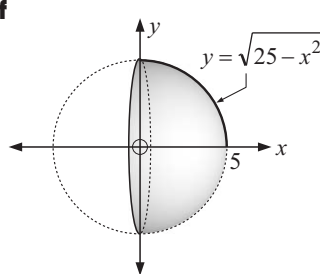
$$\begin{aligned}
 \text{Volume} &= \pi \int_1^4 \left(x^{\frac{3}{2}} \right)^2 dx \\
 &= \pi \int_1^4 x^3 dx \\
 &= \pi \left[\frac{x^4}{4} \right]_1^4 \\
 &= \pi \left(\frac{256}{4} - \frac{1}{4} \right) \\
 &= \frac{255\pi}{4} \text{ units}^3
 \end{aligned}$$

e



$$\begin{aligned}
 \text{Volume} &= \pi \int_2^4 (x^2)^2 dx \\
 &= \pi \int_2^4 x^4 dx \\
 &= \pi \left[\frac{x^5}{5} \right]_2^4 \\
 &= \pi \left[\frac{1024}{5} - \frac{32}{5} \right] \\
 &= \frac{992\pi}{5} \text{ units}^3
 \end{aligned}$$

f



$$\begin{aligned}
 \text{Volume} &= \pi \int_0^5 (25 - x^2) dx \\
 &= \pi \left[25x - \frac{x^3}{3} \right]_0^5 \\
 &= \pi \left[\left(125 - \frac{125}{3} \right) - 0 \right] \\
 &= \pi \left(\frac{2}{3} \right) 125 \\
 &= \frac{250\pi}{3} \text{ units}^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad V &= \pi \int_0^6 \left(\frac{x}{2} + 4 \right)^2 dx \\
 &= \pi \int_0^6 \left(\frac{1}{4}x^2 + 4x + 16 \right) dx \\
 &= \pi \left[\frac{x^3}{12} + \frac{4x^2}{2} + 16x \right]_0^6 \\
 &= \pi(18 + 72 + 96) - 0 \\
 &= 186\pi \text{ units}^3
 \end{aligned}$$

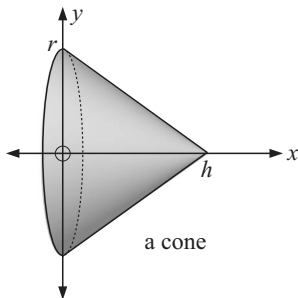
$$\begin{aligned}
 \mathbf{b} \quad V &= \pi \int_1^2 (x^2 + 3)^2 dx \\
 &= \pi \int_1^2 (x^4 + 6x^2 + 9) dx \\
 &= \pi \left[\frac{x^5}{5} + \frac{6x^3}{3} + 9x \right]_1^2 \\
 &= \pi \left(\left\{ \frac{32}{5} + 16 + 18 \right\} - \left\{ \frac{1}{5} + 2 + 9 \right\} \right) \\
 &= \pi \left(\frac{146}{5} \right) \\
 &= \frac{146\pi}{5} \text{ units}^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad V &= \pi \int_0^4 (e^x)^2 dx \\
 &= \pi \int_0^4 e^{2x} dx \\
 &= \pi \left[\frac{1}{2}e^{2x} \right]_0^4 \\
 &= \pi \left(\frac{1}{2}e^8 - \frac{1}{2} \right) \\
 &= \frac{\pi}{2}(e^8 - 1) \text{ units}^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad \text{Volume} &= \pi \int_5^8 y^2 dx \\
 &= \pi \int_5^8 (64 - x^2) dx \\
 &= \pi \left[64x - \frac{x^3}{3} \right]_5^8 \\
 &= \pi \left(\left\{ 512 - \frac{512}{3} \right\} - \left\{ 320 - \frac{125}{3} \right\} \right) \\
 &= 63\pi \text{ units}^3
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad &63\pi \text{ cm}^3 \\
 &\doteq 198 \text{ cm}^3
 \end{aligned}$$

4 a a cone of base r and height h



b

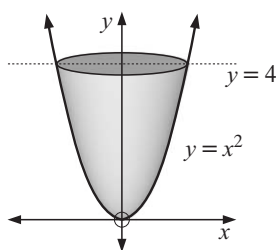
$$\begin{aligned}
 \text{slope} &= \frac{r-0}{0-h} \\
 &= -\frac{r}{h}
 \end{aligned}$$

$$\therefore \text{equation is } y = -\frac{r}{h}x + r$$

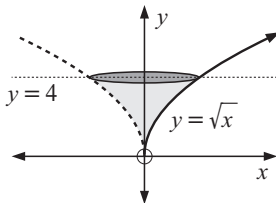
$$\begin{aligned}
 \mathbf{c} \quad V &= \pi \int_0^h \left(\frac{-r}{h}x + r \right)^2 dx \\
 &= \pi r^2 \int_0^h \left(-\frac{x}{h} + 1 \right)^2 dx \\
 &= \pi r^2 \int_0^h \left(\frac{x^2}{h^2} - \frac{2x}{h} + 1 \right) dx \\
 &= \pi r^2 \left[\frac{x^3}{3h^2} - \frac{2x^2}{2h} + x \right]_0^h \\
 &= \pi r^2 \left(\left\{ \frac{h}{3} - h + h \right\} - 0 \right) \\
 &= \frac{1}{3}\pi r^2 h \text{ units}^3
 \end{aligned}$$

5 a a sphere of radius r

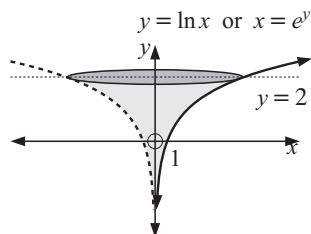
$$\begin{aligned}
 \mathbf{b} \quad V &= \pi \int_{-r}^r y^2 dx \\
 &= 2\pi \int_0^r (r^2 - x^2) dx \\
 &= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r \\
 &= 2\pi \left(r^3 - \frac{r^3}{3} - 0 \right) \\
 &= 2\pi \times \frac{2}{3} r^3 \\
 &= \frac{4}{3} \pi r^3 \text{ units}^3
 \end{aligned}$$

6 a

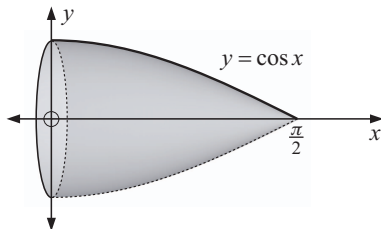
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^4 x^2 dy \\
 &= \pi \int_0^4 y dy \\
 &= \pi \left[\frac{y^2}{2} \right]_0^4 \\
 &= \pi(8 - 0) \\
 &= 8\pi \text{ units}^3
 \end{aligned}$$

b

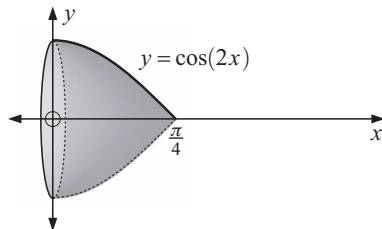
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^4 x^2 dy \\
 &= \pi \int_0^4 y^4 dy \\
 &= \pi \left[\frac{y^5}{5} \right]_0^4 \\
 &= \pi \left(\frac{4^5}{5} - 0 \right) \\
 &= \frac{1024\pi}{5} \text{ units}^3
 \end{aligned}$$

c

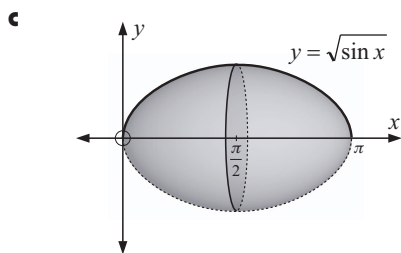
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^2 x^2 dy \\
 &= \pi \int_0^2 (e^y)^2 dy \\
 &= \pi \int_0^2 e^{2y} dy \\
 &= \pi \left[\frac{1}{2} e^{2y} \right]_0^2 \\
 &= \pi \left(\frac{e^4}{2} - \frac{1}{2} \right) \\
 &= \frac{\pi}{2} (e^4 - 1) \text{ units}^3
 \end{aligned}$$

EXERCISE 28A.2**1 a**

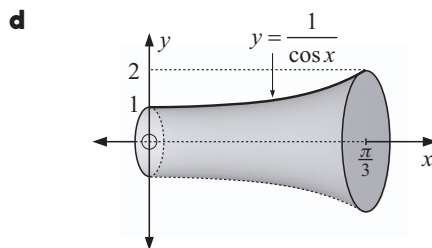
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\frac{\pi}{2}} (\cos x)^2 dx \\
 &= \pi \int_0^{\frac{\pi}{2}} \cos^2 x dx \\
 &= \pi \int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2} \cos(2x) dx \\
 &= \pi \left[\frac{1}{2} x + \frac{1}{2} \left(\frac{1}{2} \right) \sin(2x) \right]_0^{\frac{\pi}{2}} \\
 &= \pi \left[\frac{\pi}{4} + \frac{1}{4} \sin \pi - 0 \right] \\
 &= \frac{\pi^2}{4} \text{ units}^3
 \end{aligned}$$

b

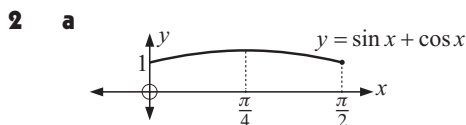
$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\frac{\pi}{4}} \cos^2(2x) dx \\
 &= \pi \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2} \cos(4x) \right) dx \\
 &= \pi \left[\frac{1}{2} x + \frac{1}{2} \left(\frac{1}{4} \right) \sin(4x) \right]_0^{\frac{\pi}{4}} \\
 &= \pi \left[\frac{\pi}{8} + \frac{1}{8} \sin \pi - 0 \right] \\
 &= \frac{\pi^2}{8} \text{ units}^3
 \end{aligned}$$



$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\pi} \sin x \, dx \\
 &= \pi [-\cos x]_0^{\pi} \\
 &= \pi [-\cos \pi - (-\cos 0)] \\
 &= \pi(2) \\
 &= 2\pi \text{ units}^3
 \end{aligned}$$

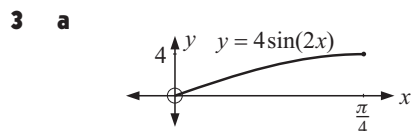


$$\begin{aligned}
 \text{Volume} &= \pi \int_0^{\pi/3} \frac{1}{\cos^2 x} \, dx \\
 &= \pi [\tan x]_0^{\pi/3} \\
 &= \pi \left(\tan \frac{\pi}{3} - \tan 0 \right) \\
 &= \pi \times (\sqrt{3} - 0) \\
 &= \pi\sqrt{3} \text{ units}^3
 \end{aligned}$$



b Volume

$$\begin{aligned}
 &= \pi \int_0^{\pi/4} (\sin x + \cos x)^2 \, dx \\
 &= \pi \int_0^{\pi/4} (\sin^2 x + 2 \sin x \cos x + \cos^2 x) \, dx \\
 &= \pi \int_0^{\pi/4} (1 + \sin(2x)) \, dx \\
 &= \pi \left[x - \frac{1}{2} \cos(2x) \right]_0^{\pi/4} \\
 &= \pi \left\{ \frac{\pi}{4} - \frac{1}{2} \cos \left(\frac{\pi}{2} \right) \right\} - \left\{ 0 - \frac{1}{2} (1) \right\} \\
 &= \pi \left(\frac{\pi}{4} + \frac{1}{2} \right) \text{ units}^3
 \end{aligned}$$



b Volume

$$\begin{aligned}
 &= \pi \int_0^{\pi/4} (4 \sin(2x))^2 \, dx \\
 &= 16\pi \int_0^{\pi/4} \sin^2(2x) \, dx \\
 &= 16\pi \int_0^{\pi/4} \left(\frac{1}{2} - \frac{1}{2} \cos(4x) \right) \, dx \\
 &= 16\pi \left[\frac{x}{2} - \frac{1}{2} \left(\frac{1}{4} \right) \sin(4x) \right]_0^{\pi/4} \\
 &= 16\pi \left\{ \frac{\pi}{8} - \frac{1}{8} \sin \pi - \left(0 - \frac{1}{8} \sin 0 \right) \right\} \\
 &= 2\pi^2 \text{ units}^3
 \end{aligned}$$

EXERCISE 28B

1 a The graphs meet where

$$\begin{aligned}
 4 - x^2 &= 3 \\
 \therefore x^2 &= 1 \\
 \therefore x &= \pm 1
 \end{aligned}$$

$\therefore$ A is $(-1, 3)$
and B is $(1, 3)$

b $V = 2\pi \int_0^1 ((4 - x^2)^2 - 3^2) \, dx$

$$\begin{aligned}
 &= 2\pi \int_0^1 (16 - 8x^2 + x^4 - 9) \, dx \\
 &= 2\pi \int_0^1 (x^4 - 8x^2 + 7) \, dx \\
 &= 2\pi \left[\frac{x^5}{5} - \frac{8x^3}{3} + 7x \right]_0^1 \\
 &= 2\pi \left\{ \frac{1}{5} - \frac{8}{3} + 7 - 0 \right\} \\
 &= \frac{136\pi}{15} \text{ units}^3
 \end{aligned}$$

- 2 a**
- The graphs meet where

$$e^{\frac{x}{2}} = e$$

$$\therefore e^{\frac{x}{2}} = e^1$$

$$\therefore \frac{x}{2} = 1$$

$$\therefore x = 2$$

i.e., at A(2, e)

$$\begin{aligned} \mathbf{b} \quad V &= \pi \int_0^2 \left(e^2 - \left(e^{\frac{x}{2}} \right)^2 \right) dx \\ &= \pi \int_0^2 (e^2 - e^x) dx \\ &= \pi [e^2 x - e^x]_0^2 \\ &= \pi [2e^2 - e^2 - (0 - 1)] \\ &= \pi [e^2 + 1] \text{ units}^3 \end{aligned}$$

- 3 a**
- The graphs meet where

$$x = \frac{1}{x}$$

$$\therefore x^2 = 1$$

$$\therefore x = \pm 1$$

$$\therefore x = 1 \quad \{\text{as } x > 0\}$$

$$\therefore A \text{ is } (1, 1)$$

$$\begin{aligned} \mathbf{b} \quad V &= \pi \int_1^2 \left(x^2 - \left(\frac{1}{x} \right)^2 \right) dx \\ &= \pi \int_1^2 (x^2 - x^{-2}) dx \\ &= \pi \left[\frac{x^3}{3} - \frac{x^{-1}}{-1} \right]_1^2 \\ &= \pi \left\{ \left(\frac{8}{3} + \frac{1}{2} \right) - \left(\frac{1}{3} + 1 \right) \right\} \\ &= \frac{11\pi}{6} \text{ units}^3 \end{aligned}$$

- 4 a**
- The curves meet where

$$\sqrt{x-4} = 1$$

$$\therefore x - 4 = 1$$

$$\therefore x = 5$$

$$\therefore A \text{ is } (5, 1)$$

$$\begin{aligned} \mathbf{b} \quad V &= \pi \int_5^8 \left((\sqrt{x-4})^2 - 1^2 \right) dx \\ &= \pi \int_5^8 (x - 4 - 1) dx \\ &= \pi \int_5^8 (x - 5) dx \\ &= \pi \left[\frac{x^2}{2} - 5x \right]_5^8 \\ &= \pi \left\{ (32 - 40) - \left(\frac{25}{2} - 25 \right) \right\} \\ &= \pi \left(\frac{9}{2} \right) = \frac{9\pi}{2} \text{ units}^3 \end{aligned}$$

- 5 a**
- $x^2 + (y-3)^2 = 4$

$$\therefore (y-3)^2 = 4 - x^2$$

$$\therefore y - 3 = \pm \sqrt{4 - x^2}$$

$$\therefore y = 3 \pm \sqrt{4 - x^2}$$

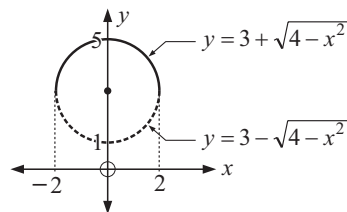
$$\mathbf{c} \quad V = \pi \int_{-2}^2 \left\{ \left(3 + \sqrt{4 - x^2} \right)^2 - \left(3 - \sqrt{4 - x^2} \right)^2 \right\} dx$$

$$= 2\pi \int_0^2 \left\{ \left(3 + \sqrt{4 - x^2} \right)^2 - \left(3 - \sqrt{4 - x^2} \right)^2 \right\} dx$$

$$= 2\pi \int_0^2 \left\{ (9 + 6\sqrt{4 - x^2} + 4 - x^2) - (9 - 6\sqrt{4 - x^2} + 4 - x^2) \right\} dx$$

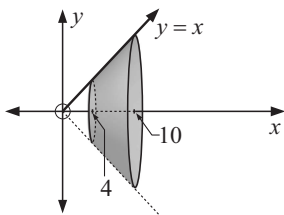
$$= 2\pi \int_0^2 12\sqrt{4 - x^2} dx$$

$$= 24\pi \int_0^2 \sqrt{4 - x^2} dx = 24\pi^2 \text{ units}^3 \quad (\doteq 236.9 \text{ units}^3) \quad \{\text{using technology}\}$$

b

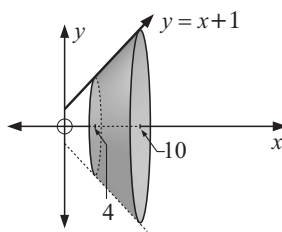
REVIEW SET 28

1 a



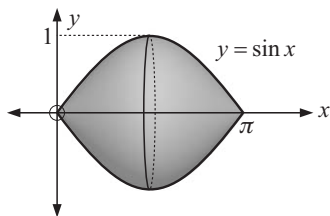
$$\begin{aligned}
 V &= \pi \int_4^{10} x^2 dx \\
 &= \pi \left[\frac{x^3}{3} \right]_4^{10} \\
 &= \pi \left(\frac{1000}{3} - \frac{64}{3} \right) \\
 &= \frac{936\pi}{3} \\
 &= 312\pi \text{ units}^3
 \end{aligned}$$

b



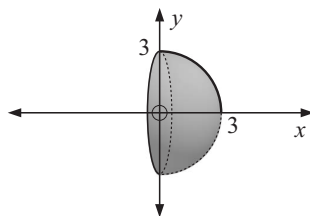
$$\begin{aligned}
 V &= \pi \int_4^{10} (x+1)^2 dx \\
 &= \pi \left[\left(\frac{1}{3} \right) \frac{(x+1)^3}{3} \right]_4^{10} \\
 &= \pi \left(\frac{11^3}{3} - \frac{5^3}{3} \right) \\
 &= \frac{1206\pi}{3} = 402\pi \text{ units}^3
 \end{aligned}$$

c



$$\begin{aligned}
 V &= \pi \int_0^{\pi} \sin^2 x dx \\
 &= \pi \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2} \cos(2x) \right) dx \\
 &= \pi \left[\frac{1}{2}x - \frac{1}{2} \left(\frac{1}{2} \right) \sin(2x) \right]_0^{\pi} \\
 &= \pi \left[\frac{1}{2}\pi - \frac{1}{4} \sin 2\pi - 0 \right] \\
 &= \frac{\pi^2}{2} \text{ units}^3
 \end{aligned}$$

d

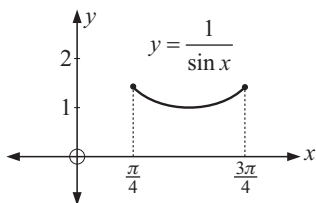


$$\begin{aligned}
 V &= \pi \int_0^3 (9 - x^2) dx \\
 &= \pi \left[9x - \frac{x^3}{3} \right]_0^3 \\
 &= \pi \left\{ 27 - \frac{27}{3} - 0 \right\} \\
 &= 18\pi \text{ units}^3
 \end{aligned}$$

- 2 a $y = \cos(2x)$ meets the x -axis where
 $2x = \frac{\pi}{2}$
 $\therefore x = \frac{\pi}{4}$

$$\begin{aligned}
 \text{so, } V &= \pi \int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \cos^2(2x) dx \\
 &= \pi \int_{\frac{\pi}{16}}^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2} \cos(4x) \right) dx \\
 &= \pi \left[\frac{1}{2}x + \frac{1}{8} \sin(4x) \right]_{\frac{\pi}{16}}^{\frac{\pi}{4}} \\
 &= \pi \left\{ \left(\frac{\pi}{8} + \frac{1}{8} \sin \pi \right) - \left(\frac{\pi}{32} + \frac{1}{8} \sin \left(\frac{\pi}{4} \right) \right) \right\} \\
 &= \pi \left\{ \frac{\pi}{8} - \frac{\pi}{32} - \frac{1}{8} \left(\frac{1}{\sqrt{2}} \right) \right\} \\
 &= \pi \left\{ \frac{3\pi}{32} - \frac{1}{8\sqrt{2}} \right\} \text{ units}^3
 \end{aligned}$$

$$\begin{aligned}
 \text{b } V &= \pi \int_0^2 (e^{-x} + 4)^2 dx \\
 &= \pi \int_0^2 (e^{-2x} + 8e^{-x} + 16) dx \\
 &= \pi \left[\frac{1}{-2} e^{-2x} + \frac{8}{-1} e^{-x} + 16x \right]_0^2 \\
 &= \pi \left\{ \left(-\frac{1}{2} e^{-4} - 8e^{-2} + 32 \right) - \left(-\frac{1}{2} - 8 \right) \right\} \\
 &= \pi \left\{ \frac{81}{2} - \frac{1}{2e^4} - \frac{8}{e^2} \right\} \text{ units}^3 \quad (\doteq 123.8 \text{ units}^3)
 \end{aligned}$$

3

$$\begin{aligned}
 V &= \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(\frac{1}{\sin x} \right)^2 dx \\
 &= \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \operatorname{cosec}^2 x \, dx \\
 &= \pi \left[-\cot x \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\
 &= \pi \left\{ -\cot \left(\frac{3\pi}{4} \right) - \left(-\cot \left(\frac{\pi}{4} \right) \right) \right\} \\
 &= \pi \{ -(-1) + 1 \} \\
 &= 2\pi \text{ units}^3
 \end{aligned}$$

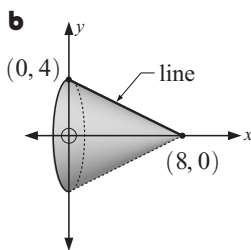
- 4** They meet where $x^2 = 4$
 $\therefore x = \pm 2$
 but $x > 0 \therefore x = 2$
 i.e., at $(2, 4)$

$$\begin{aligned}
 V &= \pi \int_0^2 (4^2 - (x^2)^2) dx \\
 &= \pi \int_0^2 (16 - x^4) dx \\
 &= \pi \left[16x - \frac{x^5}{5} \right]_0^2 \\
 &= \pi \left(32 - \frac{32}{5} - 0 \right) \\
 &= \frac{128\pi}{5} \text{ units}^3
 \end{aligned}$$

- 5** They meet where $\sin x = \cos x$
 $\therefore \frac{\sin x}{\cos x} = 1$
 $\therefore \tan x = 1$
 $\therefore x = \frac{\pi}{4}$

$$\begin{aligned}
 V &= \pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x) dx \\
 &= \pi \int_0^{\frac{\pi}{4}} \cos(2x) dx \\
 &= \pi \left[\frac{1}{2} \sin(2x) \right]_0^{\frac{\pi}{4}} \\
 &= \pi \left(\frac{1}{2} \sin \left(\frac{\pi}{2} \right) - \frac{1}{2} \sin 0 \right) \\
 &= \pi \left(\frac{1}{2}(1) - 0 \right) \\
 &= \frac{\pi}{2} \text{ units}^3
 \end{aligned}$$

6 a $V = \frac{1}{3}\pi r^2 h$
 $= \frac{1}{3}\pi \times 4^2 \times 8$
 $= \frac{1}{3}\pi \times 128$
 $= \frac{128\pi}{3} \text{ units}^3$



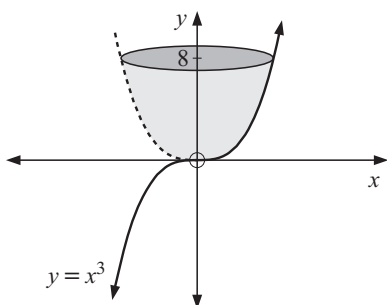
$$\text{slope} = \frac{0-4}{8-0} = -\frac{1}{2}$$

$\therefore$ line has equation

$$y = -\frac{1}{2}x + 4$$

$$\begin{aligned} \therefore V &= \pi \int_0^8 \left(-\frac{1}{2}x + 4\right)^2 dx \\ &= \pi \int_0^8 \left(\frac{x^2}{4} - 4x + 16\right) dx \\ &= \pi \left[\frac{x^3}{12} - \frac{4x^2}{2} + 16x \right]_0^8 \\ &= \pi \left\{ \frac{128}{3} - 128 + 128 - 0 \right\} \\ &= \frac{128\pi}{3} \text{ units}^3 \end{aligned}$$

7



$$\begin{aligned} \text{Volume} &= \pi \int_0^8 x^2 dy \\ &= \pi \int_0^8 y^{\frac{2}{3}} dy && x^3 = y \\ &= \pi \left[\frac{y^{\frac{5}{3}}}{\frac{5}{3}} \right]_0^8 && \therefore (x^3)^{\frac{2}{3}} = y^{\frac{2}{3}} \\ &= \frac{3\pi}{5} \left(8^{\frac{5}{3}} - 0^{\frac{5}{3}} \right) && \text{i.e., } x^2 = y^{\frac{2}{3}} \\ &= \frac{3\pi}{5} \times (2^3)^{\frac{5}{3}} \\ &= \frac{3\pi}{5} \times 2^5 \\ &= \frac{96\pi}{5} \text{ units}^3 \end{aligned}$$

Chapter 29

FURTHER INTEGRATION AND DIFFERENTIAL EQUATIONS

EXERCISE 29A

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \int \frac{4}{\sqrt{1-x^2}} dx \\
 &= 4 \int \frac{1}{\sqrt{1-x^2}} dx \\
 &= 4 \arcsin\left(\frac{x}{1}\right) + c \\
 &= 4 \arcsin(x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \frac{3}{\sqrt{4-x^2}} dx \\
 &= 3 \int \frac{1}{\sqrt{4-x^2}} dx \\
 &= 3 \arcsin\left(\frac{x}{2}\right) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int \frac{1}{x^2+16} dx \\
 &= \frac{1}{4} \arctan\left(\frac{x}{4}\right) + c
 \end{aligned}$$

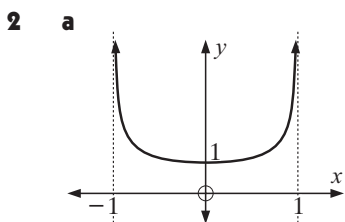
$$\begin{aligned}
 \mathbf{d} \quad & \int \frac{1}{4x^2+1} dx \\
 &= \frac{1}{4} \int \frac{1}{x^2+(\frac{1}{2})^2} dx \\
 &= \frac{1}{4} \left(\frac{1}{\frac{1}{2}}\right) \arctan\left(\frac{x}{\frac{1}{2}}\right) + c \\
 &= \frac{1}{2} \arctan(2x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{e} \quad & \int \frac{1}{\sqrt{1-4x^2}} dx \\
 &= \frac{1}{2} \int \frac{1}{\sqrt{\frac{1}{4}-x^2}} dx \\
 &= \frac{1}{2} \arcsin\left(\frac{x}{\frac{1}{2}}\right) + c \\
 &= \frac{1}{2} \arcsin(2x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{f} \quad & \int \frac{2}{\sqrt{4-9x^2}} dx \\
 &= 2 \int \frac{1}{\sqrt{4-9x^2}} dx \\
 &= \frac{2}{3} \int \frac{1}{\sqrt{\frac{4}{9}-x^2}} dx \\
 &= \frac{2}{3} \arcsin\left(\frac{3x}{2}\right) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{g} \quad & \int \frac{1}{4+2x^2} dx \\
 &= \frac{1}{2} \int \frac{1}{2+x^2} dx \\
 &= \frac{1}{2} \left(\frac{1}{\sqrt{2}}\right) \arctan\left(\frac{x}{\sqrt{2}}\right) + c \\
 &= \frac{1}{2\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{h} \quad & \int \frac{5}{9+4x^2} dx \\
 &= \frac{5}{4} \int \frac{1}{\frac{9}{4}+x^2} dx \\
 &= \frac{5}{4} \left(\frac{1}{\frac{3}{2}}\right) \arctan\left(\frac{x}{\frac{3}{2}}\right) + c \\
 &= \frac{5}{6} \arctan\left(\frac{2x}{3}\right) + c
 \end{aligned}$$



b i If $f(x) = \frac{1}{\sqrt{1-x^2}}$ then $f(-x) = \frac{1}{\sqrt{1-(-x)^2}}$

$$\begin{aligned}
 &= \frac{1}{\sqrt{1-x^2}} \\
 &= f(x) \quad \text{for all } x
 \end{aligned}$$

$\therefore f(x)$ is an even function
and so is symmetric about the y -axis

c Area = $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} dx$

$$\begin{aligned}
 &= [\arcsin(x)]_0^{\frac{1}{2}} \\
 &= \arcsin\left(\frac{1}{2}\right) - \arcsin(0) \\
 &= \frac{\pi}{6} - 0 \\
 &= \frac{\pi}{6} \text{ units}^2
 \end{aligned}$$

ii $f(x)$ is defined when $1-x^2 > 0$

$$\begin{aligned}
 \therefore x^2 - 1 &< 0 \\
 \therefore (x+1)(x-1) &< 0
 \end{aligned}$$

$$\begin{array}{c}
 + \quad | \quad - \quad | \quad + \\
 \hline
 -1 \qquad \qquad 1
 \end{array}$$

$$\therefore x \in]-1, 1[$$

EXERCISE 29B**1 a** Let $u = x - 3$

$$\text{so } \frac{du}{dx} = 1 \quad \text{and} \quad x = u + 3$$

$$\begin{aligned} \therefore \int x\sqrt{x-3} \, dx &= \int (u+3)\sqrt{u} \frac{du}{dx} \, dx \\ &= \int \left(u^{\frac{3}{2}} + 3u^{\frac{1}{2}} \right) du \\ &= \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{3u^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= \frac{2}{5}u^{\frac{5}{2}} + 2u^{\frac{3}{2}} + c \\ &= \frac{2}{5}(x-3)^{\frac{5}{2}} + 2(x-3)^{\frac{3}{2}} + c \end{aligned}$$

c Let $u = 3 - x^2$ so $\frac{du}{dx} = -2x$

$$\begin{aligned} \therefore \int x^3\sqrt{3-x^2} \, dx &= \int x^2\sqrt{3-x^2} x \, dx \\ &= \int x^2\sqrt{3-x^2} \left(-\frac{1}{2} \frac{du}{dx} \right) dx \\ &= -\frac{1}{2} \int (3-u)\sqrt{u} \, du \\ &= -\frac{1}{2} \int \left(3u^{\frac{1}{2}} - u^{\frac{3}{2}} \right) du \\ &= -\frac{1}{2} \left[\frac{3u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{5}{2}}}{\frac{5}{2}} \right] + c \\ &= -\frac{1}{2} \left[2u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}} \right] + c \\ &= -u^{\frac{3}{2}} + \frac{1}{5}u^{\frac{5}{2}} + c \\ &= -(3-x^2)^{\frac{3}{2}} + \frac{1}{5}(3-x^2)^{\frac{5}{2}} + c \end{aligned}$$

e Let $u = \sqrt{x-1}$

$$\begin{aligned} \text{so } \frac{du}{dx} &= \frac{1}{2}(x-1)^{-\frac{1}{2}} (1) \\ &= \frac{1}{2\sqrt{x-1}} \end{aligned}$$

$$\text{and } x = u^2 + 1$$

b Let $u = x + 1$

$$\text{so } \frac{du}{dx} = 1 \quad \text{and} \quad x = u - 1$$

$$\begin{aligned} \therefore \int x^2\sqrt{x+1} \, dx &= \int (u-1)^2\sqrt{u} \frac{du}{dx} \, dx \\ &= \int (u^2 - 2u + 1)u^{\frac{1}{2}} \, du \\ &= \int \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}} \right) du \\ &= \frac{u^{\frac{7}{2}}}{\frac{7}{2}} - \frac{2u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c \\ &= \frac{2}{7}u^{\frac{7}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + c \\ &= \frac{2}{7}(x+1)^{\frac{7}{2}} - \frac{4}{5}(x+1)^{\frac{5}{2}} + \frac{2}{3}(x+1)^{\frac{3}{2}} + c \end{aligned}$$

d Let $u = t^2 + 2$ so $\frac{du}{dt} = 2t$

$$\begin{aligned} \therefore \int t^3\sqrt{t^2+2} \, dt &= \int t^2\sqrt{t^2+2} t \, dt \\ &= \int t^2\sqrt{t^2+2} \left(\frac{1}{2} \frac{du}{dt} \right) dt \\ &= \frac{1}{2} \int (u-2)\sqrt{u} \, du \\ &= \frac{1}{2} \int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}} \right) du \\ &= \frac{1}{2} \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2u^{\frac{3}{2}}}{\frac{3}{2}} \right] + c \\ &= \frac{1}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c \\ &= \frac{1}{5}(t^2+2)^{\frac{5}{2}} - \frac{2}{3}(t^2+2)^{\frac{3}{2}} + c \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{\sqrt{x-1}}{x} \, dx &= \int \frac{x-1}{x} \frac{1}{\sqrt{x-1}} \, dx \\ &= \int \frac{u^2}{u^2+1} \left(2 \frac{du}{dx} \right) dx \\ &= \int \frac{2u^2}{u^2+1} \, du \\ &= \int \frac{2(u^2+1)-2}{u^2+1} \, du \\ &= \int 2 - \frac{2}{u^2+1} \, du \\ &= 2u - 2 \arctan u + c \\ &= 2\sqrt{x-1} - 2 \arctan \sqrt{x-1} + c \end{aligned}$$

2 a Let $u = x - 1$ so $\frac{du}{dx} = 1$

When $x = 4$, $u = 3$
and when $x = 3$, $u = 2$

$$\begin{aligned} \therefore \int_3^4 x\sqrt{x-1} \, dx &= \int_2^3 (u+1)\sqrt{u} \, du \\ &= \int_2^3 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du \\ &= \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^3 \\ &= \left(\frac{2}{5}(3)^{\frac{5}{2}} + \frac{2}{3}(3)^{\frac{3}{2}}\right) - \left(\frac{2}{5}(2)^{\frac{5}{2}} + \frac{2}{3}(2)^{\frac{3}{2}}\right) \\ &= \frac{2}{5}(9\sqrt{3}) + \frac{2}{3}(3\sqrt{3}) - \frac{2}{5}(4\sqrt{2}) - \frac{2}{3}(2\sqrt{2}) \\ &= \left(\frac{18}{5} + 2\right)\sqrt{3} - \left(\frac{8}{5} + \frac{4}{3}\right)\sqrt{2} \\ &= \frac{28\sqrt{3}}{5} - \frac{44\sqrt{2}}{15} \end{aligned}$$

c Let $u = x - 2$ so $\frac{du}{dx} = 1$

When $x = 5$, $u = 3$
and when $x = 2$, $u = 0$

$$\begin{aligned} \therefore \int_2^5 x^2\sqrt{x-2} \, dx &= \int_0^3 (u+2)^2\sqrt{u} \, du \\ &= \int_0^3 (u^2 + 4u + 4)\sqrt{u} \, du \\ &= \int_0^3 \left(u^{\frac{5}{2}} + 4u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\right) du \end{aligned}$$

3 a Let $x = 3 \tan \theta$ so $\frac{dx}{d\theta} = 3 \sec^2 \theta$

$$\begin{aligned} \therefore \int \frac{x^2}{9+x^2} \, dx &= \int \frac{9 \tan^2 \theta}{9+9 \tan^2 \theta} 3 \sec^2 \theta \, d\theta \\ &= 3 \int \frac{\tan^2 \theta}{1+\tan^2 \theta} \sec^2 \theta \, d\theta \\ &= 3 \int \tan^2 \theta \, d\theta \quad \{\text{as } \sec^2 \theta = 1 + \tan^2 \theta\} \\ &= 3 \int (\sec^2 \theta - 1) \, d\theta \\ &= 3 \tan \theta - 3\theta + c \\ &= x - 3 \arctan\left(\frac{x}{3}\right) + c \end{aligned}$$

b Let $u = x + 6$ so $\frac{du}{dx} = 1$

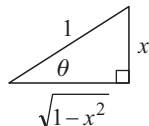
When $x = 3$, $u = 9$
and when $x = 0$, $u = 6$

$$\begin{aligned} \therefore \int_0^3 x\sqrt{x+6} \, dx &= \int_6^9 (u-6)\sqrt{u} \, du \\ &= \int_6^9 \left(u^{\frac{3}{2}} - 6u^{\frac{1}{2}}\right) du \\ &= \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{6u^{\frac{3}{2}}}{\frac{3}{2}}\right]_6^9 \\ &= \left(\frac{2}{5}(9)^{\frac{5}{2}} - 4(9)^{\frac{3}{2}}\right) - \left(\frac{2}{5}(6)^{\frac{5}{2}} - 4(6)^{\frac{3}{2}}\right) \\ &= \frac{2}{5}(3^5) - 4(3^3) - \frac{2}{5} \times 36\sqrt{6} + 4 \times 6\sqrt{6} \\ &= \frac{486}{5} - 108 - \frac{72}{5}\sqrt{6} + 24\sqrt{6} \\ &= -\frac{54}{5} + \frac{48}{5}\sqrt{6} \\ &= \frac{1}{5}(48\sqrt{6} - 54) \end{aligned}$$

$$\begin{aligned} &= \left[\frac{u^{\frac{7}{2}}}{\frac{7}{2}} + \frac{4u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{4u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3 \\ &= \frac{2}{7}(3)^{\frac{7}{2}} + \frac{8}{5}(3)^{\frac{5}{2}} + \frac{8}{3}(3)^{\frac{3}{2}} - 0 \\ &= \frac{2}{7}(27\sqrt{3}) + \frac{8}{5}(9\sqrt{3}) + \frac{8}{3}(3\sqrt{3}) \\ &= \frac{1054}{35}\sqrt{3} \end{aligned}$$

b Let $x = \sin \theta$ so $\frac{dx}{d\theta} = \cos \theta$

$$\begin{aligned} \therefore \int \frac{x^2}{\sqrt{1-x^2}} \, dx &= \int \frac{\sin^2 \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta \, d\theta \\ &= \int \frac{\sin^2 \theta}{\cos \theta} \cos \theta \, d\theta \\ &= \int \sin^2 \theta \, d\theta \\ &= \int \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta\right) d\theta \\ &= \frac{1}{2}\theta - \frac{1}{2}\left(\frac{1}{2}\right) \sin 2\theta + c \\ &= \frac{1}{2} \arcsin x - \frac{1}{2} \sin \theta \cos \theta + c \\ &= \frac{1}{2} \arcsin x - \frac{1}{2} x \sqrt{1-x^2} + c \\ &\quad \{\text{since } \cos \theta = \sqrt{1-\sin^2 \theta}\} \end{aligned}$$



$$\begin{aligned} \mathbf{c} \quad & \int \frac{2x}{x^2+9} dx \\ &= \ln|x^2+9| + c \quad \left\{ \text{form } \int \frac{f'(x)}{f(x)} dx \right\} \\ &= \ln(x^2+9) + c \quad \left\{ \text{as } x^2+9 > 0 \right. \\ & \quad \left. \text{for all } x \right\} \end{aligned}$$

$$\begin{aligned} \mathbf{e} \quad & \text{Let } x = 2 \sec \theta \quad \text{so} \quad \frac{dx}{d\theta} = 2 \sec \theta \tan \theta \\ & \therefore \int \frac{\sqrt{x^2-4}}{x} dx \\ &= \int \frac{\sqrt{4 \sec^2 \theta - 4}}{2 \sec \theta} 2 \sec \theta \tan \theta d\theta \\ &= \frac{2}{2} \int \frac{\sqrt{\sec^2 \theta - 1}}{\sec \theta} \times 2 \sec \theta \tan \theta d\theta \\ &= 2 \int \sqrt{\sec^2 \theta - 1} \tan \theta d\theta \\ &= 2 \int \tan \theta \tan \theta d\theta \quad \{ \sec^2 \theta - 1 = \tan^2 \theta \} \\ &= 2 \int \tan^2 \theta d\theta \\ &= 2 \int (\sec^2 \theta - 1) d\theta \quad \begin{array}{c} x \\ \theta \quad \quad \sqrt{x^2-4} \\ 2 \end{array} \\ &= 2 \tan \theta - 2\theta + c \\ &= 2 \frac{\sqrt{x^2-4}}{2} - 2 \arccos\left(\frac{2}{x}\right) + c \\ &= \sqrt{x^2-4} - 2 \arccos\left(\frac{2}{x}\right) + c \end{aligned}$$

$$\begin{aligned} \mathbf{g} \quad & \int \frac{1}{\sqrt{9-4x^2}} dx \\ &= \frac{1}{2} \int \frac{1}{\sqrt{\frac{9}{4}-x^2}} dx \\ \text{Let } x &= \frac{3}{2} \sin \theta, \quad \text{so} \quad \frac{dx}{d\theta} = \frac{3}{2} \cos \theta \\ \therefore \text{ the integral} \\ &= \frac{1}{2} \int \frac{1}{\sqrt{\frac{9}{4}-\frac{9}{4} \sin^2 \theta}} \frac{3}{2} \cos \theta d\theta \\ &= \frac{1}{2} \int \frac{1}{\frac{3}{2} \sqrt{1-\sin^2 \theta}} \times \frac{3}{2} \cos \theta d\theta \\ &= \frac{1}{2} \int \frac{\cos \theta}{\cos \theta} d\theta \\ &= \frac{1}{2} \int 1 d\theta \\ &= \frac{1}{2} \theta + c \\ &= \frac{1}{2} \arcsin\left(\frac{2x}{3}\right) + c \quad \{ \text{since } \sin \theta = \frac{2x}{3} \} \end{aligned}$$

$$\begin{aligned} \mathbf{d} \quad & \text{Let } u = \ln x \quad \text{so} \quad \frac{du}{dx} = \frac{1}{x} \\ & \therefore \int \frac{4 \ln x}{x(1+[\ln x]^2)} dx \\ &= \int \frac{4u}{1+u^2} \frac{du}{dx} dx \\ &= 2 \int \frac{2u}{1+u^2} du \\ &= 2 \ln|1+u^2| + c \quad \left\{ \text{form } \int \frac{f'(x)}{f(x)} dx \right\} \\ &= 2 \ln(1+u^2) + c \quad \{1+u^2 > 0\} \\ &= 2 \ln(1+[\ln x]^2) + c \end{aligned}$$

$$\begin{aligned} \mathbf{f} \quad & \int \sin x \cos 2x dx \\ &= \int \sin x (2 \cos^2 x - 1) dx \\ &= 2 \int \cos^2 x \sin x dx - \int \sin x dx \\ &= -2 \int [\cos x]^2 (-\sin x) dx - \int \sin x dx \\ &= -2 \frac{[\cos x]^3}{3} - (-\cos x) + c \\ &= -\frac{2}{3} \cos^3 x + \cos x + c \\ &= \cos x - \frac{2}{3} \cos^3 x + c \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \int \frac{x^3}{1+x^2} dx \\ &= \int \frac{x(1+x^2)-x}{1+x^2} dx \\ &= \int \left(x - \frac{x}{1+x^2} \right) dx \\ &= \int \left(x - \frac{1}{2} \left(\frac{2x}{1+x^2} \right) \right) dx \\ &= \frac{x^2}{2} - \frac{1}{2} \ln|1+x^2| + c \\ &= \frac{x^2}{2} - \frac{1}{2} \ln(1+x^2) + c \\ & \quad \{ \text{as } 1+x^2 > 0 \} \end{aligned}$$

i Let $u = \ln x$ so $\frac{du}{dx} = \frac{1}{x}$

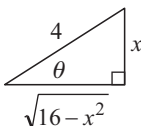
$$\begin{aligned} \therefore \int \frac{1}{x(9+4[\ln x]^2)} dx \\ &= \int \frac{1}{9+4u^2} \frac{du}{dx} dx \\ &= \int \frac{1}{9+4u^2} du \\ &= \frac{1}{4} \int \frac{1}{u^2 + \frac{9}{4}} du \\ &= \frac{1}{4} \left(\frac{1}{\frac{3}{2}} \right) \arctan \left(\frac{u}{\frac{3}{2}} \right) + c \\ &= \frac{1}{6} \arctan \left(\frac{2u}{3} \right) + c \\ &= \frac{1}{6} \arctan \left(\frac{2 \ln x}{3} \right) + c \end{aligned}$$

j Let $x = 4 \tan \theta$ so $\frac{dx}{d\theta} = 4 \sec^2 \theta$

$$\begin{aligned} \therefore \int \frac{1}{x(x^2+16)} dx \\ &= \int \frac{1}{4 \tan \theta (16 \tan^2 \theta + 16)} \times 4 \sec^2 \theta d\theta \\ &= \int \frac{1}{4 \tan \theta \times 16 \sec^2 \theta} \times 4 \sec^2 \theta d\theta \\ &= \frac{1}{16} \int \frac{1}{\tan \theta} d\theta \quad \begin{array}{c} \sqrt{x^2+16} \\ \theta \\ 4 \end{array} \quad \begin{array}{c} x \\ \square \end{array} \\ &= \frac{1}{16} \int \frac{\cos \theta}{\sin \theta} d\theta \\ &= \frac{1}{16} \ln |\sin \theta| + c \quad \left\{ \text{form } \frac{f'(\theta)}{f(\theta)} \right\} \\ &= \frac{1}{16} \ln \left| \frac{x}{\sqrt{x^2+16}} \right| + c \\ &= \frac{1}{16} \ln \left(\frac{|x|}{\sqrt{x^2+16}} \right) + c \end{aligned}$$

k Let $x = 4 \sin \theta$ so $\frac{dx}{d\theta} = 4 \cos \theta$

$$\begin{aligned} \therefore \int \frac{1}{x^2 \sqrt{16-x^2}} dx \\ &= \int \frac{1}{16 \sin^2 \theta \sqrt{16-16 \sin^2 \theta}} 4 \cos \theta d\theta \\ &= \int \frac{1}{16 \sin^2 \theta \times 4 \cos \theta} 4 \cos \theta d\theta \\ &= \frac{1}{16} \int \frac{1}{\sin^2 \theta} d\theta \\ &= \frac{1}{16} \int \operatorname{cosec}^2 \theta d\theta \\ &= \frac{1}{16} (-\cot \theta) + c \\ &= -\frac{1}{16} \cot \theta + c \\ &= -\frac{1}{16} \frac{\sqrt{16-x^2}}{x} + c \\ &= -\frac{\sqrt{16-x^2}}{16x} + c \end{aligned}$$



l Let $x = 2 \sin \theta$ so $\frac{dx}{d\theta} = 2 \cos \theta$

$$\begin{aligned} \therefore \int x^2 \sqrt{4-x^2} dx \\ &= \int 4 \sin^2 \theta \sqrt{4-4 \sin^2 \theta} 2 \cos \theta d\theta \\ &= \int 4 \sin^2 \theta 2 \cos \theta 2 \cos \theta d\theta \\ &= 4 \int 4 \sin^2 \theta \cos^2 \theta d\theta \\ &= 4 \int \sin^2(2\theta) d\theta \\ &= 4 \int \left(\frac{1}{2} - \frac{1}{2} \cos(4\theta) \right) d\theta \quad \begin{array}{c} 2 \\ \theta \\ \sqrt{4-x^2} \end{array} \quad \begin{array}{c} x \\ \square \end{array} \\ &= 2\theta - 2\left(\frac{1}{4}\right) \sin(4\theta) \\ &= 2\theta - \frac{1}{2} \sin(4\theta) + c \end{aligned}$$

Now $\sin \theta = \frac{x}{2}$, $\cos \theta = \frac{\sqrt{4-x^2}}{2}$

$$\therefore \sin 2\theta = 2 \left(\frac{x}{2} \right) \frac{\sqrt{4-x^2}}{2} = \frac{x\sqrt{4-x^2}}{2}$$

$$\begin{aligned} \text{and } \cos 2\theta &= \cos^2 \theta - \sin^2 \theta = \frac{4-x^2}{4} - \frac{x^2}{4} \\ &= \frac{4-2x^2}{4} \end{aligned}$$

$$\begin{aligned} \sin 4\theta &= 2 \left(\frac{x\sqrt{4-x^2}}{2} \right) \left(\frac{4-2x^2}{4} \right) \\ &= \frac{x\sqrt{4-x^2}(2-x^2)}{2} \end{aligned}$$

$$\begin{aligned} \therefore \int x^2 \sqrt{4-x^2} dx \\ &= 2 \arcsin \left(\frac{x}{2} \right) - \frac{1}{4} x \sqrt{4-x^2} (2-x^2) + c \end{aligned}$$

EXERCISE 29C

1 a $\int x e^x dx$ has $u = x \quad v' = e^x$
 $u' = 1 \quad v = e^x$

$$\begin{aligned}\therefore \int x e^x dx &= x e^x - \int 1 e^x dx \\ &= x e^x - e^x + c\end{aligned}$$

C $\int x^2 \ln x \, dx$ has $u = \ln x$ $v' = x^2$
 $u' = \frac{1}{x}$ $v = \frac{x^3}{3}$

$$\begin{aligned} \therefore \int x^2 \ln x \, dx &= \ln x \left(\frac{x^3}{3} \right) - \int \frac{1}{x} \frac{x^3}{3} \, dx \\ &= \frac{x^3 \ln x}{3} - \frac{1}{3} \int x^2 \, dx \\ &= \frac{x^3 \ln x}{3} - \frac{1}{3} \frac{x^3}{3} + c \\ &= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + c \end{aligned}$$

e $\int x \cos 2x \, dx$ has $u = x$ $v' = \cos 2x$
 $u' = 1$ $v = \frac{1}{2} \sin 2x$

$$\begin{aligned} \therefore \int x \cos 2x \, dx &= x\left(\frac{1}{2} \sin 2x\right) - \int \frac{1}{2} \sin 2x \, dx \\ &= \frac{1}{2}x \sin 2x - \frac{1}{2}\left(-\frac{1}{2}\right) \cos 2x + c \\ &= \frac{1}{2}x \sin 2x + \frac{1}{4} \cos 2x + c \end{aligned}$$

g $\int \ln x \, dx = \int 1 \times \ln x \, dx$,
which has $u = \ln x \quad v' = 1$
 $u' = \frac{1}{x} \quad v = x$

$$\begin{aligned} \therefore \int \ln x \, dx &= x \ln x - \int \left(\frac{1}{x}\right) x \, dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + c \end{aligned}$$

i $\int \arctan x \, dx = \int 1 \arctan x \, dx$
 which has $u = \arctan x \quad v' = 1$
 $u' = \frac{1}{x^2 + 1} \quad v = x$

$$\begin{aligned} \therefore \int \arctan x \, dx &= \\ &= x \arctan x - \int \frac{x}{x^2 + 1} \, dx \\ &= x \arctan x - \frac{1}{2} \int \frac{2x}{x^2 + 1} \, dx \\ &= x \arctan x - \frac{1}{2} \ln |x^2 + 1| + c \\ &= x \arctan x - \frac{1}{2} \ln(x^2 + 1) + c \quad \{\text{as } x^2 + 1 > 0\} \end{aligned}$$

b $\int x \sin x \, dx$ has $u = x \quad v' = \sin x$
 $u' = 1 \quad v = -\cos x$

$$\begin{aligned} \therefore \int x \sin x \, dx &= x(-\cos x) - \int 1(-\cos x) \, dx \\ &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \sin x + c \end{aligned}$$

d $\int x \sin 3x \, dx$ has $u = x \quad v' = \sin 3x$
 $u' = 1 \quad v = -\frac{1}{3} \cos 3x$

$$\begin{aligned} \therefore \int x \sin 3x \, dx &= x(-\tfrac{1}{3} \cos 3x) - \int (-\tfrac{1}{3} \cos 3x) \, dx \\ &= -\tfrac{1}{3}x \cos 3x + \tfrac{1}{3}(\tfrac{1}{3}) \sin 3x + c \\ &= -\tfrac{1}{3}x \cos 3x + \tfrac{1}{9} \sin 3x + c \end{aligned}$$

f $\int x \sec^2 x \, dx$ has $u = x \quad v' = \sec^2 x$
 $u' = 1 \quad v = \tan x$

$$\begin{aligned} \therefore \int x \sec^2 x \, dx &= x \tan x - \int \tan x \, dx \\ &= x \tan x - \int \frac{\sin x}{\cos x} \, dx \\ &= x \tan x + \int \frac{-\sin x}{\cos x} \, dx \quad \left\{ \text{form } \int \frac{f'(x)}{f(x)} \, dx \right\} \\ &= x \tan x + \ln |\cos x| + c \end{aligned}$$

$$\begin{aligned} \mathbf{h} \quad & \int (\ln x)^2 dx \\ &= \int (\ln x) (\ln x) dx \\ \text{which has } & u = \ln x \quad v' = \ln x \\ & u' = \frac{1}{x} \quad v = x \ln x - x \end{aligned}$$

{using **g**}

$$\begin{aligned} \therefore \int (\ln x)^2 dx &= \ln x(x \ln x - x) - \int \frac{1}{x}(x \ln x - x) dx \\ &= x(\ln x)^2 - x \ln x - \int (\ln x - 1) dx \\ &= x(\ln x)^2 - x \ln x - [x \ln x - x] + x + c \\ &= x(\ln x)^2 - 2x \ln x + 2x + c \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \int x^2 e^{-x} dx \quad \left\{ \begin{array}{ll} u = x^2 & v' = e^{-x} \\ u' = 2x & v = -e^{-x} \end{array} \right\} \\
 &= -x^2 e^{-x} - \int 2x(-e^{-x}) dx \\
 &= -x^2 e^{-x} + 2 \int x e^{-x} dx \quad \left\{ \begin{array}{ll} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{array} \right\} \\
 &= -x^2 e^{-x} + 2 \left[x(-e^{-x}) - \int -e^{-x} dx \right] \\
 &= -x^2 e^{-x} - 2x e^{-x} + 2 \int e^{-x} dx \\
 &= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int e^x \cos x dx \quad \left\{ \begin{array}{ll} u = e^x & v' = \cos x \\ u' = e^x & v = \sin x \end{array} \right\} \\
 &= e^x \sin x - \int e^x \sin x dx \quad \left\{ \begin{array}{ll} u = e^x & v' = \sin x \\ u' = e^x & v = -\cos x \end{array} \right\} \\
 &= e^x \sin x - [-e^x \cos x - \int e^x(-\cos x) dx] \\
 &= e^x \sin x + e^x \cos x - \int e^x \cos x dx + c_1 \\
 \therefore 2 \int e^x \cos x dx &= e^x(\sin x + \cos x) + c_1 \\
 \therefore \int e^x \cos x dx &= \frac{1}{2}e^x(\sin x + \cos x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \int e^{-x} \sin x dx \quad \left\{ \begin{array}{ll} u = e^{-x} & v' = \sin x \\ u' = -e^{-x} & v = -\cos x \end{array} \right\} \\
 &= -e^{-x} \cos x - \int -e^{-x}(-\cos x) dx \\
 &= -e^{-x} \cos x - \int e^{-x} \cos x dx \quad \left\{ \begin{array}{ll} u = e^{-x} & v' = \cos x \\ u' = -e^{-x} & v = \sin x \end{array} \right\} \\
 &= -e^{-x} \cos x - [e^{-x} \sin x - \int -e^{-x} \sin x dx] \\
 &= -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x dx + c_1 \\
 \therefore 2 \int e^{-x} \sin x dx &= -e^{-x}(\sin x + \cos x) + c_1 \\
 \therefore \int e^{-x} \sin x dx &= -\frac{1}{2}e^{-x}(\sin x + \cos x) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int x^2 \sin x dx \quad \left\{ \begin{array}{ll} u = x^2 & v' = \sin x \\ u' = 2x & v = -\cos x \end{array} \right\} \\
 &= -x^2 \cos x - \int -2x \cos x dx \\
 &= -x^2 \cos x + \int 2x \cos x dx \quad \left\{ \begin{array}{ll} u = 2x & v' = \cos x \\ u' = 2 & v = \sin x \end{array} \right\} \\
 &= -x^2 \cos x + [2x \sin x - \int 2 \sin x dx] \\
 &= -x^2 \cos x + 2x \sin x - 2 \int \sin x dx + c \\
 &= -x^2 \cos x + 2x \sin x - 2(-\cos x) + c \\
 &= -x^2 \cos x + 2x \sin x + 2 \cos x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & \int u^2 e^u du \quad \left\{ \begin{array}{ll} a = u^2 & b' = e^u \\ a' = 2u & b = e^u \end{array} \right\} \\
 &= u^2 e^u - \int 2u e^u du \\
 &= u^2 e^u - 2 \int u e^u du \quad \left\{ \begin{array}{ll} a = u & b' = e^u \\ a' = 1 & b = e^u \end{array} \right\} \\
 &= u^2 e^u - 2 [u e^u - \int e^u du] + c \\
 &= u^2 e^u - 2u e^u + 2e^u + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \text{Let } u = \ln x \text{ so } \frac{du}{dx} = \frac{1}{x} = \frac{1}{e^u} \\
 & \therefore \int (\ln x)^2 dx \\
 &= \int u^2 e^u du \\
 &= u^2 e^u - 2u e^u + 2e^u + c \quad \{\text{using } \mathbf{a}\} \\
 &= (\ln x)^2 e^{\ln x} - 2 \ln x e^{\ln x} + 2e^{\ln x} + c \\
 &= x(\ln x)^2 - 2x \ln x + 2x + c
 \end{aligned}$$

4 a $\int u \sin u \, du$
 $= -u \cos u - \int -\cos u \, du$
 $= -u \cos u + \sin u + c$

$\left\{ \begin{array}{l} a = u \\ a' = 1 \end{array} \quad \begin{array}{l} b' = \sin u \\ b = -\cos u \end{array} \right\}$

b If $u^2 = 2x$, $2u \frac{du}{dx} = 2$
 $\therefore \frac{du}{dx} = \frac{1}{u}$

$\therefore \int \sin \sqrt{2x} \, dx$
 $= \int \sin u (u \, du)$
 $= \int u \sin u \, du$
 $= -u \cos u + \sin u + c$
 $= -\sqrt{2x} \cos \sqrt{2x} + \sin \sqrt{2x} + c$

5 $u^2 = 3x$ $\int \cos \sqrt{3x} \, dx$
 $\therefore 2u \frac{du}{dx} = 3$ $= \int \cos u \left(\frac{2u}{3} \right) du$
 $\therefore \frac{du}{dx} = \frac{3}{2u}$ $= \frac{2}{3} \int u \cos u \, du$ $\left\{ \begin{array}{l} a = u \\ a' = 1 \end{array} \quad \begin{array}{l} b' = \cos u \\ b = \sin u \end{array} \right\}$
 $= \frac{2}{3} \left[u \sin u - \int \sin u \, du \right]$
 $= \frac{2}{3} u \sin u - \frac{2}{3} (-\cos u) + c$
 $= \frac{2}{3} \sqrt{3x} \sin \sqrt{3x} + \frac{2}{3} \cos \sqrt{3x} + c$

EXERCISE 29D.1

1 a $\frac{dy}{dx} = 5y$
 $\therefore \frac{1}{y} \frac{dy}{dx} = 5$
 $\therefore \int \frac{1}{y} \frac{dy}{dx} \, dx = \int 5 \, dx$
 $\therefore \int \frac{1}{y} \, dy = \int 5 \, dx$
 $\therefore \ln |y| = 5x + c$
 $\therefore y = Ae^{5x}$

b $\frac{dM}{dt} = -2M$
 $\therefore \frac{1}{M} \frac{dM}{dt} = -2$
 $\therefore \int \frac{1}{M} \frac{dM}{dt} \, dt = \int -2 \, dt$
 $\therefore \int \frac{1}{M} \, dM = \int -2 \, dt$
 $\therefore \ln |M| = -2t + c$
 $\therefore M = Ae^{-2t}$

c $\frac{dy}{dx} = \frac{2}{y}$
 $\therefore y \frac{dy}{dx} = 2$
 $\therefore \int y \frac{dy}{dx} \, dx = \int 2 \, dx$
 $\therefore \int y \, dy = \int 2 \, dx$
 $\therefore \frac{1}{2} y^2 = 2x + a$
 $\therefore x = \frac{y^2}{4} + c$

d $\frac{dP}{dt} = 3\sqrt{P}$
 $\therefore \frac{1}{\sqrt{P}} \frac{dP}{dt} = 3$
 $\therefore \int \frac{1}{\sqrt{P}} \frac{dP}{dt} \, dt = \int 3 \, dt$
 $\therefore \int \frac{1}{\sqrt{P}} \, dP = \int 3 \, dt$
 $\therefore 2\sqrt{P} = 3t + a$
i.e., $\sqrt{P} = \frac{3}{2}t + c$

e $\frac{dQ}{dt} = 2Q + 3$ $\therefore \int \frac{1}{2Q+3} \, dQ = \int 1 \, dt$
 $\therefore \frac{1}{2Q+3} \frac{dQ}{dt} = 1$ $\therefore \frac{1}{2} \ln |2Q+3| = t + a$
 $\ln |2Q+3| = 2t + 2a$
 $\therefore 2Q+3 = ce^{2t}$
 $\therefore Q = Ae^{2t} - \frac{3}{2}$

$$\mathbf{f} \quad \frac{dQ}{dt} = \frac{1}{2Q+3}$$

$$\therefore (2Q+3) \frac{dQ}{dt} = 1$$

$$\therefore \int (2Q+3) \frac{dQ}{dt} dt = \int 1 dt$$

$$\therefore \int (2Q+3) dQ = \int 1 dt$$

$$\therefore Q^2 + 3Q = t + a$$

$$\therefore t = Q^2 + 3Q + c$$

$$\mathbf{2} \quad \mathbf{a} \quad \frac{dy}{dx} = 4y$$

$$\therefore \frac{1}{y} \frac{dy}{dx} = 4$$

$$\therefore \int \frac{1}{y} \frac{dy}{dx} dx = \int 4 dx$$

$$\therefore \int \frac{1}{y} dy = \int 4 dx$$

$$\therefore \ln|y| = 4x + c$$

$$\therefore y = Ae^{4x}$$

$$\text{But } y = 10 \text{ when } x = 0,$$

$$\text{so } A = 10$$

$$\therefore y = 10e^{4x}$$

$$\mathbf{b} \quad \frac{dM}{dt} = -3M$$

$$\therefore \frac{1}{M} \frac{dM}{dt} = -3$$

$$\therefore \int \frac{1}{M} \frac{dM}{dt} dt = \int -3 dt$$

$$\therefore \int \frac{1}{M} dM = \int -3 dt$$

$$\therefore \ln|M| = -3t + c$$

$$\therefore M = Ae^{-3t}$$

$$\text{But } M(0) = 20,$$

$$\text{so } A = 20$$

$$\therefore M = 20e^{-3t}$$

$$\mathbf{c} \quad \frac{dy}{dt} = \frac{\sqrt{y}}{3}$$

$$\therefore \frac{1}{\sqrt{y}} \frac{dy}{dt} = \frac{1}{3}$$

$$\therefore \int \frac{1}{\sqrt{y}} \frac{dy}{dt} dt = \int \frac{1}{3} dt$$

$$\therefore \int \frac{1}{\sqrt{y}} dy = \int \frac{1}{3} dt$$

$$\therefore 2\sqrt{y} = \frac{1}{3}t + c$$

$$\therefore 6\sqrt{y} = t + 3c$$

$$\text{But } y = 9 \text{ when } t = 24,$$

$$\text{so } 18 = 24 + 3c$$

$$\therefore 3c = -6$$

$$\therefore 6\sqrt{y} = t - 6$$

$$\therefore \sqrt{y} = \frac{1}{6}t - 1$$

$$\mathbf{d} \quad \frac{dP}{dn} = 2P + 3$$

$$\therefore \frac{1}{2P+3} \frac{dP}{dn} = 1$$

$$\therefore \int \frac{1}{2P+3} \frac{dP}{dn} dn = \int 1 dn$$

$$\therefore \int \frac{1}{2P+3} dP = \int 1 dn$$

$$\therefore \frac{1}{2} \ln|2P+3| = n + c$$

$$\therefore \ln|2P+3| = 2n + 2c$$

$$\therefore 2P+3 = Ae^{2n}$$

$$\therefore 2P = Ae^{2n} - 3$$

$$\therefore P = Be^{2n} - \frac{3}{2}$$

$$\text{But } P = 2 \text{ when } n = 0,$$

$$\text{so } 2 = B - \frac{3}{2}$$

$$\therefore B = \frac{7}{2}$$

$$\therefore P = \frac{7}{2}e^{2n} - \frac{3}{2}$$

$$\mathbf{e} \quad \frac{dy}{dx} = k\sqrt{y}$$

$$\therefore \frac{1}{\sqrt{y}} \frac{dy}{dx} = k$$

$$\therefore \int \frac{1}{\sqrt{y}} \frac{dy}{dx} dx = \int k dx$$

$$\therefore \int \frac{1}{\sqrt{y}} dy = \int k dx$$

$$\therefore 2\sqrt{y} = kx + c$$

$$\text{Now } y(4) = 1,$$

$$\text{so } 4k + c = 2$$

$$\text{and } y(5) = 4,$$

$$\text{so } 5k + c = 4$$

$$\therefore k = 2$$

$$\text{and } c = -6$$

$$\therefore 2\sqrt{y} = 2x - 6$$

$$\therefore y = (x-3)^2$$

$$\begin{aligned}
 \mathbf{3} \quad \frac{dy}{dx} &= 2y & \therefore \int \frac{1}{y} dy &= \int 2 dx \\
 \therefore \frac{1}{y} \frac{dy}{dx} &= 2 & \therefore \ln|y| &= 2x + c \\
 & & \therefore y &= Ae^{2x} \\
 \therefore \int \frac{1}{y} \frac{dy}{dx} dx &= \int 2 dx & \text{i.e., the curve is an exponential function}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad \frac{dp}{dt} &= -\frac{1}{2}p & \mathbf{b} \quad \frac{dM}{dr} &= 8 - 2M \\
 \therefore \frac{1}{p} \frac{dp}{dt} &= -\frac{1}{2} & \therefore \frac{1}{8 - 2M} \frac{dM}{dr} &= 1 \\
 \therefore \int \frac{1}{p} \frac{dp}{dt} dt &= \int -\frac{1}{2} dt & \therefore \int \frac{1}{8 - 2M} \frac{dM}{dr} dr &= \int 1 dr \\
 \therefore \int \frac{1}{p} dp &= \int -\frac{1}{2} dt & \therefore \int \frac{1}{8 - 2M} dM &= \int 1 dr \\
 \therefore \ln|p| &= -\frac{1}{2}t + c & \therefore r &= -\frac{1}{2} \ln|8 - 2M| + c \\
 \therefore p &= Ae^{-\frac{1}{2}t} & \text{But when } r = 0, \quad M = 2 & \\
 \text{when } t = 0, \quad p = 10, & & \therefore -\frac{1}{2} \ln|8 - 4| + c = 0 & \\
 \text{so } A = 10 & & \therefore c = \frac{1}{2} \ln 4 = \ln 4^{\frac{1}{2}} = \ln 2 & \\
 \therefore p &= 10e^{-\frac{1}{2}t} & \text{Hence when } M = 3.5, & \\
 \therefore \text{when } t = 2, \quad p = \frac{10}{e} & & r = -\frac{1}{2} \ln|8 - 7| + \ln 2 & \\
 & & \therefore r = -\frac{1}{2} \ln 1 + \ln 2 = \ln 2 &
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad \frac{ds}{dt} + ks &= 0 & \therefore \ln|s| &= -kt + c \\
 \therefore \frac{ds}{dt} &= -ks & \therefore s &= Ae^{-kt} \quad \dots (*) \\
 \therefore \frac{1}{s} \frac{ds}{dt} &= -k & \text{Now } s(0) = 50, \text{ so } A = 50 \text{ and } s(3) = 20, & \\
 \therefore \int \frac{1}{s} \frac{ds}{dt} dt &= \int -k dt & \text{so } 50e^{-3k} = 20 & \\
 \therefore \frac{1}{s} ds &= \int -k dt & \therefore e^{-3k} = \frac{20}{50} = 0.4 & \\
 & & \text{But from } (*), \quad s = 50(e^{-3k})^{\frac{t}{3}}, & \\
 & & \therefore s = 50(0.4)^{\frac{t}{3}} \text{ as required} &
 \end{aligned}$$

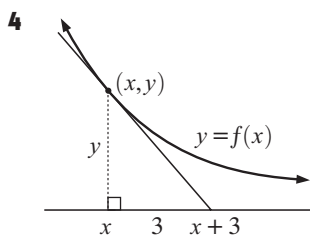
EXERCISE 29D.2

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad xy' &= 3y & \mathbf{b} \quad xy &= 4y' & \mathbf{c} \quad y' &= ye^x \\
 \therefore x \frac{dy}{dx} &= 3y & \therefore xy &= 4 \frac{dy}{dx} & \therefore \frac{dy}{dx} &= ye^x \\
 \therefore \frac{1}{y} \frac{dy}{dx} &= \frac{3}{x} & \therefore x &= \frac{4}{y} \frac{dy}{dx} & \therefore \frac{1}{y} \frac{dy}{dx} &= e^x \\
 \therefore \int \frac{1}{y} \frac{dy}{dx} dx &= \int \frac{3}{x} dx & \therefore \int x dx &= \int \frac{4}{y} \frac{dy}{dx} dx & \therefore \int \frac{1}{y} \frac{dy}{dx} dx &= \int e^x dx \\
 \therefore \int \frac{1}{y} dy &= \int \frac{3}{x} dx & \therefore \int x dx &= \int \frac{4}{y} dy & \therefore \int \frac{1}{y} dy &= \int e^x dx \\
 \therefore \ln|y| &= 3 \ln|x| + c & \therefore \frac{x^2}{2} &= 4 \ln|y| + c & \therefore \ln|y| &= e^x + c \\
 &= \ln|x^3| + c & \therefore \frac{x^2}{8} + d &= \ln|y| & \therefore y &= Ae^{e^x} \\
 \therefore y &= Ax^3 & \therefore y &= Ae^{\frac{x^2}{8}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & y' = xe^y \\
 & \therefore \frac{dy}{dx} = xe^y \\
 & \therefore e^{-y} \frac{dy}{dx} = x \\
 & \therefore \int e^{-y} \frac{dy}{dx} dx = \int x dx \\
 & \therefore \int e^{-y} dy = \int x dx \\
 & \therefore -e^{-y} = \frac{x^2}{2} + d \\
 & \therefore e^{-y} = -\frac{x^2}{2} - d \\
 & \therefore -y = \ln\left(-\frac{x^2}{2} + c\right) \\
 & \therefore y = -\ln\left(-\frac{x^2}{2} + c\right)
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad & \frac{dz}{dr} = z + zr^2 = z(1+r^2) \\
 & \therefore \frac{1}{z} \frac{dz}{dr} = 1 + r^2 \\
 & \therefore \int \frac{1}{z} \frac{dz}{dr} dr = \int (1+r^2) dr \\
 & \therefore \int \frac{1}{z} dz = \int (1+r^2) dr \\
 & \therefore \ln|z| = r + \frac{r^3}{3} + c \\
 & \text{But } z(0) = 1, \\
 & \text{so } \ln 1 = 0 + 0 + c \\
 & \therefore c = 0 \\
 & \therefore \ln|z| = r + \frac{r^3}{3} \\
 & \therefore z = e^{r + \frac{r^3}{3}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad & \frac{dy}{dx} = -2xy \\
 & \therefore \frac{1}{y} \frac{dy}{dx} = -2x \\
 & \therefore \int \frac{1}{y} \frac{dy}{dx} dx = \int -2x dx \\
 & \therefore \int \frac{1}{y} dy = \int -2x dx \\
 & \therefore \ln|y| = -x^2 + c \\
 & \therefore y = Ae^{-x^2} \\
 & \text{But } y(0) = 1, \\
 & \text{so } A = 1 \\
 & \therefore y = e^{-x^2}
 \end{aligned}$$



$$\begin{aligned}
 & \frac{dy}{dx} = -\frac{y}{3} \\
 & \therefore \frac{1}{y} \frac{dy}{dx} = -\frac{1}{3} \\
 & \therefore \int \frac{1}{y} \frac{dy}{dx} dx = \int -\frac{1}{3} dx \\
 & \therefore \int \frac{1}{y} dy = \int -\frac{1}{3} dx \\
 & \therefore \ln|y| = -\frac{1}{3}x + c \\
 & \therefore y = Ae^{-\frac{1}{3}x} \\
 & \text{But } y(0) = 2, \\
 & \text{so } 2 = Ae^0 \\
 & \therefore A = 2 \\
 & \text{So, } y = 2e^{-\frac{x}{3}}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{5} \quad & \frac{dy}{dx} = \frac{x}{y} \\
 & \therefore y \frac{dy}{dx} = x \\
 & \therefore \int y \frac{dy}{dx} dx = \int x dx \\
 & \therefore \int y dy = \int x dx \\
 & \therefore \frac{y^2}{2} = \frac{x^2}{2} + c \\
 & \text{i.e., } x^2 - y^2 = -2c
 \end{aligned}$$

The curve passes through $(5, -4)$,

$$\text{so } 5^2 - (-4)^2 = -2c$$

$$\text{i.e., } -2c = 9$$

$$\therefore x^2 - y^2 = 9$$

If $(a, 3)$ lies on the curve,

$$\text{then } a^2 - 3^2 = 9$$

$$\therefore a^2 = 18$$

$$\therefore a = \pm 3\sqrt{2}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad & \frac{dy}{dx} = y^2(1+x) \\
 & \therefore \frac{1}{y^2} \frac{dy}{dx} = 1+x \\
 & \therefore \int \frac{1}{y^2} \frac{dy}{dx} dx = \int (1+x) dx
 \end{aligned}$$

$$\begin{aligned}
 & \therefore \int y^{-2} dy = \int (1+x) dx \\
 & \therefore -\frac{1}{y} = x + \frac{1}{2}x^2 + c \\
 & \therefore y = \frac{1}{-x - \frac{1}{2}x^2 - c}
 \end{aligned}$$

But $(1, 2)$ lies on the curve,

$$\text{so } \frac{1}{-1 - \frac{1}{2} - c} = 2$$

$$\therefore \frac{1}{2} = -\frac{3}{2} - c$$

$$\therefore c = -2$$

$$\therefore y = \frac{1}{2 - x - \frac{1}{2}x^2}$$

$$\text{i.e., } y = \frac{-2}{x^2 + 2x - 4}$$

b When $x^2 + 2x - 4 = 0$,

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(-4)}}{2}$$

$$\therefore x = \frac{-2 \pm 2\sqrt{5}}{2} = -1 \pm \sqrt{5}$$

$$\therefore \text{VAs are } x = -1 + \sqrt{5}, \quad x = -1 - \sqrt{5}$$

$$\text{HA is } y = 0 \quad \{\text{as } |x| \rightarrow \infty, \quad y \rightarrow 0\}$$

7 a At any point (x, y) , the product of the slopes of the curves is $\left(-\frac{x}{y}\right)\left(\frac{y}{x}\right) = -1$
 $\therefore$ the solution curves are always at right angles.

b For $\frac{dy}{dx} = -\frac{x}{y}$

$$\therefore \int y \frac{dy}{dx} dx = \int -x dx$$

$$\therefore \int y dy = \int -x dx$$

$$\therefore \frac{y^2}{2} = -\frac{x^2}{2} + a$$

$$\text{i.e., } x^2 + y^2 = c$$

This represents a circle with centre $(0, 0)$

and radius $\sqrt{c}$ (1)

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\therefore \int \frac{1}{y} \frac{dy}{dx} dx = \int \frac{1}{x} dx$$

$$\therefore \int \frac{1}{y} dy = \int \frac{1}{x} dx$$

$$\therefore \ln |y| = \ln |x| + a$$

$$\therefore \ln |y| - \ln |x| = a$$

$$\therefore \ln \left| \frac{y}{x} \right| = a$$

$$\therefore \frac{y}{x} = \pm e^a = k, \quad \text{say}$$

$$\therefore y = kx \quad \text{..... (2)}$$

This represents a line with slope k that passes through $(0, 0)$.

Any line passing through the centre of a circle will include a diameter of the circle and therefore be normal to the circle. This is why the solution curves in **a** were at right angles.

8 acceleration $a \propto$ velocity v

$$\therefore \frac{dv}{dt} = kv \quad \text{for some } k$$

$$\therefore \frac{1}{v} \frac{dv}{dt} = k$$

$$\therefore \int \frac{1}{v} \frac{dv}{dt} dt = \int k dt$$

$$\therefore \int \frac{1}{v} dv = \int k dt$$

$$\therefore \ln |v| = kt + c$$

$$\text{when } t = 0, \quad v = 4$$

$$\text{so } c = \ln 4$$

Also, when $t = 4$, $v = 6$

$$\therefore 4k + \ln 4 = \ln 6$$

$$\therefore 4k = \ln \frac{6}{4}$$

$$\therefore k = \frac{1}{4} \ln \frac{3}{2}$$

$$\therefore \ln |v| = \frac{t}{4} \ln \frac{3}{2} + \ln 4$$

$$= \ln \left[\left(\frac{3}{2} \right)^{\frac{t}{4}} 4 \right]$$

$$\therefore v = 4 \left(\frac{3}{2} \right)^{\frac{t}{4}} \text{ ms}^{-1}$$

and when $t = 5$

$$v = 4 \left(\frac{3}{2} \right)^{\frac{5}{4}}$$

$$\text{i.e., } v \doteq 6.64 \text{ ms}^{-1}$$

9

$$\begin{aligned}\frac{dw}{dt} &\propto w \\ \therefore \frac{dw}{dt} &= kw \quad \text{for some } k \\ \therefore \frac{1}{w} \frac{dw}{dt} &= k \\ \therefore \int \frac{1}{w} \frac{dw}{dt} dt &= \int k dt \\ \therefore \int \frac{1}{w} dw &= \int k dt \\ \therefore \ln w &= kt + c \quad (\text{since } w \geq 0) \\ \therefore w &= Ae^{kt}\end{aligned}$$

If there was weight w_0 of sugar at time 0,

$$\begin{aligned}\text{then } A &= w_0 \\ \text{i.e., } w &= w_0 e^{kt}\end{aligned}$$

$$\text{When } t = 10, \quad w = 0.2w_0,$$

$$\text{so } w_0 e^{10k} = 0.2w_0$$

$$\therefore e^{10k} = 0.2$$

$$\text{But } w = w_0 (e^{10k})^{\frac{t}{10}}$$

$$\text{so } w = w_0 \times 0.2^{\frac{t}{10}}$$

$$\therefore \text{ when } t = 30, \quad w = w_0 \times 0.2^3$$

$$\therefore w = 0.008w_0$$

i.e., 0.8% is remaining

10

$$\begin{aligned}\frac{dI}{dt} &= -kI \\ \therefore \frac{1}{I} \frac{dI}{dt} &= -k \\ \therefore \int \frac{1}{I} \frac{dI}{dt} dt &= -\int k dt \\ \therefore \int \frac{1}{I} dI &= -\int k dt \\ \therefore \ln |I| &= -kt + c\end{aligned}$$

If $I(0) = I_0$, then $c = I_0$

$$\therefore \ln I = -kt + \ln I_0$$

$$\therefore I = I_0 e^{-kt}$$

$$\text{When } t = 1, \quad I = 0.1I_0,$$

$$\text{so } I_0 e^{-k} = 0.1I_0$$

$$\therefore e^{-k} = 0.1$$

$$\therefore I = I_0 0.1^t$$

$$\therefore \text{ when } I = 0.001I_0,$$

$$I_0 0.1^t = 0.001I_0$$

$$\text{i.e., } t = 3 \text{ seconds}$$

11 a

$$\begin{aligned}\frac{dv}{dt} &= g - 4v \\ \therefore \frac{1}{g - 4v} \frac{dv}{dt} &= 1 \\ \therefore \int \frac{1}{g - 4v} \frac{dv}{dt} dt &= \int 1 dt \\ \therefore \int \frac{1}{g - 4v} dv &= \int 1 dt \\ \therefore -\frac{1}{4} \ln |g - 4v| &= t + c \\ \therefore \ln |g - 4v| &= -4t - 4c \\ \therefore g - 4v &= Ae^{-4t} \\ \text{i.e., } 4v &= g - Ae^{-4t}\end{aligned}$$

Since the metal is released from rest,

$$v(0) = 0$$

$$\therefore g - A = 0$$

$$\therefore A = g$$

$$\therefore 4v = g - ge^{-4t}$$

$$\therefore v = \frac{g}{4}(1 - e^{-4t}) \quad \text{as required}$$

$$\text{as } t \rightarrow \infty, \quad e^{-4t} \rightarrow 0 \quad \text{and } \therefore v(t) \rightarrow \frac{g}{4} \text{ ms}^{-1}$$

$$\therefore \text{ there is a limiting velocity of } \frac{g}{4} \text{ ms}^{-1}$$

$$\text{b } v = \frac{g}{10} \quad \text{when } \frac{g}{10} = \frac{g}{4}(1 - e^{-4t})$$

$$\therefore 1 - e^{-4t} = \frac{4}{10}$$

$$\therefore e^{-4t} = \frac{3}{5}$$

$$\therefore -4t = \ln \frac{3}{5}$$

$$\therefore t = -\frac{1}{4} \ln \frac{3}{5} \doteq 0.128 \text{ sec}$$

- 12 a** Suppose V_0 is the initial volume of water in the lake, and $V(t)$ is the total amount of water that has evaporated at time t . Then the volume of water remaining in the lake is $(V_0 - V)$.

Hence $\frac{dV}{dt} \propto (V_0 - V)$ i.e., $\frac{dV}{dt} = k(V_0 - V)$

b $\frac{1}{V_0 - V} \frac{dV}{dt} = k$

$$\therefore \int \frac{1}{V_0 - V} \frac{dV}{dt} dt = \int k dt$$

$$\therefore \int \frac{1}{V_0 - V} dV = \int k dt$$

$$\therefore -\ln|V_0 - V| = kt + c$$

$$\therefore \ln|V_0 - V| = -kt - c$$

$$\therefore V_0 - V = Ae^{-kt}$$

But $V(0) = 0$, so $A = V_0$

After 20 days, $V = \frac{1}{2}V_0$

$$\text{so } V_0 - \frac{1}{2}V_0 = V_0 e^{-20k}$$

$$\therefore e^{-20k} = \frac{1}{2}$$

$$\therefore V_0 - V = V_0 (e^{-20k})^{\frac{t}{20}} = V_0 \left(\frac{1}{2}\right)^{\frac{t}{20}}$$

$$\therefore V = V_0 \left(1 - 0.5^{\frac{t}{20}}\right)$$

Hence after 50 days without rain,

$$V = V_0(1 - 0.5^{2.5}) \doteq 0.8232V_0$$

$\therefore$ 82.3% has evaporated and 17.7% remains.

- 13** Let V be the volume of water in the tank and h be the water depth.

Then $\frac{dV}{dt} \propto \sqrt{h}$,

i.e., $\frac{dV}{dt} = k\sqrt{h}$ for some k

We also know that $V = \pi r^2 h$

where $r = 2$ m is the radius of the tank

Hence $V = 4\pi h$

and $\frac{dV}{dh} = 4\pi$

Now $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ {Chain Rule}

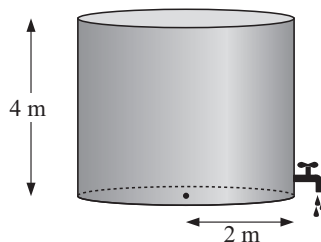
so $k\sqrt{h} = 4\pi \times \frac{dh}{dt}$

$$\therefore \frac{1}{\sqrt{h}} \frac{dh}{dt} = \frac{k}{4\pi}$$

$$\therefore \int \frac{1}{\sqrt{h}} \frac{dh}{dt} dt = \int \frac{k}{4\pi} dt$$

$$\therefore \int \frac{1}{\sqrt{h}} dh = \int \frac{k}{4\pi} dt$$

$$\therefore 2\sqrt{h} = \frac{k}{4\pi} t + c$$



Initially the tank is full,

so $h(0) = 4$

$$\therefore 4 = 0 + c$$

i.e., $2\sqrt{h} = \frac{k}{4\pi} t + 4$

Also, $h(2) = 1$

so $2 = \frac{k}{4\pi} \times 2 + 4$

$$\therefore \frac{k}{4\pi} = -1$$

$$\therefore k = -4\pi$$

$$\therefore 2\sqrt{h} = 4 - t$$

$\therefore$ the tank is empty when $t = 4$

i.e., it takes 4 hours to empty.

14 a $V = \frac{1}{3}\pi h^2(3r - h)$
 $= \pi h^2 r - \frac{1}{3}\pi h^3$

$\therefore$ for fixed r , $\frac{dV}{dh} = 2\pi hr - \pi h^2$

Now $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ {Chain Rule}

$\therefore$ since $\frac{dV}{dt} = -r^2$,

$$-r^2 = (2\pi hr - \pi h^2) \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{r^2}{\pi h^2 - 2\pi hr}$$

$$\mathbf{b} \quad (\pi h^2 - 2\pi hr) \frac{dh}{dt} = r^2$$

$$\therefore \int (\pi h^2 - 2\pi hr) \frac{dh}{dt} dt = \int r^2 dt$$

$$\therefore \int (\pi h^2 - 2\pi hr) dh = \int r^2 dt$$

$$\therefore \frac{1}{3}\pi h^3 - \pi h^2 r = r^2 t + c$$

If $r = 10$ cm and the bowl is initially full,

i.e., $h = 10$ cm,

$$\text{then } \frac{1000}{3}\pi - 1000\pi = c$$

$$\text{i.e., } c = -\frac{2000}{3}\pi$$

$$\therefore 100t = \frac{1}{3}\pi h^3 - 10\pi h^2 + \frac{2000}{3}\pi$$

$$\therefore t = \frac{\pi}{300}h^3 - \frac{\pi}{10}h^2 + \frac{2000}{300}\pi$$

$$\therefore t = \frac{\pi}{300}[h^3 - 30h^2 + 2000]$$

$$t = \frac{1375\pi}{300} \div 14.4 \text{ hours}$$

$$\mathbf{15} \quad \frac{dT}{dt} \propto (T - T_m)$$

$$\therefore \frac{dT}{dt} = k(T - T_m) \text{ for some } k$$

$$\therefore \frac{1}{T - T_m} \frac{dT}{dt} = k$$

$$\therefore \int \frac{1}{T - T_m} dT = \int k dt$$

$$\therefore \ln|T - T_m| = kt + c$$

$$\therefore T - T_m = Ae^{kt}$$

$$\text{Given } T_m = 5^\circ\text{C}$$

$$\text{and that } T = 100^\circ\text{C} \text{ when } t = 0,$$

$$Ae^0 = 100 - 5$$

$$\therefore A = 95$$

$$\text{Also, since } T(1) = 80,$$

$$95e^k = 80 - 5 = 75$$

$$\therefore e^k = \frac{15}{19}$$

$$\therefore T - 5 = 95\left(\frac{15}{19}\right)^t$$

$$\text{i.e., } T = 5 + 95\left(\frac{15}{19}\right)^t$$

$$\text{Now } T = 10^\circ\text{C}$$

$$\text{when } 10 = 5 + 95\left(\frac{15}{19}\right)^t$$

$$\therefore \left(\frac{15}{19}\right)^t = \frac{5}{95} = \frac{1}{19}$$

$$\therefore t \ln \frac{15}{19} = \ln \frac{1}{19}$$

$$\therefore t = \frac{\ln \frac{1}{19}}{\ln \frac{15}{19}}$$

$$\therefore t \div 12.5 \text{ minutes}$$

16 From question **15**, a general solution to Newton's Law of Cooling is $T - T_m = Ae^{kt}$.

Assume $t = 0$ to be 6 am, so $T(0) = 13$.

Then since $T_m = 5^\circ\text{C}$

$$Ae^0 = 13 - 5 = 8$$

$$\therefore A = 8$$

$$\therefore T = 5 + 8e^{kt}$$

$$\text{Also, } T(3) = 9,$$

$$\text{so } 8e^{3k} = 4$$

$$\therefore e^{3k} = \frac{1}{2}$$

$$\therefore T = 5 + 8(e^{3k})^{\frac{t}{3}}$$

$$= 5 + 8\left(\frac{1}{2}\right)^{\frac{t}{3}}$$

i.e., time of death was 6 hours before 6 am,

i.e., midnight.

$\therefore$ the temperature was 37°C when

$$5 + 8\left(\frac{1}{2}\right)^{\frac{t}{3}} = 37$$

$$\therefore \left(\frac{1}{2}\right)^{\frac{t}{3}} = \frac{32}{8} = 4$$

$$\therefore \frac{t}{3} = -2$$

$$\therefore t = -6 \text{ hours}$$

17 a $z = r \cos \theta + ir \sin \theta$

Treating r and i as constants,

$$\begin{aligned}\frac{dz}{d\theta} &= r(-\sin \theta) + ir \cos \theta \\ &= i(r \cos \theta + ir \sin \theta) \\ &= iz\end{aligned}$$

c Thus, $r \cos \theta + ir \sin \theta = re^{i\theta}$
 $\therefore \cos \theta + i \sin \theta = e^{i\theta}$
 $\therefore \text{cis } \theta = e^{i\theta}$

d i If $z = |z|\text{cis } \theta$ and $w = |w|\text{cis } \phi$
 then $z = |z|e^{i\theta}$ and $w = |w|e^{i\phi}$
 So, $zw = |z||w|e^{i\theta+i\phi}$
 i.e., $zw = |z||w|e^{i(\theta+\phi)}$
 So, $|zw| = |z||w|$
 and $\arg(zw) = \theta + \phi$
 $= \arg z + \arg w$

ii $\frac{\text{cis } \theta}{\text{cis } \phi} = \frac{e^{i\theta}}{e^{i\phi}}$
 $= e^{i\theta-i\phi}$
 $= e^{i(\theta-\phi)}$
 $= \text{cis } (\theta - \phi)$

b Since $\frac{dz}{d\theta} = iz$

then $\frac{1}{z} \frac{dz}{d\theta} = i$

$$\therefore \int \frac{1}{z} \frac{dz}{d\theta} d\theta = \int i d\theta$$

$$\therefore \int \frac{1}{z} dz = \int i d\theta$$

$$\therefore \ln |z| = i\theta + c$$

$$\therefore z = Ae^{i\theta}$$

But when $\theta = 0$, $z = r$

$$\therefore z = re^{i\theta}$$

iii Using $e^{i\theta} = \cos \theta + i \sin \theta$

and letting $\theta = \pi$,

$$\begin{aligned}e^{i\pi} &= \cos \pi + i \sin \pi \\ &= -1 + i(0) \\ &= -1\end{aligned}$$

$$\therefore e^{i\pi} + 1 = 0$$

EXERCISE 29D.3

1 $\frac{dy}{dx} - 2xe^x = y$ so $\frac{dy}{dx} = 2xe^x + y$

Now letting $y = ue^x$, $\frac{dy}{dx} = \frac{du}{dx}e^x + ue^x$

$$\therefore 2xe^x + ue^x = \frac{du}{dx}e^x + ue^x$$

$$\therefore \frac{du}{dx}e^x = 2xe^x$$

$$\therefore \frac{du}{dx} = 2x$$

$$\therefore \int \frac{du}{dx} dx = \int 2x dx$$

$$\therefore \int 1 du = \int 2x dx$$

$$\therefore u = x^2 + c$$

$$\therefore \frac{y}{e^x} = x^2 + c$$

$$\therefore y = e^x(x^2 + c)$$

2 If $y = ue^x$, then $\frac{dy}{dx} = \frac{du}{dx}e^x + ue^x$

But $\left(\frac{dy}{dx}\right)^2 = y^2 + 2e^xy + e^{2x}$

$$\therefore \left[\frac{du}{dx}e^x + ue^x\right]^2 = u^2e^{2x} + 2e^x(ue^x) + e^{2x}$$

$$\begin{aligned}\therefore \left[\frac{du}{dx}e^x\right]^2 + 2ue^{2x}\frac{du}{dx} + u^2e^{2x} \\ = u^2e^{2x} + 2ue^{2x} + e^{2x}\end{aligned}$$

$$\therefore \left(\frac{du}{dx}\right)^2 e^{2x} + 2ue^{2x} \left(\frac{du}{dx}\right) = 2ue^{2x} + e^{2x}$$

$$\therefore \left(\frac{du}{dx}\right)^2 + 2u \frac{du}{dx} = 2u + 1$$

$$\therefore \left(\frac{du}{dx}\right)^2 + 2u \frac{du}{dx} - 2u - 1 = 0$$

$$\therefore \left(\frac{du}{dx} - 1\right)\left(\frac{du}{dx} + 2u + 1\right) = 0$$

$$\therefore \frac{du}{dx} = 1 \text{ or } -2u - 1$$

$$\begin{array}{ll}
 \text{If } \frac{du}{dx} = 1 & \text{then} \qquad \qquad \text{or,} \qquad \qquad \text{if } \frac{du}{dx} = -2u - 1 & \text{then} \\
 \int \frac{du}{dx} dx = \int 1 dx & \qquad \qquad \int \frac{1}{2u+1} \frac{du}{dx} dx = \int -1 dx \\
 \therefore \int 1 du = x + c & \qquad \qquad \therefore \int \frac{1}{2u+1} du = \int -1 dx \\
 \therefore u = x + c & \qquad \qquad \therefore \frac{1}{2} \ln |2u+1| = -x + c \\
 \text{i.e., } \frac{y}{e^x} = x + c & \qquad \qquad \therefore 2u+1 = Ae^{-2x} \\
 \therefore y = e^x(x+c) & \qquad \qquad \text{i.e., } \frac{2y}{e^x} + 1 = Ae^{-2x} \\
 & \qquad \qquad \therefore 2y + e^x = Ae^{-x} \\
 & \qquad \qquad \therefore y = \frac{Ae^{-x} - e^x}{2}
 \end{array}$$

Hence $y = e^x(x+c)$ or $y = \frac{Ae^{-x} - e^x}{2}$

$$\begin{array}{ll}
 \mathbf{3} & 4xy \frac{dy}{dx} = -x^2 - y^2, \quad x > 0 \quad \therefore \int \frac{4u}{5u^2+1} \frac{du}{dx} dx = \int -\frac{1}{x} dx \\
 \text{Let } y = ux & \text{so } \frac{dy}{dx} = \frac{du}{dx} x + u \\
 \therefore 4x(ux) \left[\frac{du}{dx} x + u \right] & = -x^2 - u^2 x^2 \\
 \therefore 4ux^2 \left[\frac{du}{dx} x + u \right] & = -x^2 - u^2 x^2 \\
 \therefore 4u \left[\frac{du}{dx} x + u \right] & = -1 - u^2 \\
 \therefore x \frac{du}{dx} + u & = \frac{-u^2 - 1}{4u} \\
 \therefore x \frac{du}{dx} & = \frac{-u^2 - 1 - 4u^2}{4u} \\
 \therefore x \frac{du}{dx} & = \frac{-5u^2 - 1}{4u} \\
 \therefore \frac{4u}{5u^2+1} \frac{du}{dx} & = -\frac{1}{x} \\
 & \therefore \int \frac{4u}{5u^2+1} du = -\int \frac{1}{x} dx \\
 & \therefore \frac{4}{10} \int \frac{10u}{5u^2+1} du = -\int \frac{1}{x} dx \\
 & \therefore \frac{2}{5} \ln |5u^2+1| = -\ln |x| + c \\
 & \therefore \ln |5u^2+1| = -\frac{5}{2} \ln |x| + c \\
 & \therefore \frac{5y^2}{x^2} + 1 = Ax^{-\frac{5}{2}} \\
 & \therefore \frac{x^2 + 5y^2}{x^2} = Ax^{-\frac{5}{2}} \\
 & \therefore \sqrt{x} (x^2 + 5y^2) = A \\
 & \therefore x (x^2 + 5y^2)^2 = k
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{4} & x \frac{dy}{dx} - y = 4x^2 y \\
 \text{Let } y = ux, & \text{so } \frac{dy}{dx} = \frac{du}{dx} x + u \\
 \therefore x \left[\frac{du}{dx} x + u \right] - ux & = 4x^2 (ux) \\
 \therefore \frac{du}{dx} x^2 + ux - ux & = 4ux^3 \\
 \therefore \frac{du}{dx} x^2 & = 4ux^3 \\
 \therefore \frac{du}{dx} & = 4ux \\
 & \therefore \frac{1}{u} \frac{du}{dx} = 4x \\
 & \therefore \int \frac{1}{u} \frac{du}{dx} dx = \int 4x dx \\
 & \therefore \int \frac{1}{u} du = \int 4x dx \\
 & \therefore \ln |u| = \frac{4x^2}{2} + c \\
 & \therefore u = Ae^{2x^2} \\
 \text{But } y = ux, & \text{so } \frac{y}{x} = Ae^{2x^2} \\
 & \therefore y = Axe^{2x^2}
 \end{array}$$

REVIEW SET 29A

1 Let $u = 4 - x$ so $\frac{du}{dx} = -1$

$$\begin{aligned} \therefore \int x^2 \sqrt{4-x} \, dx &= \int (4-u)^2 \sqrt{u} \left(-\frac{du}{dx}\right) dx \\ &= - \int (16 - 8u + u^2) u^{\frac{1}{2}} \, du \\ &= - \int \left(16u^{\frac{1}{2}} - 8u^{\frac{3}{2}} + u^{\frac{5}{2}}\right) du \\ &= - \left[\frac{16u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{8u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{7}{2}}}{\frac{7}{2}} \right] + c \\ &= -\frac{32}{3} u^{\frac{3}{2}} + \frac{16}{5} u^{\frac{5}{2}} - \frac{2}{7} u^{\frac{7}{2}} + c \\ &= -\frac{32}{3} (4-x)^{\frac{3}{2}} + \frac{16}{5} (4-x)^{\frac{5}{2}} - \frac{2}{7} (4-x)^{\frac{7}{2}} + c \end{aligned}$$

2 $\int \arctan x \, dx$

$$\begin{aligned} &= \int 1 \arctan x \, dx \quad \left\{ \begin{array}{l} u = \arctan x \quad v' = 1 \\ u' = \frac{1}{x^2+1} \quad v = x \end{array} \right\} \\ &= x \arctan x - \int \frac{1}{x^2+1} (x) \, dx \\ &= x \arctan x - \frac{1}{2} \int \frac{2x}{x^2+1} \, dx \\ &= x \arctan x - \frac{1}{2} \ln |x^2+1| + c \\ &= x \arctan x - \frac{1}{2} \ln(x^2+1) + c \quad \{x^2+1 > 0\} \end{aligned}$$

3 a $\int e^{-x} \cos x \, dx \quad \left\{ \begin{array}{l} u = e^{-x} \quad v' = \cos x \\ u' = -e^{-x} \quad v = \sin x \end{array} \right\}$

$$\begin{aligned} &= e^{-x} \sin x - \int -e^{-x} \sin x \, dx \\ &= e^{-x} \sin x + \int e^{-x} \sin x \, dx \quad \left\{ \begin{array}{l} u = e^{-x} \quad v' = \sin x \\ u' = -e^{-x} \quad v = -\cos x \end{array} \right\} \\ &= e^{-x} \sin x + e^{-x} (-\cos x) - \int (-e^{-x}) (-\cos x) \, dx \\ &= e^{-x} \sin x - e^{-x} \cos x - \int e^{-x} \cos x \, dx + c_1 \\ \therefore 2 \int e^{-x} \cos x \, dx &= e^{-x} (\sin x - \cos x) + c_1 \\ \therefore \int e^{-x} \cos x \, dx &= \frac{1}{2} e^{-x} (\sin x - \cos x) + c \end{aligned}$$

c Let $u = 9 - x^2$ so $\frac{du}{dx} = -2x$

$$\begin{aligned} \therefore \int \frac{x^3}{\sqrt{9-x^2}} \, dx &= \int \frac{x^2}{\sqrt{9-x^2}} x \, dx \\ &= \int \frac{1}{\sqrt{u}} (9-u) \left(-\frac{1}{2} \frac{du}{dx}\right) dx \\ &= -\frac{1}{2} \int \frac{9-u}{\sqrt{u}} \, du \\ &= -\frac{1}{2} \int \left(9u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right) du \\ &= -\frac{1}{2} \left[\frac{9u^{\frac{1}{2}}}{\frac{1}{2}} - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right] + c \\ &= -9u^{\frac{1}{2}} + \frac{1}{3} u^{\frac{3}{2}} + c \\ &= -9\sqrt{9-x^2} + \frac{1}{3} (9-x^2)^{\frac{3}{2}} + c \end{aligned}$$

b $\int x^2 e^x \, dx \quad \left\{ \begin{array}{l} u = x^2 \quad v' = e^x \\ u' = 2x \quad v = e^x \end{array} \right\}$

$$\begin{aligned} &= x^2 e^x - \int 2x e^x \, dx \quad \left\{ \begin{array}{l} u = 2x \quad v' = e^x \\ u' = 2 \quad v = e^x \end{array} \right\} \\ &= x^2 e^x - [2x e^x - \int 2e^x \, dx] \\ &= x^2 e^x - 2x e^x + 2 \int e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + c \\ &= e^x (x^2 - 2x + 2) + c \end{aligned}$$

4 $y' = -\frac{2e^x}{y}$

$$\begin{aligned} \therefore y \frac{dy}{dx} &= -2e^x \\ \therefore \int y \, dy &= \int -2e^x \, dx \\ \therefore \frac{y^2}{2} &= -2e^x + c \end{aligned}$$

But $y(0) = 4$,

$$\begin{aligned} \text{so } \frac{16}{2} &= -2e^0 + c \\ \therefore c &= 10 \\ \therefore \frac{y^2}{2} &= -2e^x + 10 \\ \therefore y^2 &= 20 - 4e^x \end{aligned}$$

$$5 \quad L \frac{dI}{dt} = E - RI$$

Using $R = 4$, $L = 0.2$ and $E = 20$,

$$\frac{1}{5} \frac{dI}{dt} = 20 - 4I = 4(5 - I)$$

$$\therefore \frac{1}{5 - I} \frac{dI}{dt} = 20$$

$$\therefore \int \frac{1}{5 - I} \frac{dI}{dt} dt = \int 20 dt$$

$$\therefore \int \frac{1}{5 - I} dI = \int 20 dt$$

$$\therefore -\ln |5 - I| = 20t + c$$

But $I(0) = 0$, so $c = -\ln 5$

$$\therefore 20t = \ln 5 - \ln |5 - I|$$

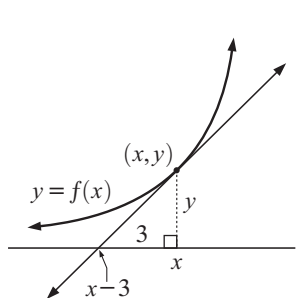
$$\therefore \therefore t = \frac{1}{20} \ln \left| \frac{5}{5 - I} \right|$$

The current is therefore 0.5 amps at time

$$t = \frac{1}{20} \ln \left(\frac{5}{4.5} \right)$$

$$t \doteq 0.00527 \text{ sec}$$

6



$$\frac{dy}{dx} = \frac{y}{3}$$

$$\therefore \frac{1}{y} \frac{dy}{dx} = \frac{1}{3}$$

$$\therefore \int \frac{1}{y} \frac{dy}{dx} dx = \int \frac{1}{3} dx$$

$$\therefore \int \frac{1}{y} dy = \int \frac{1}{3} dx$$

$$\therefore \ln |y| = \frac{1}{3}x + c$$

$$\therefore y = Ae^{\frac{1}{3}x}$$

But when $x = 0$, $y = 3$

$$\therefore 3 = Ae^0$$

$$\therefore A = 3$$

$$\text{So, } y = 3e^{\frac{1}{3}x}$$

$$7 \quad \text{Letting } y = ux, \quad \frac{dy}{dx} = \frac{du}{dx}x + u$$

So, $2xy \frac{dy}{dx} = x^2 + y^2$ becomes

$$2x(ux) \left[\frac{du}{dx}x + u \right] = x^2 + (ux)^2$$

$$\therefore 2ux^2 \left[x \frac{du}{dx} + u \right] = x^2 + u^2x^2$$

$$\therefore 2u \left[x \frac{du}{dx} + u \right] = 1 + u^2 \quad \{\div \text{ each term by } x^2\}$$

$$\therefore x \frac{du}{dx} + u = \frac{1 + u^2}{2u}$$

$$\therefore x \frac{du}{dx} = \frac{1 - u^2}{2u}$$

$$\therefore \frac{2u}{1 - u^2} \frac{du}{dx} = \frac{1}{x}$$

$$\therefore \int \frac{2u}{1 - u^2} \frac{du}{dx} dx = \int \frac{1}{x} dx$$

$$\therefore \int \frac{2u}{1 - u^2} du = \int \frac{1}{x} dx$$

$$\therefore -\int \frac{-2u}{1 - u^2} du = \int \frac{1}{x} dx$$

$$\therefore -\ln |1 - u^2| = \ln |x| + c$$

$$\therefore \ln |x(1 - u^2)| = -c$$

$$\therefore x(1 - u^2) = A$$

$$\therefore x \left(1 - \frac{y^2}{x^2} \right) = A$$

$$\therefore x \left(\frac{x^2 - y^2}{x^2} \right) = A$$

$$\text{i.e., } x^2 - y^2 = Ax$$

$$8 \quad \mathbf{a} \quad \text{cis } \theta \text{ cis } \phi$$

$$= e^{i\theta} \times e^{i\phi}$$

$$= e^{i\theta + i\phi}$$

$$= e^{i(\theta + \phi)}$$

$$= \text{cis } (\theta + \phi)$$

$$\mathbf{b} \quad (\text{cis } \theta)^n$$

$$= (e^{i\theta})^n$$

$$= e^{i\theta n}$$

$$= e^{i(n\theta)}$$

$$= \text{cis } n\theta$$

REVIEW SET 29B

$$\begin{aligned}
 \mathbf{1} \quad \mathbf{a} \quad & \int \frac{5}{\sqrt{9-x^2}} dx \\
 &= 5 \int \frac{1}{\sqrt{3^2-x^2}} dx \\
 &= 5 \arcsin\left(\frac{x}{3}\right) + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \frac{1}{9+4x^2} dx \\
 &= \frac{1}{4} \int \frac{1}{\frac{9}{4}+x^2} dx \\
 &= \frac{1}{4} \left(\frac{1}{\frac{3}{2}}\right) \arctan\left(\frac{x}{\frac{3}{2}}\right) + c \\
 &= \frac{1}{6} \arctan\left(\frac{2x}{3}\right) + c
 \end{aligned}$$

$$\mathbf{c} \quad \text{Let } u = x - 5 \quad \text{so} \quad \frac{du}{dx} = 1$$

$$\begin{aligned} \text{When } x = 10, \quad u &= 5 \\ \text{and when } x = 7, \quad u &= 2 \end{aligned}$$

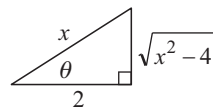
$$\begin{aligned}
 \therefore \int_7^{10} x\sqrt{x-5} dx &= \int_2^5 (u+5)\sqrt{u} du \\
 &= \int_2^5 \left(u^{\frac{3}{2}} + 5u^{\frac{1}{2}}\right) du \\
 &= \left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{5u^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^5
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{5} \left(5^{\frac{5}{2}}\right) + \frac{10}{3} \left(5^{\frac{3}{2}}\right) - \left[\frac{2}{5} \left(2^{\frac{5}{2}}\right) + \frac{10}{3} \left(2^{\frac{3}{2}}\right)\right] \\
 &= \frac{2}{5} (25\sqrt{5}) + \frac{10}{3} (5\sqrt{5}) - \frac{2}{5} (4\sqrt{2}) - \frac{10}{3} (2\sqrt{2}) \\
 &= 10\sqrt{5} + \frac{50}{3}\sqrt{5} - \frac{8}{5}\sqrt{2} - \frac{20}{3}\sqrt{2} \\
 &= \frac{80}{3}\sqrt{5} - \frac{124}{15}\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & \int x \cos x dx \\
 \text{has } u &= x \quad v' = \cos x \\
 u' &= 1 \quad v = \sin x \\
 \therefore \int x \cos x dx &= x \sin x - \int \sin x dx \\
 &= x \sin x - (-\cos x) + c \\
 &= x \sin x + \cos x + c
 \end{aligned}$$

$$\mathbf{b} \quad \text{Let } x = 2 \sec \theta \quad \text{so} \quad \frac{dx}{d\theta} = 2 \sec \theta \tan \theta$$

$$\begin{aligned}
 \therefore \int \frac{\sqrt{x^2-4}}{x} dx &= \int \frac{\sqrt{4 \sec^2 \theta - 4}}{2 \sec \theta} \times 2 \sec \theta \tan \theta d\theta \\
 &= \int \sqrt{4(\sec^2 \theta - 1)} \tan \theta d\theta \\
 &= \int 2 \tan \theta \tan \theta d\theta \\
 &= 2 \int \tan^2 \theta d\theta \\
 &= 2 \int (\sec^2 \theta - 1) d\theta \\
 &= 2[\tan \theta - \theta] + c \\
 &= 2 \tan \theta - 2\theta + c \\
 &= 2 \left(\frac{\sqrt{x^2-4}}{2} \right) - 2 \arccos\left(\frac{2}{x}\right) + c \\
 &= \sqrt{x^2-4} - 2 \arccos\left(\frac{2}{x}\right) + c
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{3} \quad & \frac{dy}{dx} = \frac{1}{y+2} \\
 \therefore (y+2) \frac{dy}{dx} &= 1 \\
 \therefore \int (y+2) dy &= \int 1 dx \\
 \therefore \frac{y^2}{2} + 2y &= x + c
 \end{aligned}$$

$$\begin{aligned} \text{But when } x = 0, \quad y &= 0 \\ \therefore c &= 0 \end{aligned}$$

$$\therefore \frac{y^2}{2} + 2y = x$$

$$\therefore y^2 + 4y = 2x$$

$$\therefore (y+2)^2 = 2x+4$$

$$\therefore y+2 = \pm\sqrt{2x+4}$$

However, since $y = 0$ when $x = 0$,

$$y+2 = +\sqrt{2x+4}$$

$$\therefore y = \sqrt{2x+4} - 2$$

$$4 \quad \frac{dy}{dx} = \frac{2x}{\cos y}$$

$$\therefore \cos y \frac{dy}{dx} = 2x$$

$$\therefore \int \cos y \, dy = \int 2x \, dx$$

$$\therefore \sin y = x^2 + c$$

$$\text{But } y(1) = \frac{\pi}{2}$$

$$\therefore \sin\left(\frac{\pi}{2}\right) = 1 + c$$

$$\therefore c = 0$$

$$\therefore \sin y = x^2$$

$$\text{i.e., } y = \arcsin(x^2) \text{ as required}$$

- 5 a Since the uniformly inclined plane has highest point P and a 5 m horizontal base,

$$\text{it has slope } \frac{dy}{dx} = -\frac{y}{5}$$

$$\therefore \frac{1}{y} \frac{dy}{dx} = -\frac{1}{5}$$

$$\therefore \int \frac{1}{y} \frac{dy}{dx} dx = \int -\frac{1}{5} dx$$

$$\therefore \int \frac{1}{y} dy = \int -\frac{1}{5} dx$$

$$\therefore \ln|y| = -\frac{1}{5}x + c$$

$$\therefore y = Ae^{-\frac{1}{5}x}$$

$$\text{Now when } x = 0, y = 10,$$

$$\text{so } A = 10$$

$$\therefore y = 10e^{-\frac{1}{5}x} \text{ as required.}$$

b $x = 30$ m at L

$$\therefore y = 10e^{-\frac{30}{5}}$$

$$= 10e^{-6}$$

$$\div 0.0248 \text{ m}$$

c Since $\frac{dy}{dx} = -\frac{y}{5},$

$$\text{slope at H} = -\frac{10}{5} = -2$$

$$\text{and slope at L} \div -\frac{0.02479}{5}$$

$$\div -0.00496$$

- 6 a Letting $N(t)$ be the number of bacteria at time t ,

$$\frac{dN}{dt} = kN \text{ for some } k$$

b Using a, $\frac{1}{N} \frac{dN}{dt} = k$

$$\therefore \int \frac{1}{N} dN = \int k dt$$

$$\therefore \ln|N| = kt + c$$

$$\therefore N = Ae^{kt}$$

$$\text{The initial population was } N(0) = 10^5, \therefore A = 10^5$$

$$\text{i.e., } N = 10^5 e^{kt}$$

Since the population doubles every 37 minutes,

$$10^5 e^{k(t+37)} = 2 \times 10^5 e^{kt}$$

$$\therefore e^{kt} e^{37k} = 2e^{kt}$$

$$\therefore (e^k)^{37} = 2$$

$$\therefore e^k = 2^{\frac{1}{37}}$$

$$\therefore N = 10^5 (2^{\frac{1}{37}})^t$$

$$\text{i.e., } N = 10^5 \times 2^{\frac{t}{37}}$$

$$\text{So after 4 hours, or 240 mins, the population is } N(240) = 10^5 \times 2^{\frac{240}{37}}$$

$$\div 8.97 \times 10^6$$

$$\begin{aligned}
 \mathbf{7 \quad a} \quad & \frac{dy}{dx} = (y-1)^2(2+x) & \therefore -2 + \frac{1}{2} + c = -1 \\
 & \therefore (y-1)^{-2} \frac{dy}{dx} = 2+x & \therefore c = \frac{1}{2} \\
 & \therefore \int (y-1)^{-2} \frac{dy}{dx} dx = \int (2+x) dx & \therefore -\frac{1}{y-1} = 2x + \frac{x^2}{2} + \frac{1}{2} \\
 & \therefore \int (y-1)^{-2} dy = \int (2+x) dx & \therefore y-1 = -\frac{1}{\frac{x^2}{2} + 2x + \frac{1}{2}} \\
 & \therefore -(y-1)^{-1} = 2x + \frac{x^2}{2} + c & = -\frac{2}{x^2 + 4x + 1} \\
 & \text{But the curve passes through } (-1, 2) & \therefore y = 1 - \frac{2}{x^2 + 4x + 1}
 \end{aligned}$$

b As $x \rightarrow \pm\infty$, $y \rightarrow 1$.

$\therefore y = 1$ is a horizontal asymptote.

The function is undefined when $x^2 + 4x + 1 = 0$

$$\text{i.e., } x = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

$\therefore$ vertical asymptotes are $x = -2 + \sqrt{3}$ and $x = -2 - \sqrt{3}$

8 If V is the volume of water in the tank, then

$$\frac{dV}{dt} \propto V$$

$$\therefore \frac{1}{V} \frac{dV}{dt} = k \quad \text{for some } k$$

$$\therefore \int \frac{1}{V} dV = \int k dt$$

$$\therefore \ln |V| = kt + c$$

$$\text{i.e., } V = Ae^{kt}$$

At time zero, the tank is full, and we call this volume V_0 .

Then $A = V_0$,

and $V = V_0 e^{kt}$

But after 20 minutes, the tank is half full

$$\therefore V_0 e^{20k} = \frac{1}{2} V_0$$

$$\therefore (e^k)^{20} = \frac{1}{2}$$

$$\text{i.e., } e^k = \left(\frac{1}{2}\right)^{\frac{1}{20}}$$

$$\therefore V = V_0 \left(\frac{1}{2}\right)^{\frac{t}{20}}$$

So, after one hour, the volume present is $V(60) = V_0 \left(\frac{1}{2}\right)^3 = \frac{1}{8} V_0$

Hence $\frac{1}{8}$ of the water remains.

Chapter 30

STATISTICAL DISTRIBUTIONS

EXERCISE 30A

- 1
 - a The quantity of fat in a lamb chop is a continuous random variable.
 - b The mark out of 50 for the Geography test is a discrete random variable.
 - c The weight of seventeen year-old students is a continuous random variable.
 - d The volume of water in a cup of coffee is a continuous random variable.
 - e The number of trout in a lake is a discrete random variable.
 - f The number of hairs on a cat is a discrete random variable.
 - g The length of hairs on a horse is a continuous random variable.
 - h The height of a sky-scraper is a continuous random variable.
- 2
 - a
 - i The random variable is the height of water in the rain gauge.
 - ii $0 \leq x \leq 200$ mm
 - iii The variable is a continuous random variable.
 - b
 - i The random variable is the stopping distance.
 - ii $0 \leq x \leq 50$ m
 - iii The variable is a continuous random variable.
 - c
 - i The random variable could be the time taken for the switch to fail or it could be the number of times that the switch is turned on or off before it fails.
 - ii $0 \leq x \leq 10\,000$ hours if we are measuring time, or $1 \leq x \leq 2000$ if we are counting on/off.
 - iii The variable is a continuous random variable in the first case and a discrete random variable in the second.

- 3
 - a Since x is the number of weighing devices that are accurate, $x = 0, 1, 2, 3$ or 4 .

b

		YYNN		
		YNNY		
	YYYN	YNNY	NNNY	
	YYNY	NNYY	NNYN	
	YNNY	NNYY	NNYN	
YYYY	YYYY	YYNN	YNNN	NNNN
$(x = 4)$	$(x = 3)$	$(x = 2)$	$(x = 1)$	$(x = 0)$

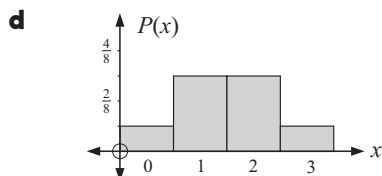
- c
 - i If two are accurate then $x = 2$
 - ii If at least two are accurate then 2, 3 or 4 are accurate i.e., $x = 2, 3$ or 4
- 4
 - a If 3 coins are tossed then the number of heads x can be 0, 1, 2 or 3.
 - b Suppose H represents heads, T represents tails.

		HHT	TTH	
		HTH	THT	
	HHH	THH	HTT	TTT
$(x = 3)$	$(x = 2)$	$(x = 1)$	$(x = 0)$	

c

$$P(x = 0) = \frac{1}{8} \quad P(x = 1) = \frac{3}{8}$$

$$P(x = 2) = \frac{3}{8} \quad P(x = 3) = \frac{1}{8}$$



EXERCISE 30B

$$\begin{array}{ll}
 \mathbf{1} \quad \mathbf{a} & \sum_{x=0}^2 P(x) = 1 \\
 & \therefore 0.3 + k + 0.5 = 1 \\
 & \therefore k = 0.2 \\
 \mathbf{b} & \sum_{x=0}^3 P(x) = 1 \\
 & \therefore k + 2k + 3k + k = 1 \\
 & \therefore 7k = 1 \\
 & \therefore k = \frac{1}{7}
 \end{array}$$

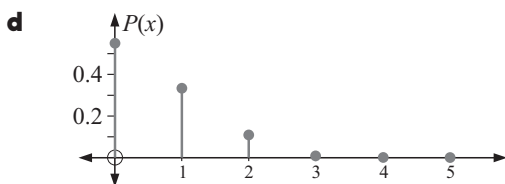
$$\begin{array}{ll}
 \mathbf{2} \quad \mathbf{a} & \text{Since this is a probability distribution } \sum P(i) = 1 \\
 & \therefore a + 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.0000 = 1 \\
 & \therefore a + 0.4512 = 1 \\
 & \therefore a = 0.5488
 \end{array}$$

$P(0)$ is the probability that Jason does not hit a home run in a game.

$$\mathbf{b} \quad P(2) = 0.1088 \quad (\text{from table})$$

$$\begin{array}{l}
 \mathbf{c} \quad P(1) + P(2) + P(3) + P(4) + P(5) = 0.3333 + 0.1088 + 0.0084 + 0.0007 + 0.00 \\
 \quad \quad \quad = 0.4512
 \end{array}$$

This represents the probability that Jason hits *at least one* home run in a game.



$$\begin{array}{ll}
 \mathbf{3} \quad \mathbf{a} & \text{The probabilities all lie in } 0 \leq P(i) \leq 1, \therefore \text{OK} \\
 & \text{Sum of probabilities } \sum P(i) = 0.2 + 0.3 + 0.4 + 0.2 = 1.1 \\
 & \therefore \text{sum of probabilities} \neq 1 \\
 & \therefore \text{this is not a valid probability distribution.}
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{b} & \text{Notice that } P(5) = -0.2, \therefore \text{not all of the probabilities lie in } 0 \leq P(i) \leq 1 \\
 & \therefore \text{this is not a valid probability distribution.}
 \end{array}$$

$$\mathbf{4} \quad \mathbf{a} \quad \text{The random variable represents the number of hits that Sally has in each game.}$$

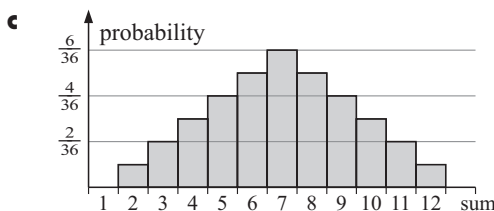
$$\begin{array}{ll}
 \mathbf{b} & 0.7 + 0.14 + k + 0.46 + 0.08 + 0.02 = 1 \quad \{\text{since } \sum p(i) = 1\} \\
 & \therefore k + 0.77 = 1 \\
 & \therefore k = 0.23
 \end{array}$$

$$\begin{array}{ll}
 \mathbf{c} \quad \mathbf{i} & P(x \geq 2) \\
 & = P(x = 2 \text{ or } x = 3 \text{ or } x = 4 \text{ or } x = 5) \\
 & = P(2) + P(3) + P(4) + P(5) \\
 & = 0.23 + 0.46 + 0.08 + 0.02 \\
 & = 0.79 \\
 \mathbf{ii} & P(1 \leq x \leq 3) \\
 & = P(1) + P(2) + P(3) \\
 & = 0.14 + 0.23 + 0.46 \\
 & = 0.83
 \end{array}$$

$$\mathbf{5} \quad \mathbf{a} \quad \text{Rolling a die twice, sample space:}$$

	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
roll 1	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
		1	2	3	4	5	6
		roll 2					

$$\begin{array}{ll}
 \mathbf{b} \quad P(0) = 0 & P(1) = 0 \\
 P(2) = \frac{1}{36} & P(3) = \frac{2}{36} \\
 P(4) = \frac{3}{36} & P(5) = \frac{4}{36} \\
 P(6) = \frac{5}{36} & P(7) = \frac{6}{36} \\
 P(8) = \frac{5}{36} & P(9) = \frac{4}{36} \\
 P(10) = \frac{3}{36} & P(11) = \frac{2}{36} \\
 P(12) = \frac{1}{36}
 \end{array}$$



6 a $P(x) = k(x+2), \quad x = 1, 2, 3$

$$\therefore P(1) = 3k, \quad P(2) = 4k, \quad P(3) = 5k$$

and since this is a probability distribution, $\sum P(i) = 3k + 4k + 5k$

$$\text{and } 12k = 1 \quad \{\text{as } \sum P(i) = 1\}$$

$$\therefore k = \frac{1}{12}$$

b $P(x) = \frac{k}{x+1} \quad x = 0, 1, 2, 3$

$$\therefore P(0) = k, \quad P(1) = \frac{k}{2}, \quad P(2) = \frac{k}{3}, \quad P(3) = \frac{k}{4}$$

$$\text{Now } k + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} = 1 \quad \{\text{as } \sum P(i) = 1\}$$

$$\therefore \frac{12k + 6k + 4k + 3k}{12} = 1$$

$$\therefore \frac{25k}{12} = 1$$

$$\therefore k = \frac{12}{25}$$

7 a $P(x) = k \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}, \quad x = 0, 1, 2, 3, 4$

$$P(0) = k \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^4 = \frac{16k}{81} \quad P(1) = k \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^3 = \frac{8k}{81} \quad P(2) = k \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 = \frac{4k}{81}$$

$$P(3) = k \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^1 = \frac{2k}{81} \quad P(4) = k \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^0 = \frac{k}{81}$$

b As $\sum P(i) = 1$, $\therefore P(x \geq 2) = P(2) + P(3) + P(4)$

$$\therefore \frac{16k}{81} + \frac{8k}{81} + \frac{4k}{81} + \frac{2k}{81} + \frac{k}{81} = 1 \quad = \frac{4k}{81} + \frac{2k}{81} + \frac{k}{81}$$

$$\therefore \frac{31k}{81} = 1 \quad = \frac{7k}{81}$$

$$\therefore k = \frac{81}{31} \quad = \frac{7 \times 2.6130}{81}$$

$$\therefore k \doteq 2.6130$$

$$\doteq 0.2258$$

8 a P(no faulty component)

$$= P(x = 0)$$

$$= P(0)$$

$$= C_0^{10} (0.04)^0 (0.96)^{10-0}$$

$$= C_0^{10} (0.96)^{10}$$

$$= (0.96)^{10}$$

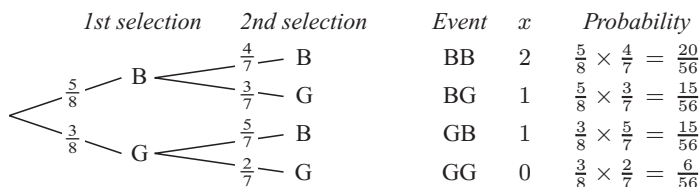
$$\doteq 0.665$$

b P(at least one faulty component)

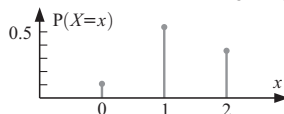
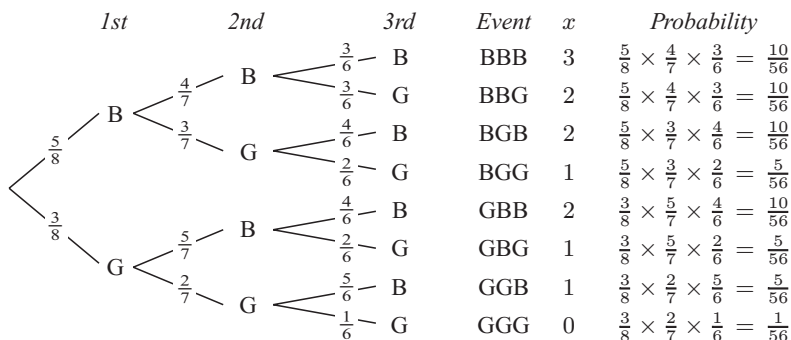
$$= 1 - P(\text{none are faulty})$$

$$= 1 - 0.6648$$

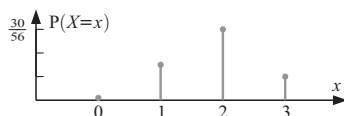
$$\doteq 0.335$$

9 a

x	0	1	2
$P(X=x)$	$\frac{3}{28}$	$\frac{15}{28}$	$\frac{10}{28}$

**b**

x	0	1	2	3
$P(X=x)$	$\frac{1}{56}$	$\frac{15}{56}$	$\frac{30}{56}$	$\frac{10}{56}$

**10 a**

Die 2

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Die 1

36 sample points

$$\mathbf{b} \quad P(D=7) = \frac{6}{36} = \frac{1}{6}$$

c

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

d

$$\begin{aligned}
 &P(D \geq 8 \mid D \geq 6) \\
 &= \frac{P(D \geq 8 \cap D \geq 6)}{P(D \geq 6)} \\
 &= \frac{P(D \geq 8)}{P(D \geq 6)} \\
 &= \frac{15}{36} \div \frac{26}{36} \\
 &= \frac{15}{26}
 \end{aligned}$$

$$\mathbf{11} \quad P(X=x) = \frac{(0.2)^x e^{-0.2}}{x!}$$

a i

$$\begin{aligned}
 P(X=0) &= \frac{(0.2)^0 e^{-0.2}}{0!} \\
 &= 0.8187 \dots \\
 &\div 0.819
 \end{aligned}$$

ii

$$\begin{aligned}
 P(X=1) &= \frac{(0.2)^1 e^{-0.2}}{1!} \\
 &= 0.1637 \dots \\
 &\div 0.164
 \end{aligned}$$

iii

$$\begin{aligned}
 P(X=2) &= \frac{(0.2)^2 e^{-0.2}}{2!} \\
 &= 0.01637 \dots \\
 &\div 0.0164
 \end{aligned}$$

b

$$\begin{aligned}
 &P(X \geq 3) \\
 &= 1 - P(X \leq 2) \\
 &= 1 - P(X=0 \text{ or } X=1 \text{ or } X=2) \\
 &\div 1 - (0.8187 + 0.1637 + 0.0164) \\
 &\div 0.0012
 \end{aligned}$$

EXERCISE 30C

- 1** $P(\text{rain}) = 0.28$ $\therefore$ would expect rain on 0.28×365.25 days a year
i.e., $\div 102$ days.

- 2 a** $P(\text{HHH}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ **b** For 200 tosses, we expect $200 \times \frac{1}{8}$ to be '3 heads'
i.e., 25 to be '3 heads'

- 3** $P(\text{double}) = P(1, 1 \text{ or } 2, 2 \text{ or } 3, 3 \text{ or } 4, 4 \text{ or } 5, 5 \text{ or } 6, 6)$
 $= \frac{6}{36}$ {6 of the possible 36 outcomes}
 $= \frac{1}{6}$

$\therefore$ when rolling 180 times we expect $180 \times \frac{1}{6} = 30$ doubles.

- 4 a i** $P(\text{wins \$10}) = P(\text{rolls a 6}) = \frac{1}{6}$ **ii** $P(\text{wins \$4}) = P(\text{rolls 4 or 5}) = \frac{2}{6}$ (or $\frac{1}{3}$) **iii** $P(\text{wins \$1}) = P(\text{rolls 1, 2 or 3}) = \frac{3}{6}$ (or $\frac{1}{2}$)
- b i** Expectation $= \frac{2}{6} \times \$4 \div \1.33 **ii** Expectation $= \frac{3}{6} \times \$1 = \0.50 **iii** Expectation $= \frac{1}{6} \times \$10 + \frac{2}{6} \times \$4 + \frac{3}{6} \times \$1 = \frac{1}{6}(\$21) = \$3.50$

- c** It costs \$4 to play and the expected return is \$3.50.

$\therefore$ you expect to lose \$0.50 per game.

- d** So, over 100 games you expect to lose $100 \times \$0.50 = \50 .

- 5** Expect to see snow falling on $\frac{3}{7} \times 5 \times 7$ days = 15 days

- 6 a** $165 + 87 + 48 = 300$ **i** $P(\text{votes A}) \div \frac{165}{300} = 0.55$ **ii** $P(\text{votes B}) \div \frac{87}{300} = 0.29$ **iii** $P(\text{votes C}) \div \frac{48}{300} = 0.16$

- b i** Expect $7500 \times 0.55 = 4125$ to vote A **ii** Expect $7500 \times 0.29 = 2175$ to vote B
iii Expect $7500 \times 0.16 = 1200$ to vote C

- 7 a** Expect to win $\frac{1}{6} \times \$1 + \frac{1}{6} \times \$2 + \frac{1}{6} \times \$3 + \frac{1}{6} \times \$4 + \frac{1}{6} \times \$5 + \frac{1}{6} \times \6
 $= \frac{1}{6} \times \$21$
 $= \$3.50$

- b** No, as on each occasion he would expect to lose \$0.50 (on average).

8

result	win
H	\$2
T	-\$1

For playing *once*,

expect to win $\frac{1}{2} \times \$2 + \frac{1}{2} \times (-\$1) = \$0.50$

$\therefore$ for 3 games, expect to win \$1.50.

9

result	win
HH	\$10
HT or TH	\$3
TT	-\$5

- a** Expectation $= \frac{1}{4} \times \$10 + \frac{2}{4} \times \$3 + \frac{1}{4} \times (-\$5) = \$2.75$

- b** Expected win per game (payout) = \$2.75
 $\therefore$ organiser would charge $\$2.75 + \1.00
 $= \$3.75$ to play each game.

EXERCISE 30D**1**

x_i	0	1	2	3	4	5	> 5
$P(x_i)$	0.54	0.26	0.15	k	0.01	0.01	0.00

$$\mathbf{a} \quad 0.54 + 0.26 + 0.15 + k + 0.01 + 0.01 = 1$$

$$\therefore k + 0.97 = 1$$

$$\therefore k = 0.03$$

$$\mathbf{b} \quad \mu = \sum x_i p_i$$

$$= 0 \times 0.54 + 1 \times 0.26 + \dots + 5 \times 0.01$$

$$= 0.26 + 0.30 + 0.09 + 0.04 + 0.05$$

$$= 0.74 \quad \text{i.e., over a long period the mean number of deaths per dozen crayfish is 0.74.}$$

$$\mathbf{c} \quad \sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$$

$$= \sqrt{(0 - 0.74)^2 \times 0.54 + (1 - 0.74)^2 \times 0.26 + \dots + (5 - 0.74)^2 \times 0.01}$$

$$\div 0.9962$$

$$\mathbf{2} \quad P(x) = \frac{x^2 + x}{20} \quad \text{for } x = 1, 2, 3$$

$$\mu = \sum x_i p_i$$

$$= 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.6$$

$$= 2.5$$

x_i	1	2	3
$P(x_i) = p_i$	$\frac{2}{20} = 0.1$	$\frac{6}{20} = 0.3$	$\frac{12}{20} = 0.6$

$$\text{and } \sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$$

$$= \sqrt{(1 - 2.5)^2 \times 0.1 + (2 - 2.5)^2 \times 0.3 + (3 - 2.5)^2 \times 0.6}$$

$$\div 0.6708$$

$$\mathbf{3} \quad \mathbf{a} \quad P(x) = C_x^3 (0.4)^x (0.6)^{3-x} \quad \text{for } x = 0, 1, 2, 3$$

$$\begin{aligned} \therefore P(0) &= C_0^3 (0.4)^0 (0.6)^3 \\ &= (0.6)^3 \\ &= 0.216 \end{aligned}$$

$$\begin{aligned} P(1) &= C_1^3 (0.4)^1 (0.6)^2 \\ &= 3(0.4)(0.6)^2 \\ &= 0.432 \end{aligned}$$

$$\begin{aligned} P(2) &= C_2^3 (0.4)^2 (0.6)^1 \\ &= 3(0.16)(0.6) \\ &= 0.288 \end{aligned}$$

$$\begin{aligned} P(3) &= C_3^3 (0.4)^3 (0.6)^0 \\ &= 1(0.4)^3 \\ &= 0.064 \end{aligned}$$

x_i	0	1	2	3
$P(x_i)$	0.216	0.432	0.288	0.064

$$\mathbf{b} \quad \mu = \sum x_i p_i = 0(0.216) + 1(0.432) + 2(0.288) + 3(0.064) = 1.2$$

$$\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$$

$$= \sqrt{(0 - 1.2)^2 (0.216) + (1 - 1.2)^2 (0.432) + (2 - 1.2)^2 \times 0.288 + (3 - 1.2)^2 \times 0.064}$$

$$\div 0.8485$$

4

$$\sigma = \sqrt{\sum (x_i - \mu)^2 p_i}$$

$$\therefore \sigma^2 = \sum (x_i - \mu)^2 p_i$$

$$= (x_1 - \mu)^2 p_1 + (x_2 - \mu)^2 p_2 + \dots + (x_n - \mu)^2 p_n$$

$$= (x_1^2 - 2x_1\mu + \mu^2)p_1 + (x_2^2 - 2x_2\mu + \mu^2)p_2 + \dots + (x_n^2 - 2x_n\mu + \mu^2)p_n$$

$$= (x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 + \dots + x_n^2 p_n) - 2\mu(x_1 p_1 + x_2 p_2 + \dots + x_n p_n)$$

$$+ \mu^2(p_1 + p_2 + p_3 + \dots + p_n)$$

Now $p_1 + p_2 + \dots + p_n = 1$

$$\begin{aligned}\therefore \sigma^2 &= \sum x_i^2 p_i - 2\mu(\sum x_i p_i) + \mu^2(1) \\ &= \sum x_i^2 p_i - 2\mu(\mu) + \mu^2 \quad \{\text{since } \sum x_i^2 p_i = \mu\} \\ &= \sum x_i^2 p_i - \mu^2\end{aligned}$$

5 a

x_i	1	2	3	4	5
$P(x_i)$	0.1	0.2	0.4	0.2	0.1

$$\begin{aligned}\mathbf{b} \quad \mu &= \sum x_i p_i \\ &= 1(0.1) + 2(0.2) + \dots + 5(0.1) \\ &= 0.1 + 0.4 + 1.2 + 0.8 + 0.5 \\ &= 3\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\ &= \sqrt{\sum x_i^2 p_i - \mu^2} \\ &= \sqrt{1^2(0.1) + 2^2(0.2) + \dots + 5^2(0.1) - (3.0)^2} \\ &= \sqrt{0.1 + 0.8 + 3.6 + 3.2 + 2.5 - 9} \\ &= \sqrt{1.2} \\ &\div 1.095\end{aligned}$$

$$\begin{aligned}\mathbf{c} \quad \mathbf{i} \quad &P(\mu - \sigma < x < \mu + \sigma) \\ &= P(3 - 1.095 < x < 3 + 1.095) \\ &= P(1.905 < x < 4.095) \\ &\div P(x = 2, 3, 4) \\ &\div 0.2 + 0.4 + 0.2 \\ &\div 0.8\end{aligned}$$

$$\begin{aligned}\mathbf{ii} \quad &P(\mu - 2\sigma < x < \mu + 2\sigma) \\ &= P(3 - 2.19 < x < 3 + 2.19) \\ &= P(0.81 < x < 5.19) \\ &\div P(x = 1, 2, 3, 4 \text{ or } 5) \\ &\div 0.1 + 0.2 + 0.4 + 0.2 + 0.1 \\ &\div 1.0\end{aligned}$$

6 Let x be the payout values, then $x = \$20\,000$, $\$8000$, or $\$0$

$\therefore$ the probability distribution is

x_i	20 000	8000	0
$P(x_i) = p_i$	0.0025	0.03	0.9675

$$\begin{aligned}\text{The expectation is } \mu &= \sum x_i p_i = 20\,000(0.0025) + 8000(0.03) + 0(0.9675) \\ &= \$290\end{aligned}$$

i.e., the company expects to pay out \$290 on average in the long run

$\therefore$ the company should charge $\$290 + \$100 = \$390$

7

7

	Die 2						
	1	2	3	4	5	6	
Die 1	1	1	2	3	4	5	6
	2	2	2	3	4	5	6
	3	3	3	3	4	5	6
	4	4	4	4	4	5	6
	5	5	5	5	5	5	6
	6	6	6	6	6	6	6

a

m_i	1	2	3	4	5	6
$P(m_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

b

$$\begin{aligned}\mu &= \sum m_i p_i \\ &= 1\left(\frac{1}{36}\right) + 2\left(\frac{3}{36}\right) + 3\left(\frac{5}{36}\right) + \dots + 6\left(\frac{11}{36}\right) \\ &= \frac{1}{36} + \frac{6}{36} + \frac{15}{36} + \frac{28}{36} + \frac{45}{36} + \frac{66}{36} \\ &= \frac{161}{36} \\ &\div 4.472\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{\sum m_i^2 p_i - \mu^2} \\ &= \sqrt{1^2\left(\frac{1}{36}\right) + 2^2\left(\frac{3}{36}\right) + \dots + 6^2\left(\frac{11}{36}\right) - (4.4722)^2} \\ &= \sqrt{1.97165} \\ &\div 1.404\end{aligned}$$

8 Examples are:

- (1) Tossing one coin, where X is the number of 'heads' resulting. $X = 0$ or 1

x	0	1
$P(x)$	$\frac{1}{2}$	$\frac{1}{2}$

- (2) Rolling one die, where X is the number on the uppermost face. $X = 1, 2, 3, 4, 5$ or 6

x	1	2	3	4	5	6
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

EXERCISE 30E.1

- 1 a** mean of $X = E(X)$

$$\begin{aligned}
 &= \sum xp_x \\
 &= 2(0.3) + 3(0.3) + 4(0.2) + 5(0.1) + 6(0.1) \\
 &= 3.4
 \end{aligned}$$

b $E(X^2) = \sum x^2 p_x = 4(0.3) + 9(0.3) + 16(0.2) + 25(0.1) + 36(0.1) = 13.2$

$$\begin{aligned}
 \text{Now } \text{Var}(X) &= E(X^2) - \{E(X)\}^2 \\
 &= 13.2 - (3.4)^2 \\
 &= 1.64
 \end{aligned}$$

c $\sigma = \sqrt{\text{Var } X} \doteq 1.28$

2 a $\sum p_x = 1$

$$\begin{aligned}
 \therefore 0.2 + k + 0.4 + 0.1 &= 1 \\
 \therefore k &= 0.3
 \end{aligned}$$

b $E(X) = \sum xp_x$

$$\begin{aligned}
 &= 5(0.2) + 6(0.3) + 7(0.4) + 8(0.1) \\
 &= 6.4
 \end{aligned}$$

c $\text{Var}(X) = \sum x^2 p_x - \{E(X)\}^2$

$$\begin{aligned}
 &= 25(0.2) + 36(0.3) + 49(0.4) + 64(0.1) - 6.4^2 \\
 &= 0.84
 \end{aligned}$$

3 a $E(X)$

$$\begin{aligned}
 &= \sum xp_x \\
 &= 1(0.4) + 2(0.3) + 3(0.2) + 4(0.1) \\
 &= 2
 \end{aligned}$$

b $E(X^2)$

$$\begin{aligned}
 &= \sum x^2 p_x \\
 &= 1(0.4) + 4(0.3) + 9(0.2) + 16(0.1) \\
 &= 5
 \end{aligned}$$

c $\text{Var}(X)$

$$\begin{aligned}
 &= E(X^2) - \{E(X)\}^2 \\
 &= 5 - 2^2 \\
 &= 1
 \end{aligned}$$

d $\sigma = \sqrt{\text{Var}(X)}$

$$\begin{aligned}
 &= \sqrt{1} \\
 &= 1
 \end{aligned}$$

e $E(X + 1)$

$$\begin{aligned}
 &= E(X) + E(1) \\
 &= 2 + 1 \\
 &= 3
 \end{aligned}$$

f $\text{Var}(X + 1)$

$$\begin{aligned}
 &= E((X + 1)^2) - \{E(X + 1)\}^2 \\
 &= E(X^2 + 2X + 1) - 3^2 \\
 &= E(X^2) + 2E(X) + E(1) - 9 \\
 &= 5 + 2(2) + 1 - 9 \\
 &= 1
 \end{aligned}$$

g $E(2X^2 + 3X - 7)$

$$\begin{aligned}
 &= 2E(X^2) + 3E(X) - E(7) \\
 &= 2(5) + 3(2) - 7 \\
 &= 9
 \end{aligned}$$

4 $E(X) = 2.8$ $\therefore 1(0.2) + 2a + 3(0.3) + 4b = 2.8$

$$\begin{aligned}
 &\therefore 0.2 + 2a + 0.9 + 4b = 2.8 \\
 &\therefore 2a + 4b = 1.7 \quad \dots\dots (1)
 \end{aligned}$$

$$\begin{aligned}\text{Now } \text{Var}(X) &= E(X^2) - \{E(X)\}^2 \\ &= 1(0.2) + 4a + 9(0.3) + 16b - (2.8)^2\end{aligned}$$

$$\therefore 0.2 + 4a + 2.7 + 16b - 2.8^2 = 1.26$$

$$\therefore 4a + 16b = 6.2$$

$$\therefore 2a + 8b = 3.1 \quad \dots\dots (2)$$

$$\text{Solving (1) and (2) gives } 1.7 - 4b = 3.1 - 8b \quad \therefore 4b = 1.4$$

$$\therefore b = 0.35$$

$$\text{and in (1), } 2a + 1.4 = 1.7$$

$$\therefore 2a = 0.3$$

$$\therefore a = 0.15 \quad \text{i.e., } a = 0.15, \quad b = 0.35$$

5 a $P(X = 0) = a(0) = 0$

$$P(X = 1) = a(-7) = -7a$$

$$P(X = 2) = a(-12) = -12a$$

$$P(X = 3) = a(-15) = -15a$$

$$P(X = 4) = a(-16) = -16a$$

$$P(X = 5) = a(-15) = -15a$$

$$P(X = 6) = a(-12) = -12a$$

$$P(X = 7) = a(-7) = -7a$$

$$P(X = 8) = a(0) = 0$$

$$\therefore 2(-7a - 12a - 15a) - 16a = 1$$

$$\therefore a(-84) = 1$$

$$\therefore a = -\frac{1}{84}$$

b $E(X)$

$$= \sum xp_x$$

$$= 1\left(\frac{7}{84}\right) + 2\left(\frac{12}{84}\right) + 3\left(\frac{15}{84}\right) + 4\left(\frac{16}{84}\right) + 5\left(\frac{15}{84}\right) + 6\left(\frac{12}{84}\right) + 7\left(\frac{7}{84}\right)$$

$$= \frac{336}{84}$$

$$= 4$$

c $E(X^2)$

$$= \sum x^2 p_x$$

$$= 1\left(\frac{7}{84}\right) + 4\left(\frac{12}{84}\right) + 9\left(\frac{15}{84}\right) + 16\left(\frac{16}{84}\right) + 25\left(\frac{15}{84}\right) + 36\left(\frac{12}{84}\right) + 49\left(\frac{7}{84}\right)$$

$$= \frac{1596}{84}$$

$$= 19$$

$$\therefore \text{Var}(X) = E(X^2) - \{E(X)\}^2$$

$$= 19 - 4^2$$

$$= 3$$

$$\therefore \sigma = \sqrt{\text{Var}(X)} = \sqrt{3}$$

6 a $\left(\frac{1}{2} + \frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 + 6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4$

$$= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16} \quad \text{i.e., binomial}$$

So, the probability distribution for X ,
the number of heads occurring is

x	0	1	2	3	4
$P(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

b i $E(X) = \sum xp_x = 0\left(\frac{1}{16}\right) + 1\left(\frac{4}{16}\right) + 2\left(\frac{6}{16}\right) + 3\left(\frac{4}{16}\right) + 4\left(\frac{1}{16}\right) = 2$

$$\therefore \text{mean} = 2$$

ii $E(X^2) = \sum x^2 p_x = 0\left(\frac{1}{16}\right) + 1\left(\frac{4}{16}\right) + 4\left(\frac{6}{16}\right) + 9\left(\frac{4}{16}\right) + 16\left(\frac{1}{16}\right) = 5$

$$\sigma = \sqrt{\text{Var}(X)} = \sqrt{E(X^2) - \{E(X)\}^2} = \sqrt{5 - 2^2} = 1$$

$$\begin{aligned}
 \mathbf{7} \quad & \text{P}(0 \text{ bitter, } 3 \text{ not bitter}) & \text{P}(1 \text{ bitter, } 2 \text{ not}) & \text{P}(2 \text{ bitter, } 1 \text{ not}) \\
 & = \frac{C_0^2 C_3^8}{C_3^{10}} & = \frac{C_1^2 C_2^8}{C_3^{10}} & = \frac{C_2^2 C_1^8}{C_3^{10}} \\
 & = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} \times \frac{3 \times 2 \times 1}{10 \times 9 \times 8} & = 2 \times \frac{8 \times 7}{2 \times 1} \times \frac{3 \times 2 \times 1}{10 \times 9 \times 8} & = 8 \times \frac{3 \times 2 \times 1}{10 \times 9 \times 8} \\
 & = \frac{42}{90} & = \frac{7}{15} & = \frac{6}{90} \\
 & = \frac{7}{15} & & = \frac{1}{15}
 \end{aligned}$$

x	0	1	2
$P(x)$	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

$$\begin{aligned}
 \mathbf{a} \quad & E(X) \\
 & = 0 \left(\frac{7}{15} \right) + 1 \left(\frac{7}{15} \right) + 2 \left(\frac{1}{15} \right) \\
 & = \frac{9}{15} \\
 & = 0.6 \\
 & \therefore \text{mean} = 0.6 \text{ bitter almonds}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & E(X^2) = 0^2 \left(\frac{7}{15} \right) + 1^2 \left(\frac{7}{15} \right) + 2^2 \left(\frac{1}{15} \right) = \frac{11}{15} \\
 & \therefore \text{Var}(X) = E(X^2) - \{E(X)\}^2 \\
 & = \frac{11}{15} - \left(\frac{3}{5} \right)^2 \\
 & = \frac{11}{15} - \frac{9}{25} \\
 & \div 0.3733 \quad \text{and so} \quad \sigma \div 0.611
 \end{aligned}$$

EXERCISE 30E.2

- 1** X has mean 6 and standard deviation 2.

$$\begin{aligned}
 E(Y) &= E(2X + 5) & \text{Var}(2X + 5) \\
 &= 2E(X) + E(5) & = 2^2 \text{Var}(X) \\
 &= 2 \times 6 + 5 & = 4 \times 2^2 \\
 &= 17 & = 16
 \end{aligned}$$

$\therefore$ mean of Y distribution is 17

$\therefore$ standard deviation of Y distribution is $\sqrt{16} = 4$

$$\begin{aligned}
 \mathbf{2} \quad \mathbf{a} \quad & E(aX + b) = E(aX) + E(b) \quad \{\text{using } E(A + B) = E(A) + E(B)\} \\
 & = aE(X) + E(b) \quad \{\text{using } E(kX) = kE(X)\} \\
 & = aE(X) + b \quad \{\text{using } E(k) = k, \quad k \text{ a constant}\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad \mathbf{i} \quad & E(3X + 4) & \mathbf{ii} \quad E(-2X + 1) & \mathbf{iii} \quad E\left(\frac{4X - 2}{3}\right) \\
 & = 3E(X) + 4 & = -2E(X) + 1 & = E\left(\frac{4}{3}X - \frac{2}{3}\right) \\
 & = 3(3) + 4 & = -2(3) + 1 & = \frac{4}{3}E(X) - \frac{2}{3} \\
 & = 13 & = -5 & = \frac{4}{3}(3) - \frac{2}{3} \\
 & & & = 3\frac{1}{3}
 \end{aligned}$$

- 3** X has mean 5 and standard deviation 2.

$$\begin{aligned}
 \mathbf{a} \quad \mathbf{i} \quad & E(Y) = E(2X + 3) = 2E(X) + 3 = 2 \times 5 + 3 = 13 \\
 \mathbf{ii} \quad & \text{Var}(Y) = \text{Var}(2X + 3) = 2^2 \text{Var}(X) = 4 \times 2^2 = 16 \\
 \mathbf{b} \quad \mathbf{i} \quad & E(Y) = E(-2X + 3) = -2E(X) + 3 = -2 \times 5 + 3 = -7 \\
 \mathbf{ii} \quad & \text{Var}(Y) = \text{Var}(-2X + 3) = (-2)^2 \text{Var}(X) = 4 \times 2^2 = 16 \\
 \mathbf{c} \quad & Y = \frac{X - 5}{2} = \frac{1}{2}X - \frac{5}{2} \\
 \mathbf{i} \quad & E(Y) = E\left(\frac{1}{2}X - \frac{5}{2}\right) = \frac{1}{2}E(X) - \frac{5}{2} = \frac{1}{2} \times 5 - \frac{5}{2} = 0 \\
 \mathbf{ii} \quad & \text{Var}(Y) = \text{Var}\left(\frac{1}{2}X - \frac{5}{2}\right) = \left(\frac{1}{2}\right)^2 \text{Var}(X) = \frac{1}{4} \times 2^2 = 1
 \end{aligned}$$

- 4** $Y = 2X + 3$

$$\begin{aligned}
 \mathbf{a} \quad & E(Y) \\
 & = E(2X + 3) \\
 & = 2E(X) + 3 \\
 \mathbf{b} \quad & E(Y^2) \\
 & = E(4X^2 + 12X + 9) \\
 & = 4E(X^2) + 12E(X) + 9
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad \text{Var}(Y) &= E(Y^2) - \{E(Y)\}^2 \\
 &= 4E(X^2) + 12E(X) + 9 - [4\{E(X)\}^2 + 12E(X) + 9] \\
 &= 4E(X^2) - 4\{E(X)\}^2 \\
 \text{5} \quad V(aX + b) &= E((aX + b)^2) - \{E(aX + b)\}^2 \\
 &= E(a^2X^2 + 2abX + b^2) - \{aE(X) + b\}^2 \\
 &= a^2E(X^2) + 2abE(X) + b^2 - [a^2\{E(X)\}^2 + 2abE(X) + b^2] \\
 &= a^2E(X^2) + 2abE(X) + b^2 - a^2\{E(X)\}^2 - 2abE(X) - b^2 \\
 &= a^2(E(X^2) - \{E(X)\}^2) \\
 &= a^2 \text{Var}(X)
 \end{aligned}$$

EXERCISE 30F

- 1**
- a** The binomial distribution applies, as tossing a coin has one of two possible outcomes ($X = 0$ for a tail and $X = 1$ for a head) and each toss is independent of every other toss.
 - b** The binomial distribution applies, as this is equivalent to tossing one coin 100 times.
 - c** The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.
 - d** The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.
 - e** The binomial distribution does not apply, assuming that ten bolts are drawn without replacement. We do not have a repetition of independent trials.
- 2** X is the random variable for the number working night-shift.
 $\therefore X = 0, 1, 2, 3, 4, 5, 6, 7$ and X is Bin (7, 0.35).
- | | |
|---|---|
| <p>a $P(X = 3)$
 $= \text{binompdf}(7, 0.35, 3)$
 $\doteq 0.268$</p> <p>c $P(\text{at least 4 work night-shift})$
 $= P(X \geq 4)$
 $= 1 - P(X \leq 3)$
 $\doteq 1 - 0.800$
 $\doteq 0.200$</p> | <p>b $P(X < 4)$
 $= P(X \leq 3)$
 $= \text{binomcdf}(7, 0.35, 3)$
 $\doteq 0.800$</p> |
|---|---|
- 3** X is the number of faulty items.
 $\therefore X = 0, 1, 2, 3, \dots, 12$ and X is Bin (12, 0.06).
- | | |
|---|---|
| <p>a $P(X = 0)$
 $= \text{binompdf}(12, 0.06, 0)$
 $\doteq 0.476$</p> <p>c $P(\text{at least 2 are faulty})$
 $= P(X \geq 2)$
 $= 1 - P(X \leq 1)$
 $\doteq 0.160$ {from b}</p> | <p>b $P(\text{at most one is faulty})$
 $= P(X \leq 1)$
 $= \text{binomcdf}(12, 0.06, 1)$
 $\doteq 0.840$</p> <p>d $P(\text{less than 4 are faulty})$
 $= P(X < 4)$
 $= 1 - P(X \leq 3)$
 $= \text{binomcdf}(12, 0.06, 3)$
 $= 0.996$</p> |
|---|---|

- 4** X is the random variable for the number of times in a week when the bus is on time.

$\therefore X = 0, 1, 2, 3, 4, 5, 6$ or 7 and X is Bin $(7, 0.6)$ {late 2 in 5, on time 3 in 5}

a $P(X = 7)$

$$= \text{binompdf}(7, 0.6, 7)$$

$$\div 0.0280$$

b $P(\text{on time only on Monday})$

$$= 0.6 \times (0.4)^6$$

$$= 0.00246$$

c $P(X = 6)$

$$= \text{binompdf}(7, 0.6, 6)$$

$$\div 0.131$$

d $P(X \geq 4)$

$$= 1 - P(X \leq 3)$$

$$= 1 - \text{binomcdf}(7, 0.6, 3)$$

$$\div 0.710$$

- 5** X is the random variable for the number with flu

$\therefore X = 0, 1, 2, 3, \dots, 25$ and X is Bin $(25, 0.3)$

a $P(X \geq 2)$

$$= 1 - P(X \leq 1)$$

$$= 1 - \text{binomcdf}(25, 0.3, 1)$$

$$\div 0.998$$

b $P(\text{test cancelled})$

$$= P(X \geq 6) \quad \{20\% \text{ of } 25 = 5\}$$

$$= 1 - P(X \leq 5)$$

$$= 1 - \text{binomcdf}(25, 0.3, 5)$$

$$\div 0.8065$$

EXERCISE 30G

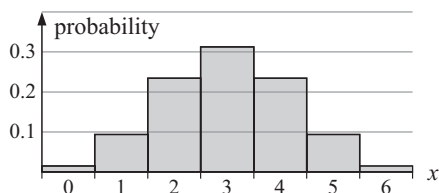
- 1** x is distributed Bin $(6, p)$

- a** If $p = 0.5$, x is distributed Bin $(6, 0.5)$

i $\mu = np = 6 \times 0.5 = 3$ and $\sigma = \sqrt{npq} = \sqrt{6 \times 0.5 \times 0.5} \div 1.225$

ii $P(x = 0)$	$P(x = 1)$	$P(x = 2)$	$P(x = 3)$
$= C_0^6(0.5)^0(0.5)^6$	$= C_1^6(0.5)^1(0.5)^5$	$= C_2^6(0.5)^2(0.5)^4$	$= C_3^6(0.5)^3(0.5)^3$
$= 0.0156$	$= 0.0938$	$= 0.2344$	$= 0.3125$
$P(x = 4)$	$P(x = 5)$	$P(x = 6)$	
$= C_4^6(0.5)^4(0.5)^2$	$= C_5^6(0.5)^5(0.5)^1$	$= C_6^6(0.5)^6(0.5)^0$	
$= 0.2344$	$= 0.0938$	$= 0.0156$	

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.0156	0.0938	0.2344	0.3125	0.2344	0.0938	0.0156



- iii** The distribution is bell-shaped.

- b** If $p = 0.2$, x is distributed Bin $(6, 0.2)$

i $\mu = np$ and $\sigma = \sqrt{npq}$

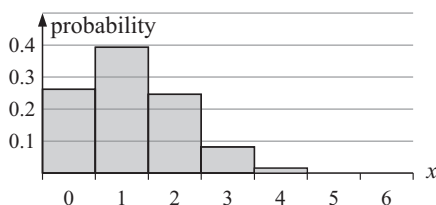
$$= 6 \times 0.2 = \sqrt{6 \times 0.2 \times 0.8}$$

$$= 1.2 \div 0.980$$

ii $P(x = 0)$	$P(x = 1)$	$P(x = 2)$	$P(x = 3)$
$= C_0^6(0.2)^0(0.8)^6$	$= C_1^6(0.2)^1(0.8)^5$	$= C_2^6(0.2)^2(0.8)^4$	$= C_3^6(0.2)^3(0.8)^3$
$= 0.2621$	$= 0.3932$	$= 0.2458$	$= 0.0819$

$$\begin{aligned}
 P(x=4) &= C_4^6 (0.2)^4 (0.8)^2 = 0.0154 \\
 P(x=5) &= C_5^6 (0.2)^5 (0.8)^1 = 0.0015 \\
 P(x=6) &= C_6^6 (0.2)^6 (0.8)^0 = 0.0001
 \end{aligned}$$

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.2621	0.3932	0.2458	0.0819	0.0154	0.0015	0.0001



iii The distribution is skewed to the right, i.e., positively skewed.

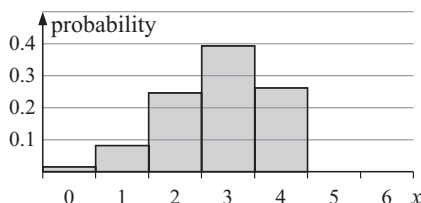
c If $p = 0.8$, x is distributed $\text{Bin}(6, 0.8)$

$$\begin{aligned}
 \text{i } \mu &= np = 6 \times 0.8 = 4.8 \\
 \sigma &= \sqrt{npq} = \sqrt{6 \times 0.8 \times 0.2} \div 0.980
 \end{aligned}$$

$$\begin{aligned}
 \text{ii } P(x=0) &= C_0^6 (0.8)^0 (0.2)^6 = 0.0001 \\
 P(x=1) &= C_1^6 (0.8)^1 (0.2)^5 = 0.0015 \\
 P(x=2) &= C_2^6 (0.8)^2 (0.2)^4 = 0.0154 \\
 P(x=3) &= C_3^6 (0.8)^3 (0.2)^3 = 0.0819 \\
 P(x=4) &= C_4^6 (0.8)^4 (0.2)^2 = 0.2458 \\
 P(x=5) &= C_5^6 (0.8)^5 (0.2)^1 = 0.3932 \\
 P(x=6) &= C_6^6 (0.8)^6 (0.2)^0 = 0.2621
 \end{aligned}$$

Notice that this is the reverse of **b**.

x_i	0	1	2	3	4	5	6
$P(x_i)$	0.0001	0.0015	0.0154	0.0819	0.2458	0.3932	0.2621



iii This distribution is the exact reflection of **b**. It is skewed to the left or negatively skewed.

2 Number of tosses, $n = 10$

X is the number of heads obtained. $\therefore x$ is distributed $\text{Bin}(10, 0.5)$

$$\begin{aligned}
 \mu &= np = 10 \times 0.5 = 5 \\
 \sigma &= \sqrt{npq} = \sqrt{10 \times 0.5 \times 0.5} \div 1.58
 \end{aligned}$$

3 a x is distributed $\text{Bin}(3, p)$

$$\begin{aligned}
 P(x=0) &= C_0^3 p^0 q^3 = (1-p)^3 \\
 P(x=1) &= C_1^3 p^1 q^2 = 3p(1-p)^2 \\
 P(x=2) &= C_2^3 p^2 q^1 = 3p^2(1-p)
 \end{aligned}$$

$$\begin{aligned}
 P(x=3) &= C_3^3 p^3 q^0 = p^3
 \end{aligned}$$

x_i	0	1	2	3
$P(x_i)$	$(1-p)^3$	$3p(1-p)^2$	$3p^2(1-p)$	p^3

$$\begin{aligned}
 \text{b} \quad \mu &= \sum x_i p_i \\
 &= 0(1-p)^3 + 1 \times 3p(1-p)^2 + 2 \times 3(1-p) + 3p^3 \\
 &= 3p(1-p)^2 + 6p^2(1-p) + 3p^3 \\
 &= 3p(1-2p+p^2) + 6p^2 - 6p^3 + 3p^3 \\
 &= 3p - 6p^2 + 3p^3 + 6p^2 - 6p^3 + 3p^3 \\
 &= 3p \quad \text{as required} \\
 \\
 \text{c} \quad \sigma^2 &= \sum x_i^2 p_i - \mu^2 \\
 &= 0^2 \times (1-p)^3 + 1^2 \times 3p(1-p)^2 + 2^2 \times 3p^2(1-p) + 3^2 p^3 - (3p)^2 \\
 &= 3p(1-p)^2 + 12p^2(1-p) + 9p^2(p-1) \\
 &= (1-p)[3p(1-p) + 12p^2 - 9p^2] \\
 &= (1-p)[3p - 3p^2 + 3p^2] \\
 &= 3p(1-p) \\
 &= 3pq \\
 \therefore \sigma &= \sqrt{3pq} \quad \text{as required}
 \end{aligned}$$

- 4 X is the number of defective bolts in the sample.

$$\begin{array}{llll}
 X \text{ is distributed } \text{Bin}(30, 0.04) & \mu = np & \text{and} & \sigma = \sqrt{npq} \\
 & = 30 \times 0.04 & & = \sqrt{30 \times 0.04 \times 0.96} \\
 & = 1.2 & & \div 1.07
 \end{array}$$

- 5 X is the number of groups that do not arrive.

$$\begin{array}{llll}
 X \text{ is distributed } \text{Bin}(5, 0.13) & \mu = np & \text{and} & \sigma = \sqrt{npq} \\
 & = 5 \times 0.13 & & = \sqrt{5 \times 0.13 \times 0.87} \\
 & = 0.65 & & = 0.752
 \end{array}$$

EXERCISE 30H

1 a $\bar{x} = \frac{\sum fx}{\sum f} = \frac{0 + 18 + 24 + 18 + 12 + 0 + 6}{52} = \frac{78}{52} = 1.5$

b Using $m = 1.5$, $p_x = \frac{(1.5)^x e^{-1.5}}{x!}$

$$\begin{array}{llll}
 \therefore p_0 = e^{-1.5} & \therefore p_1 = \frac{1.5e^{-1.5}}{1} & \therefore p_2 = \frac{(1.5)^2 e^{-1.5}}{2!} & \therefore p_3 = \frac{(1.5)^3 e^{-1.5}}{3!} \\
 \therefore p_0 \div 0.22313 & \therefore p_1 \div 0.3347 & \therefore p_2 \div 0.2510 & \therefore p_3 \div 0.1255 \\
 \therefore 52p_0 \div 11.6 & \therefore 52p_1 \div 17.4 & \therefore 52p_2 \div 13.0 & \therefore 52p_3 \div 6.5 \\
 \\
 \therefore p_4 = \frac{(1.5)^4 e^{-1.5}}{4!} & \therefore p_5 = \frac{(1.5)^5 e^{-1.5}}{5!} & \therefore p_6 = \frac{(1.5)^6 e^{-1.5}}{6!} \\
 \therefore p_4 \div 0.0471 & \therefore p_5 \div 0.0144 & \therefore p_6 \div 0.0035 \\
 \therefore 52p_4 \div 2.4 & \therefore 52p_5 \div 0.7 & \therefore 52p_6 \div 0.2
 \end{array}$$

Comparison:

x	0	1	2	3	4	5	6
f	12	18	12	6	3	0	1
$52p_x$	11.6	17.4	13.0	6.5	2.4	0.7	0.2

2 a Standard deviation = 2.67

$$\text{i} \quad \therefore \text{Var}(X) = 2.67^2 \\ \div 7.13$$

$$\text{ii} \quad \text{Since } \text{Var}(X) = m,$$

$$m \div 7.13$$

$$\therefore p_x \div \frac{(7.13)^x e^{-7.13}}{x!}$$

where $x = 0, 1, 2, 3, 4, 5, \dots$

$$\text{b i} \quad P(X = 2) \\ = \text{poissonpdf}(7.13, 2) \\ \div 0.0204$$

$$\text{ii} \quad P(X \leq 3) \\ = \text{poissoncdf}(7.13, 3) \\ \div 0.0752$$

$$\text{iii} \quad P(X \geq 5) \\ = 1 - P(X \leq 4) \\ = 1 - \text{poissoncdf}(7.13, 4) \\ \div 0.839$$

$$\text{iv} \quad P(X \geq 3 \mid x \geq 1) \\ = \frac{P(X \geq 3 \cap X \geq 1)}{P(X \geq 1)} \\ = \frac{P(X \geq 3)}{P(X \geq 1)} \\ = \frac{1 - P(X \leq 2)}{1 - P(X = 0)} \\ = \frac{1 - \text{poissoncdf}(7.13, 2)}{1 - \text{poissonpdf}(7.13, 0)} \\ \div 0.974$$

$$\text{3 a} \quad \bar{x} = \frac{1 \times 156 + 2 \times 132 + 3 \times 75 + 4 \times 33 + 5 \times 9 + 6 \times 3 + 7 \times 1}{91 + 156 + 132 + 75 + 33 + 9 + 3 + 1} \\ = \frac{847}{500} \\ \div 1.694$$

$$\text{b} \quad \text{Using } m = 1.694, \quad p_x = \frac{(1.694)^x e^{-1.694}}{x!} \quad \text{where } x = 0, 1, 2, 3, 4, \dots$$

$$500p_0 = 500 \times 1.694^0 \times e^{-1.694} \times \frac{1}{0!} \div 91.9$$

$$500p_1 = 500 \times 1.694^1 \times e^{-1.694} \times \frac{1}{1!} \div 155.7$$

$$500p_2 = 500 \times 1.694^2 \times e^{-1.694} \times \frac{1}{2!} \div 131.8$$

$$500p_3 = 500 \times 1.694^3 \times e^{-1.694} \times \frac{1}{3!} \div 74.4$$

$$500p_4 = 500 \times 1.694^4 \times e^{-1.694} \times \frac{1}{4!} \div 31.5$$

$$500p_5 = 500 \times 1.694^5 \times e^{-1.694} \times \frac{1}{5!} \div 10.7$$

$$500p_6 = 500 \times 1.694^6 \times e^{-1.694} \times \frac{1}{6!} \div 3.0$$

$$500p_7 = 500 \times 1.694^7 \times e^{-1.694} \times \frac{1}{7!} \div 0.7$$

Comparison:

x	0	1	2	3	4	5	6	7
f	91	156	132	75	33	9	3	1
$500p_x$	92	156	132	74	32	11	3	1

The fit is excellent.

$$\begin{aligned}
 \text{c} \quad & \text{Var}(X) \\
 &= E(X^2) - \{E(X)\}^2 \\
 &= \sum x^2 p_x - (1.694)^2 \\
 &= 1 \times \frac{156}{500} + 4 \times \frac{132}{500} + 9 \times \frac{75}{500} + 16 \times \frac{33}{500} + 25 \times \frac{9}{500} + 36 \times \frac{3}{500} + 49 \times \frac{1}{500} - (1.694)^2 \\
 &= 1.6683 \\
 &\therefore s \doteq 1.29 \quad \text{and} \quad \sqrt{m} = \sqrt{1.694} \doteq 1.30 \quad \therefore s \text{ is very close to } \sqrt{m}
 \end{aligned}$$

$$4 \quad p_x = \frac{3^x e^{-3}}{x!} \quad \text{where } x = 0, 1, 2, 3, 4, 5, \dots$$

$$\begin{aligned}
 \text{a} \quad & P(X = 0) \\
 &= \text{poissonpdf}(3, 0) \\
 &\doteq 0.0498
 \end{aligned}$$

$$\begin{aligned}
 \text{b} \quad & P(\text{at least 3 are rented}) \\
 &= P(X = 3 \text{ or } 4) \\
 &= \text{poissonpdf}(3, 3) + \text{poissonpdf}(3, 4) \\
 &\doteq 0.392
 \end{aligned}$$

$$\begin{aligned}
 \text{c} \quad & P(\text{some requests are refused}) \\
 &= P(X \geq 5) \\
 &= 1 - P(X \leq 4) \\
 &= 1 - \text{poissoncdf}(3, 4) \\
 &\doteq 0.185
 \end{aligned}$$

$$\begin{aligned}
 \text{d} \quad & P(X = 4 \mid X \geq 2) \\
 &= \frac{P(X = 4 \cap X \geq 2)}{P(X \geq 2)} \\
 &= \frac{P(X = 4)}{1 - P(X \leq 1)} \\
 &= \frac{\text{poissonpdf}(3, 4)}{1 - \text{poissoncdf}(3, 1)} \doteq 0.210
 \end{aligned}$$

$$5 \quad P(X = x) = \frac{m^x e^{-m}}{x!} \quad \text{where } x = 0, 1, 2, 3, 4, \dots$$

$$\text{a} \quad \text{If } P(X = 1) + P(X = 2) = P(X = 3), \text{ then}$$

$$\begin{aligned}
 \frac{m e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!} &= \frac{m^3 e^{-m}}{3!} \\
 \therefore m + \frac{m^2}{2} &= \frac{m^3}{6} \quad \{\text{dividing each term by } e^{-m}\} \\
 \therefore 6m + 3m^2 &= m^3 \\
 \therefore m(m^2 - 3m - 6) &= 0 \quad \text{where } m \neq 0 \\
 \therefore m^2 - 3m - 6 &= 0 \\
 \therefore m &= \frac{3 \pm \sqrt{9 - 4(1)(-6)}}{2} = \frac{3 \pm \sqrt{33}}{2} \\
 \text{But } m > 0, \text{ so } m &= \frac{3 + \sqrt{33}}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{b i} \quad & P(X \geq 3) \\
 &= 1 - P(X \leq 2) \\
 &= 1 - \text{poissoncdf}(2.7, 2) \\
 &\doteq 0.506
 \end{aligned}$$

$$\begin{aligned}
 \text{ii} \quad & P(X \leq 4 \mid X \geq 2) \\
 &= \frac{P(X \leq 4 \cap X \geq 2)}{P(X \geq 2)} \\
 &= \frac{P(X = 2, 3 \text{ or } 4)}{P(X \geq 2)} \\
 &= \frac{P(X \leq 4) - P(X \leq 1)}{1 - P(X \leq 1)} \\
 &= \frac{\text{poissoncdf}(2.7, 4) - \text{poissoncdf}(2.7, 1)}{1 - \text{poissoncdf}(2.7, 1)} \\
 &\doteq 0.818
 \end{aligned}$$

EXERCISE 30I

$$1 \quad a \quad \int_0^4 ax(x-4)dx = 1$$

$$\therefore a \int_0^4 (x^2 - 4x)dx = 1$$

$$\therefore a \left[\frac{x^3}{3} - \frac{4x^2}{2} \right]_0^4 = 1$$

$$\therefore a \left(\frac{64}{3} - 32 \right) = 1$$

$$\therefore a \left(\frac{-32}{3} \right) = 1$$

$$\therefore a = -\frac{3}{32}$$

$$\begin{aligned} c \quad i \quad \mu &= \int_0^4 x f(x) dx \\ &= \int_0^4 -\frac{3}{32} x^2 (x-4) dx \\ &= -\frac{3}{32} \int_0^4 x^2 (x-4) dx \\ &= 2 \quad \{\text{technology}\} \end{aligned}$$

$$ii \quad \text{mode} = 2 \\ \{\text{symmetry of graph}\}$$

$$\begin{aligned} iv \quad &\int_0^4 x^2 f(x) dx \\ &= \int_0^4 -\frac{3}{32} x^3 (x-4) dx \\ &= 4.8 \quad \{\text{technology}\} \end{aligned}$$

$$\begin{aligned} \therefore \text{Var}(X) &= 4.8 - 2^2 \\ &= 0.8 \end{aligned}$$

$$2 \quad a \quad \int_0^b -0.2x(x-b) dx = 1$$

$$\therefore -0.2 \int_0^b (x^2 - bx) dx = 1$$

$$\therefore \left[\frac{x^3}{3} - \frac{bx^2}{2} \right]_0^b = -5$$

$$\therefore \frac{b^3}{3} - \frac{b^3}{2} - 0 = -5$$

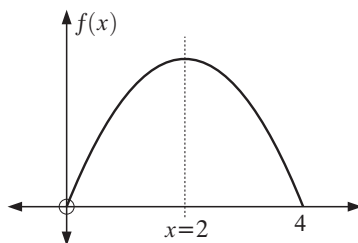
$$\therefore 2b^3 - 3b^3 = -30$$

$$\therefore -b^3 = -30$$

$$\therefore b^3 = 30$$

$$\therefore b = \sqrt[3]{30}$$

$$b \quad f(x) = -\frac{3}{32}x(x-4), \quad 0 \leq x \leq 4$$



$$iii \quad \text{If } \int_0^m -\frac{3}{32}x(x-4) dx = \frac{1}{2}$$

$$\text{then } \int_0^m (x^2 - 4x) dx = -\frac{16}{3}$$

$$\therefore \left[\frac{x^3}{3} - \frac{4x^2}{2} \right]_0^m = -\frac{16}{3}$$

$$\therefore \frac{m^3}{3} - 2m^2 - 0 = -\frac{16}{3}$$

$$\therefore m^3 - 6m^2 = -16$$

$$\therefore m^3 - 6m^2 + 16 = 0$$

$$2 \quad \begin{array}{ccc|c} 1 & -6 & 0 & 16 \\ 0 & 2 & -8 & -16 \\ \hline 1 & -4 & -8 & 0 \end{array}$$

$$\therefore m = 2 \quad \text{or} \quad \frac{4 \pm \sqrt{16 + 32}}{2}$$

$$\text{i.e., } m = 2 \quad \text{or} \quad 2 \pm 2\sqrt{3}$$

$$\therefore m = 2 \quad \{\text{as } 0 < m < 4\}$$

$$\text{i.e., the median is 2.}$$

$$\begin{aligned} b \quad i \quad \mu &= \int_0^{\sqrt[3]{30}} -0.2x^2(x - \sqrt[3]{30}) dx \\ &= 1.5536 \dots \quad \{\text{technology}\} \\ &\doteq 1.55 \end{aligned}$$

$$\begin{aligned} ii \quad &\int_0^{\sqrt[3]{30}} x^2 f(x) dx \\ &= \int_0^{\sqrt[3]{30}} -0.2x^3(x - \sqrt[3]{30}) dx \\ &= 2.8964 \dots \quad \{\text{technology}\} \\ \therefore \text{Var}(X) &\doteq 2.8964 - \mu^2 \\ &\doteq 0.483 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \quad \mathbf{a} \quad & \int_0^3 k e^{-x} dx = 1 \\
 \therefore & k \int_0^3 e^{-x} dx = 1 \\
 \therefore & k \left[\frac{e^{-x}}{-1} \right]_0^3 = 1 \\
 \therefore & k(-e^{-3} - (-1)) = 1 \\
 \therefore & k(1 - e^{-3}) = 1 \\
 \therefore & k \div 1.0524
 \end{aligned}$$

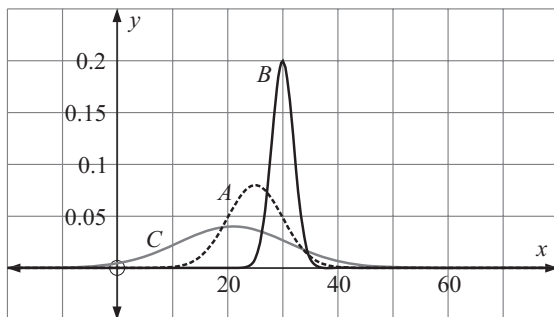
$$\begin{aligned}
 \mathbf{b} \quad & \text{If } m \text{ is the median then} \\
 & \int_0^m k e^{-x} dx = \frac{1}{2} \\
 \therefore & \int_0^m e^{-x} dx = \frac{1}{2k} \\
 \therefore & \left[\frac{e^{-x}}{-1} \right]_0^m = \frac{1}{2k} \\
 \therefore & -e^{-m} - (-1) = \frac{1}{2k} \\
 \therefore & e^{-m} \div 1 - \frac{1}{2(1.0524)} \\
 \therefore & e^{-m} \div 0.52489 \\
 \therefore & -m \div \ln(0.52489) \\
 \therefore & m \div 0.645
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \quad \mathbf{a} \quad & \int_0^5 k x^2(x-6) dx = 1 \\
 \therefore & k \int_0^5 (x^3 - 6x^2) dx = 1 \\
 \therefore & k \left[\frac{x^4}{4} - \frac{6x^3}{3} \right]_0^5 = 1 \\
 \therefore & k \left(\frac{625}{4} - 250 \right) = 1 \\
 \therefore & k \left(\frac{-375}{4} \right) = 1 \\
 \therefore & k = -\frac{4}{375}
 \end{aligned}$$

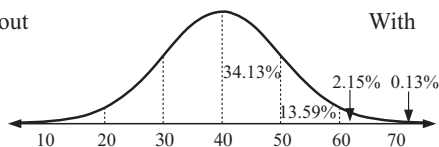
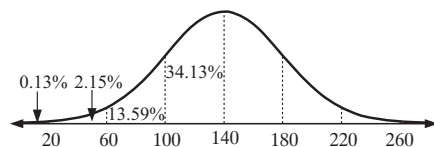
$$\begin{aligned}
 \mathbf{b} \quad & f(x) = -\frac{4}{375} x^2(x-6) \\
 & = -\frac{4}{375} (x^3 - 6x^2) \\
 \therefore & f'(x) = -\frac{4}{375} (3x^2 - 12x) \\
 & \text{which is 0 when } 3x(x-4) = 0 \\
 & \quad \text{i.e., } x = 0 \text{ or } 4 \\
 \therefore & \text{the mode is 4.}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & \text{If } m \text{ is the median,} \\
 & \int_0^m -\frac{4}{375} x^2(x-6) dx = \frac{1}{2} \\
 \therefore & \int_0^m (x^3 - 6x^2) dx = -\frac{375}{8} \\
 \therefore & \left[\frac{x^4}{4} - \frac{6x^3}{3} \right]_0^m = -\frac{375}{8} \\
 \therefore & \frac{m^4}{4} - 2m^3 = -\frac{375}{8} \\
 \therefore & 2m^4 - 16m^3 + 375 = 0 \\
 \therefore & m \div 3.455 \\
 & \{\text{technology}\}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \mu = \int_0^5 x f(x) dx \\
 & = \int_0^5 -\frac{4}{375} x^3(x-6) dx \\
 & = 3\frac{1}{3} \quad \{\text{technology}\} \\
 \mathbf{e} \quad & E(X^2) = \int_0^5 x^2 f(x) dx \\
 & = \int_0^5 -\frac{4}{375} x^4(x-6) dx \\
 & = 12\frac{2}{9} \\
 \therefore & \text{Var}(X) = 12\frac{2}{9} - \left(3\frac{1}{3}\right)^2 \\
 & = 1\frac{1}{9}
 \end{aligned}$$

EXERCISE 30J.1**1****2 a/b/c/d**

The mean volume (or life time, weight, diameter etc) is likely to occur most often with variations around the mean occurring symmetrically as a result of random variations in the production process.

3 Without**With**

a $P(\text{without and } < 50)$

$$\div 50\% + 34.13\%$$

$$\div 84.1\%$$

c $P(\text{with and } 20 \leq x \leq 60)$

$$\div 2.15\%$$

d $P(\text{with and } x \geq 60)$

$$\div 13.59\% + 34.13\% + 50\%$$

$$\div 97.7\%$$

b $P(\text{with and } < 60)$

$$\div 0.13\% + 2.15\%$$

$$\div 2.28\%$$

$$\div 2.3\%$$

$P(\text{without and } 20 \leq x \leq 60)$

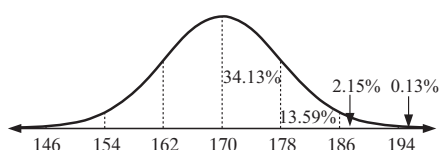
$$\div 2(34.13\% + 13.59\%)$$

$$\div 95.4\%$$

$P(\text{without and } x \geq 60)$

$$\div 2.15\% + 0.13\%$$

$$\div 2.3\%$$

4

a $P(162 < x < 170)$

$$\div 34.1\%$$

b $P(170 < x < 186)$

$$\div 34.13\% + 13.59\%$$

$$\div 47.7\%$$

c $P(178 < x < 186)$

$$\div 13.59\%$$

$$\div 0.136$$

d $P(x < 162)$

$$\div 1 - (0.5 + 0.3413)$$

$$\div 0.159$$

e $P(x < 154)$

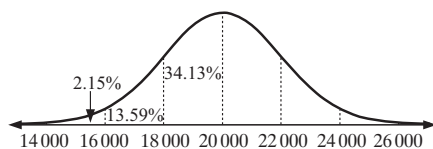
$$\div 0.0215 + 0.0013$$

$$\div 0.0228$$

f $P(x > 162)$

$$\div 1 - 0.159 \quad \{\text{using d}\}$$

$$\div 0.841$$

5

b $P(x > 16000)$

$$\div 0.1359 + 0.3413 + 0.5$$

$$\div 0.9772$$

$$\therefore \text{expect } 260 \times 0.9772 = 254 \text{ days}$$

a $P(x < 18000)$

$$\div 1 - 0.5 - 0.3413$$

$$\div 0.1587$$

$$\therefore \text{expect } 260 \times 0.1587 = 41 \text{ days}$$

c $P(18000 \leq x \leq 24000)$

$$\div 0.3413 \times 2 + 0.1359$$

$$\div 0.8185$$

$$\therefore \text{expect } 0.8185 \times 260 = 213 \text{ days}$$

EXERCISE 30J.2 Using Technology

1 a 0.341 **b** 0.383 **c** 0.106

2 a 0.341 **b** 0.264 **c** 0.212 **d** 0.945 **e** 0.579 **f** 0.383

EXERCISE 30K.1

1 a $E\left(\frac{X - \mu}{\sigma}\right) = E\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right)$

$$= \frac{1}{\sigma}E(X) - \frac{\mu}{\sigma}$$

$$= \frac{1}{\sigma}\mu - \frac{\mu}{\sigma}$$

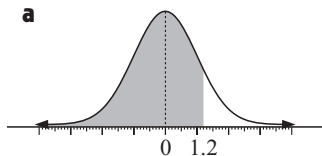
$$= 0$$

b $\text{Var}\left(\frac{X - \mu}{\sigma}\right) = \text{Var}\left(\frac{1}{\sigma}X - \frac{\mu}{\sigma}\right)$

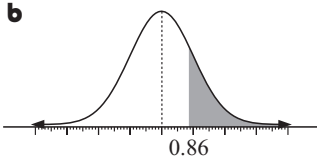
$$= \left(\frac{1}{\sigma}\right)^2 \text{Var}(X)$$

$$= \frac{1}{\sigma^2} \times \sigma^2$$

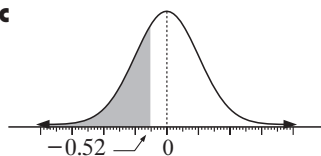
$$= 1$$

2 a

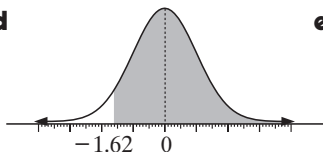
$$\begin{aligned} P(z \leq 1.2) \\ = 0.8849 \\ \doteq 0.885 \end{aligned}$$

b

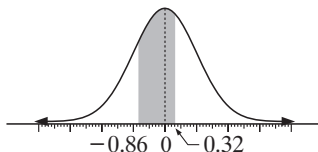
$$\begin{aligned} P(z \geq 0.86) \\ = 1 - P(z \leq 0.86) \\ = 1 - 0.8051 \\ \doteq 0.195 \end{aligned}$$

c

$$\begin{aligned} P(z \leq -0.52) \\ \doteq 0.3015 \end{aligned}$$

d

$$\begin{aligned} P(z \geq -1.62) \\ = 1 - P(z \leq -1.62) \\ \doteq 1 - 0.0526 \\ \doteq 0.947 \end{aligned}$$

e

$$\begin{aligned} P(-0.86 < z < 0.32) \\ = P(z < 0.32) - P(z < -0.86) \\ = 0.6255 - 0.1949 \\ \doteq 0.431 \end{aligned}$$

3 a

$$\begin{aligned} P(z \geq 0.837) \\ = 1 - P(z \leq 0.837) \\ \doteq 1 - \text{normalcdf}(-E99, 0.837) \\ \doteq 0.201 \end{aligned}$$

b

$$\begin{aligned} P(z \leq 0.0614) \\ = \text{normalcdf}(-E99, 0.0614) \\ \doteq 0.524 \end{aligned}$$

c

$$\begin{aligned} P(z \geq -0.876) \\ = 1 - P(z \leq -0.876) \\ \doteq 0.809 \end{aligned}$$

d

$$\begin{aligned} P(-0.3862 \leq z \leq 0.2506) \\ = \text{normalcdf}(-0.3862, 0.2506) \\ \doteq 0.249 \end{aligned}$$

e

$$\begin{aligned} P(-2.367 \leq z \leq -0.6503) \\ = \text{normalcdf}(-2.367, -0.6503) \\ \doteq 0.249 \end{aligned}$$

4 a

$$\begin{aligned} P(-0.5 < z < 0.5) \\ = \text{normalcdf}(-0.5, 0.5) \\ \doteq 0.383 \end{aligned}$$

b

$$\begin{aligned} P(-1.960 < z < 1.960) \\ = \text{normalcdf}(-1.96, 1.96) \\ \doteq 0.950 \end{aligned}$$

5 a

$$\begin{aligned} P(z \leq a) &= 0.95 \\ \therefore a &\doteq 1.645 \\ \text{\{searching in tables\}} \end{aligned}$$

b

$$\begin{aligned} P(z \geq a) &= 0.90 \\ \therefore 1 - P(z \leq a) &= 0.90 \\ \therefore P(z \leq a) &= 0.1 \\ \therefore a &\doteq -1.28 - \frac{3}{18}(0.01) \\ \therefore a &\doteq -1.282 \end{aligned}$$

6 a

$$\text{For Physics, } Z = \frac{83 - 78}{10.8} \doteq 0.463$$

$$\text{For Chemistry, } Z = \frac{77 - 72}{11.6} \doteq 0.431$$

$$\text{For Maths, } Z = \frac{84 - 74}{10.1} \doteq 0.990$$

$$\text{For German, } Z = \frac{91 - 86}{9.6} \doteq 0.521$$

$$\text{For Biology, } Z = \frac{72 - 62}{12.2} \doteq 0.820$$

b Maths, Biology, German, Physics, Chemistry

$$7 \quad Z\text{-score for algebra} = \frac{56 - 50.2}{15.8} \div 0.3671 \quad Z\text{-score for geometry} = \frac{x - 58.7}{18.7}$$

$$\therefore \text{ we need to solve } \frac{x - 58.7}{18.7} = 0.3671$$

$$\therefore x - 58.7 = 6.9$$

$$\therefore x = 65.6 \quad \text{i.e., a result of 65.6\%}$$

EXERCISE 30K.2

1 X is normal with mean 70, standard deviation 4.

a $P(x \geq 74)$

$$= \text{normalcdf}(74, \text{E99}, 70, 4) \\ \div 0.159$$

b $P(x \leq 68)$

$$= \text{normalcdf}(-\text{E99}, 68, 70, 4) \\ \div 0.309$$

c $P(60.6 \leq x \leq 68.4)$

$$= \text{normalcdf}(60.6, 68.4, 70, 4) \\ \div 0.335$$

2 X is normal with mean 58.3 and standard deviation 8.96.

a $P(x \geq 61.8)$

$$= \text{normalcdf}(61.8, \text{E99}, 58.3, 8.96) \\ \div 0.348$$

b $P(x \leq 54.2)$

$$= \text{normalcdf}(-\text{E99}, 54.2, 58.3, 8.96) \\ \div 0.324$$

c $P(50.67 \leq x \leq 68.92)$

$$= \text{normalcdf}(50.67, 68.92, 58.3, 8.96) \\ \div 0.685$$

3 L is normal with mean 50.2 mm and standard deviation 0.93 mm.

a $P(l \geq 50)$

$$= \text{normalcdf}(50, \text{E99}, 50.2, 0.93) \\ \div 0.585$$

b $P(l \leq 51)$

$$= \text{normalcdf}(-\text{E99}, 51, 50.2, 0.93) \\ = 0.805$$

c $P(49 \leq l \leq 50.5)$

$$= \text{normalcdf}(49, 50.5, 50.2, 0.93) \\ \div 0.528$$

EXERCISE 30K.3

1 **a** $P(z \leq k) = 0.81$

$$\therefore k \div 0.87 + \frac{22}{28}(0.01)$$

$$\therefore k \div 0.878$$

b $P(z \leq k) = 0.58$

$$\therefore k \div 0.20 + \frac{7}{39}(0.01)$$

$$\therefore k \div 0.202$$

c $P(z \leq k) = 0.17$

$$\therefore k \div -0.96 + \frac{15}{26}(0.01)$$

$$\therefore k \div -0.954$$

2 **a** $P(z \leq k) = 0.384$

$$\therefore k \div \text{invNorm}(0.384)$$

$$\therefore k \div -0.295$$

b $P(z \leq k) = 0.878$

$$\therefore k \div \text{invNorm}(0.878)$$

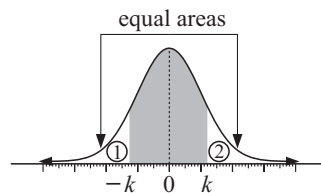
$$\therefore k \div 1.165$$

c $P(z \leq k) = 0.1384$

$$\therefore k \div \text{invNorm}(0.1384)$$

$$\therefore k \div -1.088$$

$$\begin{aligned}
 3 \quad a \quad & P(-k \leq z \leq k) \\
 &= P(z \leq k) - P(z \leq -k) \\
 &= P(z \leq k) - P(z \geq k) \quad \{\text{as area 1} = \text{area 2}\} \\
 &= P(z \leq k) - [1 - P(z \leq k)] \\
 &= P(z \leq k) - 1 + P(z \leq k) \\
 &= 2P(z \leq k) - 1
 \end{aligned}$$



$$\begin{aligned}
 b \quad i \quad & P(-k \leq z \leq k) = 0.238 \\
 \therefore & 2P(z \leq k) - 1 = 0.238 \\
 \therefore & 2P(z \leq k) = 1.238 \\
 \therefore & P(z \leq k) = 0.619 \\
 \therefore & k = \text{invNorm}(0.619) \\
 \therefore & k \doteq 0.303
 \end{aligned}$$

$$\begin{aligned}
 ii \quad & P(-k \leq z \leq k) = 0.7004 \\
 \therefore & 2P(z \leq k) - 1 = 0.7004 \\
 \therefore & 2P(z \leq k) = 1.7004 \\
 \therefore & P(z \leq k) = 0.8502 \\
 \therefore & k = \text{invNorm}(0.8502) \\
 \therefore & k \doteq 1.037
 \end{aligned}$$

$$\begin{aligned}
 4 \quad a \quad & P(X \leq k) = 0.9 \\
 \therefore & k = \text{invNorm}(0.9, 56, \sqrt{18}) \\
 \therefore & k \doteq 61.4
 \end{aligned}$$

$$\begin{aligned}
 b \quad & P(X \geq k) = 0.8 \\
 \therefore & P(X \leq k) = 0.2 \\
 \therefore & k = \text{invNorm}(0.2, 38.7, \sqrt{8.8}) \\
 \therefore & k \doteq 36.2
 \end{aligned}$$

EXERCISE 30L

- 1 Let X be the length (in cm) of a bolt.
Then X is normally distributed with $\mu = 19.8$ and $\sigma = 0.3$.
 $P(19.7 < x < 20) = \text{normalcdf}(19.7, 20, 19.8, 0.3)$
 $\doteq 0.378$

- 2 Let X be the money collected (in \$)
Then X is normally distributed with $\mu = 40$ and $\sigma = 6$.

$$\begin{array}{ll}
 a \quad P(30.00 < x < 50.00) & b \quad P(x \geq 50) \\
 = \text{normalcdf}(30, 50, 40, 6) & = \text{normalcdf}(50, \text{E99}, 40, 6) \\
 \doteq 0.904 & \doteq 0.0478 \\
 \doteq 90.4\% & \doteq 4.78\%
 \end{array}$$

- 3 Let X be the result of the Physics test.
Then X is normally distributed with $\mu = 46$ and $\sigma = 25$.

$$\begin{aligned}
 & \text{We need to find } k \text{ such that } P(x \geq k) = 0.07 \\
 & \text{i.e., } 1 - P(x \leq k) = 0.07 \\
 & \text{i.e., } P(x \leq k) = 0.93 \\
 & \therefore k = \text{invNorm}(0.93, 46, 25) \\
 & \therefore k = 82.894\dots\dots \\
 & \therefore k \doteq 83 \quad \{k \text{ assumed to be an integer}\}
 \end{aligned}$$

i.e., lowest score would be 83 to get an A.

- 4 Let X be the length of an eel (in cm)
Then X is normally distributed with $\mu = 41$ and $\sigma = \sqrt{11}$

$$\begin{array}{ll}
 a \quad P(x \geq 50) & b \quad P(40 \leq x \leq 50) \\
 = \text{normalcdf}(50, \text{E99}, 41, \sqrt{11}) & = \text{normalcdf}(40, 50, 41, \sqrt{11}) \\
 \doteq 0.00333 & \doteq 0.615 \\
 & \doteq 61.5\% \\
 c \quad P(x \geq 45) & \\
 = \text{normalcdf}(45, \text{E99}, 41, \sqrt{11}) & \\
 \doteq 0.114 & \therefore \text{we expect } 200 \times 0.114 \doteq 23 \text{ eels}
 \end{array}$$

5

$$\begin{aligned}
 P(X \geq 35) &= 0.32 & P(X \leq 8) &= 0.26 \\
 \therefore P(X \leq 35) &= 0.68 & \therefore P\left(\frac{X - \mu}{\sigma} \leq \frac{8 - \mu}{\sigma}\right) &= 0.26 \\
 \therefore P\left(\frac{X - \mu}{\sigma} \leq \frac{35 - \mu}{\sigma}\right) &= 0.68 & \therefore P\left(Z \leq \frac{8 - \mu}{\sigma}\right) &= 0.26 \\
 \therefore P\left(Z \leq \frac{35 - \mu}{\sigma}\right) &= 0.68 & \therefore \frac{8 - \mu}{\sigma} &= \text{invNorm}(0.26) \\
 \therefore \frac{35 - \mu}{\sigma} &= \text{invNorm}(0.68) & \therefore 8 - \mu &\div -0.6433\sigma \dots\dots (2) \\
 \therefore \frac{35 - \mu}{\sigma} &\div 0.4677 & & \\
 \therefore 35 - \mu &\div 0.4677\sigma \dots\dots (1) & &
 \end{aligned}$$

Solving (1) and (2), $35 - 0.4677\sigma = 8 + 0.6433\mu$

$$\begin{aligned}
 \therefore 27 &= 1.111\sigma \\
 \therefore \sigma &\div 24.3 \quad \text{and} \quad \mu = 35 - 0.4677 \times 24.3 \\
 &\quad \text{i.e., } \mu \div 23.6 \\
 \text{i.e., } \mu &\div 23.6 \quad \text{and} \quad \sigma \div 24.3
 \end{aligned}$$

6 a Let the mean be μ and standard deviation be σ .

$$\begin{aligned}
 \text{Then } P(x \geq 80) &= 0.1 & \text{and } P(x \leq 30) &= 0.15 \\
 \therefore P(x \leq 80) &= 0.9 & \therefore P\left(\frac{x - \mu}{\sigma} \leq \frac{30 - \mu}{\sigma}\right) &= 0.15 \\
 \therefore P\left(\frac{x - \mu}{\sigma} \leq \frac{80 - \mu}{\sigma}\right) &= 0.9 & \therefore P\left(z \leq \frac{30 - \mu}{\sigma}\right) &= 0.15 \\
 \therefore P\left(z \leq \frac{80 - \mu}{\sigma}\right) &= 0.9 & \therefore \frac{30 - \mu}{\sigma} &= \text{invNorm}(0.15) \\
 \therefore \frac{80 - \mu}{\sigma} &= \text{invNorm}(0.9) & \therefore 30 - \mu &\div -1.0364\sigma \dots\dots (2) \\
 \therefore 80 - \mu &\div 1.2816\sigma \dots\dots (1) & &
 \end{aligned}$$

$$\begin{aligned}
 \text{From (1) and (2)} \quad (80 - \mu) - (30 - \mu) &\div 1.2816\sigma + 1.0364\sigma \\
 50 &\div 2.318\sigma
 \end{aligned}$$

$$\therefore \sigma \div \frac{50}{2.318} \div 21.57$$

$$\text{and in (1)} \quad 80 - \mu \div 1.2816 \times 21.57 \div 27.6$$

$$\therefore \mu \div 52.4$$

$$\text{i.e., } \mu \div 52.4 \quad \text{and} \quad \sigma \div 21.6$$

b X is the result of the Maths Exam. X is normally distributed with mean μ and standard deviation σ .Then $P(x \geq 80) = 0.1$ and $P(x \leq 30) = 0.15$ So, from **a** $\mu \div 52.4$ and $\sigma \div 21.6$

If part marks can be given,

$$\begin{aligned}
 P(x > 50) \\
 &= \text{normalcdf}(50, E99, 52.4, 21.6) \\
 &\div 0.544 \\
 &\div 54.4\%
 \end{aligned}$$

or For integer marks only,

$$\begin{aligned}
 P(x \geq 51) \\
 &= \text{normalcdf}(51, E99, 52.4, 21.6) \\
 &\div 0.526 \\
 &\div 52.6\%
 \end{aligned}$$

- 7 a** Let the mean be μ and standard deviation be σ and X be the diameter (in cm).

$$\begin{aligned} \therefore P(x < 1.94) &= 0.02 \quad \text{and} \quad P(x > 2.06) = 0.03 \\ \therefore P\left(\frac{x - \mu}{\sigma} < \frac{1.94 - \mu}{\sigma}\right) &= 0.02 & \therefore P\left(\frac{x - \mu}{\sigma} > \frac{2.06 - \mu}{\sigma}\right) &= 0.03 \\ \therefore P\left(z < \frac{1.94 - \mu}{\sigma}\right) &= 0.02 & \therefore P\left(z > \frac{2.06 - \mu}{\sigma}\right) &= 0.03 \\ \therefore \frac{1.94 - \mu}{\sigma} &= \text{invNorm}(0.02) & \text{i.e., } P\left(z < \frac{2.06 - \mu}{\sigma}\right) &= 0.97 \\ \therefore 1.94 - \mu &\div -2.054\sigma \dots\dots (1) & \frac{2.06 - \mu}{\sigma} &= \text{invNorm}(0.97) \\ & & 2.06 - \mu &\div 1.881\sigma \dots\dots (2) \end{aligned}$$

$$\text{From (1) and (2)} \quad (2.06 - \mu) - (1.94 - \mu) = 1.881\sigma + 2.054\sigma$$

$$\therefore 3.935\sigma = 0.12$$

$$\therefore \sigma \div 0.0305$$

$$\text{and in (1)} \quad 1.94 - \mu = -2.054 \times 0.0305 \div -0.0626$$

$$\therefore \mu \div 2.00$$

$$\text{i.e., } \mu \div 2.00 \quad \text{and} \quad \sigma \div 0.0305$$

- b** This is a binomial situation with the probability $p = 0.02 + 0.03 = 0.05$ of failure to operate and $n = 20$

$$\therefore P(\text{less than 2 will operate})$$

$$= P(x \leq 1)$$

$$= \text{binomcdf}(20, 0.05, 1)$$

$$\div 0.736$$

REVIEW SET 30A

- 1 a** $P(x) = \frac{a}{x^2 + 1}$ for $a = 0, 1, 2, 3$

x_i	0	1	2	3
$P(x_i)$	a	$\frac{a}{2}$	$\frac{a}{5}$	$\frac{a}{10}$

$$\text{Now } a + \frac{a}{2} + \frac{a}{5} + \frac{a}{10} = 1 \quad \{\text{as } \sum P(x_i) = 1\}$$

$$\therefore 10a + 5a + 2a + a = 10 \quad \{\times \text{ each term by } 10\}$$

$$\therefore 18a = 10$$

$$\therefore a = \frac{5}{9}$$

$$\begin{aligned} \text{b } P(x \geq 1) &= P(x = 1, x = 2 \text{ or } x = 3) & \text{or } P(x \geq 1) &= 1 - P(x < 1) \\ &= P(x = 1) + P(x = 2) + P(x = 3) & &= 1 - P(x = 0) \\ &= \frac{5}{18} + \frac{1}{9} + \frac{5}{90} & &= 1 - \frac{5}{9} \\ &= \frac{4}{9} & &= \frac{4}{9} \end{aligned}$$

$$\begin{aligned} \text{2 a } P(x) &= C_x^4 \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} & P(2) &= C_2^4 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = 0.375 \\ \therefore P(0) &= C_0^4 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 = 0.0625 & P(3) &= C_3^4 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1 = 0.25 \\ P(1) &= C_1^4 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = 0.25 & P(4) &= C_4^4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0 = 0.0625 \end{aligned}$$

i.e.,

x_i	0	1	2	3	4
$P(x_i)$	0.0625	0.25	0.375	0.25	0.0625

$$\mathbf{b} \quad \mu = \sum x_i P(x_i)$$

$$= 0 \times 0.0625 + 1 \times 0.25 + 2 \times 0.375 + 3 \times 0.25 + 4 \times 0.0625$$

$$= 2$$

$$\sigma = \sqrt{\sum (x_i - \mu)^2 P(x_i)}$$

$$= \sqrt{(-2)^2(0.0625) + (-1)^2(0.25) + 0^2(0.375) + 1^2(0.25) + 2^2(0.0625)}$$

$$= 1$$

3 X is the number of defectives. Then X is Bin (10, 0.18). $X = 0, 1, 2, 3, \dots, 10$.

$$\mathbf{a} \quad P(X = 1)$$

$$= \text{binompdf}(10, 0.18, 1)$$

$$\doteq 0.302$$

$$\mathbf{b} \quad P(X = 2)$$

$$= \text{binompdf}(10, 0.18, 2)$$

$$\doteq 0.298$$

$$\mathbf{c} \quad P(X \geq 2) = 1 - P(X \leq 1)$$

$$= 1 - \text{binomcdf}(10, 0.18, 1)$$

$$\doteq 0.561$$

4 Let X be the number of defective toothbrushes.

X is distributed Bin(120, 0.04), $n = 120$, $p = 0.04$, $q = 0.96$,

$$\mathbf{a} \quad \mu = np$$

$$= 120 \times 0.04$$

$$= 4.8$$

$$\mathbf{b} \quad \sigma = \sqrt{npq}$$

$$= \sqrt{120 \times 0.4 \times 0.6}$$

$$\doteq 2.15$$

5

Result	Pays
1, 3, 5	\$2
2	\$3
4	\$6
6	\$9

$$\mathbf{a} \quad \text{Expected return} = \frac{3}{6} \times \$2 + \frac{1}{6} \times \$3 + \frac{1}{6} \times \$6 + \frac{1}{6} \times \$9$$

$$= \frac{1}{6}(\$24)$$

$$= \$4$$

b For a \$5 amount to play the game the club expects a \$1 return/game
 $\therefore$ for 75 people, the return expected is \$75.

6 If X is the arm length random variable (in cm) then X is normally distributed with $\mu = 64$ and $\sigma = 4$.

$$\mathbf{a} \quad \mathbf{i} \quad P(60 < x < 72)$$

$$= \text{normalcdf}(60, 72, 64, 4)$$

$$\doteq 0.8186$$

$$\doteq 81.9\%$$

$$\mathbf{ii} \quad P(X > 60)$$

$$= \text{normalcdf}(60, \text{E99}, 64, 4)$$

$$\doteq 0.841$$

$$\doteq 84.1\%$$

$$\mathbf{b} \quad P(56 < x < 68) = \text{normalcdf}(56, 68, 64, 4)$$

$$\doteq 0.819$$

$$\doteq 81.9\%$$

7 X is the rod length (in mm)

X is normally distributed
 with mean μ and $\sigma = 3$

$$\text{Now } P(x < 25) = 0.02$$

$$\therefore P\left(\frac{x - \mu}{3} < \frac{25 - \mu}{3}\right) = 0.02$$

$$\therefore P\left(z < \frac{25 - \mu}{3}\right) = 0.02$$

$$\therefore \frac{25 - \mu}{3} = \text{invNorm}(0.02)$$

$$\therefore \frac{25 - \mu}{3} \doteq -2.0537$$

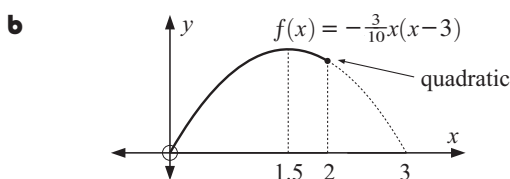
$$\therefore 25 - \mu \doteq -6.161$$

$$\therefore \mu \doteq 31.2$$

$$\begin{aligned}
 \mathbf{8} \quad \mathbf{a} \quad & \int_0^2 ax(x-3) dx = 1 \\
 \therefore & a \int_0^2 (x^2 - 3x) dx = 1 \\
 \therefore & a \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_0^2 = 1 \\
 \therefore & a \left[\frac{8}{3} - 6 \right] = 1 \\
 \therefore & a \left(-\frac{10}{3} \right) = 1 \\
 \therefore & a = -\frac{3}{10}
 \end{aligned}$$

iii If the median is m , then

$$\begin{aligned}
 & \int_0^m f(x) dx = \frac{1}{2} \\
 & \int_0^m -0.3x(x-3) dx = \frac{1}{2} \\
 \therefore & \int_0^m (x^2 - 3x) dx = -\frac{5}{3} \\
 & \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_0^m = -\frac{5}{3} \\
 \therefore & \frac{m^3}{3} - \frac{3m^2}{2} + \frac{5}{3} = 0 \\
 \therefore & 2m^3 - 9m^2 + 10 = 0 \\
 \therefore & m \doteq 1.24
 \end{aligned}$$



$$\begin{aligned}
 \mathbf{c} \quad \mathbf{i} \quad & \mu = \int_0^2 x f(x) dx \\
 & = \int_0^2 -\frac{3}{10} x^2(x-3) dx \\
 & = 1.2 \quad \{\text{technology}\}
 \end{aligned}$$

ii mode = 1.5 {at max. value of $f(x)$ }

$$\begin{aligned}
 \mathbf{iv} \quad & E(X^2) = \int_0^2 x^2 f(x) dx \\
 & = \int_0^2 -0.3x^3(x-3) dx \\
 & = 1.2
 \end{aligned}$$

$$\begin{aligned}
 \therefore \text{Var}(X) &= E(X^2) - \{E(X)\}^2 \\
 &= 1.68 - (1.2)^2 \\
 &\doteq 0.24
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{v} \quad & P(1 \leq x \leq 2) \\
 &= \int_1^2 -0.3x(x-3) dx \\
 &= 0.65
 \end{aligned}$$

REVIEW SET 30B

1 a $P(x_i) = k \left(\frac{3}{4}\right)^x \left(\frac{1}{4}\right)^{3-x}$ for $x = 0, 1, 2, 3$

$$P(0) = k \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^3 = \frac{k}{64}$$

$$P(1) = k \left(\frac{3}{4}\right)^1 \left(\frac{1}{4}\right)^2 = \frac{3k}{64}$$

$$P(2) = k \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^1 = \frac{9k}{64}$$

$$P(3) = k \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^0 = \frac{27k}{64}$$

x_i	0	1	2	3
$P(x_i)$	$\frac{k}{64}$	$\frac{3k}{64}$	$\frac{9k}{64}$	$\frac{27k}{64}$

Now $\frac{k}{64} + \frac{3k}{64} + \frac{9k}{64} + \frac{27k}{64} = 1$ {as $\sum p(x_i) = 1$ }

$$\therefore \frac{40k}{64} = 1$$

$$\therefore k = 1.6$$

$$\begin{aligned}
 \mathbf{b} \quad P(x \geq 1) &= 1 - P(x = 0) = 1 - \frac{k}{64} \\
 &= 1 - \frac{1.6}{64} \\
 &= 0.975
 \end{aligned}$$

2 X is the number of hits, then $X = 0, 1, 2, 3, 4$ and X is Bin $(4, 0.96)$

$$\begin{aligned}
 \mathbf{a} \quad & P(X = 4) \\
 &= \text{binompdf}(4, 0.96, 4) \\
 &\doteq 0.849
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{c} \quad & P(X \geq 3) \\
 &= 1 - P(X \leq 2) \\
 &= 1 - \text{binomcdf}(4, 0.96, 2) \\
 &\doteq 0.991
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & P(X = 0) \\
 &= \text{binompdf}(4, 0.96, 0) \\
 &\doteq 0.000
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & P(X = 1) \\
 &= \text{binompdf}(4, 0.96, 1) \\
 &\doteq 0.000246
 \end{aligned}$$

- 3** X is the contents of the container (in mL).

X is normally distributed with $\mu = 377$ and $\sigma = 4.2$

a i $P(X < 368.6)$

$$= \text{normalcdf}(-E99, 368.6, 377, 4.2)$$

$$\div 0.0228$$

$$\div 2.28\%$$

ii $P(372.8 < X < 389.6)$

$$= \text{normalcdf}(372.8, 389.6, 377, 4.2)$$

$$\div 0.840$$

$$\div 84.0\%$$

b $P(364.4 < X < 381.2) = \text{normalcdf}(364.4, 381.2, 377, 4.2)$

$$\div 0.840$$

- 4** X is the life of a battery (in weeks)

X is normally distributed with $\mu = 33.2$, $\sigma = 2.8$

a $P(X \geq 35) = \text{normalcdf}(35, E99, 33.2, 2.8)$

$$\div 0.260$$

b We need to find k such that $P(X \geq k) = 0.08$

$$\text{i.e., } P(X \leq k) = 0.92$$

$$\therefore k = \text{invNorm}(0.92, 33.2, 2.8)$$

$$\therefore k \div 37.134$$

So, the manufacturer can expect that no more than 8% will fail for a maximum of 37 weeks and 1 day, i.e., 260 days.

5

x_i	0	1	2	3	4
$P(x_i)$	0.10	0.30	0.45	0.10	k

a If this is a probability distribution then $\sum p(x_i) = 1$

$$\therefore 1 = 0.1 + 0.3 + 0.45 + 0.1 + k$$

$$\therefore k = 1 - 0.95$$

$$\therefore k = 0.05$$

b $\mu = \sum x_i p_i$

$$= 0(0.1) + 1(0.3) + 2(0.45) + 3(0.1) + 4(0.05)$$

$$= 0 + 0.3 + 0.9 + 0.3 + 0.2$$

$$= 1.7$$

$$\sigma^2 = \sum x_i^2 p_i - (\mu)^2$$

$$= 0^2(0.1) + 1^2(0.3) + 2^2(0.45) + 3^2(0.1) + 4^2(0.05) - (1.7)^2$$

$$= 0.3 + 1.8 + 0.9 + 0.8 - 2.89$$

$$= 0.91$$

$$\therefore \sigma = \sqrt{0.91} \div 0.954$$

- 6** Let X denote the number of cases of netballers needing knee surgery.

x is distributed $\text{Bin}(487, 0.0132)$, $n = 487$, $p = 0.0132$, $q = 0.9868$

$$\therefore \mu = np$$

$$= 487 \times 0.0132$$

$$\div 6.43$$

$$\sigma = \sqrt{npq}$$

$$= \sqrt{487 \times 0.0132 \times 0.9868}$$

$$\div 2.52$$

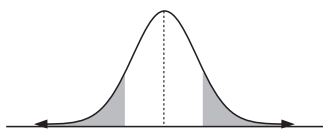
- 7** Let X denote the mass of a Coffin Bay Oyster. X is distributed normally with a mean of 38.6 and a standard deviation of 6.3.

a $P(38.6 - a \leq x \leq 38.6 + a) = 0.6826$

$$P\left(\frac{38.6 - a - 38.6}{6.3} \leq \frac{x - 38.6}{6.3} \leq \frac{38.6 + a - 38.6}{6.3}\right) = 0.6826$$

$$P\left(-\frac{a}{6.3} \leq z \leq \frac{a}{6.3}\right) = 0.6826$$

$$\therefore \text{ by symmetry, } P\left(z \leq -\frac{a}{6.3}\right) = \frac{1 - 0.6826}{2} = 0.1587$$



$$\therefore -\frac{a}{6.3} = \text{invNorm}(0.1587)$$

$$\therefore -\frac{a}{6.3} \doteq -0.9998 \quad \text{and} \quad \therefore a \doteq 6.3 \text{ gms}$$

b $P(x \geq b) = 0.8413$

$$\therefore P(x \leq b) = 0.1587$$

$$P\left(\frac{x - 38.6}{6.3} \leq \frac{b - 38.6}{6.3}\right) = 0.1587$$

$$\text{i.e., } P\left(z \leq \frac{b - 38.6}{6.3}\right) = 0.1587$$

$$\therefore \frac{b - 38.6}{6.3} \doteq \text{invNorm}(0.1587)$$

$$\therefore b - 38.6 \doteq 6.3 \times -0.9998$$

$$\therefore b \doteq -6.298 + 38.6$$

$$\therefore b \doteq 32.3 \text{ gms}$$

8 $f(x) = ax^2(2 - x)$ for $0 < x < 2$

a Since $f(x)$ is a probability distribution function the area under the curve is 1

$$\text{i.e., } \int_0^2 ax^2(2 - x) dx = 1$$

$$\therefore a \int_0^2 (2x^2 - x^3) dx = 1$$

$$\therefore a \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2 = 1$$

$$\therefore a \left(\frac{16}{3} - \frac{16}{4} \right) - 0 = 1$$

$$\therefore a \left(\frac{4}{3} \right) = 1$$

$$\therefore a = \frac{3}{4}$$

b The mode is the most frequently occurring score, i.e., the value of x when $f(x)$ is a maximum.

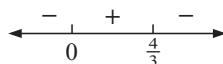
$$f(x) = \frac{3}{4}(2x^2 - x^3)$$

$$f'(x) = \frac{3}{4}(4x - 3x^2)$$

$$= \frac{3}{4}x(4 - 3x)$$

which has sign

diagram:



$\therefore$ is a maximum when $x = \frac{4}{3}$

$\therefore$ the mode is $\frac{4}{3}$.

c If the median is m , then,

$$\int_0^m \frac{3}{4}x^2(2 - x) dx = \frac{1}{2}$$

$$\frac{3}{4} \int_0^m (2x^2 - x^3) dx = \frac{1}{2}$$

$$\therefore \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^m = \frac{2}{3}$$

$$\therefore \frac{2m^3}{3} - \frac{m^4}{4} = \frac{2}{3}$$

$$\therefore m \doteq 1.2285$$

{using technology}

i.e., the median is approximately 1.23.

d $P(0.6 < x < 1.2) = \int_{0.6}^{1.2} \frac{3}{4}x^2(2 - x) dx$
 $= 0.3915$
 {using technology}

